

**WEAKLY CLEAN AND RELATED RINGS**

**QUA KIAT TAT**

**INSTITUTE OF MATHEMATICAL SCIENCES  
FACULTY OF SCIENCE  
UNIVERSITY OF MALAYA  
KUALA LUMPUR**

**2015**

**WEAKLY CLEAN AND RELATED RINGS**

**QUA KIAT TAT**

**THESIS SUBMITTED IN FULFILMENT OF THE  
REQUIREMENTS FOR THE DEGREE OF DOCTOR OF  
PHILOSOPHY**

**INSTITUTE OF MATHEMATICAL SCIENCES  
FACULTY OF SCIENCE  
UNIVERSITY OF MALAYA  
KUALA LUMPUR**

**2015**

**UNIVERSITY OF MALAYA**  
**ORIGINAL LITERARY WORK DECLARATION**

Name of Candidate: QUA KIAT TAT (I.C/Passport No: 841210-01-5187)

Registration/Matric No: SHB110001

Name of Degree: Doctor of Philosophy (Mathematics & Science Philosophy)

Title of Project Paper/Research Report/Dissertation/Thesis ("this Work"):

WEAKLY CLEAN AND RELATED RINGS

Field of Study: Ring Theory

I do solemnly and sincerely declare that:

- (1) I am the sole author/writer of this Work;
- (2) This Work is original;
- (3) Any use of any work in which copyright exists was done by way of fair dealing and for permitted purposes and any excerpt or extract from, or reference to or reproduction of any copyright work has been disclosed expressly and sufficiently and the title of the Work and its authorship have been acknowledged in this Work;
- (4) I do not have any actual knowledge nor do I ought reasonably to know that the making of this work constitutes an infringement of any copyright work;
- (5) I hereby assign all and every rights in the copyright to this Work to the University of Malaya ("UM"), who henceforth shall be owner of the copyright in this Work and that any reproduction or use in any form or by any means whatsoever is prohibited without the written consent of UM having been first had and obtained;
- (6) I am fully aware that if in the course of making this Work I have infringed any copyright whether intentionally or otherwise, I may be subject to legal action or any other action as may be determined by UM.

Candidate's Signature

Date:

Subscribed and solemnly declared before,

Witness's Signature

Date:

Name:

Designation:

# Abstract

Let  $R$  be an associative ring with identity. Let  $Id(R)$  and  $U(R)$  denote the set of idempotents and the set of units in  $R$ , respectively. An element  $x \in R$  is said to be weakly clean if  $x$  can be written in the form  $x = u + e$  or  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . If  $x$  is represented uniquely in this form, whether  $x = u + e$  or  $x = u - e$ , then  $x$  is said to be uniquely weakly clean. We say that  $x \in R$  is pseudo weakly clean if  $x$  can be written in the form  $x = u + e + (1 - e)rx$  or  $x = u - e + (1 - e)rx$  for some  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$ . For any positive integer  $n$ , an element  $x \in R$  is  $n$ -weakly clean if  $x = u_1 + \cdots + u_n + e$  or  $x = u_1 + \cdots + u_n - e$  for some  $u_1, \dots, u_n \in U(R)$  and  $e \in Id(R)$ . The ring  $R$  is said to be weakly clean (uniquely weakly clean, pseudo weakly clean,  $n$ -weakly clean) if all of its elements are weakly clean (uniquely weakly clean, pseudo weakly clean,  $n$ -weakly clean). Let  $g(x)$  be a polynomial in  $Z(R)[x]$  where  $Z(R)$  denotes the centre of  $R$ . An element  $r \in R$  is  $g(x)$ -clean if  $r = u + s$  for some  $u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$  in  $R$ . The ring  $R$  is said to be  $g(x)$ -clean if all of its elements are  $g(x)$ -clean. In this dissertation we investigate weakly clean and related rings. We determine some characterisations and properties of weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean and  $g(x)$ -clean rings for certain types of  $g(x) \in Z(R)[x]$ . Some generalisations of results on clean and related rings are also obtained.

# Abstrak

Biar  $R$  suatu gelanggang dengan identiti. Biar  $Id(R)$  dan  $U(R)$  menandakan set semua idempoten dan set semua unit dalam  $R$ , masing-masing. Unsur  $x \in R$  dikatakan bersih secara lemah jika  $x$  boleh ditulis dalam bentuk  $x = u + e$  atau  $x = u - e$  bagi sesuatu  $u \in U(R)$  dan  $e \in Id(R)$ . Jika perwakilan  $x$  dalam bentuk tersebut adalah unik, sama ada  $x = u + e$  atau  $x = u - e$ , maka  $x$  dikatakan bersih secara lemah berunik. Kita katakan bahawa  $x \in R$  adalah bersih secara lemah pseudo jika  $x$  boleh ditulis dalam bentuk  $x = u + e + (1 - e)rx$  atau  $x = u - e + (1 - e)rx$  bagi sesuatu  $u \in U(R)$ ,  $e \in Id(R)$  dan  $r \in R$ . Bagi sebarang integer positif  $n$ , sesuatu unsur  $x \in R$  adalah bersih secara  $n$ -lemah jika  $x = u_1 + \dots + u_n + e$  atau  $x = u_1 + \dots + u_n - e$  bagi sesuatu  $u_1, \dots, u_n \in U(R)$  dan  $e \in Id(R)$ . Gelanggang  $R$  dikatakan bersih secara lemah (bersih secara lemah berunik, bersih secara lemah pseudo, bersih secara  $n$ -lemah) jika semua unsurnya adalah bersih secara lemah (bersih secara lemah berunik, bersih secara lemah pseudo, bersih secara  $n$ -lemah). Biar  $g(x)$  suatu polinomial dalam  $Z(R)[x]$  yang mana  $Z(R)$  menandakan pusat bagi  $R$ . Unsur  $r \in R$  adalah  $g(x)$ -bersih jika  $r = u + s$  bagi sesuatu  $u \in U(R)$  dan  $s \in R$  dengan  $g(s) = 0$  dalam  $R$ . Gelanggang  $R$  dikatakan  $g(x)$ -bersih jika semua unsur dalamnya adalah  $g(x)$ -bersih. Dalam disertasi ini, kita mengkaji gelanggang bersih secara lemah dan gelanggang berkaitan. Kita menentukan beberapa cirian dan sifat bagi gelanggang bersih secara lemah, bersih secara lemah pseudo, bersih secara lemah berunik, bersih secara  $n$ -lemah dan  $g(x)$ -bersih bagi jenis tertentu  $g(x) \in Z(R)[x]$ . Beberapa pengitlakan hasil untuk gelanggang bersih dan gelanggang berkaitan juga diperolehi.

# Acknowledgment

Foremost, I would like to express my sincerest gratitude to my supervisor, Professor Dr. Angelina Chin Yan Mui, for her support, encouragement, guidance and infinite patience throughout the course of my study. All the advice and suggestions that she has given me has made me a better researcher.

I would also like to thank all my friends, especially Hee Song, Jeffery, Kok Haur, Choung Min, Kok Bin, Ke Xin and Sheena. Many thanks to them for sharing and listening, and for their patience, tolerance and friendship. Finally, it is with deepest gratitude that I acknowledge the support of my family.

# Table of Contents

<b>Abstract</b>	iii
<b>Abstrak</b>	iv
<b>Acknowledgement</b>	v
<b>Chapter 1 Introduction and Preliminaries</b>	<b>1</b>
1.1 Introduction . . . . .	1
1.1.1 Clean and strongly clean rings . . . . .	1
1.1.2 Weakly clean rings . . . . .	4
1.1.3 Uniquely clean rings and uniquely strongly clean rings . . . . .	5
1.1.4 Pseudo clean rings . . . . .	7
1.1.5 $n$ -clean rings . . . . .	7
1.1.6 $g(x)$ -clean rings . . . . .	8
1.1.7 Some other related rings . . . . .	9
1.2 Thesis organisation . . . . .	14
<b>Chapter 2 Weakly Clean Rings</b>	<b>16</b>
2.1 Introduction . . . . .	16
2.2 Some characterisations of weakly clean rings . . . . .	17
2.3 Some properties of weakly clean rings . . . . .	22
2.4 Strongly weakly clean rings . . . . .	33

<b>Chapter 3</b>	<b>Uniquely Weakly Clean Rings</b>	<b>40</b>
3.1	Introduction . . . . .	40
3.2	Some properties of uniquely weakly clean rings . . . . .	41
3.3	Some characterisations of uniquely weakly clean rings . . . . .	45
3.4	Uniquely weakly clean group rings . . . . .	48
<b>Chapter 4</b>	<b>Some Results on <math>n</math>-Weakly Clean Rings</b>	<b>51</b>
4.1	Introduction . . . . .	51
4.2	Some properties of $n$ -weakly clean rings . . . . .	52
4.3	Some results on $n$ -weakly clean group rings . . . . .	61
4.4	Some results on $n$ -weakly clean matrices . . . . .	65
<b>Chapter 5</b>	<b>Pseudo Weakly Clean Rings</b>	<b>68</b>
5.1	Introduction . . . . .	68
5.2	An example . . . . .	69
5.3	Some characterisations of pseudo weakly clean rings . . . . .	71
5.4	Some properties of pseudo weakly clean rings . . . . .	76
5.5	Pseudo weakly clean in non-unital rings . . . . .	87
<b>Chapter 6</b>	<b>Some Results on <math>g(x)</math>-clean Rings</b>	<b>91</b>
6.1	Introduction . . . . .	91
6.2	Some properties of $g(x)$ -clean rings . . . . .	92
6.3	Some results on $x(x - c)$ -clean rings . . . . .	95
6.4	More equivalent conditions for $x(x - c)$ -clean rings . . . . .	104
<b>Chapter 7</b>	<b>Summary of Properties</b>	<b>112</b>
7.1	Corners . . . . .	112
7.2	Direct products . . . . .	113
7.3	Centres . . . . .	114



7.4 Homomorphic Images . . . . .	114
7.5 Polynomial rings . . . . .	115
7.6 Power series rings . . . . .	115
<b>Bibliography</b>	<b>116</b>

# Chapter 1

## Introduction and Preliminaries

### 1.1 Introduction

This dissertation is mainly concerned with weakly clean and related rings. All rings considered in this dissertation are associative with identity unless stated otherwise and all modules are unitary. For any ring  $R$ , by an  $R$ -module  $M$  we mean a right  $R$ -module and we sometimes write  $M$  as  $M_R$ . Given the ring  $R$ , let  $Z(R)$  denote the centre of  $R$ ,  $N(R)$  the set of nilpotent elements in  $R$ ,  $J(R)$  the Jacobson radical of  $R$ ,  $U(R)$  the set of units in  $R$  and  $Id(R)$  the set of idempotents in  $R$ . The notation  $M_n(R)$  as usual denotes the ring of  $n \times n$  matrices over  $R$  ( $n \geq 1$ ). In the remainder of this chapter, we shall give some background on some of the rings studied in this dissertation. We also discuss how clean rings are related to some other rings.

#### 1.1.1 Clean and strongly clean rings

Let  $R$  be a ring. An element  $x \in R$  is clean if  $x = u + e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . The ring  $R$  is a clean ring if all of its elements are clean. Clean rings were first introduced by Nicholson [50] as a class of exchange rings. An element  $x \in R$  is called exchange if there exists an idempotent  $e \in xR$  such that  $1 - e \in (1 - x)R$ . The ring  $R$  is said to be exchange if all of its elements are

exchange.

Nicholson [50, Proposition 1.8] observed an interesting relation between elements in  $R$ :

$$x \text{ is clean in } R \Rightarrow f - x \in R(x - x^2) \text{ for some } f^2 = f \in R. \quad (1.1)$$

An element  $x \in R$  satisfying the condition on the right-hand side of (1.1) is said to be suitable. The ring  $R$  is said to be suitable if all of its elements are suitable. Consequently, every clean ring is suitable. In [9], Camillo, *et al* have shown that every ring can be embedded in a clean ring. This implies that an investigation of clean rings can lead to information on other rings. In a later paper, Burgess and Raphael [7, Theorem 2.1] showed that every ring can be embedded as an essential ring extension of a clean ring; thus adding importance to the study of clean rings.

The ring  $R$  is semiperfect if  $R/J(R)$  is Artinian and every idempotent in  $R/J(R)$  can be lifted to an idempotent of  $R$ . It has been shown in [11, Theorem 9] and [32, Corollary 4] that every semiperfect ring is clean. Let  $n$  be a positive integer. The ring  $R$  is said to be  $n$ -good if every element of  $R$  can be written as a sum of  $n$  units in  $R$  (see [34]). Camillo and Yu [11, Proposition 10] have shown that if  $R$  is a clean ring with  $2 \in U(R)$ , then every element of  $R$  is the sum of a unit and a square root of 1. It follows that clean rings with 2 invertible are 2-good.

The ring  $R$  is said to be semipotent if every right (equivalently, left) ideal  $T \not\subseteq J(R)$  contains a nonzero idempotent (see [52]). Han and Nicholson [32, Proposition 1] proved that every clean ring is semipotent. In [32], Han and Nicholson proved that if  $e \in Id(R)$  and the corner rings  $eRe$  and  $(1 - e)R(1 - e)$

are clean, then  $R$  is also clean. This implies that the matrix ring  $M_n(R)$  over a clean ring is clean. However, corner rings of clean rings need not be clean. This has been shown by Ster [58] by constructing a non-clean corner ring of a clean ring.

There have been several other results relating matrices to clean rings. In [32], Han and Nicholson have stated that by using induction, it can be shown that for each integer  $n \geq 1$ , a ring  $R$  is clean if and only if the ring of all  $n \times n$  upper triangular matrices over  $R$  is clean. Khurana and Lam in [39] showed that for a commutative ring  $R$ , if  $a \in R$  is clean, then for any  $b \in R$ ,  $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$  is clean in  $M_2(R)$ . In [70, Theorem 2.9], Yang and Zhou showed that for a commutative local ring  $R$  with  $2 \in U(R)$ , the matrix ring  $M_t(R)$  is clean if and only if  $M_t(RC_2)$  is clean, where  $C_2$  is the cyclic group of order 2 and  $t \geq 2$ .

An element  $x \in R$  is strongly clean if  $x = u + e$  for some  $u \in U(R)$  and  $e \in Id(R)$  with  $eu = ue$ . The ring  $R$  is strongly clean if all of its elements are strongly clean. Strongly clean rings were first introduced by Nicholson in [51] as a natural generalisation of strongly  $\pi$ -regular rings. The ring  $R$  is said to be strongly  $\pi$ -regular if for each  $x \in R$  there exist a positive integer  $n$ , depending on  $x$ , and an element  $y \in R$  such that  $x^n = x^{n+1}y$  and  $xy = yx$ . Nicholson in [51, Theorem 1] showed that every strongly  $\pi$ -regular ring is strongly clean but the converse is not necessarily true.

“Is the centre of a clean ring also clean?” This problem was raised in a survey paper by Nicholson and Zhou [54]. Note that the centre of a regular ring is regular, as shown by Goodearl in [31]. In [36, Example 2.7], Hong, Kim and Lee gave an example to show that the centre of an exchange ring need not be exchange. However, in the same paper it was shown that the centre of an

abelian exchange ring is exchange (see [36, Corollary 2.6]). (A ring  $R$  is said to be abelian if all of its idempotents are central.) Since clean rings are exchange, it is of interest to know whether the centre of a clean ring is necessarily clean. Burgess and Raphael answered this in the negative in [7, Proposition 2.5]. Note, however, that a non-clean ring may have a clean centre. For example, in [36, Proposition 2.5], Hong, Kim and Lee proved that the centre of an exchange ring is clean, although it is known by [11] that an exchange ring need not be clean.

In the following we show that the centre of a strongly  $\pi$ -regular ring is strongly  $\pi$ -regular; hence, strongly clean and therefore, clean.

**Proposition 1.1.1.** *The centre of a strongly  $\pi$ -regular ring is strongly  $\pi$ -regular.*

**Proof.** Let  $R$  be a strongly  $\pi$ -regular ring and let  $x \in Z(R)$ . Then there exist a positive integer  $n$  and an element  $y \in R$  such that  $x^n = x^{n+1}y$  and  $xy = yx$ . Let  $z = x^n y^{n+1}$ . Then  $x^n = x^{n+1}y = x^{n+2}y^2 = \dots = x^{2n}y^n = x^{2n+1}y^{n+1} = x^{n+1}x^n y^{n+1} = x^{n+1}z$  and  $xz = zx$ . For any  $r \in R$ ,  $zr = (x^n y^{n+1})r = y^{n+1}r x^n = y^{n+1}r x^{2n+1}y^{n+1} = x^{2n+1}y^{n+1}r y^{n+1} = x^n r y^{n+1} = r x^n y^{n+1} = rz$ . Hence  $z \in Z(R)$  and it follows that  $Z(R)$  is strongly  $\pi$ -regular.  $\square$

### 1.1.2 Weakly clean rings

Let  $R$  be a ring. An element  $x \in R$  is weakly clean if  $x = u + e$  or  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . The ring  $R$  is weakly clean if all of its elements are weakly clean. Clearly,  $R$  is weakly clean if for any  $x \in R$ , either  $x$  or  $-x$  is clean. Weakly clean rings first appeared in Ahn's Ph.D. thesis [1]. Further work on these rings can be found in [2] where weakly clean analogues of several results on clean rings were obtained. It is also shown in [2] that if  $R$  is a weakly clean

ring with no nontrivial idempotents, then  $R$  has exactly two maximal ideals and  $2 \in U(R)$ .

An element  $x \in R$  is called weakly exchange if there exists an idempotent  $e \in xR$  such that  $1 - e \in (1 - x)R$  or  $1 - e \in (1 + x)R$ . The ring  $R$  is said to be weakly exchange if all of its elements are weakly exchange. It is clear that exchange elements are weakly exchange. In [22], Chin and Qua found an element-wise characterisation of abelian weakly clean rings. By checking carefully the proof of [22, Theorem 2.1], it follows that weakly clean elements are weakly exchange. A ring  $R$  is said to be NLI if for any  $x \in N(R)$  and  $y \in R$ ,  $xy - yx \in N(R)$ . In [63], Wei has proven that NLI weakly exchange rings are weakly clean. In particular, if  $R$  is an abelian ring, then  $R$  is weakly clean if and only if  $R$  is weakly exchange (by [63, Corollary 2.3]).

### 1.1.3 Uniquely clean rings and uniquely strongly clean rings

Let  $R$  be a ring. An element  $x \in R$  is uniquely clean if  $x = u + e$  for some  $u \in U(R)$  and  $e \in Id(R)$  and this representation is unique. The ring  $R$  is uniquely clean if all of its elements are uniquely clean. Uniquely clean rings were first considered by Anderson and Camillo [3] for the commutative case. In the non-commutative case, uniquely clean rings first appeared in a paper by Nicholson and Zhou [53]. The ring  $R$  is said to be local if  $R$  has a unique maximal right ideal. Nicholson and Zhou [53, Theorem 15] showed that a local ring  $R$  is uniquely clean if and only if  $R/J(R) \cong \mathbb{Z}_2$ . In [53, Lemma 4], Nicholson and Zhou also proved that every idempotent in a uniquely clean ring is central; thus a uniquely clean ring is strongly clean. Another consequence of this result is that

a uniquely clean ring is directly finite, where the ring  $R$  is said to be directly finite if for any  $a, b \in R$ ,  $ab = 1$  implies that  $ba = 1$ .

The ring  $R$  is said to be a Boolean ring if  $x^2 = x$  for all  $x \in R$ .  $R$  is said to be a left (respectively, right) quasi-duo ring if every maximal left (respectively, right) ideal of  $R$  is two-sided. In [53, Theorem 20], Nicholson and Zhou proved that  $R$  is uniquely clean if and only if  $R/J(R)$  is Boolean and idempotents lift uniquely modulo  $J(R)$ . In particular,  $R$  is Boolean if  $R$  is uniquely clean and  $J(R) = \{0\}$ . In [53, Proposition 23], Nicholson and Zhou also proved that every uniquely clean ring is left and right quasi-duo.

An element  $x$  in a ring  $R$  is said to be nil clean if  $x$  can be written as the sum of an idempotent and a nilpotent element of  $R$ . The ring  $R$  is said to be nil clean if every element in  $R$  is nil clean (see [28]). In [27], Danchev and McGovern have shown that if  $R$  is nil clean, then it is uniquely clean and therefore clean. The converse is not true by considering the ring  $R = \mathbb{Z}_{(2)}$ , where  $\mathbb{Z}_{(2)}$  is the localization of the integers at the prime 2.

An element  $x$  in the ring  $R$  is said to be uniquely strongly clean if  $x = u + e$  uniquely for some  $u \in U(R)$  and  $e \in Id(R)$  with  $eu = ue$ . That is, if  $x = u_1 + e_1 = u_2 + e_2$  for some  $u_1, u_2 \in U(R)$  and  $e_1, e_2 \in Id(R)$  with  $e_i u_i = u_i e_i$  ( $i = 1, 2$ ), then  $u_1 = u_2$  and  $e_1 = e_2$ . The ring  $R$  is uniquely strongly clean if all of its elements are uniquely strongly clean. The notion of uniquely strongly clean rings first appeared in a paper by Wang and Chen [60]. Further work on these rings can be found in a paper by Chen, Wang and Zhou [18].

In [18, Example 4], Chen, Wang and Zhou showed that uniquely clean rings are uniquely strongly clean and the converse holds when idempotents are central. In [18, Example 8], Chen, Wang and Zhou also gave an example of a ring that

is uniquely strongly clean but not uniquely clean and in [18, Corollary 7], they showed that a strongly clean ring is not necessarily uniquely strongly clean.

#### 1.1.4 Pseudo clean rings

Let  $R$  be a ring. An element  $x \in R$  is said to be pseudo clean if there exist  $e \in Id(R)$  and  $u \in U(R)$  such that  $x - e - u \in (1 - e)Rx$ . The ring  $R$  is pseudo clean if all of its elements are pseudo clean. Clearly, every clean ring is pseudo clean. Pseudo clean rings were first introduced by Ster in [58] as a subclass of exchange rings. In the same paper, Ster also constructed an example of a non-clean pseudo clean ring (see [58, Example 3.1]). By using the same example, Ster also showed that corners of clean rings need not be clean. Further work on pseudo clean rings can be found in [59]. In [59], Ster also considered the notion of pseudo clean in non-unital rings and gave an example of a non-pseudo clean exchange ring.

#### 1.1.5 $n$ -clean rings

Let  $R$  be a ring. Given a positive integer  $n$ , an element  $x \in R$  is said to be  $n$ -clean if  $x = u_1 + \cdots + u_n + e$  for some  $u_1, \dots, u_n \in U(R)$  and  $e \in Id(R)$ . The ring  $R$  is said to be  $n$ -clean if each of its elements is  $n$ -clean. The notion of  $n$ -cleanness first appeared in [67].

In [68], Xiao and Tong obtained the following result which tells us that being clean implies being  $n$ -clean for any positive integer  $n$ .

**Proposition 1.1.2.** *[68, Lemma 2.1] Let  $R$  be a ring and let  $n, m$  be positive integers with  $n < m$ . If  $R$  is  $n$ -clean, then  $R$  is also  $m$ -clean.*

In [67, Corollary 2.12], Xiao and Tong showed that the ring of square matrices



over an  $n$ -clean ring is  $n$ -clean. They also showed that for each integer  $t \geq 2$ , the ring  $UT_t(R)$  (respectively,  $LT_t(R)$ ) of all  $t \times t$  upper (respectively, lower) triangular matrices over the ring  $R$  is  $n$ -clean if and only if  $R$  is  $n$ -clean.

### 1.1.6 $g(x)$ -clean rings

Let  $R$  be a ring and let  $g(x)$  be a polynomial in  $Z(R)[x]$ . In [61], an element  $r \in R$  is called  $g(x)$ -clean if  $r = u + s$  for some  $u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ .  $R$  is said to be  $g(x)$ -clean if every element of  $R$  is  $g(x)$ -clean. In the same paper, Wang and Chen [61] proved that if  $g_1(x) = (x - a)(x - b)$  with  $a, b \in Z(R)$ , then  $R$  is  $g_1(x)$ -clean if and only if  $R$  is clean and  $b - a$  is a unit (see [61, Theorem 2.1]). Now let  $g_1(x) = (x - a)(x - b)$  and  $g_2(x) \in (x - a)(x - b)Z(R)[x]$ , where  $a, b \in Z(R)$  and  $b - a \in U(R)$ . By [61, Remark 2.3],  $R$  is clean if and only if  $R$  is  $g_1(x)$ -clean, and in this case,  $R$  is also  $g_2(x)$ -clean. On the other hand, if  $R$  is  $g_2(x)$ -clean for any  $g_2(x) \in (x - a)(x - b)Z(R)[x]$ , then  $R$  is clean.

In [29], Fan and Yang investigated  $g(x)$ -clean rings where  $g(x) = x^n - x$  with  $n \in \mathbb{N}$ . They showed that  $(x^n - x)$ -clean rings are 2-clean (hence,  $m$ -clean for  $m \geq 2$ ) ([29, Proposition 4.5]). In the same paper, Fan and Yang gave an example to show that the polynomial ring over a  $g(x)$ -clean ring is not necessarily  $g(x)$ -clean. However, the formal power series ring over a  $g(x)$ -clean ring is  $g(x)$ -clean.

The ring  $R$  is said to be strongly  $g(x)$ -clean if every element  $r \in R$  can be written as  $r = u + s$  for some  $u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ , and  $us = su$ . The ring  $R$  is said to be  $(n, g(x))$ -clean if every element  $r \in R$  can be written as  $r = u_1 + \cdots + u_n + s$  for some  $u_1, \dots, u_n \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ . If every element  $r \in R$  can be written as  $r = u + s$  or  $r = u - s$  for some

$u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ , then  $R$  is said to be weakly  $g(x)$ -clean. The definitions of strongly  $g(x)$ -clean,  $(n, g(x))$ -clean and weakly  $g(x)$ -clean rings can in fact be found in [30], [33] and [4], respectively. These papers also contain results which are analogous to those on  $g(x)$ -clean rings obtained by Fan and Yang in [29].

### 1.1.7 Some other related rings

Let  $R$  be a ring. An element  $x \in R$  is said to be quasiregular if  $1 - x$  is a unit in  $R$ . Nilpotent elements are clearly quasiregular. We first note some examples of clean elements which have been stated in [48].

**Example 1.1.1.** Units, idempotents, quasiregular elements and nilpotent elements in a ring are clean.

By the example above, it follows that units, idempotents, quasiregular elements and nilpotent elements in a ring are weakly clean. The ring  $R$  is said to be a division ring if every non-zero element in  $R$  has a multiplicative inverse (that is, for any  $x \in R$  with  $x \neq 0$ , there exists an element  $y \in R$  with  $xy = yx = 1$ ). It follows by Example 1.1.1 that division rings, Boolean rings and local rings are clean (hence, weakly clean).

A ring  $R$  is said to be semisimple if  $R$  is Artinian and  $J(R) = \{0\}$ . It is well known that every semisimple ring is isomorphic to a finite direct product of full matrix rings over division rings. Since every division ring is clean and a full matrix ring over a clean ring is clean (by [32, Theorem]), we have that semisimple rings are clean.

**Proposition 1.1.3.** *A ring  $R$  is local if and only if it is clean and has no nontrivial idempotents.*

**Proof.** ( $\Rightarrow$ ): Let  $R$  be a local ring and let  $x \in R$ . If  $x \in J(R)$ , then  $x - 1 = -(1 - x) \in U(R)$ . Thus,  $x = 1 + (x - 1)$ . Hence,  $x$  is clean. If  $x \notin J(R)$ , then  $x \in U(R)$  and we have  $x = 0 + x$ . It follows that  $x$  is clean. Since  $x$  is arbitrary, this shows that  $R$  is clean. Next, if  $e^2 = e \in R$ , then  $e(e - 1) = 0$ . Since  $e$  or  $1 - e$  is a unit, hence,  $e = 0$  or  $e = 1$ .

( $\Leftarrow$ ): Let  $R$  be a clean ring with no nontrivial idempotents. Suppose that  $x \in R$  and  $x$  is not a unit. Since  $R$  is clean and the only idempotents in  $R$  are 0 and 1, so  $x = 1 + u$  for some unit  $u \in R$ . It follows that  $1 - x = -u \in U(R)$ . Therefore,  $R$  is a local ring.  $\square$

As a consequence of Proposition 1.1.3, we have the following:

**Corollary 1.1.1.** *Local rings are strongly clean.*

Strongly clean rings are however not necessarily local. For example, let  $R = UT_2(\mathbb{Z}_3)$ , the ring of  $2 \times 2$  upper triangular matrices over  $\mathbb{Z}_3$ . By [17, Theorem 3.9],  $R$  is strongly clean. Since  $R$  contains the nontrivial idempotent  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ , it follows by Proposition 1.1.3 that  $R$  is not a local ring.

Note that a commutative ring  $R$  is said to be quasilocal if it has a unique maximal ideal. By Proposition 1.1.3, quasilocal rings are clean (hence, weakly clean). This fact has also been proven in [3, Proposition 2(1)].

Let  $R$  be a ring. A right  $R$ -module  $M_R$  is said to have the exchange property if for any right  $R$ -module  $A_R$  and any two decompositions of  $A_R$ ,  $A_R = M'_R \oplus N_R = \bigoplus_{i \in I} A_i$  with  $M'_R \cong M_R$ , there exist submodules  $A'_i \subseteq A_i$  such that  $A_R = M'_R \oplus (\bigoplus_{i \in I} A'_i)$ . If this condition is satisfied whenever the index set  $I$  is finite, then the right  $R$ -module  $M_R$  is said to have the finite exchange property. The ring  $R$  is said to be an exchange ring if the right regular module  $R_R$  has

the finite exchange property and this definition is left-right symmetric. In [50], Nicholson has shown that  $R$  is an exchange ring if and only if idempotents can be lifted modulo every left (right) ideal of  $R$  if and only if  $R$  is suitable. Hence, every clean ring is an exchange ring. An exchange ring with all idempotents central is a clean ring (by [50, Proposition 1.8]).

The ring  $R$  is strongly exchange if for every  $x \in R$ , there exist  $e \in Id(R)$  and  $a, b \in R$  such that  $e = ax = xa$  and  $1 - e = b(1 - x) = (1 - x)b$ . In [19, Theorem 2.2], Chen has shown that strongly exchange rings are strongly clean; hence, clean.

**Proposition 1.1.4.** *An exchange ring with no nontrivial idempotents is local.*

**Proof.** Suppose that  $R$  is an exchange ring with no nontrivial idempotents. Hence  $R$  is clean by [13, Corollary 2.2]. It follows by Proposition 1.1.3 that  $R$  is local. □

An element  $x$  in the ring  $R$  is said to be right uniquely exchange if there exists a unique  $e \in Id(R)$  such that  $e \in xR$  and  $1 - e \in (1 - x)R$ . The ring  $R$  is said to be right uniquely exchange if all of its elements are right uniquely exchange. It has been shown by Lee and Zhou [41, Example 8] that every uniquely clean ring is a right uniquely exchange ring. They also showed that the converse is not necessarily true (see [41, Example 9]).

The ring  $R$  is said to be von Neumann regular (or just regular) if for any element  $x \in R$ , there exists  $y \in R$  such that  $x = xyx$ . In [3, Theorem 10], Anderson and Camillo have shown that a commutative regular ring is clean. The ring  $R$  is said to be strongly regular if for every  $x \in R$ , there exists  $y \in R$  such that  $x = x^2y$  and  $xy = yx$ . It is known that a ring is strongly regular if and

only if it is regular and abelian. It is clear that strongly regular rings are strongly  $\pi$ -regular. The ring  $R$  is said to be unit regular if for every element  $x \in R$ , there exists a unit  $u \in R$  such that  $x = xux$ . It is known that every strongly regular ring is unit regular (see [31]). Camillo and Khurana in [8, Theorem 1] have proved that every unit regular ring is clean (hence, weakly clean). Conversely, a weakly clean ring is unit regular if for any element  $x$ ,  $x = u + e$  or  $x = u - e$  in  $R$ ,  $xR \cap eR = \{0\}$ , where  $u \in U(R)$  and  $e \in Id(R)$  (see [22, Theorem 2.2]).

Let  $S$  be a subset of the ring  $R$ . If for any sequence of elements  $\{a_1, a_2, a_3, \dots\} \subseteq S$ , there exists an integer  $n \geq 1$  such that  $a_n \dots a_3 a_2 a_1 = 0$  ( $a_1 a_2 a_3 \dots a_n = 0$ ), then  $S$  is right (left)  $T$ -nilpotent. The ring  $R$  is said to be right (respectively, left) perfect if  $R/J(R)$  is semisimple and  $J(R)$  is right (respectively, left)  $T$ -nilpotent. A perfect ring is a ring which is both left and right perfect. Nicholson has stated that left (right) perfect rings are strongly clean in [51, Corollary], but the converse is not necessarily true.

The ring  $R$  is said to be right (respectively, left) topologically boolean if for every pair of distinct maximal right (respectively, left) ideals of  $R$  there is an idempotent in exactly one of them (see [21], or see [25] for the commutative case). In [47, Theorem 1.7], McGovern has shown that a commutative ring is clean if and only if it is topologically boolean. This was later extended by Chin [21, Theorem 3.1] who showed that an abelian ring is clean if and only if it is right (left) topologically boolean. A commutative ring  $R$  is said to be a  $pm$ -ring if every prime ideal of  $R$  is contained in a unique maximal ideal of  $R$ . In [3, Corollary 4], Anderson and Camillo showed that a commutative clean ring is a  $pm$ -ring.

We illustrate the relations between clean and other rings discussed above in

Figure 1. Some of the rings mentioned in Figure 1 will be defined in subsequent chapters.

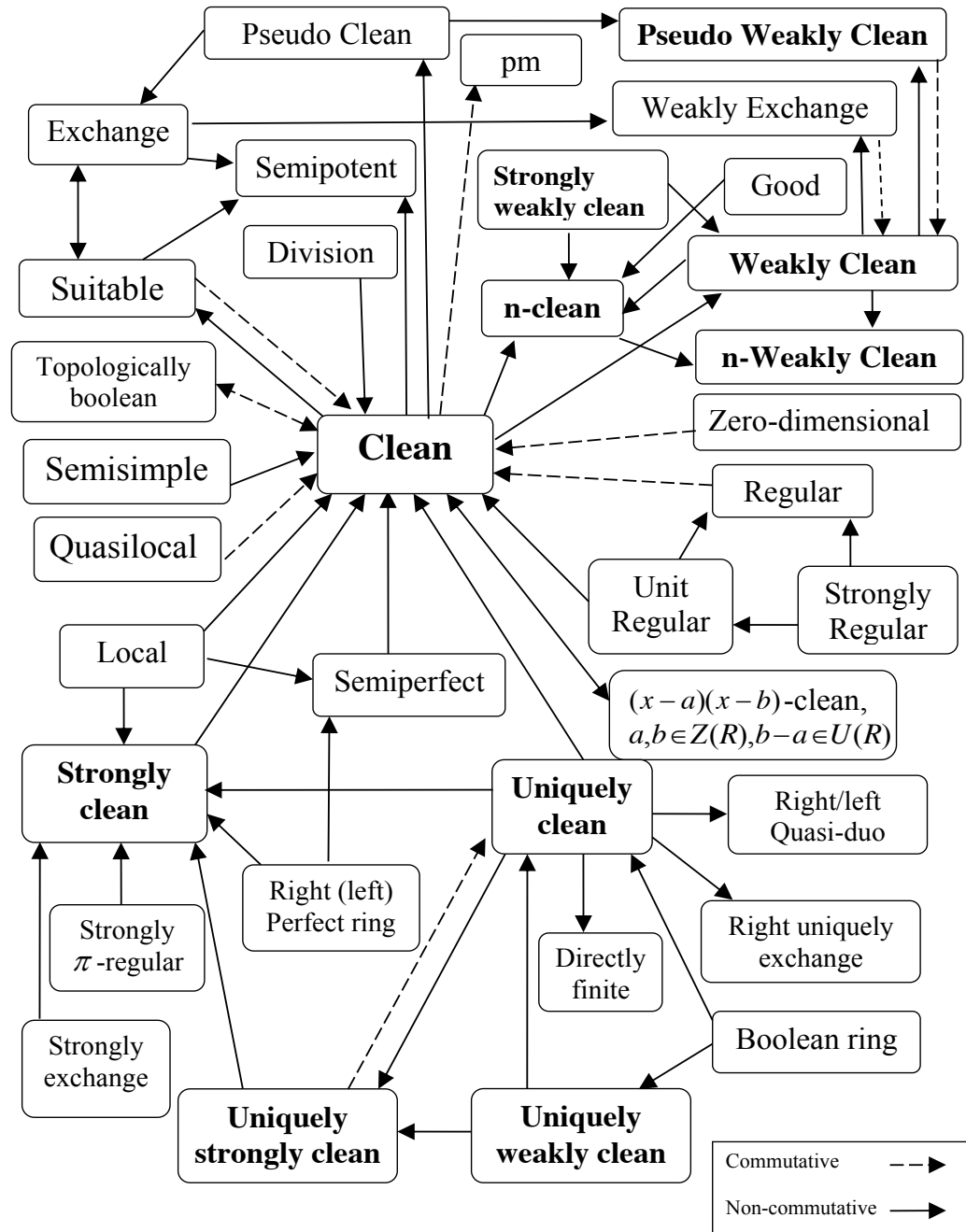


Figure 1: Clean and related rings

## 1.2 Thesis organisation

Let  $R$  be a ring. We give here a brief description of the succeeding chapters in this dissertation. In Chapter 2 we present some characterisations and properties of weakly clean rings. We also study centres of weakly clean rings and obtain some sufficient conditions for the centre of a weakly clean ring to be weakly clean. We also provide an example to show that the centre of a weakly clean ring is not necessarily weakly clean. In the last section of the chapter, we consider the notion of strongly weakly clean rings and determine some of their properties.

In Chapter 3 we study uniquely weakly clean rings. We obtain some properties and characterisations of uniquely weakly clean rings. We also extend some known results on uniquely clean group rings to uniquely weakly clean group rings.

In Chapter 4 we investigate  $n$ -weakly clean rings, where  $n$  is a positive integer. We extend some results on  $n$ -clean rings and weakly clean rings to  $n$ -weakly clean rings. We also give some necessary or sufficient conditions for a group ring to be  $n$ -weakly clean. In the last section of the chapter, we consider the  $n$ -weakly clean condition on matrices.

In Chapter 5 we define pseudo weakly clean rings. This class of rings contains the pseudo clean rings and we generalise some known results on pseudo clean rings to pseudo weakly clean rings. We also consider the pseudo weakly clean condition in non-unital rings.

In Chapter 6 we study  $g(x)$ -clean rings where  $g(x) \in Z(R)[x]$  and obtain some of their properties. We also define  $c$ -topologically boolean rings and show, via set-theoretic topology, that among conditions equivalent to  $R$  being an  $x(x - c)$ -clean ring where  $c \in U(R) \cap Z(R)$  is that  $R$  is right (left)  $c$ -topologically boolean.

Finally, in the last chapter, we give a summary of some basic properties of clean, weakly clean, pseudo weakly clean, uniquely weakly clean and  $n$ -weakly clean ( $n \geq 2$ ) rings.



# Chapter 2

## Weakly Clean Rings

### 2.1 Introduction

Let  $R$  be a ring. An element  $x \in R$  is weakly clean if  $x = u + e$  or  $x = u - e$  for some unit  $u$  and idempotent  $e$  in  $R$ . In other words, the element  $x \in R$  is weakly clean if either  $x$  or  $-x$  is clean. The ring  $R$  is weakly clean if all of its elements are weakly clean. Weakly clean rings first appeared in Ahn's Ph.D. thesis [1]. Further work on these rings can be found in [2] where weakly clean analogues of several results on clean rings were obtained. Clearly, clean rings are weakly clean but the converse is not necessarily true, as shown in the following example.

**Example 2.1.1.** Let  $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \{\frac{a}{b} \in \mathbb{Q} \mid 3 \nmid b, 5 \nmid b\}$ . It is clear that  $R$  has no nontrivial idempotents. Let  $x \in R$ . Then  $x = \frac{a}{b}$  for some  $a, b \in \mathbb{Z}$  where  $b \neq 0$ ,  $3 \nmid b$  and  $5 \nmid b$ . If  $\frac{a}{b}$  is a unit, then  $x = \frac{a}{b} = \frac{a}{b} + 0$  which is clean; hence, weakly clean. If  $\frac{a}{b}$  is not a unit, then it can be shown by some elementary number theory that either  $\frac{a}{b} - 1$  or  $\frac{a}{b} + 1$  is a unit. Therefore,  $x$  can be written as the sum or the difference of a unit and an idempotent as follows:

$$x = \begin{cases} \left(\frac{a}{b} - 1\right) + 1, & \text{if } \frac{a}{b} - 1 \text{ is a unit;} \\ \left(\frac{a}{b} + 1\right) - 1, & \text{if } \frac{a}{b} + 1 \text{ is a unit.} \end{cases}$$

Thus,  $x \in R$  is weakly clean. However, note that  $\frac{3}{8} \in R$  is not clean in  $R$  since  $\frac{3}{8} = \frac{3}{8} + 0 = -\frac{5}{8} + 1$  but  $\frac{3}{8}, \frac{5}{8}$  are not units in  $R$ .

Another example of a weakly clean ring is a nil clean ring. An element  $x$  in a ring  $R$  is said to be nil clean if  $x$  can be written as the sum of an idempotent and a nilpotent element of  $R$ . The ring  $R$  is said to be nil clean if every element in  $R$  is nil clean. Note that a nil clean element is weakly clean. Indeed, if  $x \in R$  is nil clean, then  $x = e + z$  for some  $e \in Id(R)$  and  $z \in N(R)$ . Let  $n$  be the smallest positive integer such that  $z^n = 0$ . Then  $(1+z)(1-z+z^2-z^3+\dots+(-1)^{n-1}z^{n-1}) = 1$ , that is,  $1+z \in U(R)$ . Thus,  $x = e + z = (1+z) - (1-e)$  is weakly clean in  $R$ .

In this chapter, we first present some characterisations of weakly clean rings in Section 2.2. In Section 2.3, we investigate some further properties of weakly clean rings. Among the questions that will be addressed in this section is whether the centre of a weakly clean ring is weakly clean. In Section 2.4, we define strongly weakly clean rings and obtain some properties of these rings.

## 2.2 Some characterisations of weakly clean rings

We begin with an element-wise characterisation of weakly clean rings.

**Proposition 2.2.1.** *Let  $R$  be a weakly clean ring. Then for any  $x \in R$  there exists  $e \in Id(R)$  such that  $e \in xR$  and  $1-e \in (1-x)R$  or  $1-e \in (1+x)R$ .*

**Proof.** Let  $x \in R$ . Then  $x = u + f$  or  $x = u - f$  for some idempotent  $f$  and unit  $u$  in  $R$ . Set  $e = u(1-f)u^{-1}$ . Then  $e \in Id(R)$ . For  $x = u + f$ , we have

$$\begin{aligned} (x-e)u &= (u+f-u(1-f)u^{-1})u = u^2 + fu - u + uf \\ &= x^2 - x. \end{aligned}$$

Therefore,

$$e = x - (x^2 - x)u^{-1} = x(1 - (x-1)u^{-1}) \in xR$$

and

$$1 - e = (1 - x) + (x^2 - x)u^{-1} = (1 - x)(1 - xu^{-1}) \in (1 - x)R.$$

For  $x = u - f$ ,

$$\begin{aligned} (x + e)u &= (u - f + u(1 - f)u^{-1})u = u^2 - fu + u - uf \\ &= x^2 + x. \end{aligned}$$

We thus have

$$e = (x^2 + x)u^{-1} - x = x((x + 1)u^{-1} - 1) \in xR$$

and

$$1 - e = (1 + x) - (x^2 + x)u^{-1} = (1 + x)(1 - xu^{-1}) \in (1 + x)R.$$

□

We next show that the converse of Proposition 2.2.1 also holds if  $R$  is an abelian ring. This result has in fact been proven in [22] but we provide a proof here for the sake of completeness.

**Theorem 2.2.1.** *Let  $R$  be an abelian ring. Then  $R$  is weakly clean if and only if for any  $x \in R$  there exists  $e \in Id(R)$  such that  $e \in xR$  and  $1 - e \in (1 - x)R$  or  $1 - e \in (1 + x)R$ .*

**Proof.** ( $\Rightarrow$ ): This follows by Proposition 2.2.1.

( $\Leftarrow$ ): Let  $x \in R$  and let  $e^2 = e \in xR$  with  $1 - e \in (1 - x)R$  or  $1 - e \in (1 + x)R$ .

Then  $e = xa'$  for some  $a' \in R$  and we have  $a'xa' = a'e = ea' = xa'^2$ . Let  $a = a'e$ .

Note that  $ae = (a'e)e = a'e = a$  and  $axa = (a'e)x(a'e) = (a'xa')e = (xa'^2)e =$

$(xa')a'e = ea'e = a'e = a$ . Since  $(ax)^2 = (axa)x = ax$ ,  $(xa)^2 = x(axa) = xa$  and

idempotents are central in  $R$ , then

$$ax = (axa)x = a(xa)x = (xa)ax = xa(ax) = x(ax)a = xa.$$

Suppose first that  $1 - e \in (1 - x)R$ . Then  $1 - e = (1 - x)b'$  for some  $b' \in R$ . By letting  $b = b'(1 - e)$ , we have  $b(1 - e) = b$ ,  $1 - e = (1 - x)b$  and  $b(1 - x) = (1 - x)b$ . Note that  $a - b$  is the inverse of  $x - (1 - e)$  because

$$\begin{aligned}
(a - b)(x - (1 - e)) &= ax - a(1 - e) - bx + b(1 - e) \\
&= xa - a + ae - bx + b \\
&= e - a + a + b(1 - x) \\
&= e + (1 - x)b \\
&= e + (1 - e) \\
&= 1 \\
&= (x - (1 - e))(a - b).
\end{aligned}$$

Therefore,  $x = (x - (1 - e)) + (1 - e)$  is clean; hence, weakly clean.

Now suppose that  $1 - e \in (1 + x)R$ . Then  $1 - e = (1 + x)c'$  for some  $c' \in R$ . By letting  $c = c'(1 - e)$ , we have  $c(1 - e) = c$ ,  $1 - e = (1 + x)c$  and  $c(1 + x) = (1 + x)c$ . Note that  $a + c$  is the inverse of  $x + (1 - e)$  because

$$\begin{aligned}
(a + c)(x + (1 - e)) &= ax + a(1 - e) + cx + c(1 - e) \\
&= xa + a - ae + cx + c = e + a - a + c(1 + x) \\
&= e + (1 + x)c = e + (1 - e) \\
&= 1 \\
&= (x + (1 - e))(a + c).
\end{aligned}$$

It follows that  $x = (x + (1 - e)) - (1 - e)$  is weakly clean.  $\square$

We now extend [20, Theorem 3.9] on clean elements to weakly clean elements.

**Proposition 2.2.2.** *Let  $R$  be a ring and let  $x \in R$ . Then the following statements are equivalent:*

(a)  $x$  is weakly clean.

(b) There exist  $e \in Id(R)$  and  $u \in U(R)$  such that  $e = uxe$  and  $1 - e = u(x - 1)(1 - e)$  or  $1 - e = u(x + 1)(1 - e)$ .

(c) There exist  $e \in Id(R)$  and  $u \in U(R)$  such that  $e = exu$  and  $1 - e = (1 - e)(x - 1)u$  or  $1 - e = (1 - e)(x + 1)u$ .

**Proof.** (a)  $\Rightarrow$  (b): Let  $x$  be weakly clean in  $R$ . Then  $x = v + f$  or  $x = v - f$  for some  $v \in U(R)$  and  $f \in Id(R)$ . It follows that  $xf = vf + f$  or  $xf = vf - f$ . If  $xf = vf + f$ , then  $(x - 1)f = vf$ . Hence,  $f = v^{-1}(x - 1)f$ . Similarly, if  $xf = vf - f$ , then  $(x + 1)f = vf$  and thus,  $f = v^{-1}(x + 1)f$ . Let  $e = 1 - f$  and  $u = v^{-1}$ . Then

$$1 - e = \begin{cases} u(x - 1)(1 - e) & \text{if } x = v + f, \\ u(x + 1)(1 - e) & \text{if } x = v - f. \end{cases} \quad (2.1)$$

For  $x = v + f$ , we have  $xf = vf + f$  and therefore,  $xf = vf + (x - v)$ . It follows that  $x(1 - f) = v(1 - f)$  and hence,  $1 - f = v^{-1}x(1 - f)$ . Similarly, for  $x = v - f$ , we have  $xf = vf - f$  and hence,  $xf = vf + (x - v)$ . It follows that  $x(1 - f) = v(1 - f)$ . Thus,  $1 - f = v^{-1}x(1 - f)$ , that is,  $e = uxe$ .

(b)  $\Rightarrow$  (a): Suppose that there exist  $e \in Id(R)$  and  $u \in U(R)$  such that  $e = uxe$  and  $1 - e = u(x - 1)(1 - e)$  or  $1 - e = u(x + 1)(1 - e)$ . If  $1 - e = u(x - 1)(1 - e)$ , then we have  $1 - e = ux - uxe - u + ue = ux - e - u + ue$ . It follows that  $ux = 1 + u - ue$ ; hence,  $x = u^{-1} + (1 - e)$ . Thus,  $x$  is clean (hence, weakly clean).

For  $1 - e = u(x + 1)(1 - e)$ , we have  $1 - e = ux - uxe + u - ue = ux - e + u - ue$ . It follows that  $ux = 1 - u + ue$ , hence,  $x = u^{-1} - (1 - e)$ . Thus,  $x$  is weakly clean.

(a)  $\Leftrightarrow$  (c): This may be proven using arguments similar to those in the proof of

(a)  $\Leftrightarrow$  (b). □

Let  $R$  be a ring. In [63], Wei defined  $R$  to be weakly Abel if  $eR(1-e) \subseteq J(R)$  for each  $e \in Id(R)$ . It was also noted in [63] that a weakly Abel weakly exchange ring is weakly clean. In [64], Wei defined the ring  $R$  to be generalized weakly symmetric (GWS) if for any  $x, y, z \in R$ ,  $xyz = 0$  implies  $yxz \in N(R)$ . By the proof of Theorem 2.13(a) in [64], it is known that GWS rings are weakly Abel. A GWS ring which is weakly exchange is therefore weakly clean. The ring  $R$  is said to be 2-primal if  $N(R) = P(R)$  where  $P(R)$  denotes the prime radical of  $R$  (see [5]). We say that  $R$  is  $NI$  if  $N(R)$  forms an ideal of  $R$  (see [42]). If  $MN(R) \subseteq M$  for each maximal left ideal  $M$  of  $R$ , then  $R$  is said to be left  $NQD$  (see [65]). By [63, Proposition 2.5], 2-primal and  $NI$ -rings are left  $NQD$ .

By Theorem 2.2.1, [63, Theorem 2.2] and [63, Corollary 2.4], we have the following corollary.

**Corollary 2.2.1.** *Let  $R$  be a weakly exchange ring. If  $R$  satisfies any one of the following, then  $R$  is weakly clean:*

- (a)  $R$  is abelian.
- (b)  $R$  is GWS.
- (c)  $R$  is weakly-abel.
- (d)  $R$  is left quasi-duo.
- (e)  $R$  is 2-primal.
- (f)  $R$  is  $NI$ .
- (g)  $R$  is left  $NQD$ .

We end this section with another characterisation of weakly clean elements.

**Proposition 2.2.3.** *Let  $R$  be a ring and let  $x \in R$ . Then  $x$  is weakly clean in  $R$  if and only if there exist an idempotent  $f \in R$  and a unit  $v \in R$  such that  $vx = fv + 1$  or  $vx + fv = 1$ .*

**Proof.** Let  $x \in R$  be weakly clean. Then  $x = u + e$  or  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . Suppose that  $x = u + e$ . Then by multiplying  $u^{-1}$  on the left of both sides of the equation, we get  $u^{-1}x = 1 + u^{-1}e = 1 + (u^{-1}eu)u^{-1}$ . Thus,  $vx = fv + 1$  where  $v = u^{-1} \in U(R)$  and  $f = u^{-1}eu \in Id(R)$ . Now suppose that  $x = u - e$ . Then by multiplying  $u^{-1}$  on the left of both sides of the equation, we get  $u^{-1}x = 1 - u^{-1}e = 1 - (u^{-1}eu)u^{-1}$ . Hence,  $vx + fv = 1$  where  $v = u^{-1} \in U(R)$  and  $f = u^{-1}eu \in Id(R)$ .

Conversely, suppose that there exist an idempotent  $f \in Id(R)$  and a unit  $v \in U(R)$  such that  $vx = fv + 1$  or  $vx + fv = 1$ . If  $vx = fv + 1$ , then we have  $x = v^{-1} + v^{-1}fv$ , where  $v^{-1}$  is a unit and  $v^{-1}fv$  is an idempotent. If  $vx + fv = 1$ , then we have  $x = v^{-1} - v^{-1}fv$ , where  $v^{-1}$  is a unit and  $v^{-1}fv$  is an idempotent. Hence,  $x$  is weakly clean. □

## 2.3 Some properties of weakly clean rings

We have seen in Example 2.1.1 that a weakly clean ring is not necessarily clean. However, a weakly clean ring is  $n$ -clean for  $n \geq 2$  as shown in the following:

**Proposition 2.3.1.** *A weakly clean ring is  $n$ -clean for  $n \geq 2$ .*

**Proof.** Let  $R$  be a weakly clean ring and let  $x \in R$ . Then  $x = u + e$  or  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . If  $x = u + e$ , then we may write  $x = u + (2e - 1) + (1 - e)$  which implies that  $x$  is 2-clean. If  $x = u - e$ , then

$x = u + (-1) + (1 - e)$  is also 2-clean. Since  $x$  is arbitrary, it follows that  $R$  is 2-clean and hence, by Proposition 1.1.2,  $R$  is  $n$ -clean for  $n \geq 2$ .  $\square$

Let  $k$  be a positive integer. A ring  $R$  is said to be  $k$ -good if every element in  $R$  can be written as a sum of  $k$  units in  $R$  (see [34]). In the following proposition we show that weakly clean rings with 2 invertible are either 2-good or 3-good.

**Proposition 2.3.2.** *Let  $R$  be a ring in which 2 is invertible. Then  $R$  is weakly clean if and only if for every element  $x \in R$ ,  $x = u + z$  or  $x = -2 + u - z$  for some  $u, z \in U(R)$  where  $z$  is a square root of 1.*

**Proof.** Let  $R$  be weakly clean. Then for  $x \in R$ , we have  $2^{-1}(x + 1) = v + e$  or  $2^{-1}(x + 1) = v - e$  for some  $v \in U(R)$  and  $e \in Id(R)$ . It follows that  $x = 2v + (2e - 1)$  or  $x = 2v - 2e - 1 = 2v - 2 - (2e - 1)$ . Let  $u = 2v$  and  $z = 2e - 1$ . Then  $u, z \in U(R)$  and  $z^2 = 1$ , as required. Conversely, for  $x \in R$ , we have  $2x - 1 = u + z$  or  $2x - 1 = -2 + u - z$  for some  $u, z \in U(R)$  with  $z^2 = 1$ . For  $2x - 1 = u + z$ , we have  $x = 2^{-1}u + 2^{-1}(z + 1)$ , where  $(2^{-1}(z + 1))^2 = 2^{-1}(z + 1) \in Id(R)$  and  $2^{-1}u \in U(R)$ . Thus,  $x$  is clean (hence, weakly clean). For  $2x - 1 = -2 + u - z$ , we have  $x = -1 + 2^{-1}u + 2^{-1}(1 - z) = 2^{-1}u - (1 - 2^{-1}(1 - z))$  where  $2^{-1}u \in U(R)$ . We note that  $(2^{-1}(1 - z))^2 = 2^{-1}(1 - z) \in Id(R)$ , thus  $1 - 2^{-1}(1 - z) \in Id(R)$ . It follows that  $x$  is weakly clean.  $\square$

For weakly clean rings where both 2 and 3 are invertible, we have the following:

**Proposition 2.3.3.** *Let  $R$  be a weakly clean ring with  $2, 3 \in U(R)$ . Then  $R$  is 2-good.*

**Proof.** Let  $x \in R$ . Then  $\frac{x+1}{2} = u + e$  or  $\frac{x+1}{2} = u - e$  for some unit  $u$  and idempotent  $e \in R$ . If  $\frac{x+1}{2} = u + e$ , then  $x = 2u + (2e - 1)$  where  $(2e - 1)^2 = 1$ .



If  $\frac{x+1}{2} = u - e$ , then  $x = 2u - (1 + 2e)$  where  $(1 + 2e)(1 - \frac{2}{3}e) = 1$ , that is,  $1 + 2e$  is a unit. In both cases,  $x$  is 2-good.  $\square$

**Proposition 2.3.4.** *Let  $R$  be a weakly clean ring with  $2 \in U(R)$ . Then  $R$  is 3-good.*

**Proof.** Let  $x \in R$ . Since  $R$  is weakly clean ring, we have  $x = u + e$  or  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . If  $x = u + e$ , then we may also write  $x = u + (1 + e) + (-1)$  where  $1 + e$  is a unit because  $(1 + e)(1 - \frac{1}{2}e) = 1$ . If  $x = u - e$ , then we have  $x = u + (-(1 + e)) + 1$  where  $-(1 + e)$  is a unit because  $-(1 + e)(-1 + \frac{1}{2}e) = 1$ . Thus,  $x$  is 3-good.  $\square$

In [2, Theorem 1.9], Ahn and Anderson have shown that polynomial rings are never weakly clean. In the same paper, it was also proven that power series rings over commutative weakly clean rings are weakly clean. We next state two more basic properties of weakly clean rings which have been proven by Ahn and Anderson in [2].

**Proposition 2.3.5.** *[2, Lemma 1.2] If  $R$  is weakly clean, then so is every homomorphic image of  $R$ .*

**Proposition 2.3.6.** *[2, Theorem 1.7] Let  $\{R_i\}$  be a family of commutative rings. Then the direct product  $\prod R_i$  of rings is weakly clean if and only if each  $R_i$  is weakly clean and at most one is not a clean ring.*

Let  $R$  be a ring and let  $I$  be an ideal of  $R$  with  $I \subseteq J(R)$ . It is known that  $R$  is clean if and only if  $R/I$  is clean and idempotents can be lifted modulo  $I$  (see [32]). It is natural to consider the corresponding lifting idempotent property for weakly clean rings. We first note the following two propositions:

**Proposition 2.3.7.** *Let  $R$  be a ring, let  $x \in R$  and let  $I$  be an ideal of  $R$ . The following conditions are equivalent:*

(a) *If  $x^2 - x \in I$  and  $x = u + e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , then there exists  $f^2 = f \in Id(R)$  such that  $f - x \in I$ .*

(b) *If  $x^2 + x \in I$  and  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , then there exists  $f^2 = f \in Id(R)$  such that  $f + x \in I$ .*

**Proof.** (a)  $\Rightarrow$  (b): Assume (a). Let  $x \in R$  such that  $x^2 + x \in I$  and  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . Then  $(-x)^2 - (-x) = x^2 + x \in I$  and  $-x = (-u) + e$ . By the assumption (a), it follows that there exists  $f^2 = f \in Id(R)$  such that  $f - (-x) \in I$ , that is,  $f + x \in I$ . Thus, (b) holds.

(b)  $\Rightarrow$  (a): Assume (b). Let  $x \in R$  such that  $x^2 - x \in I$  and  $x = u + e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . Then  $(-x)^2 + (-x) = x^2 - x \in I$  and  $(-x) = (-u) - e$ . By the assumption (b), it follows that there exists  $f^2 = f \in Id(R)$  such that  $f + (-x) \in I$ , that is,  $f - x \in I$ . Thus, (a) holds.

By using arguments similar to those in Proposition 2.3.7, we also have the following:

**Proposition 2.3.8.** *Let  $R$  be a ring, let  $x \in R$  and let  $I$  be an ideal of  $R$ . The following conditions are equivalent:*

(a) *If  $x^2 - x \in I$  and  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , then there exists  $f^2 = f \in Id(R)$  such that  $f - x \in I$ .*

(b) *If  $x^2 + x \in I$  and  $x = u + e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , then there exists  $f^2 = f \in Id(R)$  such that  $f + x \in I$ .*

Now, for weakly clean rings we have the following:

**Proposition 2.3.9.** *Let  $R$  be a weakly clean ring, let  $x \in R$  and let  $I$  be an ideal of  $R$ . If  $x^2 - x \in I$  and  $x = u + e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , then there exists  $f^2 = f \in Id(R)$  such that  $x - f \in I$ . Moreover, if  $x^2 + x \in I$  and  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , then there exists  $f'^2 = f' \in Id(R)$  such that  $x + f' \in I$ .*

**Proof.** For the first assertion, let  $f = u(1 - e)u^{-1}$ . Then  $f^2 = f$  and  $(x - f)u = (x - u(1 - e)u^{-1})u = eu + ue + u^2 - u = (u + e)^2 - (u + e) = x^2 - x \in I$ . It follows that  $x - f \in I$ . The second assertion follows from the first and Proposition 2.3.7. □

**Proposition 2.3.10.** *Let  $R$  be a ring and let  $I$  be an ideal of  $R$  such that  $I \subseteq J(R)$ . Then  $R$  is weakly clean if and only if  $R/I$  is weakly clean and for any  $x = u + e \in R$  such that  $x^2 - x \in I$  where  $u \in U(R)$  and  $e \in Id(R)$ , there exists  $f^2 = f \in R$  such that  $x - f \in I$ .*

**Proof.** ( $\Rightarrow$ ): Assume that  $R$  is weakly clean. Then so is  $R/I$ , being a homomorphic image of  $R$ . Let  $x = u + e \in R$  such that  $x^2 - x \in I$  where  $u \in U(R)$  and  $e \in Id(R)$ . By Proposition 2.3.9, we readily have that there exists  $f^2 = f \in Id(R)$  such that  $x - f \in I$ .

( $\Leftarrow$ ): For the converse, let  $x \in R$ . Then  $x + I \in R/I$  and since  $R/I$  is weakly clean, we have that  $x + I = u + e + I$  or  $x + I = u - e + I$  for some  $u + I \in U(R/I)$  and  $e + I \in Id(R/I)$ . Now since  $e^2 - e \in I$  and  $e = (2e - 1) + (1 - e)$  where  $2e - 1 \in U(R)$  and  $1 - e \in Id(R)$ , it follows by the assumption that there exists  $f^2 = f \in R$  such that  $e - f \in I$ . We thus have that  $(x - f) - u \in I \subseteq J(R)$  or  $(x + f) - u \in I \subseteq J(R)$ ,

that is,  $x - f + J(R) \in U(R/J(R))$  or  $x + f + J(R) \in U(R/J(R))$ . Hence,  $x - f$  or  $x + f$  is a unit in  $R$ . It follows that  $x = v + f$  or  $x = v - f$  for some  $v \in U(R)$ . Thus,  $x$  is weakly clean.  $\square$

In [58], it was shown that corners of clean rings need not be clean. We now consider corners of weakly clean rings. It is clear that if  $R$  is an abelian weakly clean ring, then so is  $eRe$  for any  $e \in Id(R)$ . On the other hand, being weakly clean in a corner of the ring  $R$  implies being weakly clean in  $R$ , as shown in the following:

**Proposition 2.3.11.** *Let  $R$  be a ring and let  $e \in Id(R)$ . If  $x \in eRe$  is weakly clean in  $eRe$ , then  $x \in eRe$  is weakly clean in  $R$ .*

**Proof.** Suppose that  $x \in eRe$  is weakly clean in  $eRe$ . Then  $x = v + f$  or  $x = v - f$ , where  $f^2 = f \in eRe$  and  $v \in eRe$  such that  $vw = e = wv$  for some  $w \in eRe$ . For  $x = v + f$ , let  $u = v - (1 - e)$ . Then  $u$  is a unit in  $R$  with  $u^{-1} = w - (1 - e)$ . Hence,  $x - u = x - (v - (1 - e)) = f + (1 - e)$ , an idempotent in  $R$ . For  $x = v - f$ , let  $u = v + (1 - e)$ . Then  $u$  is a unit in  $R$  with  $u^{-1} = w + (1 - e)$ . Hence,  $x - u = x - (v + (1 - e)) = -f - (1 - e) = -(f + (1 - e))$ , where  $f + (1 - e)$  is an idempotent in  $R$ . Thus,  $x$  is weakly clean in  $R$ .  $\square$

By Proposition 2.3.11 and the fact that corners of abelian weakly clean rings are weakly clean, we have the following:

**Corollary 2.3.1.** *Let  $R$  be an abelian ring and let  $e \in Id(R)$ . Then  $x \in eRe$  is weakly clean in  $R$  if and only if  $x \in eRe$  is weakly clean in  $eRe$ .*

We next investigate some conditions for a weakly clean ring to be clean. First we state the following result by Danchev [26, Proposition 2.6].

**Proposition 2.3.12.** [26, Proposition 2.6] *Suppose that  $R$  is a ring with  $2 \in J(R)$ . Then  $R$  is weakly clean if and only if  $R$  is clean.*

**Proposition 2.3.13.** *Let  $R$  be a weakly clean ring and let  $M$  and  $N$  be a pair of distinct maximal right ideals of  $R$ . If  $2 \in M$  or  $N$ , then there is an idempotent in exactly one of  $M$  or  $N$ .*

**Proof.** Without loss of generality, we assume that  $2 \in N$ . Let  $a \in M \setminus N$ . Then  $N + aR = R$  and hence,  $1 - ax \in N$  for some  $x \in R$ . Let  $r = ax$ . Then  $1 - r \in N$  and  $r \in M \setminus N$ . Since  $2 \in N$ , we have  $(1 + r) + N = (1 - r) + N = N$  and hence,  $1 + r \in N$ . Since  $R$  is weakly clean, there exist an idempotent  $e$  and a unit  $u$  in  $R$  such that  $r = u + e$  or  $r = u - e$ . If  $e \in M$ , then  $u = r - e \in M$  or  $u = r + e \in M$ . It follows that  $M = R$ ; a contradiction. Thus  $e \notin M$ . If  $e \notin N$ , then  $1 - e \in N$  and hence,  $u + N = r - e + N = r - 1 + N = N$  or  $u + N = r + e + N = r + 1 + N = N$ . But this implies that  $u \in N$ ; a contradiction. Thus, we have that  $e$  is an idempotent belonging to  $N$  only.  $\square$

By Proposition 2.3.12 (or Proposition 2.3.13) and the fact that being clean is equivalent to being topologically boolean in commutative rings, we readily have the following:

**Proposition 2.3.14.** *Let  $R$  be a commutative ring with  $\text{char } R = 2$ . The following are equivalent:*

- (a)  $R$  is clean.
- (b)  $R$  is weakly clean.
- (c)  $R$  is topologically boolean.

**Corollary 2.3.2.** *Let  $R$  be a weakly clean ring such that  $R$  has a maximal right ideal  $M$  with  $2 \in M$ . If  $R$  has no nontrivial idempotents, then  $R$  is clean.*

**Proof.** Suppose that  $R$  has another maximal right ideal  $M'$ . By Proposition 2.3.13, there exists  $e^2 = e \in R$  such that  $e \in M$ ,  $e \notin M'$ . Since  $R$  has no nontrivial idempotents,  $e = 0$  or  $1$ . If  $e = 0$ , then  $e \in M'$ ; a contradiction. If  $e = 1$ , then  $1 \in M$  and hence,  $M = R$ ; a contradiction. Therefore,  $R$  has exactly one maximal right ideal which implies that  $R$  is local; hence clean.  $\square$

Is the centre of a weakly clean ring also weakly clean? The corresponding question for clean rings has been raised in a survey paper by Nicholson and Zhou [54] and answered in the negative in [7, Proposition 2.5]. In the following we investigate some conditions under which the centre of a weakly clean ring is weakly clean and show that, in general, the centre of a weakly clean ring is not necessarily weakly clean.

First we note some of the obvious. A subring of a weakly clean ring need not be weakly clean. For example, the ring of rational numbers  $\mathbb{Q}$  is clean (hence, weakly clean) but the ring of integers  $\mathbb{Z}$  which is a subring of  $\mathbb{Q}$  is not weakly clean. A proper ideal  $I$  of a weakly clean ring  $R$  is never weakly clean; otherwise,  $I$  would contain a unit which contradicts the fact that  $I$  is proper.

We first consider some conditions under which the centre of a weakly clean ring is weakly clean. The next four lemmas are well-known, but we give a proof here for the sake of completeness.

**Lemma 2.3.1.** *Let  $R$  be a ring and let  $e \in Id(R)$ . Suppose that  $ex = 0$  if and only if  $xe = 0$  for all  $x \in R$ . Then  $e \in Z(R)$ .*

**Proof.** Let  $x \in R$ . Note that  $e(x - ex) = 0$  and  $(x - xe)e = 0$ . By the hypothesis,

$(x - ex)e = 0$  and  $e(x - xe) = 0$ . Hence,  $xe = exe = ex$ . Thus,  $e \in Z(R)$ .  $\square$

An element  $x \in R$  is said to be anti-commutative if  $xy = -yx$  for all  $y \in R$ .

**Lemma 2.3.2.** *Let  $R$  be a ring and let  $e$  be an anti-commutative idempotent in  $R$ . Then  $e \in Z(R)$ .*

**Proof.** Since  $e \in Id(R)$  is anti-commutative, we have  $ex = -xe$  for any  $x \in R$ . In particular, if  $ex = 0$ , then  $xe = -ex = 0$  and vice versa. It follows by Lemma 2.3.1 that  $e \in Z(R)$ .  $\square$

**Lemma 2.3.3.** *Let  $R$  be a ring. If  $N(R) \subseteq Z(R)$ , then  $R$  is abelian.*

**Proof.** Let  $e$  be an idempotent of  $R$ . Then for any  $x \in R$ ,

$$(ex - exe)^2 = exex - exexe - exe(ex) + (exe)(exe) = 0,$$

hence  $ex - exe \in N(R)$ . Since  $N(R) \subseteq Z(R)$ , we have  $e(ex - exe) = (ex - exe)e = 0$ , that is,  $ex = exe$ . Similarly,  $xe = exe$  for any  $x \in R$ . Thus,  $ex = xe$  for any  $x \in R$  and hence,  $e \in Z(R)$ .  $\square$

It is obvious that if the idempotents in a ring  $R$  are central and  $R$  is weakly clean, then the centre  $Z(R)$  is also weakly clean. In the following proposition, we obtain other conditions for the centre of a weakly clean ring to be weakly clean.

**Proposition 2.3.15.** *Let  $R$  be a weakly clean ring. Then the centre  $Z(R)$  of  $R$  is weakly clean if any one of the following conditions is satisfied:*

(a) *For all  $e \in Id(R)$  and  $x \in R$ ,  $ex = 0$  if and only if  $xe = 0$ .*

(b) *For all  $e \in Id(R)$ ,  $e$  is anti-commutative.*

(c)  *$N(R) \subseteq Z(R)$ .*

(d) *The idempotents in  $R$  commute with one another.*

(e)  *$R$  has no zero divisors.*

**Proof.** (a) Assume that (a) holds. Then by Lemma 2.3.1, we know that the idempotents in  $R$  are central. It thus follows that  $Z(R)$  is weakly clean.

(b) Assume that  $e$  is anti-commutative for all  $e \in Id(R)$ . Then it follows by Lemma 2.3.2 that  $e \in Z(R)$  for all  $e \in Id(R)$ . Hence,  $Z(R)$  is weakly clean.

(c) Assume that (c) holds. Then by Lemma 2.3.3 we know that the idempotents in  $R$  are central. Thus, as in parts (a) and (b), we have that  $Z(R)$  is weakly clean.

(d) Let  $e \in Id(R)$ . Note that for any  $x \in R$ , we have  $(e + ex(1 - e))^2 = e^2 + ex(1 - e) + ex(1 - e)e + ex(1 - e)ex(1 - e) = e + ex(1 - e)$  and similarly,  $(e + (1 - e)xe)^2 = e + (1 - e)xe$ . Thus,  $e + ex(1 - e)$  and  $e + (1 - e)xe$  are idempotents for all  $x$  in  $R$ . Since idempotents in  $R$  commute with one another, so  $e(e + ex(1 - e)) = (e + ex(1 - e))e$  and  $e(e + (1 - e)xe) = (e + (1 - e)xe)e$ . Expanding these and simplifying, we have that  $ex = exe$  and  $xe = exe$  for any  $x \in R$ . Hence,  $ex = xe$  for all  $x \in R$  which shows that every idempotent in  $R$  is central. It thus follows that  $Z(R)$  is weakly clean.

(e) Suppose that  $R$  has no zero divisors. Then  $Id(R) = \{0, 1\} \subseteq Z(R)$  and thus,  $Z(R)$  is weakly clean.

**Lemma 2.3.4.** *Let  $R$  be a ring. If  $U(R) \subseteq Z(R)$ , then  $N(R) \subseteq Z(R)$  and  $R$  is abelian.*

**Proof.** We first show that  $N(R) \subseteq Z(R)$ . Let  $x \in N(R)$ . Then  $x^n = 0$  for some  $n \in \mathbb{N}$  and we have

$$(1 - x)(1 + x + \cdots + x^{n-1}) = 1 = (1 + x + \cdots + x^{n-1})(1 - x),$$



that is,  $1 - x \in U(R) \subseteq Z(R)$ . It follows that for any  $y \in R$ ,  $(1 - x)y = y(1 - x)$  from which we have  $xy = yx$ . Hence,  $x \in Z(R)$ . By Lemma 2.3.3, it follows that  $R$  is abelian.  $\square$

**Proposition 2.3.16.** *Any weakly clean ring with commuting units is commutative.*

**Proof.** If  $R$  is a weakly clean ring with  $U(R) \subseteq Z(R)$ , then by Lemma 2.3.4,  $Id(R) \subseteq Z(R)$ . Then since every  $x \in R$  can be written as  $x = u + e$  or  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , it follows that  $R$  is commutative.  $\square$

In [7, Theorem 2.1], Burgess and Raphael showed that every ring can be embedded as an essential ring extension of a clean ring. By using an example in [7], we will show that the centre of a weakly clean ring is not necessarily weakly clean.

A ring  $R$  is a right (respectively, left) Kasch ring if every simple right (respectively, left)  $R$ -module can be embedded in  $R_R$  (respectively,  ${}_R R$ ). The ring  $R$  is called a Kasch ring if it is both right and left Kasch.

**Lemma 2.3.5.** *A ring which is its own complete ring of quotients is not necessarily weakly clean.*

**Proof.** Let  $S$  be a commutative ring which is not weakly clean and let  $M$  be the direct sum of a copy of each simple  $S$ -module. Then the trivial extension  $R = S \oplus M$  is a Kasch ring ([40, Proposition 8.30]), hence, its own complete ring of quotients. Since  $S$  is a homomorphic image of  $R$  and  $S$  is not weakly clean, it follows that  $R$  is also not weakly clean.  $\square$

For a ring  $R$ , let  $Q_{max}(R)$  denote the complete ring of quotients of  $R$ .

**Proposition 2.3.17.** *The centre of a weakly clean ring is not necessarily weakly clean.*

**Proof.** By taking  $S = \mathbb{Z}$  in Lemma 2.3.5, we have that the extension  $R = S \oplus M$  is a Kasch ring and therefore,  $Q_{max}(R) = R$  is not weakly clean. By [7, Proposition 2.4],  $Q_{max}(R)$  is the centre of a clean (hence, weakly clean) ring. This shows that the centre of a weakly clean ring is not necessarily weakly clean.  $\square$

## 2.4 Strongly weakly clean rings

Let  $R$  be a ring. An element  $x \in R$  is strongly weakly clean if  $x = u + e$  or  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$  such that  $ue = eu$ . The ring  $R$  is said to be strongly weakly clean if all of its elements are strongly weakly clean. Clearly, if  $x \in R$  is strongly weakly clean, then either  $x$  or  $-x$  is strongly clean.

In [19], Chen used the notion of strongly exchange rings to show that corners of strongly clean rings are strongly clean. Here, we give another (more elementary) proof of this result and use it to deduce that corners of strongly weakly clean rings are strongly weakly clean.

Let  $R$  be a ring and let  $x \in R$ . The left annihilator of  $x$  in  $R$  is  $ann_l(x) = \{r \in R | rx = 0\}$  whereas the right annihilator is  $ann_r(x) = \{r \in R | xr = 0\}$ .

**Proposition 2.4.1.** *Let  $R$  be a ring and let  $x \in R$  be strongly clean. If  $x = u + e$  for some  $u \in U(R)$ ,  $e \in Id(R)$  with  $ue = eu$ , then  $ann_l(x) \subseteq Re$  and  $ann_r(x) \subseteq eR$ .*

**Proof.** Let  $r \in ann_l(x)$ . Then  $0 = rx = r(u + e)$ , that is,  $ru = -re$  and hence,  $r = -reu^{-1} = -ru^{-1}e \in Re$ . Thus,  $ann_l(x) \subseteq Re$ . Similarly, it may be shown that  $ann_r(x) \subseteq eR$ .  $\square$

The following lemma is also well-known but we give a proof here for the sake of completeness.

**Lemma 2.4.1.** *Let  $R$  be a ring and let  $e \in Id(R)$ . Then  $ann_l(1 - e) = Re$  and  $ann_r(1 - e) = eR$ .*

**Proof.** Since  $e(1 - e) = 0 = (1 - e)e$ , the inclusions  $Re \subseteq ann_l(1 - e)$  and  $eR \subseteq ann_r(1 - e)$  clearly hold. For the reverse inclusion, if  $x \in ann_l(1 - e)$ , then  $0 = x(1 - e)$ , that is,  $x = xe \in Re$ . Hence,  $ann_l(1 - e) \subseteq Re$  and the equality  $ann_l(1 - e) = Re$  thus follows. Similarly, it may be shown that  $ann_r(1 - e) \subseteq eR$  and therefore,  $ann_r(1 - e) = eR$ .  $\square$

**Theorem 2.4.1.** *Let  $R$  be a ring and let  $e \in Id(R)$ . Then  $x \in eRe$  is strongly clean in  $R$  if and only if  $x \in eRe$  is strongly clean in  $eRe$ .*

**Proof.** ( $\Leftarrow$ ): This follows readily by [51, Proposition 3].

( $\Rightarrow$ ): Let  $x \in eRe$  be strongly clean in  $R$ . Then  $x = u + f$  for some  $u \in U(R)$  and  $f \in Id(R)$  such that  $uf = fu$ . Since  $x \in eRe$ , it is clear that  $(1 - e)x = 0 = x(1 - e)$ , that is,  $1 - e \in ann_l(x) \cap ann_r(x)$ . By Proposition 2.4.1 and Lemma 2.4.1, we have that  $ann_l(x) \subseteq Rf = ann_l(1 - f)$  and  $ann_r(x) \subseteq fR = ann_r(1 - f)$ . Thus,  $1 - e \in ann_l(1 - f) \cap ann_r(1 - f)$  and hence,  $(1 - e)(1 - f) = 0 = (1 - f)(1 - e)$ , that is,  $ef = fe$ . Then since  $(1 - e)x = 0 = x(1 - e)$  and  $x = u + f$ , we also have that  $eu = ue$ . Note that  $x = exe = eue + efe$  where  $eue \in U(eRe)$ ,  $(efe)^2 = (ef)^2 = ef = efe \in Id(eRe)$  and  $(eue)(efe) = eufe = efue = (efe)(eue)$ . Thus  $x$  is strongly clean in  $eRe$ .  $\square$

An immediate consequence of Theorem 2.4.1 is that corner rings of strongly clean rings are strongly clean. Other than Chen [19], this fact on strongly clean

rings has in fact been proven earlier by Sánchez Campos in her unpublished manuscript [55] (see also [71]). We also have the following corollary:

**Corollary 2.4.1.** *Let  $R$  be a strongly weakly clean ring. Then so is  $eRe$  for any  $e \in Id(R)$ .*

**Proof.** Let  $x \in eRe$ . Since  $R$  is strongly weakly clean, we have that either  $x$  or  $-x$  is strongly clean in  $R$ . By Theorem 2.4.1, either  $x$  or  $-x$  is strongly clean in  $eRe$ . It follows that  $x$  is strongly weakly clean in  $eRe$ .  $\square$

**Remark.** There exists a ring  $R$  with  $x \in R$  and  $e \in Id(R)$  such that  $x$  is strongly weakly clean in  $R$  but  $exe$  is not strongly weakly clean in  $eRe$ . An example is given as follows:

**Example 2.4.1.** Let  $R = M_2(\mathbb{Z})$ , the ring of  $2 \times 2$  matrices over  $\mathbb{Z}$ . Let  $x = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \in R$  and let  $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in Id(R)$ . Then  $x = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  where  $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \in U(R)$  with  $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$ . It follows that  $x$  is strongly clean (hence, strongly weakly clean) in  $R$ . Note that  $eRe = \left\{ \begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix} \mid c \in \mathbb{Z} \right\}$ . Consider the ring isomorphism  $eRe \cong \mathbb{Z}$  with  $\begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix} \mapsto c$ . Since  $U(\mathbb{Z}) = \{-1, 1\}$  and  $Id(\mathbb{Z}) = \{0, 1\}$ , it is easy to check that 3 is not strongly weakly clean in  $\mathbb{Z}$ . It follows that  $exe = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}$  is not strongly weakly clean in  $eRe$ .

**Proposition 2.4.2.** *Let  $R$  be a ring in which 2 is invertible. Then  $R$  is strongly weakly clean if and only if for every element  $x \in R$ ,  $x = u + z$  or  $x = -2 + u - z$  for some  $u, z \in U(R)$  where  $z$  is a square root of 1 such that  $uz = zu$ .*

**Proof.** Let  $R$  be strongly weakly clean. Then for  $x \in R$ , we have  $2^{-1}(x + 1) = v + e$  or  $2^{-1}(x + 1) = v - e$  for some  $v \in U(R)$ ,  $e \in Id(R)$  with  $ve = ev$ . It follows that  $x = 2v + (2e - 1)$  or  $x = 2v - 2e - 1 = -2 + 2v - (2e - 1)$ .

Let  $u = 2v$  and  $z = 2e - 1$ . Then  $u, z \in U(R)$ ,  $z^2 = 1$  and  $uz = zu$ , as required. Conversely, for  $x \in R$ , we have  $2x - 1 = u + z$  or  $2x - 1 = -2 + u - z$  for some  $u, z \in U(R)$  with  $z^2 = 1$  and  $uz = zu$ . For  $2x - 1 = u + z$ , we have  $x = 2^{-1}u + 2^{-1}(z+1)$ , where  $(2^{-1}(z+1))^2 = 2^{-1}(z+1) \in Id(R)$  and  $2^{-1}u \in U(R)$ . Since  $uz = zu$ , it follows that  $(2^{-1}u)(2^{-1}(z+1)) = (2^{-1}(z+1))(2^{-1}u)$ . Thus,  $x$  is strongly clean (hence, strongly weakly clean). For  $2x - 1 = -2 + u - z$ , we have  $x = -1 + 2^{-1}u + 2^{-1}(1 - z) = 2^{-1}u - (1 - 2^{-1}(1 - z))$  where  $2^{-1}u \in U(R)$ . We note that  $(2^{-1}(1 - z))^2 = 2^{-1}(1 - z) \in Id(R)$ , thus  $1 - 2^{-1}(1 - z) \in Id(R)$ . Since  $uz = zu$ , we therefore have that  $(2^{-1}u)(1 - 2^{-1}(1 - z)) = (1 - 2^{-1}(1 - z))(2^{-1}u)$ . It follows that  $x$  is strongly weakly clean.  $\square$

In [51], Nicholson asked whether every semiperfect ring is strongly clean and whether the matrix ring of a strongly clean ring is strongly clean. Wang and Chen (in [60]) answered both questions in the negative. It is natural to ask whether every semiperfect ring is strongly weakly clean and whether the matrix ring of a strongly weakly clean ring is strongly weakly clean. To answer these, we use the same example as in [60, Example 1].

**Example 2.4.2.** Let  $R = \{m/n \in \mathbb{Q} \mid n \text{ is odd}\}$ . Then  $M_2(R)$  is a semiperfect ring but it is not strongly weakly clean. Indeed, since  $R$  is a commutative local ring, it follows that  $R$  is semiperfect and strongly clean (hence, strongly weakly clean). Since semiperfect rings are Morita invariant, we have that  $M_2(R)$  is semiperfect. By direct computation, all nontrivial idempotents in the matrix ring  $M_2(R)$  have the form  $\begin{pmatrix} a & b \\ c & 1 - a \end{pmatrix}$ , where  $a, b, c \in R$  and  $bc = a - a^2$ . Consider  $\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} \in M_2(R)$ . Note that  $\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix}$ ,  $\begin{pmatrix} 7 & 6 \\ 3 & 6 \end{pmatrix}$  and  $\begin{pmatrix} 9 & 6 \\ 3 & 8 \end{pmatrix}$  are not units in  $M_2(R)$ . It follows that  $\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin U(M_2(R))$  and  $\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin$

$U(M_2(R))$ . We can write

$$\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 8-a & 6-b \\ 3-c & 6+a \end{pmatrix} + \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$$

or

$$\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 8+a & 6+b \\ 3+c & 8-a \end{pmatrix} - \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$$

where  $a, b, c \in R$  and  $bc = a - a^2$ . We first consider

$$\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 8+a & 6+b \\ 3+c & 8-a \end{pmatrix} - \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}.$$

Suppose that

$$\begin{pmatrix} 8+a & 6+b \\ 3+c & 8-a \end{pmatrix} \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \begin{pmatrix} 8+a & 6+b \\ 3+c & 8-a \end{pmatrix}.$$

Then

$$\begin{aligned} & \begin{pmatrix} (8+a)a + (6+b)c & (8+a)b + (6+b)(1-a) \\ (3+c)a + (8-a)c & (3+c)b + (8-a)(1-a) \end{pmatrix} \\ = & \begin{pmatrix} a(8+a) + b(3+c) & a(6+b) + b(8-a) \\ c(8+a) + (1-a)(3+c) & c(6+b) + (1-a)(8-a) \end{pmatrix}. \end{aligned}$$

By comparing the (1, 1)-entry and the (2, 1)-entry on both sides, we obtain  $b = 2c$  and  $6a = 3 + c$ , respectively. By substituting,  $b = 2c$  and  $6a = 3 + c$  into the equation  $bc = a - a^2$ , we have  $73a^2 - 73a + 18 = 0$  which has no solutions in  $R$ .

By using similar arguments for the case

$$\begin{pmatrix} 8 & 6 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 8-a & 6-b \\ 3-c & 6+a \end{pmatrix} + \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix},$$

we will obtain the same equation  $73a^2 - 73a + 18 = 0$  which has no solutions in  $R$ . Hence,  $M_2(R)$  is not strongly weakly clean.

A ring  $R$  is said to be uniquely  $p$ -semipotent if every non-trivial principal right ideal  $I$  of  $R$  contains a unique non-zero idempotent (see [35]). Equivalently, a

ring  $R$  is said to be uniquely  $p$ -semipotent if for every non-zero and non-right invertible  $a \in R$ , there exists a unique non-zero idempotent  $e \in R$  such that  $e \in aR$ . We next extend Proposition 7 in [35] on strongly clean rings to strongly weakly clean rings.

**Proposition 2.4.3.** *Let  $R$  be a uniquely  $p$ -semipotent, strongly weakly clean ring and let  $xR$  be a non-trivial principal right ideal of  $R$  where  $x = u + (1 - e)$  or  $x = u - (1 - e)$  with  $0 \neq e \in Id(R)$  and  $u \in U(R)$  as a strongly weakly clean expression of  $x$ . Then  $x$  is a (von Neumann) regular element of  $R$ .*

**Proof.** Since  $x$  has a strongly weakly clean expression of the form  $x = u + (1 - e)$  or  $x = u - (1 - e)$  where  $0 \neq e \in Id(R)$  and  $u \in U(R)$ , it follows that  $xe = ue = eu = ex$ . Hence,  $x eu^{-1} = e$  and therefore,

$$e \in xR. \tag{2.2}$$

Next we need to show that  $x \in eR$ . It is clear that

$$x(1 - e)R \subseteq xR. \tag{2.3}$$

We show that  $x(1 - e) = 0$ . Suppose to the contrary that  $x(1 - e) \neq 0$ . It is clear that  $x(1 - e)R$  is a non-trivial principal right ideal of  $R$  where  $e \neq 0$ . Since  $R$  is uniquely  $p$ -semipotent, so  $x(1 - e)R$  contains a unique non-zero idempotent  $f$ . But by (2.2) and (2.3), we have that  $f = e$  and therefore  $e = x(1 - e)r$  for some  $r \in R$ . Since  $xe = ex$ , we then have that  $e = e^2 = ex(1 - e)r = xe(1 - e)r = 0$ ; a contradiction. Therefore,  $x(1 - e) = 0$  and hence,

$$x = xe = ex. \tag{2.4}$$

Let  $s = eu^{-1}$ . We have shown above that  $e = xs$ . Then by (2.4), we have that  $xsx = ex = x$ . Hence,  $x$  is a regular element of  $R$ .  $\square$

**Remarks.** In a search of the literature, we recently came across a paper on very clean rings by Chen, Ungor and Halicioglu [15]. In the paper, the authors defined an element  $a$  in the ring  $R$  to be very clean provided that there exists an idempotent  $e \in Id(R)$  such that  $ae = ea$  and either  $a - e$  or  $a + e$  is invertible. The ring  $R$  is said to be very clean in case every element in  $R$  is very clean. This definition of very clean rings clearly coincides with that of strongly weakly clean rings in this section. The results in this section are however different from those in [15]. The purpose of [15] was to explore very clean matrices over local rings. As a consequence of the results in [15], we have several concrete examples of strongly weakly clean rings which we list below.

**Examples of strongly weakly clean rings:**

- (a) Let  $R$  be a commutative ring with exactly two maximal ideals and suppose that  $1/2 \in R$ . Then  $R$  is strongly weakly clean (by [15, Lemma 2.3]).
- (b) Let  $p, q \neq 2$  be prime numbers. If  $(p, q) = 1$ , then the ring  $\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$  is strongly weakly clean but not strongly clean (by [15, Lemma 2.4]).
- (c) The  $2 \times 2$  upper triangular matrix ring over  $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$  is strongly weakly clean but not strongly clean (by [15, Theorem 2.5]).



# Chapter 3

## Uniquely Weakly Clean Rings

### 3.1 Introduction

Let  $R$  be a ring. We say that a weakly clean element  $x \in R$  is uniquely weakly clean if the following hold:

- (a) If  $x = u + e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , then this representation is unique.
- (b) If  $x = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , then this representation is unique.

In other words,  $x \in R$  is not uniquely weakly clean if  $x = u + e = v + f$  for some distinct units  $u, v \in R$  and some distinct idempotents  $e, f \in R$  or if  $x = u - e = v - f$  for some distinct units  $u, v \in R$  and some distinct idempotents  $e, f \in R$ . For example, the ring  $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$  is weakly clean but not uniquely weakly clean. Indeed, the element  $\frac{7}{4} \in R$  can be written as  $\frac{7}{4} = \frac{11}{4} - 1 = \frac{7}{4} - 0$  where  $\frac{11}{4}, \frac{7}{4}$  are units in  $R$ . The ring  $R$  is said to be uniquely weakly clean if all of its elements are uniquely weakly clean. Uniquely weakly clean rings do exist. For example, every Boolean ring is uniquely weakly clean.

In this chapter, we obtain some properties and characterisations of uniquely weakly clean rings in Sections 3.2 and 3.3, respectively. In Section 3.4, we extend

some known results on uniquely clean group rings to uniquely weakly clean group rings.

### 3.2 Some properties of uniquely weakly clean rings

We begin with a result which shows that uniquely weakly clean elements are not obscure and can be found easily in non-reduced commutative rings.

**Proposition 3.2.1.** *Every central nilpotent element in a ring is uniquely weakly clean.*

**Proof.** Let  $R$  be a ring and let  $x$  be a central nilpotent element in  $R$ . Then  $x^n = 0$  for some positive integer  $n$ . It follows that  $-(x - 1)(x^{n-1} + x^{n-2} + \cdots + 1) = 1$  and  $(x + 1)(x^{n-1} - x^{n-2} + x^{n-3} - \cdots + (-1)^{n-2}x + (-1)^{n-1}) = (-1)^{n-1}$ . That is,  $x - 1$  and  $x + 1$  are units in  $R$ . Then since  $x = (x - 1) + 1 = (x + 1) - 1$ , so  $x$  is weakly clean. Now suppose that  $x = (x - 1) + 1 = u + e$  where  $u \in U(R)$  and  $e \in Id(R)$ . Since  $x$  is central,  $xu = ux$  and we thus have  $eu = ue$ . Then

$$\begin{aligned} 0 = x^n &= (u + e)^n \\ &= u^n + \binom{n}{1}eu^{n-1} + \cdots + \binom{n}{n-1}e^{n-1}u + e^n. \end{aligned}$$

Hence,  $u^n = er$  for some  $r \in R$ . It follows that  $eu^n = e^2r = er = u^n$ , that is,  $(1 - e)u^n = 0$ . Since  $u$  is a unit, so is  $u^n$  and hence,  $e = 1$ . We thus have that  $x = (x - 1) + 1$  uniquely. Similarly, if  $x = (x + 1) - 1 = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ , then it may be shown that  $e = 1$  and hence,  $x = (x + 1) - 1$  uniquely. □

We next show that uniquely weakly clean rings are abelian.

**Proposition 3.2.2.** *Every idempotent in a uniquely weakly clean ring is central.*

**Proof.** Let  $R$  be a uniquely weakly clean ring and let  $e \in Id(R)$ . For any  $r \in R$ , we have that  $e + (er - ere)$  is an idempotent. Moreover,  $1 + (er - ere)$  and  $1 - (er - ere)$  are units since they are inverses of one another. Note that  $1 + [e + (er - ere)] = [1 + (er - ere)] + e$ . Since  $R$  is uniquely weakly clean, the preceding equality implies that  $e + (er - ere) = e$ . It follows that  $er = ere$ . Similarly, it may be shown that  $re = ere$ . Thus,  $er = re$  and hence,  $e$  is central in  $R$ .  $\square$

As a consequence of Proposition 3.2.2, we have the following:

**Corollary 3.2.1.** *The centre of a uniquely weakly clean ring is uniquely weakly clean.*

**Proposition 3.2.3.** *Every uniquely weakly clean ring is directly finite.*

**Proof.** Let  $R$  be a ring and let  $a, b \in R$  such that  $ab = 1$ . Then  $ba$  is an idempotent (hence, central by Proposition 3.2.2). Thus  $ba = ba(ab) = a(ba)b = 1$ .  $\square$

**Proposition 3.2.4.** *Let  $R_1, \dots, R_n$  be rings. Then the direct product  $R = \prod_{i=1}^n R_i$  is uniquely weakly clean if and only if each  $R_i$  is uniquely weakly clean.*

**Proof.** ( $\Rightarrow$ ): Assume that  $R = \prod_{i=1}^n R_i$  is uniquely weakly clean. Then each  $R_i$ , being a homomorphic image of  $R$ , is weakly clean (by Proposition 2.3.5). Suppose that  $R_i$  is not uniquely weakly clean for some  $i \in \{1, \dots, n\}$ . Then there exists  $x_i \in R_i$  such that  $x_i = u_{i1} + e_{i2} = u_{i2} + e_{i2}$  or  $x_i = u_{i1} - e_{i2} = u_{i2} - e_{i2}$

for some  $u_{i1}, u_{i2} \in U(R_i)$  and  $e_{i1}, e_{i2} \in Id(R_i)$  where  $u_{i1} \neq u_{i2}$  and  $e_{i1} \neq e_{i2}$ . Let  $r = (r_1, \dots, r_n) \in R$  such that

$$r_k = \begin{cases} 1, & \text{if } k \neq i, \\ x_i, & \text{if } k = i. \end{cases}$$

Then

$$\begin{aligned} r &= (1, \dots, 1, x_i, 1, \dots, 1) \\ &= (1, \dots, 1, u_{i1}, 1, \dots, 1) + (0, \dots, 0, e_{i1}, 0, \dots, 0) \\ &= (1, \dots, 1, u_{i2}, 1, \dots, 1) + (0, \dots, 0, e_{i2}, 0, \dots, 0) \end{aligned}$$

or

$$\begin{aligned} r &= (1, \dots, 1, x_i, 1, \dots, 1) \\ &= (1, \dots, 1, u_{i1}, 1, \dots, 1) - (0, \dots, 0, e_{i1}, 0, \dots, 0) \\ &= (1, \dots, 1, u_{i2}, 1, \dots, 1) - (0, \dots, 0, e_{i2}, 0, \dots, 0), \end{aligned}$$

which implies that  $r$  is not uniquely weakly clean; a contradiction.

( $\Leftarrow$ ): Assume that  $R_1, \dots, R_n$  are uniquely weakly clean. Suppose that  $R = \prod_{i=1}^n R_i$  is not uniquely weakly clean. Then there exists  $x = (x_1, \dots, x_n) \in R$  such that

$$\begin{aligned} x &= (u_{11}, \dots, u_{1n}) + (e_{11}, \dots, e_{1n}) \\ &= (u_{21}, \dots, u_{2n}) + (e_{21}, \dots, e_{2n}) \end{aligned}$$

or

$$\begin{aligned} x &= (u_{11}, \dots, u_{1n}) - (e_{11}, \dots, e_{1n}) \\ &= (u_{21}, \dots, u_{2n}) - (e_{21}, \dots, e_{2n}) \end{aligned}$$

for some  $(u_{i1}, \dots, u_{in}) \in U(R)$  and  $(e_{i1}, \dots, e_{in}) \in Id(R)$  ( $i = 1, 2$ ) where  $(u_{11}, \dots, u_{1n}) \neq (u_{21}, \dots, u_{2n})$  and  $(e_{11}, \dots, e_{1n}) \neq (e_{21}, \dots, e_{2n})$ . This implies that there exists some  $k \in \{1, \dots, n\}$  such that  $x_k$  is not uniquely weakly clean; a contradiction. Hence,  $R$  must be uniquely weakly clean.  $\square$

By Propositions 3.2.4 and 3.2.2, we obtain the following which tells us that corners of uniquely weakly clean rings are uniquely weakly clean.

**Corollary 3.2.2.** *Let  $R$  be a uniquely weakly clean ring and let  $e \in Id(R)$ . Then  $eRe$  is uniquely weakly clean.*

The next two results generalise Lemma 4.2 and Theorem 4.3, respectively, in [1].

**Lemma 3.2.1.** *Let  $R$  be an abelian ring. If  $e, f \in Id(R)$  and  $e - f \in J(R)$  or  $e + f \in J(R)$ , then  $e = f$ .*

**Proof.** If  $e - f \in J(R)$ , then  $e(1 - f) = (e - f)(1 - f) \in J(R)$ . Similarly, if we assume that  $e + f \in J(R)$ , then  $e(1 - f) = (e + f)(1 - f) \in J(R)$ . It is clear that  $e(1 - f)$  is an idempotent. Since  $J(R)$  has no non-zero idempotents, it follows that  $e(1 - f) = 0$ ; hence,  $e = ef$ . Similarly, since  $(1 - e)f = -(1 - e)(e - f) \in J(R)$  or  $(1 - e)f = (1 - e)(e + f) \in J(R)$ , we have that  $f = ef$ . Hence,  $e = f$ .  $\square$

**Theorem 3.2.1.** *Let  $R$  be a commutative ring with  $R/M \cong \mathbb{Z}_2$  for each maximal ideal  $M$  of  $R$ .*

(a) *If  $x \in R$  has a representation in the form  $x = u + e$  where  $u \in U(R)$  and  $e \in Id(R)$ , then this representation is unique.*

(b) *If  $x \in R$  has a representation in the form  $x = u - e$  where  $u \in U(R)$  and  $e \in Id(R)$ , then this representation is unique.*

Then a weakly clean ring  $R$  with  $R/M \cong \mathbb{Z}_2$  for each maximal ideal  $M$  of  $R$  is a uniquely weakly clean ring.

**Proof.** (a) Suppose that  $x = u_1 + e_1 = u_2 + e_2$  for some  $u_1, u_2 \in U(R)$  and  $e_1, e_2 \in Id(R)$ . Let  $M$  be a maximal ideal of  $R$ . Then  $x + M \in R/M$  and  $x + M = u_1 + e_1 + M = u_2 + e_2 + M$ . Since  $R/M \cong \mathbb{Z}_2$  and  $u_1 + M, u_2 + M$  are units in  $R/M$ , so  $u_1 + M = u_2 + M$ . It follows that  $e_1 + M = e_2 + M$  and hence,  $e_1 - e_2 \in M$ . Since  $M$  is arbitrary, we have that  $e_1 - e_2 \in J(R)$ . It then follows by Lemma 3.2.1 that  $e_1 = e_2$ . Thus,  $u_1 = u_2$  and hence, the representation is unique for  $x = u + e$ .

(b) Suppose that  $x = u_1 - e_1 = u_2 - e_2$  for some  $u_1, u_2 \in U(R)$  and  $e_1, e_2 \in Id(R)$ . Let  $M$  be a maximal ideal of  $R$ . Then  $x + M = u_1 - e_1 + M = u_2 - e_2 + M$ . Since  $R/M \cong \mathbb{Z}_2$  and  $u_1 + M, u_2 + M$  are units in  $R/M$ , so  $u_1 + M = u_2 + M$ . It follows that  $e_1 + M = e_2 + M$  and hence,  $e_1 - e_2 \in M$ . Since  $M$  is arbitrary, we have that  $e_1 - e_2 \in J(R)$ . Then it follows by Lemma 3.2.1 that  $e_1 = e_2$ . Thus,  $u_1 = u_2$  and hence, the representation is unique for  $x = u - e$ .  $\square$

A straightforward consequence of Theorem 3.2.1 is the following:

**Corollary 3.2.3.** *A commutative weakly clean ring  $R$  with  $R/M \cong \mathbb{Z}_2$  for each maximal ideal  $M$  of  $R$  is uniquely weakly clean.*

### 3.3 Some characterisations of uniquely weakly clean rings

Let  $R$  be a ring and let  $V = {}_R V_R$  be an  $R$ - $R$ -bimodule which is a general ring (possibly with no unity) in which  $(vw)r = v(wr)$ ,  $(vr)w = v(rw)$  and  $(rv)w = r(vw)$  hold for all  $v, w \in V$  and  $r \in R$ . Then the ideal extension  $I(R; V)$  of  $R$  by

$V$  is defined to be the additive abelian group  $I(R; V) = R \oplus V$  with multiplication  $(r, v)(s, w) = (rs, rw + vs + vw)$ . Note that if  $S$  is a ring and  $S = R \oplus A$ , where  $R$  is a subring of  $S$  and  $A$  is an ideal of  $R$ , then  $S \cong I(R; A)$ .

We next extend Proposition 7 in [53] to the following.

**Proposition 3.3.1.** *An ideal extension  $S = I(R; V)$  is uniquely weakly clean if the following conditions are satisfied:*

- (a)  *$R$  is uniquely weakly clean.*
- (b) *If  $e \in Id(R)$  then  $ev = ve$  for all  $v \in V$ .*
- (c) *If  $v \in V$  then  $v + w + vw = 0$  for some  $w \in V$ .*

**Proof.** Assume that (a), (b) and (c) are satisfied. Let  $s = (r, v) \in S$ . Since  $R$  is weakly clean, we may write  $r = u + e$  or  $r = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . Then  $s = (u, v) + (e, 0)$  or  $s = (u, v) - (e, 0)$ , where  $(e, 0) \in Id(S)$ . Next, we show that  $(u, v) \in U(S)$ . Since  $(0, V) = \{(0, v) \mid v \in V\} \subseteq J(S)$  (by (c)), we have that  $(1, u^{-1}v) = (1, 0) + (0, u^{-1}v) \in U(S)$ . It follows that  $(u, v) = (u, 0)(1, u^{-1}v) \in U(S)$ . Hence,  $s$  is weakly clean. Before proving uniqueness, we first show that  $Id(S) = \{(e, 0) \mid e \in Id(R)\}$ . Clearly,  $(e, 0) \in Id(S)$  for any  $e \in Id(R)$ . Now let  $(e, x) \in Id(S)$ . Then  $(e, x)^2 = (e, x)$  which gives us  $e^2 = e$  and  $x = 2ex + x^2$  (by using (b)). Multiplying  $e$  and  $x$  on both sides of  $x = 2ex + x^2$ , we have  $ex + ex^2 = 0$  and  $x^2 = 2ex^2 + x^3$ , respectively. By substituting  $x^2 = 2ex^2 + x^3$  into  $x = 2ex + x^2$  and noting that  $ex + ex^2 = 0$ , we have  $x = x^3$ . It follows that  $x^2$  is an idempotent in  $V$ . By using (c), we have  $-x^2 + y - x^2y = 0$  for some  $y \in V$ . By taking  $w = -y$ , it follows that  $x^2 + w = x^2w$ . Multiplying the last equation by  $x^2$ , we have that  $x^4 + x^2w = x^4w$ .

Then since  $x^2$  is an idempotent in  $V$ , it follows from the last equation that  $x^2 = 0$ . Hence,  $x = x^3 = 0$ .

Now for uniqueness, suppose that  $(u, v) + (e, 0) = (u', v') + (e', 0)$  where  $(u, v), (u', v') \in U(S)$  and  $(e, 0), (e', 0) \in Id(S)$ . Then  $u + e = u' + e'$  and  $v = v'$ . Since  $R$  is uniquely weakly clean, we have  $u = u'$  and  $e = e'$ . It follows that  $(u, v) = (u', v')$  and  $(e, 0) = (e', 0)$ . Similarly, it may be shown that if  $(u, v) - (e, 0) = (u', v') - (e', 0)$  where  $(u, v), (u', v') \in U(S)$  and  $(e, 0), (e', 0) \in Id(S)$ , then  $(u, v) = (u', v')$  and  $(e, 0) = (e', 0)$ .  $\square$

By Proposition 3.3.1, we are able to give a noncommutative example of a uniquely weakly clean ring as follows:

**Example 3.3.1.** Let  $R$  be a uniquely weakly clean ring and let  $S = \{(a_{ij}) \in UT_3(R) \mid a_{11} = a_{22} = a_{33}\}$  where  $UT_3(R)$  denotes the ring of  $3 \times 3$  upper triangular matrices over  $R$ . Then  $S$  is uniquely weakly clean and is noncommutative. Indeed, by taking  $V = \{(a_{ij}) \in UT_3(R) \mid a_{11} = a_{22} = a_{33} = 0\}$ , we have  $S \cong I(R; V)$ . Since  $R$  is uniquely weakly clean, condition (a) in Proposition 3.3.1 holds. Condition (b) holds because idempotents in  $R$  are central (by Proposition 3.2.2), and the idempotents in  $S$  are diagonal matrices. Since  $V$  is an upper triangular matrix with diagonal entries 0, it can be proven that  $I_3 - MVN \in U(S)$  for any  $M, N \in S$  where  $I_3$  denotes the identity matrix in  $S$ . Thus, condition (c) holds. By applying Proposition 3.3.1,  $S$  is uniquely weakly clean.

Let  $R$  be a ring and let  $\alpha : R \rightarrow R$  be a ring endomorphism. Let  $R[[x, \alpha]]$  denote the ring of skew formal power series over  $R$ . That is, all formal power series in  $x$  with coefficients from  $R$  with multiplication defined by  $xr = \alpha(r)x$  for all  $r \in R$ . In particular,  $R[[x]] = R[[x, 1_R]]$  is the ring of formal power series



over  $R$ .

**Proposition 3.3.2.** *Let  $R$  be a ring and let  $\alpha : R \rightarrow R$  be a ring endomorphism. Then  $R[[x, \alpha]]$  is uniquely weakly clean if and only if  $R$  is uniquely weakly clean and  $e = \alpha(e)$  for all  $e \in Id(R)$ .*

**Proof.** ( $\Rightarrow$ ): Note that  $R[[x, \alpha]] \cong I(R; (x))$  where  $(x)$  is the ideal of  $R$  generated by  $x$ . Thus if  $R[[x, \alpha]]$  is uniquely weakly clean, then it follows by Proposition 3.3.1 that  $R$  is uniquely weakly clean. Since  $ex = xe = \alpha(e)x$ , we thus have that  $e = \alpha(e)$  for all  $e \in Id(R)$ .

( $\Leftarrow$ ): Assume that  $R$  is uniquely weakly clean and  $e = \alpha(e)$  for all  $e \in Id(R)$ . Condition (a) in Proposition 3.3.1 then clearly holds. Condition (c) in Proposition 3.3.1 holds because  $(x) \subseteq J(R[[x, \alpha]])$ . From the assumption, we have that for any  $e \in Id(R)$ ,  $\alpha^k(e) = e$  for each  $k \geq 1$ . Since  $R$  is uniquely weakly clean, it follows by Proposition 3.2.2 that  $e$  is central. Hence,  $(ax^k)e = a(x^k e) = a(\alpha^k(e)x^k) = e(ax^k)$  for all  $a \in R$  and  $k \geq 1$ . Therefore,  $ev = ve$  for all  $v \in (x)$ , that is, condition (b) in Proposition 3.3.1 holds. We then have by Proposition 3.3.1 that  $R[[x, \alpha]] \cong I(R; (x))$  is uniquely weakly clean.  $\square$

By taking  $\alpha = 1_R$  in Proposition 3.3.2, we have the following result.

**Corollary 3.3.1.** *Let  $R$  be a ring. The formal power series ring  $R[[x]]$  is uniquely weakly clean if and only if  $R$  is uniquely weakly clean.*

## 3.4 Uniquely weakly clean group rings

In this section, we investigate conditions which are necessary for a group ring to be uniquely weakly clean. For a positive integer  $k$ , let  $C_k$  denote the cyclic group of order  $k$ .

**Proposition 3.4.1.** *Let  $R$  be a ring and let  $G$  be a group. If  $RG$  contains  $\mathbb{Z}_2C_{2n+1}$  as a subring, then  $RG$  is not uniquely weakly clean.*

**Proof.** Let  $x$  be a generator of the cyclic group  $C_{2n+1}$ . Then  $x + x^2 + \cdots + x^{2n}$  is an idempotent in  $\mathbb{Z}_2C_{2n+1}$ . In particular, if  $n = 1$ , we have that  $x + x^2$  is an idempotent in  $\mathbb{Z}_2C_3$  and it is clear that  $x^2$  is a unit in  $\mathbb{Z}_2C_3$ . Note that  $x^2$  in  $\mathbb{Z}_2C_3$  can be written as  $x^2 = x + (x + x^2) = x^2 + 0$ . That is,  $x^2$  can be written as the sum of a unit and an idempotent in two different ways. Thus,  $x$  is not uniquely weakly clean and hence,  $\mathbb{Z}_2C_3$  is not uniquely weakly clean. Now suppose that  $n \geq 2$ . Note that in  $\mathbb{Z}_2C_{2n+1}$ ,

$$(1 + x + x^2 + \cdots + x^{2n-2})(x + x^2 + x^4 + \cdots + x^{2n}) = 1$$

if  $n \geq 2$  is even and

$$(1 + x + x^2 + \cdots + x^{2n-2})(1 + x^3 + x^5 + \cdots + x^{2n-1}) = 1$$

if  $n \geq 3$  is odd. Therefore,  $1 + x + x^2 + \cdots + x^{2n-2}$  is a unit in  $\mathbb{Z}_2C_{2n+1}$  and hence, so is  $x + x^2 + x^3 + \cdots + x^{2n-1} = x(1 + x + x^2 + \cdots + x^{2n-2})$ . Note that  $x^{2n} = (x + x^2 + \cdots + x^{2n-1}) + (x + x^2 + \cdots + x^{2n-1} + x^{2n}) = x^{2n} + 0$ , that is,  $x^{2n}$  can be written as the sum of a unit and an idempotent in two different ways in  $\mathbb{Z}_2C_{2n+1}$ . Thus,  $x^{2n}$  is not uniquely weakly clean and hence,  $\mathbb{Z}_2C_{2n+1}$  is not uniquely weakly clean. It follows that  $RG$  would also not be uniquely weakly clean if it contains  $\mathbb{Z}_2C_{2n+1}$  as a subring.  $\square$

By Proposition 3.4.1 we see that  $\mathbb{Z}_2C_m$  is not uniquely weakly clean when  $m$  is divisible by some odd prime  $p$ . The same cannot be said when  $m$  is a power of 2, as shown in the following:

**Proposition 3.4.2.** *The group algebra  $\mathbb{Z}_2C_{2^k}$  is uniquely weakly clean for all integer  $k \geq 0$ .*

**Proof.** Let  $\delta : \mathbb{Z}_2C_{2^k} \rightarrow \mathbb{Z}_2$  be the augmentation map. Then  $\alpha \in U(\mathbb{Z}_2C_{2^k})$  if and only if  $\delta(\alpha) = 1$ . Indeed, the necessity part of this is clear. For the converse, if  $\delta(\alpha) = 1$ , then  $1 - \alpha \in \text{Ker } \delta = \Delta$ , the augmentation ideal of  $\mathbb{Z}_2C_{2^k}$ . Since  $\Delta = J(\mathbb{Z}_2C_{2^k})$  by [24, Proposition 16 (iv)], it follows that  $1 - \alpha \in J(\mathbb{Z}_2C_{2^k})$  and hence,  $\alpha \in U(\mathbb{Z}_2C_{2^k})$ . Let  $x$  denote a generator of  $C_{2^k}$ . We thus have that

$$U(\mathbb{Z}_2C_{2^k}) = \{\alpha = a_0 + a_1x + \cdots + a_{2^k-1}x^{2^k-1} \mid a_0, a_1, \dots, a_{2^k-1} \in \mathbb{Z}_2, \delta(\alpha) = 1\}.$$

Hence,

$$|U(\mathbb{Z}_2C_{2^k})| = \binom{2^k}{1} + \binom{2^k}{3} + \cdots + \binom{2^k}{2^k-1} = 2^{2^k-1}.$$

Now let  $e \in \text{Id}(\mathbb{Z}_2C_{2^k})$ ,  $e \neq 1$ . Then  $\delta(e) = 0$  and hence,  $e \in \Delta$  which is nil by [24, Proposition 16 (ii)]. Thus  $e = 0$  and we therefore have that  $\text{Id}(\mathbb{Z}_2C_{2^k}) = \{0, 1\}$ . Note that  $1 + u \notin U(\mathbb{Z}_2C_{2^k})$  for any  $u \in U(\mathbb{Z}_2C_{2^k})$  because  $\delta(1 + u) = 0$ . Thus  $U(\mathbb{Z}_2C_{2^k})$  and  $\{1 + u \mid u \in U(\mathbb{Z}_2C_{2^k})\}$  are mutually disjoint. We thus have that  $U(\mathbb{Z}_2C_{2^k}) + \text{Id}(\mathbb{Z}_2C_{2^k}) = \mathbb{Z}_2C_{2^k}$  which implies that  $\mathbb{Z}_2C_{2^k}$  is weakly clean. Since  $|U(\mathbb{Z}_2C_{2^k})||\text{Id}(\mathbb{Z}_2C_{2^k})| = 2^{2^k} = |\mathbb{Z}_2C_{2^k}|$ , it follows that  $\mathbb{Z}_2C_{2^k}$  must be uniquely weakly clean. □

# Chapter 4

## Some Results on $n$ -Weakly Clean Rings

### 4.1 Introduction

Let  $R$  be a ring. An element  $x \in R$  is said to be  $n$ -weakly clean if  $x = u_1 + \cdots + u_n + e$  or  $x = u_1 + \cdots + u_n - e$  for some units  $u_1, \dots, u_n \in R$  and idempotent  $e \in R$ . In other words, the element  $x \in R$  is  $n$ -weakly clean if either  $x$  or  $-x$  is  $n$ -clean. The ring  $R$  is said to be  $n$ -weakly clean if all of its elements are  $n$ -weakly clean. Clearly,  $n$ -clean rings are  $n$ -weakly clean and weakly clean rings are 1-weakly clean.

In this chapter we extend some results on  $n$ -clean rings and weakly clean rings to  $n$ -weakly clean rings. We first obtain some properties of  $n$ -weakly clean rings in Section 4.2. In Section 4.3, we determine some conditions which are necessary or sufficient for a group ring to be  $n$ -weakly clean. Finally, in Section 4.4, we obtain some conditions for a matrix over a commutative ring to be  $n$ -weakly clean.

## 4.2 Some properties of $n$ -weakly clean rings

We begin with the following result which shows that being weakly clean implies being  $n$ -weakly clean for any positive integer  $n$ .

**Proposition 4.2.1.** *Let  $R$  be a ring and let  $n$  be a positive integer. If  $x \in R$  is  $n$ -weakly clean, then  $x$  is  $(n + 1)$ -weakly clean.*

**Proof.** Let  $x \in R$  be  $n$ -weakly clean. Then  $x$  or  $-x$  is  $n$ -clean in  $R$ . By Proposition 1.1.2,  $x$  or  $-x$  is  $(n + 1)$ -clean. Thus,  $x$  is  $(n + 1)$ -weakly clean.  $\square$

By Proposition 4.2.1 and by induction, we obtain the following analogue of Proposition 1.1.2 for  $n$ -weakly clean rings.

**Corollary 4.2.1.** *Let  $m, n$  be positive integers with  $n < m$ . If  $R$  is an  $n$ -weakly clean ring, then  $R$  is  $m$ -weakly clean.*

Recall that a ring  $R$  is called an  $(S, n)$ -ring if every element in  $R$  can be written as a sum of no more than  $n$  units of  $R$ . Clearly, an  $(S, n)$ -ring is  $n$ -weakly clean.

It is known that homomorphic images of  $n$ -clean rings are  $n$ -clean (see [67]). For  $n$ -weakly clean rings we have the following:

**Proposition 4.2.2.** *Let  $n$  be a positive integer. Then every homomorphic image of an  $n$ -weakly clean ring is  $n$ -weakly clean.*

**Proof.** Let  $R$  be an  $n$ -weakly clean ring and let  $\phi : R \rightarrow S$  be a ring epimorphism. Let  $y \in S$ . Then  $y = \phi(x)$  for some  $x \in R$ . Since  $R$  is  $n$ -weakly clean, then  $x = u_1 + \cdots + u_n + e$  or  $x = u_1 + \cdots + u_n - e$  for some  $u_1, \dots, u_n \in U(R)$  and  $e \in Id(R)$ . Since  $\phi$  is an epimorphism, we then have that  $\phi(u_1), \dots, \phi(u_n) \in$

$U(S)$ ,  $\phi(e) \in Id(S)$  and  $y = \phi(x) = \phi(u_1) + \cdots + \phi(u_n) + \phi(e)$  or  $y = \phi(x) = \phi(u_1) + \cdots + \phi(u_n) - \phi(e)$ . That is,  $y$  is  $n$ -weakly clean in  $S$ . It follows that  $\phi(R) = S$  is  $n$ -weakly clean.  $\square$

We now consider direct products. For  $n$ -clean rings, we have the following:

**Proposition 4.2.3.** *Let  $n$  be a positive integer. The direct product ring  $R = \prod_{i \in I} R_i$  is  $n$ -clean if and only if each  $R_i$  is  $n$ -clean.*

**Proof.** Suppose that  $R = \prod_{i \in I} R_i$  is an  $n$ -clean ring. Then each  $R_i$  is a homomorphic image of  $R$  (via the natural projection  $\pi_i : R \rightarrow R_i$ ) and hence, each  $R_i$  is  $n$ -clean. Conversely, suppose that each  $R_i$  is an  $n$ -clean ring. Let  $x = (x_i) \in R$ . Then for each  $i$ ,  $x_i = u_{i1} + \cdots + u_{in} + e_i$  for some  $u_{i1}, \dots, u_{in} \in U(R_i)$  and  $e_i \in Id(R_i)$ . Thus,  $x = (x_i) = (u_{i1}) + \cdots + (u_{in}) + (e_i)$  with  $(u_{ij}) \in U(R)$  for  $j = 1, \dots, n$  and  $(e_i) \in Id(R)$ . Hence,  $x$  is  $n$ -clean.  $\square$

For direct products involving  $n$ -weakly clean rings, we obtain the following:

**Proposition 4.2.4.** *Let  $n$  be a positive integer. The direct product ring  $R = \prod_{k \in I} R_k$  is  $n$ -weakly clean if and only if each  $R_k$  is  $n$ -weakly clean and at most one  $R_k$  is not an  $n$ -clean ring.*

In order to prove Proposition 4.2.4, we first prove the following equivalence:

**Proposition 4.2.5.** *Let  $R$  be a ring. Then the following conditions are equivalent:*

- (a)  $R$  is an  $n$ -clean ring.
- (b) Every element  $x \in R$  has the form  $x = u_1 + \cdots + u_n - e$  where  $u_1, \dots, u_n \in U(R)$  and  $e \in Id(R)$ .

(c) Every element  $x \in R$  has the form  $x = u_1 + \cdots + u_n + e$  where  $u_1, \dots, u_n \in U(R) \cup \{0\}$  and  $e \in Id(R)$ .

(d) Every element  $x \in R$  has the form  $x = u_1 + \cdots + u_n - e$  where  $u_1, \dots, u_n \in U(R) \cup \{0\}$  and  $e \in Id(R)$ .

**Proof.** (a)  $\Rightarrow$  (b): Let  $x \in R$ . Since  $R$  is  $n$ -clean, we have  $-x = v_1 + \cdots + v_n + e$  for some  $v_1, \dots, v_n \in U(R)$  and  $e \in Id(R)$ . Hence,  $x = u_1 + \cdots + u_n - e$  where  $u_i = -v_i \in U(R)$  for  $i = 1, \dots, n$ .

(b)  $\Rightarrow$  (a): Let  $x \in R$ . Then  $-x = u_1 + \cdots + u_n - e$  for some  $u_1, \dots, u_n \in U(R)$  and  $e \in Id(R)$ . It follows that  $x = (-u_1) + \cdots + (-u_n) + e$  which shows that  $x$  is  $n$ -clean.

(c)  $\Leftrightarrow$  (d): This is similar to (a)  $\Leftrightarrow$  (b).

(a)  $\Rightarrow$  (c): This is clear by the definition of  $n$ -clean.

(c)  $\Rightarrow$  (a): Let  $x \in R$  and suppose that  $x = u_1 + u_2 + \cdots + u_n + e$  where  $u_i \in U(R) \cup \{0\}$  and  $e \in Id(R)$ . If  $u_i \neq 0$  for some  $i \in \{1, \dots, n\}$ , then we have by Proposition 1.1.2 that  $x$  is  $n$ -clean. If  $u_1 = \cdots = u_n = 0$ , then  $x = e$  and since  $e = -(1 - 2e) + (1 - e)$  where  $-(1 - 2e) \in U(R)$  and  $1 - e \in Id(R)$ , we have that  $x$  is clean. It follows by Proposition 4.2.1 and induction that  $x$  is  $n$ -clean.  $\square$

**Proof of Proposition 4.2.4.** ( $\Rightarrow$ ): Suppose that  $R = \prod_{k \in I} R_k$  is  $n$ -weakly clean. Then it follows that each  $R_k$ , being a homomorphic image of  $R$ , is  $n$ -weakly clean (by Proposition 4.2.2). Suppose that  $R_i$  and  $R_j$  ( $i \neq j$ ) are not  $n$ -clean. Since  $R_i$  is not  $n$ -clean, then by Proposition 4.2.5, there exists  $x_i \in R_i$  such that  $x_i \neq u_1 + \cdots + u_n - e$  for any  $u_1, \dots, u_n \in U(R)$  and any  $e \in Id(R)$ . But since  $R_i$  is  $n$ -weakly clean, we must have  $x_i = u_{1_i} + \cdots + u_{n_i} + e_i$  for some  $u_{1_i}, \dots, u_{n_i} \in U(R_i)$  and  $e_i \in Id(R_i)$ . Now since  $R_j$  is not  $n$ -clean but

is  $n$ -weakly clean, there is an  $x_j \in R_j$  such that  $x_j = u_{1_j} + \cdots + u_{n_j} - e_j$  for some  $u_{1_j}, \dots, u_{n_j} \in U(R_j)$  and  $e_j \in Id(R_j)$  but  $x_j \neq u_1 + \cdots + u_n + e$  for any  $u_1, \dots, u_n \in U(R_j)$  and  $e \in Id(R_j)$ . Let  $y = (y_k) \in R$  such that

$$y_k = \begin{cases} x_k, & k \in \{i, j\}, \\ 0, & k \notin \{i, j\}. \end{cases}$$

Then  $y \neq u_1 + \cdots + u_n \pm e$  for any  $u_1, \dots, u_n \in U(R)$  and  $e \in Id(R)$ , which contradicts the assumption that  $R$  is  $n$ -weakly clean. Hence, we can only have at most one  $R_i$  which is not  $n$ -clean.

( $\Leftarrow$ ): If every  $R_i$  is  $n$ -clean, then it follows by Proposition 4.2.3 that  $R = \prod_{k \in I} R_k$  is also  $n$ -clean; hence,  $n$ -weakly clean. Suppose that  $R_{i_0}$  is  $n$ -weakly clean but not  $n$ -clean and all the other  $R_i$ 's are  $n$ -clean. Let  $x = (x_i) \in R = \prod_{k \in I} R_k$ . Then for  $x_{i_0} \in R_{i_0}$ , we may write  $x_{i_0} = u_{1_{i_0}} + \cdots + u_{n_{i_0}} + e_{i_0}$  or  $x_{i_0} = u_{1_{i_0}} + \cdots + u_{n_{i_0}} - e_{i_0}$  where  $u_{1_{i_0}}, \dots, u_{n_{i_0}} \in U(R_{i_0})$  and  $e_{i_0} \in Id(R_{i_0})$ . If  $x_{i_0} = u_{1_{i_0}} + \cdots + u_{n_{i_0}} + e_{i_0}$ , then for  $i \neq i_0$ , since  $R_i$  is  $n$ -clean, we may let  $x_i = u_{1_i} + \cdots + u_{n_i} + e_i$  where  $u_{1_i}, \dots, u_{n_i} \in U(R_i)$  and  $e_i \in Id(R_i)$ . On the other hand, if  $x_{i_0} = u_{1_{i_0}} + \cdots + u_{n_{i_0}} - e_{i_0}$ , then for  $i \neq i_0$ , since  $R_i$  is  $n$ -clean, it follows by Proposition 4.2.5 that we may let  $x_i = u_{1_i} + \cdots + u_{n_i} - e_i$  where  $u_{1_i}, \dots, u_{n_i} \in U(R_i)$  and  $e_i \in Id(R_i)$ . Hence,  $x = u_1 + \cdots + u_n + e$  or  $x = u_1 + \cdots + u_n - e$  where  $u_i = (u_{i_j}) \in U(R)$  and  $e = (e_j) \in Id(R)$  ( $i = 1, \dots, n$ ). Thus,  $x$  is  $n$ -weakly clean. This completes the proof.  $\square$

Polynomial rings over  $n$ -weakly clean rings are not necessarily  $n$ -weakly clean ( $n \geq 1$ ). For example, the ring  $\mathbb{Z}_2$  is weakly clean but the polynomial ring  $\mathbb{Z}_2[x]$  is not weakly clean. However, there are examples of polynomial rings over  $n$ -weakly clean rings which are  $n$ -weakly clean for  $n \geq 2$ .



**Example 4.2.1.** Let  $\mathbb{F}$  be a field and let  $R = M_2(\mathbb{F})$ . Then  $R[x] \cong M_2(\mathbb{F}[x])$ . By [34, Theorem 11],  $R[x]$  is a 2-good ring (hence, 2-weakly clean ring). However,  $R[x]$  is not weakly clean.

Following [68], a ring  $R$  is said to satisfy (SI) if for all  $a, b \in R$ ,  $ab = 0$  implies that  $aRb = 0$ . We first note some lemmas from [68].

**Lemma 4.2.1.** [68, Lemma 3.5] *If  $R$  is a ring satisfying (SI) and  $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ , then  $f(x) \in U(R[x])$  if and only if  $a_0 \in U(R)$  and  $a_1, \dots, a_n \in N(R)$ .*

**Lemma 4.2.2.** [68, Lemma 3.6] *Let  $R$  be an abelian ring. Then  $Id(R[x]) = Id(R)$ .*

Since a ring satisfying (SI) is abelian, we have the following by Lemma 4.2.2.

**Corollary 4.2.2.** *If  $R$  is a ring satisfying (SI), then  $Id(R[x]) = Id(R)$ .*

**Proposition 4.2.6.** *If  $R$  is a ring satisfying (SI), then the polynomial ring  $R[x]$  is not  $n$ -weakly clean for any positive integer  $n$ .*

**Proof.** Let  $R$  be a ring satisfying (SI). Then by Corollary 4.2.2 and Lemma 4.2.1, we have  $Id(R[x]) = Id(R)$  and  $U(R[x]) = \{r_0 + r_1x + \cdots + r_mx^m \in R[x] \mid r_0 \in U(R), r_1, \dots, r_m \in N(R)\}$ . If  $x \in R[x]$  were  $n$ -weakly clean for some positive integer  $n$ , then  $x = \sum_{i=1}^n (u_i + r_{i1}x + \cdots + r_{im_i}x^{m_i}) + f$  or  $x = \sum_{i=1}^n (u_i + r_{i1}x + \cdots + r_{im_i}x^{m_i}) - f$ , where  $f \in Id(R)$ ,  $u_1, \dots, u_n \in U(R)$  and each  $r_{il} \in N(R) \subseteq J(R)$  ( $1 \leq l \leq m_i, 1 \leq i \leq n$ ). By comparing the coefficients of  $x$ , it follows that  $1 = \sum_{i=1}^n r_{i1} \in J(R)$ , which is a contradiction. Thus,  $R[x]$  is not  $n$ -weakly clean for any positive integer  $n$ . □

A ring  $R$  is called left (respectively, right) duo if every left (respectively, right) ideal of  $R$  is a two-sided ideal. By [44], we have that every left (right) duo ring satisfies (SI). A ring  $R$  is called reversible if for all  $a, b \in R$ ,  $ab = 0$  implies  $ba = 0$ . In general, if  $R$  is a reversible ring, then  $R$  satisfies (SI) (see [43]). By Proposition 4.2.6, we readily have the following corollary.

**Corollary 4.2.3.** *Let  $R$  be a ring. If  $R$  is left (right) duo or reversible, then the polynomial ring  $R[x]$  is not  $n$ -weakly clean for any positive integer  $n$ .*

Formal power series rings over commutative  $n$ -weakly clean rings are however  $n$ -weakly clean, as shown in the following:

**Proposition 4.2.7.** *Let  $R$  be a commutative ring and let  $n$  be a positive integer. Then the formal power series ring  $R[[x]]$  is  $n$ -weakly clean if and only if  $R$  is  $n$ -weakly clean.*

**Proof.** Suppose that  $R[[x]]$  is  $n$ -weakly clean. Then it follows by the isomorphism  $R \cong R[[x]]/(x)$  and Proposition 4.2.2 that  $R$  is an  $n$ -weakly clean ring. Conversely, suppose that  $R$  is  $n$ -weakly clean. Let  $y = \sum_{i=0}^{\infty} r_i x^i \in R[[x]]$ . Since  $R$  is  $n$ -weakly clean, we have that  $r_0 = u_1 + \cdots + u_n + e$  or  $r_0 = u_1 + \cdots + u_n - e$ , where  $u_1, \dots, u_n \in U(R)$  and  $e \in Id(R)$ . Then  $y = e + (u_1 + r_1 x + r_2 x^2 + \dots) + u_2 + \cdots + u_n$  or  $y = -e + (u_1 + r_1 x + r_2 x^2 + \dots) + u_2 + \cdots + u_n$ . Note that  $e \in Id(R) \subseteq Id(R[[x]])$ ,  $u_1 + r_1 x + r_2 x^2 + \dots \in U(R[[x]])$  and  $u_i \in U(R) \subseteq U(R[[x]])$  ( $i = 2, \dots, n$ ). Thus,  $R[[x]]$  is an  $n$ -weakly clean ring.  $\square$

We next show that being  $n$ -weakly clean in a corner of the ring  $R$  implies being  $n$ -weakly clean in  $R$ .

**Theorem 4.2.1.** *Let  $R$  be a ring and let  $e$  be an idempotent in  $R$ . For any positive integer  $n$ , if  $x \in eRe$  is  $n$ -weakly clean in  $eRe$ , then  $x$  is  $n$ -weakly clean in  $R$ .*

**Proof.** Suppose that  $x = v_1 + \cdots + v_n + f$  or  $x = v_1 + \cdots + v_n - f$ , where  $f^2 = f \in eRe$  and  $v_i \in eRe$  such that  $v_i w_i = e = w_i v_i$  for some  $w_i \in eRe$  ( $i = 1, \dots, n$ ). For  $n$  even, let

$$u_i = \begin{cases} v_i + (1 - e), & i = 1, \dots, \frac{n}{2}, \\ v_i - (1 - e), & i = \frac{n}{2} + 1, \dots, n. \end{cases}$$

Then  $u_1, \dots, u_n$  are units in  $R$  with

$$u_i^{-1} = \begin{cases} w_i + (1 - e), & i = 1, \dots, \frac{n}{2}, \\ w_i - (1 - e), & i = \frac{n}{2} + 1, \dots, n. \end{cases}$$

Hence,  $x - (u_1 + \cdots + u_{\frac{n}{2}}) - (u_{\frac{n}{2}+1} + \cdots + u_n) = f$  or  $x - (u_1 + \cdots + u_{\frac{n}{2}}) - (u_{\frac{n}{2}+1} + \cdots + u_n) = -f$  in  $R$ . That is,  $x$  is  $n$ -weakly clean in  $R$ .

For  $n$  odd and  $x = v_1 + \cdots + v_n + f$ , let

$$u_i = \begin{cases} v_i - (1 - e), & i = 1, \dots, \frac{n+1}{2}, \\ v_i + (1 - e), & i = \frac{n+3}{2}, \dots, n. \end{cases}$$

Then  $u_1, \dots, u_n$  are units in  $R$  with

$$u_i^{-1} = \begin{cases} w_i - (1 - e), & i = 1, \dots, \frac{n+1}{2}, \\ w_i + (1 - e), & i = \frac{n+3}{2}, \dots, n. \end{cases}$$

Hence,  $x - (u_1 + \cdots + u_{\frac{n+1}{2}}) - (u_{\frac{n+3}{2}} + \cdots + u_n) = x - (v_1 + \cdots + v_n) + 1 - e = f + (1 - e)$ ,

an idempotent in  $R$ . For  $n$  odd and  $x = v_1 + \cdots + v_n - f$ , let

$$u_i = \begin{cases} v_i + (1 - e), & i = 1, \dots, \frac{n+1}{2}, \\ v_i - (1 - e), & i = \frac{n+3}{2}, \dots, n. \end{cases}$$

Then  $u_1, \dots, u_n$  are units in  $R$  with

$$u_i^{-1} = \begin{cases} w_i + (1 - e), & i = 1, \dots, \frac{n+1}{2}, \\ w_i - (1 - e), & i = \frac{n+3}{2}, \dots, n. \end{cases}$$

Hence,  $x - (u_1 + \cdots + u_{\frac{n+1}{2}}) - (u_{\frac{n+3}{2}} + \cdots + u_n) = x - (v_1 + \cdots + v_n) - (1 - e) = -f - (1 - e) = -(f + (1 - e))$ , where  $f + (1 - e)$  is an idempotent in  $R$ . This shows that  $x$  is also  $n$ -weakly clean in  $R$  when  $n$  is odd. This completes the proof.  $\square$

As a consequence of Theorem 4.2.1, we show in the following that the product of an  $n$ -weakly clean element and an idempotent in an abelian ring is also  $n$ -weakly clean.

**Proposition 4.2.8.** *Let  $R$  be an abelian ring and let  $n$  be a positive integer. Let  $x \in R$  and let  $e \in Id(R)$ . Then  $xe$  is  $n$ -weakly clean in  $R$  if  $x$  is  $n$ -weakly clean in  $R$ .*

**Proof.** If  $x$  is  $n$ -weakly clean in  $R$ , then  $x = u_1 + \cdots + u_n + f$  or  $x = u_1 + \cdots + u_n - f$  for some  $u_1, \dots, u_n \in U(R)$  and  $f \in Id(R)$ . Then  $xe = u_1e + \cdots + u_n e + fe$  or  $xe = u_1e + \cdots + u_n e - fe$ . Clearly,  $u_1e, \dots, u_n e$  are units in  $eRe$  and  $fe$  is an idempotent in  $eRe$ . Hence,  $xe$  is  $n$ -weakly clean in  $eRe$ . It then follows by Theorem 4.2.1 that  $xe$  is  $n$ -weakly clean in  $R$ .  $\square$

By referring to Example 4.1 in [68], we next give an example to show that corner rings of  $n$ -weakly clean rings are not necessarily  $n$ -weakly clean. The example also shows that the converse of Theorem 4.2.1 is not necessarily true.

**Example 4.2.2.** (see [68, Example 4.1]). Let  $T = \mathbb{F}[x]$ , where  $\mathbb{F}$  is a field. By Corollary 4.2.3,  $T$  is not an  $n$ -weakly clean ring for any positive integer. Let  $R = M_2(T)$ . Then  $R$  is a 2-good ring by [34, Theorem 11] and hence,  $n$ -weakly clean for  $n \geq 2$ . Now let  $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R$ . Then  $eae = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ . We thus see that  $eRe$  is isomorphic to the ring  $T$  and hence,  $eRe$  is not  $n$ -weakly clean for any integer  $n \geq 2$ . This shows that for any integer  $n > 1$ , there exist an

$n$ -weakly clean ring  $R$  and an idempotent  $e \in R$  such that  $eRe$  is not  $n$ -weakly clean.

Finally, we see how lifting of idempotents modulo an ideal of a ring determines whether the ring is  $n$ -weakly clean.

**Proposition 4.2.9.** *Let  $R$  be a ring and let  $n$  be a positive integer. Let  $I$  be an ideal of  $R$  such that  $I \subseteq J(R)$ . If  $R/I$  is  $n$ -weakly clean and idempotents can be lifted modulo  $I$ , then  $R$  is  $n$ -weakly clean.*

**Proof.** Let  $x \in R$ . Then  $\bar{x} = x + I \in R/I$ . Since  $R/I$  is  $n$ -weakly clean,  $\bar{x} = \bar{u}_1 + \cdots + \bar{u}_n + \bar{e}$  or  $\bar{x} = \bar{u}_1 + \cdots + \bar{u}_n - \bar{e}$ , where  $\bar{u}_i = u_i + I \in U(R/I)$  for  $i = 1, \dots, n$  and  $\bar{e} = e + I \in Id(R/I)$ . Since idempotents can be lifted modulo  $I$ , we may assume that  $e^2 = e \in R$ . Since  $\bar{u}_i \in U(R/I)$ , there exists  $\bar{v}_i = v_i + I \in U(R/I)$  such that  $\bar{u}_i \bar{v}_i = 1 + I = \bar{v}_i \bar{u}_i$  for  $i = 1, \dots, n$ . Therefore,  $1 - u_i v_i, 1 - v_i u_i \in I \subseteq J(R)$  for every  $i = 1, \dots, n$ . It follows that  $u_i$  has a right inverse and a left inverse in  $R$  for every  $i = 1, \dots, n$ . Thus,  $u_i \in U(R)$  for  $i = 1, \dots, n$ . We then have  $x = u_1 + \cdots + u_n + r + e$  or  $x = u_1 + \cdots + u_n + s - e$  for some  $r, s \in I \subseteq J(R)$ . Since  $J(R) \subseteq \{a \in R \mid a + b \text{ is a unit in } R \text{ for every unit } b \in R\}$ , so  $u_n + r$  and  $u_n + s$  are units in  $R$ . It follows that  $x$  is  $n$ -weakly clean. □

A right (respectively, left) ideal of a ring is said to be a right (respectively, left) nil ideal if each of its elements is nilpotent. We say that  $N$  is a nil ideal if it is both a left and right nil ideal. It is well known that idempotents lift modulo every nil ideal of a ring. Since every nil ideal of a ring  $R$  is contained in its Jacobson radical, we thus have the following corollary of Proposition 4.2.9.

**Corollary 4.2.4.** *Let  $N$  be a nil ideal of a ring  $R$ . If  $R/N$  is  $n$ -weakly clean, then  $R$  is  $n$ -weakly clean.*

### 4.3 Some results on $n$ -weakly clean group rings

We first consider some cases where a group ring is isomorphic (as a ring) to a direct product of copies of the coefficient ring.

**Proposition 4.3.1.** *Let  $R$  be a ring and let  $2$  be a unit in  $R$ . Then  $R$  is  $n$ -clean if and only if  $RC_2$  is  $n$ -weakly clean.*

**Proof.** Let  $C_2 = \langle x \mid x^2 = 1 \rangle$  and define  $\phi : RC_2 \rightarrow R \times R$  by  $\phi(a + bx) = (a + b, a - b)$  where  $a, b \in R$ . Then  $\phi$  is a ring homomorphism. Since  $2$  is a unit in  $R$ , we have that  $\phi$  is bijective. Therefore,  $RC_2 \cong R \times R$ . If  $R$  is  $n$ -clean, it follows by Proposition 4.2.3 that  $RC_2 \cong R \times R$  is  $n$ -clean; hence  $n$ -weakly clean. Conversely, if  $RC_2$  is  $n$ -weakly clean, then it follows by Proposition 4.2.2 that  $R \times R$  is  $n$ -weakly clean. Therefore,  $R$  is  $n$ -clean by Proposition 4.2.4.  $\square$

**Proposition 4.3.2.** *Let  $R$  be a ring and let  $2$  be a unit in  $R$ . For any positive integer  $n$ ,  $R$  is  $n$ -clean if and only if  $RC_2$  is  $n$ -clean.*

**Proof.** As in the proof of Proposition 4.3.1, it may be shown that  $RC_2 \cong R \times R$ . It then follows by Proposition 4.2.3 that  $R$  is  $n$ -clean if and only if  $RC_2$  is  $n$ -clean.  $\square$

By Propositions 4.3.1 and 4.3.2 we have the following:

**Corollary 4.3.1.** *Let  $R$  be a ring and let  $2$  be a unit in  $R$ . Then for any positive integer  $n$ , the following are equivalent:*

- (a)  $R$  is  $n$ -clean.
- (b)  $RC_2$  is  $n$ -weakly clean.
- (c)  $RC_2$  is  $n$ -clean.

Corollary 4.3.1 may be extended as follows:

**Proposition 4.3.3.** *Let  $R$  be a ring and let  $2$  be a unit in  $R$ . Then for any positive integers  $k$  and  $n$ , the following are equivalent:*

- (a)  $R$  is  $n$ -clean.
- (b)  $RC_2^{(k)}$  is  $n$ -weakly clean.
- (c)  $RC_2^{(k)}$  is  $n$ -clean.

**Proof.** (a)  $\Rightarrow$  (c): Assume (a). By the isomorphism  $RC_2^{(i)} \cong (RC_2^{(i-1)})C_2$  for  $i = 1, 2, \dots$  and by induction, it suffices to show that  $RC_2$  is  $n$ -clean. But this follows readily by Corollary 4.3.1.

(c)  $\Rightarrow$  (b): This is clear.

(b)  $\Rightarrow$  (a): Assume (b). By the isomorphism  $RC_2^{(i)} \cong (RC_2^{(i-1)})C_2$ , we have that  $RC_2^{(i-1)}$  is a homomorphic image of  $RC_2^{(i)}$  for  $i = 1, 2, \dots$ . In particular,  $RC_2 = RC_2^{(1)}$  is a homomorphic image of  $RC_2^{(k)}$ . Since  $RC_2^{(k)}$  is  $n$ -weakly clean, so is  $RC_2$  by Proposition 4.2.2 and hence, by Corollary 4.3.1,  $R$  is  $n$ -clean.  $\square$

**Proposition 4.3.4.** *Let  $R$  be a ring and let  $2$  be a unit in  $R$ . Then  $M_n(R)$  is  $n$ -clean if and only if  $M_n(RC_2)$  is  $n$ -weakly clean.*

**Proof.** Since  $2 \in U(R)$ , it follows from the proof of Proposition 4.3.1 that  $RC_2 \cong R \times R$ . Therefore,  $M_n(RC_2) \cong M_n(R \times R) \cong M_n(R) \times M_n(R)$ . Now suppose

that  $M_n(R)$  is  $n$ -clean. By Proposition 4.2.3, it follows that  $M_n(R) \times M_n(R)$  is  $n$ -clean and thus, so is  $M_n(RC_2)$ . Hence,  $M_n(RC_2)$  is  $n$ -weakly clean. Conversely, if  $M_n(RC_2)$  is  $n$ -weakly clean, then so is  $M_n(R) \times M_n(R)$  by Proposition 4.2.2. It then follows by Proposition 4.2.4 that  $M_n(R)$  is  $n$ -clean.  $\square$

We next show that if a group ring is  $n$ -weakly clean locally, then it is  $n$ -weakly clean.

**Proposition 4.3.5.** *Let  $R$  be a ring and let  $G$  be a group. If  $RG$  is  $n$ -weakly clean locally, then  $RG$  is  $n$ -weakly clean.*

**Proof.** Let  $x \in RG$ . Then  $x = x_1g_1 + \cdots + x_ng_n$  for some  $x_i \in R$  and  $g_i \in G$  ( $i = 1, \dots, n$ ). Let  $S_x$  be the support group of  $x$ . Then  $S_x$  is finitely generated. It follows by the hypothesis that  $RS_x$  is  $n$ -weakly clean. Therefore,  $x \in RS_x$  is  $n$ -weakly clean in  $RS_x$  and hence,  $n$ -weakly clean in  $RG$ .  $\square$

We now obtain some necessary conditions for a commutative group ring to be  $n$ -weakly clean.

**Theorem 4.3.1.** *Let  $R$  be a commutative ring and let  $G$  be an abelian group. For any positive integer  $n$ , if  $RG$  is  $n$ -weakly clean, then  $R$  is  $n$ -weakly clean and  $G$  is locally finite.*

**Proof.** Since  $RG/\Delta \cong R$  where  $\Delta$  is the augmentation ideal of  $RG$ , it follows readily by Proposition 4.2.2 that  $R$  is  $n$ -weakly clean. Suppose that  $G$  is not locally finite. Then  $G$  is not torsion; hence,  $G/t(G)$  is nontrivial and torsion-free, where  $t(G)$  is the torsion subgroup of  $G$ . Since  $R(G/t(G)) \cong RG/R(t(G))$  is a homomorphic image of  $RG$  and  $RG$  is  $n$ -weakly clean, it follows by Proposition 4.2.2 that  $R(G/t(G))$  is  $n$ -weakly clean. We may therefore assume that  $G$  is



torsion-free. If  $G$  has rank greater than 1, then  $G$  has a torsion-free quotient  $G'$  of rank 1. But since  $RG'$  is also  $n$ -weakly clean, we can assume that  $G$  is of rank 1. Thus,  $G$  is isomorphic to a subgroup of  $(\mathbb{Q}, +)$ . Since  $R$  is commutative, then  $R/M$  is a field where  $M$  is a maximal ideal of  $R$ . Furthermore,  $(R/M)G$  is  $n$ -weakly clean because  $(R/M)G \cong RG/MG$  is a homomorphic image of  $RG$  by Proposition 4.2.2. Hence, we can assume that  $R$  is a field. Since  $G$  is torsion-free, there exists an element  $g \in G$  such that  $g^{-1} \neq g$ . Now since  $g + \cdots + g^n + g^{-1} + \cdots + g^{-n}$  is  $n$ -weakly clean in  $RG$ , there exists a finitely generated subgroup  $G_1$  of  $G$  such that  $g \in G_1$  and  $g + \cdots + g^n + g^{-1} + \cdots + g^{-n}$  is  $n$ -weakly clean in  $RG_1$ . From above,  $G_1$  is isomorphic to a finitely generated subgroup of  $(\mathbb{Q}, +)$ . Since every finitely generated subgroup of  $(\mathbb{Q}, +)$  is cyclic, so is  $G_1$ , and we can write  $G_1 = \langle h \rangle$ . Thus,  $g = h^k$ ,  $g^{-1} = h^{-k}$  for some  $k \in \mathbb{N}$ . Note that there is a natural isomorphism  $R\langle h \rangle \cong R[x, x^{-1}]$  with  $h^k + \cdots + h^{nk} + h^{-k} + \cdots + h^{-nk} \leftrightarrow x^k + \cdots + x^{nk} + x^{-k} + \cdots + x^{-nk}$ . This implies that  $x^k + \cdots + x^{nk} + x^{-k} + \cdots + x^{-nk}$  is  $n$ -weakly clean in  $R[x, x^{-1}]$  which is impossible because  $Id(R[x, x^{-1}]) \subseteq R$  and  $U(R[x, x^{-1}]) \subseteq \{ax^i \mid 0 \neq a \in R, i \in \mathbb{Z}\}$ . Hence,  $G$  must be locally finite.  $\square$

We obtain a partial converse of Theorem 4.3.1 as follows:

**Theorem 4.3.2.** *Let  $R$  be a commutative ring and let  $G$  be an abelian group. For any positive integer  $n$ , if  $R$  is  $n$ -weakly clean and  $G$  is a locally finite  $p$ -group with  $p$  nilpotent in  $R$ , then  $RG$  is  $n$ -weakly clean.*

**Proof.** Since  $G$  is a locally finite  $p$ -group with  $p$  nilpotent in  $R$ , it follows by [24, Proposition 16 (ii)] that  $\Delta$ , the augmentation ideal of  $RG$ , is nil. Then since  $RG$  is commutative, we have that  $\Delta \subseteq J(RG)$ . Now since  $RG/\Delta \cong R$  is  $n$ -weakly clean and idempotents lift modulo  $\Delta$ , Proposition 4.2.9 tells us that

$RG$  is  $n$ -weakly clean. □

## 4.4 Some results on $n$ -weakly clean matrices

We first show the following:

**Proposition 4.4.1.** *Let  $R$  be a ring and let  $n$  be a positive integer. If  $a \in R$  is  $n$ -weakly clean, then for any  $b \in R$ ,  $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$  is  $n$ -weakly clean in  $M_2(R)$ .*

**Proof.** Since  $a$  is  $n$ -weakly clean, then  $a = u_1 + \cdots + u_n + e$  or  $a = u_1 + \cdots + u_n - e$  for some  $u_i \in U(R)$  for  $i = 1, \dots, n$  and  $e \in Id(R)$ . If  $a = u_1 + \cdots + u_n + e$ , then

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} u_1 & b \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} u_2 & 0 \\ 0 & 1 \end{pmatrix} + \cdots + \begin{pmatrix} u_n & 0 \\ 0 & (-1)^n \end{pmatrix} + \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, \\ \text{if } n \text{ is odd,} \\ \begin{pmatrix} u_1 & b \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} u_2 & 0 \\ 0 & 1 \end{pmatrix} + \cdots + \begin{pmatrix} u_n & 0 \\ 0 & (-1)^n \end{pmatrix} + \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, \\ \text{if } n \text{ is even.} \end{cases}$$

Note that  $\begin{pmatrix} u_1 & b \\ 0 & -1 \end{pmatrix}$ ,  $\begin{pmatrix} u_i & 0 \\ 0 & (-1)^i \end{pmatrix}$  are invertible matrices with

$$\begin{pmatrix} u_1 & b \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} u_1^{-1} & u_1^{-1}b \\ 0 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} u_i & 0 \\ 0 & (-1)^i \end{pmatrix}^{-1} = \begin{pmatrix} u_i^{-1} & 0 \\ 0 & (-1)^i \end{pmatrix} \quad (i = 2, \dots, n).$$

It is clear that  $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$  are idempotents.

If  $a = u_1 + \cdots + u_n - e$ , then

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} u_1 & b \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} u_2 & 0 \\ 0 & -1 \end{pmatrix} + \cdots + \begin{pmatrix} u_n & 0 \\ 0 & (-1)^{n+1} \end{pmatrix} - \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, \\ \text{if } n \text{ is odd,} \\ \\ \begin{pmatrix} u_1 & b \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} u_2 & 0 \\ 0 & -1 \end{pmatrix} + \cdots + \begin{pmatrix} u_n & 0 \\ 0 & (-1)^{n+1} \end{pmatrix} - \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, \\ \text{if } n \text{ is even.} \end{cases}$$

Note that  $\begin{pmatrix} u_1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u_i & 0 \\ 0 & (-1)^{i+1} \end{pmatrix}$  are invertible matrices with

$$\begin{pmatrix} u_1 & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} u_1^{-1} & -u_1^{-1}b \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} u_i & 0 \\ 0 & (-1)^{i+1} \end{pmatrix}^{-1} = \begin{pmatrix} u_i^{-1} & 0 \\ 0 & (-1)^{i+1} \end{pmatrix} \quad (i = 2, \dots, n).$$

As above,  $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$  are clearly idempotents. Thus  $A$  is  $n$ -weakly clean in  $R$ . □

**Remark.** Proposition 4.4.1 also holds if  $A = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ .

In general, for a commutative ring  $R$ , in order for  $A \in M_t(R)$  to be  $m$ -weakly clean, we show that it suffices for all the entries in the main diagonal of  $A$  to be  $n$ -weakly clean for some  $n < m$ .

**Theorem 4.4.1.** *Let  $R$  be a commutative ring and let  $n \geq 1, t \geq 2$  be integers. If  $a_{11}, \dots, a_{tt} \in R$  are  $n$ -weakly clean, then for any  $a_{ij} \in R$  ( $i, j \in \{1, \dots, t\}, i \neq j$ ), the matrix  $A = (a_{ij}) \in M_t(R)$  is  $m$ -weakly clean for all  $m \geq n + 1$ .*

**Proof.** For each  $i \in \{1, \dots, t\}$ , since  $a_{ii}$  is  $n$ -weakly clean, so we have  $a_{ii} = u_{i1} + \cdots + u_{in} + e_i$  or  $a_{ii} = u_{i1} + \cdots + u_{in} - e_i$  for some  $u_{i1}, \dots, u_{in} \in U(R)$  and

$e_i \in Id(R)$ . Let  $U_1 = (a_{ij}^{(1)})$ ,  $U_2 = (a_{ij}^{(2)}) \in M_t(R)$  be defined as follows:

$$a_{ij}^{(1)} = \begin{cases} a_{ij}, & \text{if } i < j \\ u_{i_1}, & \text{if } i = j \\ 0, & \text{if } i > j \end{cases} \quad (i, j = 1, \dots, t).$$

$$a_{ij}^{(2)} = \begin{cases} 0, & \text{if } i < j \\ u_{i_2}, & \text{if } i = j \\ a_{ij}, & \text{if } i > j \end{cases} \quad (i, j = 1, \dots, t).$$

Let  $U_3 = (a_{ij}^{(3)}), \dots, U_n = (a_{ij}^{(n)}) \in M_t(R)$  be defined as follows:

$$a_{ij}^{(k)} = \begin{cases} 0, & \text{if } i \neq j \\ u_{i_k}, & \text{if } i = j \end{cases} \quad (k = 3, \dots, n; i, j = 1, \dots, t).$$

Now let  $L = \{i \in \{1, \dots, t\} \mid a_{ii} \text{ is } n\text{-weakly clean but not } n\text{-clean}\}$  and let

$U_{n+1} = (a_{ij}^{(n+1)}) \in M_t(R)$  be defined as follows:

$$a_{ij}^{(n+1)} = \begin{cases} 0, & \text{if } i \neq j \\ 1 - 2e_i, & \text{if } i = j, i \in L \\ 1, & \text{if } i = j, i \notin L \end{cases} \quad (i, j = 1, \dots, t).$$

Note that  $U_1, \dots, U_{n+1}$  are all invertible in  $M_t(R)$  since their determinants are units in  $R$ . Now let  $E = (e_{ij}) \in M_t(R)$  be defined as follows:

$$e_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1 - e_i, & \text{if } i = j \end{cases} \quad (i, j = 1, \dots, t).$$

Then  $E^2 = E$  and  $A = U_1 + \dots + U_{n+1} - E$ . Hence,  $A$  is  $(n+1)$ -weakly clean and by Proposition 4.2.1,  $A$  is  $m$ -weakly clean for all  $m \geq n+1$ .  $\square$

# Chapter 5

## Pseudo Weakly Clean Rings

### 5.1 Introduction

Let  $R$  be a ring. We say that an element  $x \in R$  is pseudo weakly clean if there exist an idempotent  $e$  and a unit  $u$  in  $R$  such that  $x - e - u \in (1 - e)Rx$  or  $x + e - u \in (1 - e)Rx$ . The ring  $R$  is said to be pseudo weakly clean if all of its elements are pseudo weakly clean.

Weakly clean rings are clearly pseudo weakly clean. However, a pseudo weakly clean ring is not necessarily weakly clean, as will be shown in Section 5.2.

In Sections 5.3 and 5.4, we obtain some characterisations and properties of pseudo weakly clean rings. In particular, we show that corner rings of pseudo weakly clean rings are pseudo weakly clean. In the case  $R$  is an abelian ring, we will show that pseudo weakly clean and weakly clean are equivalent conditions on  $R$ . We also obtain a relation between unit regular rings and pseudo weakly clean rings. Finally, in Section 5.5, we consider the pseudo weakly clean condition in non-unital rings. We show that if  $I$  is a non-unital subring of a ring  $R$  and  $I$  is pseudo weakly clean, then every  $x \in I$  is pseudo weakly clean in  $R$ .

## 5.2 An example

By using arguments similar to those in [58, Example 3.1], we obtain an example of a ring which is pseudo weakly clean but not weakly clean.

**Example 5.2.1.** Let  $F$  be a field and let  $M_{\mathbb{N}}(F)$  be the ring of all infinite matrices over  $F$  with finite columns. As usual, a matrix in  $M_{\mathbb{N}}(F)$  is denoted by  $A = (a_{ij})_{i,j}$ , where  $(a_{ij})_i$  are columns and  $(a_{ij})_j$  are rows in  $A$ . Let  $R = \{A = (a_{ij})_{i,j} \in M_{\mathbb{N}}(F) \mid \text{there exists a positive integer } n_A \text{ such that } a_{ij} = a_{i+1 \ j+1} \text{ for every } i \geq n_A, j \geq 1\}$ . That is,  $R$  consists of matrices of the form

$$\begin{pmatrix} \vdots & \vdots & \vdots & & & & \\ * & * & * & \dots & & & \\ & & a_1 & a_2 & a_3 & \dots & \\ & & & a_1 & a_2 & a_3 & \dots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

where the first finitely many rows are arbitrary. Then  $R$  is a ring and following [58, Example 3.1], every idempotent and every unit in  $R$  is upper triangular (ignoring the first finitely many rows). It follows that matrices that are nonzero below the main diagonal (ignoring the first finitely many rows) cannot be written as a sum or a difference of a unit and an idempotent in  $R$ . Therefore,  $R$  is not weakly clean. To show that  $R$  is pseudo weakly clean, let  $A \in R$  and suppose first that  $A$  is upper triangular, ignoring the first finitely many rows. Then we may write  $A$  as a block decomposition

$$A = \begin{pmatrix} A_0 & X \\ 0 & T \end{pmatrix},$$

where  $A_0$  is a finite matrix and  $T = (t_{ij})$  is an upper triangular matrix such that  $t_{ij} = t_{i+1 \ j+1}$  for every  $i, j$ . By Han and Nicholson [32], the ring of finite matrices over  $F$  is clean. Hence, we may write  $A_0 = U_0 + E_0$  for some unit  $U_0$  and

idempotent  $E_0$  in  $R$ . If the main diagonal of  $T$  is non-zero, then  $T$  is invertible in  $R$ , and we may write

$$A = \begin{pmatrix} U_0 & X \\ 0 & T \end{pmatrix} + \begin{pmatrix} E_0 & 0 \\ 0 & 0 \end{pmatrix}$$

where  $\begin{pmatrix} U_0 & X \\ 0 & T \end{pmatrix}$  is invertible and  $\begin{pmatrix} E_0 & 0 \\ 0 & 0 \end{pmatrix}$  is an idempotent in  $R$ . If the main diagonal of  $T$  is zero, then  $T - 1$  is invertible and we have

$$A = \begin{pmatrix} U_0 & X \\ 0 & T - 1 \end{pmatrix} + \begin{pmatrix} E_0 & 0 \\ 0 & 1 \end{pmatrix}$$

where  $\begin{pmatrix} U_0 & X \\ 0 & T - 1 \end{pmatrix}$  is invertible and  $\begin{pmatrix} E_0 & 0 \\ 0 & 1 \end{pmatrix}$  is an idempotent in  $R$ . It follows that  $A$  is clean (hence, pseudo weakly clean).

Now suppose that  $A$  is not an upper triangular matrix (ignoring the first finitely many rows). We may then write  $A$  as a block decomposition,

$$A = \begin{pmatrix} A_0 & X \\ K & T \end{pmatrix}$$

where  $A_0 \in M_n(F)$  is a finite matrix and  $n$  is chosen large enough so that we have  $n \geq n_A$ . Since  $T = (t_{ij})$  satisfies  $t_{ij} = t_{i+1, j+1}$  for every  $i, j$  and  $T$  is not strictly upper triangular, so  $T$  has a left inverse in  $R$ , say  $S$ . Then  $ST = 1$  and we note that  $S$  may be chosen such that  $SK = 0$ . Similarly, since  $1 - T$  is not strictly upper triangular, we may find  $V \in R$  such that  $V(1 - T) = 1$  and  $VK = 0$ . Since  $A_0$  is clean (by [32]), we may write  $A_0 = U_0 + E_0$  where  $U_0$  is invertible and  $E_0$  is an idempotent in  $R$ . Let

$$E = \begin{pmatrix} E_0 & E_0XV \\ 0 & 0 \end{pmatrix}, U = \begin{pmatrix} U_0 & XV \\ 0 & -1 \end{pmatrix}, Z = \begin{pmatrix} E_0 & -XV - E_0XS \\ 0 & 1 + S \end{pmatrix}.$$

It can be shown that  $E \in Id(R)$  and  $U \in U(R)$  such that  $A - E - U = (1 - E)ZA \in (1 - E)RA$ . Hence,  $A$  is pseudo weakly clean.

### 5.3 Some characterisations of pseudo weakly clean rings

We first give an element-wise characterisation of pseudo weakly clean rings as follows:

**Proposition 5.3.1.** *Let  $R$  be a pseudo weakly clean ring. Then for any  $x \in R$ , there exists an idempotent  $g \in R$  such that  $g \in Rx$  and  $1 - g \in R(1 - x)$  or  $1 - g \in R(1 + x)$ .*

**Proof.** Let  $x \in R$ . Then  $x - e - u = (1 - e)rx$  or  $x + e - u = (1 - e)rx$  for some  $e \in Id(R)$ ,  $u \in U(R)$  and  $r \in R$ . Set  $g = 1 - u^{-1}eu$ . Then  $g \in Id(R)$ . If  $x - e - u = (1 - e)rx$ , then by multiplying  $u^{-1}$  on the left of both sides of the equation, we obtain  $u^{-1}x - u^{-1}e - u^{-1}rx + u^{-1}erx = 1$ . Hence,

$$\begin{aligned}
 g = 1 - u^{-1}eu &= u^{-1}x - u^{-1}e - u^{-1}rx + u^{-1}erx - u^{-1}eu \\
 &= u^{-1}(1 - r)x + u^{-1}erx - u^{-1}(e + eu) \\
 &= u^{-1}(1 - r)x + u^{-1}erx - u^{-1}ex \quad (\because e(x - e - u) = 0) \\
 &= u^{-1}((1 - r) - e(1 - r))x \\
 &= u^{-1}(1 - e)(1 - r)x \in Rx.
 \end{aligned}$$

Note that  $1 - g = 1 - (1 - u^{-1}eu) = -u^{-1}e + u^{-1}(e + eu) = -u^{-1}e + u^{-1}(ex) = -u^{-1}e(1 - x) \in R(1 - x)$ . Now, if  $x + e - u = (1 - e)rx$ , then by multiplying  $u^{-1}$  on the left of both sides of the equation, we obtain  $u^{-1}x + u^{-1}e - u^{-1}rx + u^{-1}erx = 1$ .



Hence,

$$\begin{aligned}
g = 1 - u^{-1}eu &= u^{-1}x + u^{-1}e - u^{-1}rx + u^{-1}erx - u^{-1}eu \\
&= u^{-1}x - u^{-1}rx + u^{-1}erx + u^{-1}(e - eu) \\
&= u^{-1}(x - rx + erx + (-ex)) \quad (\because e(x + e - u) = 0) \\
&= u^{-1}((1 - r) - e(1 - r))x \\
&= u^{-1}(1 - e)(1 - r)x \in Rx.
\end{aligned}$$

We also note that  $1 - g = 1 - (1 - u^{-1}eu) = u^{-1}e - u^{-1}(e - eu) = u^{-1}e - u^{-1}(-ex) = u^{-1}e(1 + x) \in R(1 + x)$ . This completes the proof.  $\square$

**Remark.** Since a weakly clean ring is pseudo weakly clean, Proposition 5.3.1 tells us that the left analogue of Proposition 2.2.1 also holds, that is, a pseudo weakly clean ring is left weakly exchange. In fact, by using the left analogue of the arguments in the proof of Theorem 2.2.1, it may be shown that the left analogue of Theorem 2.2.1 also holds. Thus, by combining Proposition 5.3.1 and the left analogue of Theorem 2.2.1, we obtain the following:

**Theorem 5.3.1.** *Let  $R$  be an abelian ring. Then  $R$  is weakly clean if and only if  $R$  is pseudo weakly clean.*

**Remarks.**

(a) By Proposition 5.3.1, we see that weakly clean rings are right and left weakly exchange. It is however not known to us whether the weakly exchange notion is left-right symmetric.

(b) By Theorem 5.3.1, we may deduce that a pseudo weakly clean ring is not necessarily exchange. Indeed, by Theorem 5.3.1, an abelian ring which is

pseudo weakly clean is weakly clean. However, by [22], an abelian weakly clean ring is not necessarily exchange.

It is known by [8, Theorem 1] that a unit regular ring is clean (hence, pseudo weakly clean). In the following, we show that the converse is true under a certain additional condition.

**Proposition 5.3.2.** *Let  $R$  be a ring. Then  $R$  is unit regular if and only if every  $x \in R$  takes one of the following forms:*

(a)  $x = u + e + (1 - e)rx$  for some  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$  such that

$$Rx \cap R(e + (1 - e)rx) = \{0\}.$$

(b)  $x = u - e - (1 - e)rx$  for some  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$  such that

$$Rx \cap R(e + (1 - e)rx) = \{0\}.$$

**Proof.** ( $\Rightarrow$ ): This follows readily by [8, Theorem 1].

( $\Leftarrow$ ): Let  $x \in R$ . Assume that (a) occurs. Then  $x = u + e + (1 - e)rx$  for some  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$  such that  $Rx \cap R(e + (1 - e)rx) = \{0\}$ . Let  $f = 1 - e$ .

Then  $x = u + e + frx$  and we have  $(e + frx)u^{-1}x = (e + frx)u^{-1}(u + e + frx) = (1 + (e + frx)u^{-1})(e + frx) \in R(e + frx) \cap Rx = \{0\}$ . Thus,  $(e + frx)u^{-1}x = 0$

and hence,  $(x - u)u^{-1}x = 0$ . This gives us  $x = xu^{-1}x$ . Now assume that (b)

occurs. Then  $x = u - e - (1 - e)rx$  for some  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$  such that  $Rx \cap R(e + (1 - e)rx) = \{0\}$ . Let  $f = 1 - e$ . Then  $x = u - e - frx$  and

we have  $(-e - frx)u^{-1}x = (-e - frx)u^{-1}(u - e - frx) = -(e + frx)u^{-1}(u - (e + frx)) = (-1 + (e + frx)u^{-1})(e + frx) \in R(e + frx) \cap Rx = \{0\}$ . Thus, we

have  $-(e + frx)u^{-1}x = 0$  from which we get  $(x - u)u^{-1}x = 0$ . It follows that

$x = xu^{-1}x$ . Hence, in both cases we have that  $x$  is unit regular. This completes

the proof. □

**Theorem 5.3.2.** *Let  $R$  be a ring. Then the following conditions are equivalent:*

- (a)  *$R$  is unit regular.*
- (b) *Every  $x \in R$  can be written as  $x = u + f$  or  $x = u - f$  for some  $u \in U(R)$  and  $f \in Id(R)$  such that  $Rx \cap Rf = \{0\}$ .*
- (c) *Every  $x \in R$  can be written as  $x = u + e + (1 - e)rx$  for some  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$  such that  $Rx \cap R(e + (1 - e)rx) = \{0\}$  or  $x = u - e - (1 - e)rx$  for some  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$  such that  $Rx \cap R(e + (1 - e)rx) = \{0\}$ .*

**Proof.** (a)  $\Leftrightarrow$  (b): This follows readily by [22, Theorem 2.2].

(a)  $\Leftrightarrow$  (c): This follows by Proposition 5.3.2. □

We next show how a weakly clean matrix over a commutative ring is related to a pseudo weakly clean element.

**Proposition 5.3.3.** *Let  $R$  be a commutative ring and let  $a \in R$ . Then the matrix  $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$  is weakly clean in  $M_2(R)$  if and only if  $a$  is pseudo weakly clean in  $R$ .*

**Proof.** Let  $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$  be weakly clean. Then by Proposition 2.2.3, there exist  $E \in Id(M_2(R))$  and  $U \in U(M_2(R))$  such that  $UA = EU + I_2$  or  $UA + EU = I_2$  where  $I_2$  is the identity matrix  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . By multiplying  $I_2 - E \in M_2(R)$  on the left of  $UA = EU + I_2$  and  $UA + EU = I_2$ , we get  $(I_2 - E)UA = I_2 - E$ . Since  $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ , it follows that  $I_2 - E \in M_2(R)$  is an idempotent with zeros as the entries in the second column. Hence,  $I_2 - E = \begin{pmatrix} e & 0 \\ x & 0 \end{pmatrix}$  for some  $e \in Id(R)$  and  $x \in Re$ . Let  $f = 1 - e$  and let  $U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ .

From the equation  $UA = EU + I_2$ , we get

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + I_2 \quad (5.1)$$

which gives us  $\alpha a = f\alpha + 1$ ,  $f\beta = 0$ ,  $\gamma a = \gamma - x\alpha$  and  $\delta - x\beta + 1 = 0$ . Since  $U$  is invertible, so is  $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} U$ . Note that

$$\begin{aligned} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} U &= \begin{pmatrix} 1 - \beta x & \beta \\ -x & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= \begin{pmatrix} \alpha + \beta(\gamma - x\alpha) & \beta + \beta(\delta - x\beta) \\ \gamma - x\alpha & \delta - x\beta \end{pmatrix}. \end{aligned} \quad (5.2)$$

By using the relations obtained from (5.1), we reduce the matrix in (5.2) to  $\begin{pmatrix} \alpha + \beta\gamma a & 0 \\ \gamma a & -1 \end{pmatrix}$  which is invertible because  $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} U$  is invertible. It follows that  $\alpha + \beta\gamma a$  is invertible in  $R$ . Let  $u = \alpha + \beta\gamma a$ . Then substituting  $\alpha = u - \beta\gamma a$  into the equation  $\alpha a = f\alpha + 1$  obtained from (5.1), we get

$$\begin{aligned} ua - \beta\gamma a^2 &= f(u - \beta\gamma a) + 1 = fu + 1 \\ \Rightarrow u^{-1}(ua - \beta\gamma a^2) &= u^{-1}(fu + 1) \\ \Rightarrow a - u^{-1}\beta\gamma a^2 &= u^{-1}fu + u^{-1} \\ \Rightarrow a - u^{-1}fu - u^{-1} &= u^{-1}\beta\gamma a^2. \end{aligned}$$

From the relation  $f\beta = 0$  in (5.1), we have  $\beta = e\beta$ . We then obtain  $a - u^{-1}fu - u^{-1} = u^{-1}\beta\gamma a^2 = u^{-1}e\beta\gamma a^2 = u^{-1}eu(u^{-1}\beta\gamma)a^2 = (1 - u^{-1}fu)(u^{-1}\beta\gamma)a^2 \in (1 - u^{-1}fu)Ra$ , as desired.

Now consider the equation  $UA = -EU + I_2$ . We have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = - \begin{pmatrix} f & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + I_2 \quad (5.3)$$

which gives us  $\alpha a = -f\alpha + 1$ ,  $f\beta = 0$ ,  $\gamma a = x\alpha - \gamma$  and  $x\beta - \delta + 1 = 0$ . Since

$U$  is invertible, so is  $\begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} U$ . Note that

$$\begin{aligned} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} U &= \begin{pmatrix} 1 + \beta x & -\beta \\ -x & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= \begin{pmatrix} \alpha + \beta(x\alpha - \gamma) & \beta - \beta(\delta - x\beta) \\ \gamma - x\alpha & \delta - x\beta \end{pmatrix}. \end{aligned} \quad (5.4)$$

By using the relations obtained from (5.3), we reduce the matrix in (5.4) to  $\begin{pmatrix} \alpha + \beta\gamma a & 0 \\ -\gamma a & 1 \end{pmatrix}$  which is invertible because  $\begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} U$  is invertible. It follows that  $v = \alpha + \beta\gamma a$  is invertible and by using arguments similar to those above, it may be shown that  $a + v^{-1}fv - v^{-1} = v^{-1}\beta\gamma a^2 = v^{-1}e\beta\gamma a^2 = v^{-1}ev(v^{-1}\beta\gamma)a^2 = (1 - v^{-1}fv)(v^{-1}\beta\gamma)a^2 \in (1 - v^{-1}fv)Ra$ , as desired. Hence, the element  $a$  is pseudo weakly clean in  $R$ .

Conversely, suppose that  $a \in R$  is pseudo weakly clean. Since  $R$  is commutative (hence, abelian), it follows by Theorem 5.3.1 that  $a$  is weakly clean. Thus,  $a = u + e$  or  $a = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . Then  $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$  or  $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$ , that is,  $A$  is weakly clean. This completes the proof.  $\square$

## 5.4 Some properties of pseudo weakly clean rings

In this section we determine whether pseudo weakly clean rings have properties similar to those of other related rings.

**Proposition 5.4.1.** *Every homomorphic image of a pseudo weakly clean ring is pseudo weakly clean.*

**Proof.** Let  $R$  be a pseudo weakly clean ring and let  $\phi : R \rightarrow S$  be a ring epimorphism. Let  $y \in S$ . Then  $y = \phi(x)$  for some  $x \in R$ . Since  $R$  is pseudo weakly clean, we have  $x = u + e + (1 - e)rx$  or  $x = u - e + (1 - e)rx$  for some

$u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$ . Since  $\phi$  is an epimorphism, we then have that  $\phi(u) \in U(S)$ ,  $\phi(e) \in Id(S)$  and  $y = \phi(x) = \phi(u) + \phi(e) + \phi(1 - e)\phi(r)\phi(x) = \phi(u) + \phi(e) + (1 - \phi(e))\phi(r)y$  or  $y = \phi(x) = \phi(u) - \phi(e) + \phi(1 - e)\phi(r)\phi(x) = \phi(u) - \phi(e) + (1 - \phi(e))\phi(r)y$ . That is,  $y$  is pseudo weakly clean in  $S$ . It follows that  $\phi(R) = S$  is pseudo weakly clean.  $\square$

In [59, Proposition 2.3], it has been shown that the direct product ring  $R = \prod_{i \in I} R_i$  is pseudo clean if and only if each  $R_i$  is pseudo clean. For pseudo weakly clean rings we show the following:

**Proposition 5.4.2.** *The direct product ring  $R = \prod_{k \in I} R_k$  is pseudo weakly clean if and only if each  $R_k$  is pseudo weakly clean and at most one  $R_k$  is not a pseudo clean ring.*

In order to prove Proposition 5.4.2, we first prove the following equivalence:

**Proposition 5.4.3.** *Let  $R$  be a ring. Then the following conditions are equivalent:*

- (a)  $R$  is a pseudo clean ring.
- (b) Every element  $x \in R$  has the form  $x = u - e + (1 - e)rx$  where  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$ .

**Proof.** (a)  $\Rightarrow$  (b): Let  $x \in R$ . Since  $R$  is pseudo clean, we have  $-x = v + e + (1 - e)r(-x)$  for some  $v \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$ . Then  $x = u - e + (1 - e)rx$  where  $u = -v \in U(R)$ .

(b)  $\Rightarrow$  (a): Let  $x \in R$ . Then  $-x = u - e + (1 - e)r(-x)$  for some  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$ . It follows that  $x = (-u) + e + (1 - e)rx$  which shows that  $x$  is pseudo clean.  $\square$

**Proof of Proposition 5.4.2.** ( $\Rightarrow$ ): Suppose that  $R = \prod_{k \in I} R_k$  is pseudo weakly clean. By Proposition 5.4.1, it follows that each  $R_k$ , being a homomorphic image of  $R$ , is pseudo weakly clean. Suppose that  $R_i$  and  $R_j$  ( $i \neq j$ ) are not pseudo clean. Since  $R_i$  is not pseudo clean, then by Proposition 5.4.3(b), there exists  $x_i \in R_i$  such that  $x_i \neq u - e + (1 - e)rx_i$  for any  $u \in U(R_i)$ ,  $e \in Id(R_i)$  and  $r \in R_i$ . But since  $R_i$  is pseudo weakly clean, we must have  $x_i = u_i + e_i + (1 - e_i)r_i x_i$  for some  $u_i \in U(R_i)$ ,  $e_i \in Id(R_i)$  and  $r_i \in R_i$ . Now since  $R_j$  is not pseudo clean but is pseudo weakly clean, there is an  $x_j \in R_j$  such that  $x_j = u_j - e_j + (1 - e_j)r_j x_j$  for some  $u_j \in U(R_j)$ ,  $e_j \in Id(R_j)$  and  $r_j \in R_j$  but  $x_j \neq u + e + (1 - e)rx_j$  for any  $u \in U(R_j)$ ,  $e \in Id(R_j)$  and  $r \in R_j$ . Let  $y = (y_k) \in R$  such that

$$y_k = \begin{cases} x_k, & k \in \{i, j\}, \\ 0, & k \notin \{i, j\}. \end{cases}$$

Then  $y \neq u \pm e + (1 - e)ry$  for any  $u \in U(R)$ ,  $e \in Id(R)$  and  $r \in R$ , which contradicts the assumption that  $R$  is pseudo weakly clean. Hence, we can only have at most one  $R_i$  which is not pseudo clean.

( $\Leftarrow$ ): If every  $R_i$  is pseudo clean, then it follows by [59, Proposition 2.3] that  $R = \prod_{k \in I} R_k$  is also pseudo clean; hence, pseudo weakly clean. Suppose that  $R_{i_0}$  is pseudo weakly clean but not pseudo clean and all the other  $R_i$ 's are pseudo clean. Let  $x = (x_i) \in R = \prod_{k \in I} R_k$ . Then for  $x_{i_0} \in R_{i_0}$ , we may write  $x_{i_0} = u_{i_0} + e_{i_0} + (1 - e_{i_0})r_{i_0}x_{i_0}$  or  $x_{i_0} = u_{i_0} - e_{i_0} + (1 - e_{i_0})r_{i_0}x_{i_0}$  where  $u_{i_0} \in U(R_{i_0})$ ,  $e_{i_0} \in Id(R_{i_0})$  and  $r_{i_0} \in R_{i_0}$ . If  $x_{i_0} = u_{i_0} + e_{i_0} + (1 - e_{i_0})r_{i_0}x_{i_0}$ , then for  $i \neq i_0$ , since  $R_i$  is pseudo clean, we may let  $x_i = u_i + e_i + (1 - e_i)r_i x_i$  where  $u_i \in U(R_i)$ ,  $e_i \in Id(R_i)$  and  $r_i \in R_i$ . On the other hand, if  $x_{i_0} = u_{i_0} - e_{i_0} + (1 - e_{i_0})r_{i_0}x_{i_0}$ , then for  $i \neq i_0$ , since  $R_i$  is pseudo clean, it follows by Proposition 5.4.3(b) that we may let  $x_i = u_i - e_i + (1 - e_i)r_i x_i$  where  $u_i \in U(R_i)$ ,  $e_i \in Id(R_i)$  and  $r_i \in R_i$ .

Hence,  $x = u + e + (1 - e)rx$  or  $x = u - e + (1 - e)rx$  where  $u = (u_j) \in U(R)$ ,  $e = (e_j) \in Id(R)$  and  $r = (r_j) \in R$ . Thus,  $x$  is pseudo weakly clean. This completes the proof.  $\square$

In [58], it is shown that corner rings of clean rings are not necessarily clean. In the following we show that corners of pseudo weakly clean rings are pseudo weakly clean.

**Proposition 5.4.4.** *Let  $R$  be a pseudo weakly clean ring. Then so is  $eRe$  for any  $e \in Id(R)$ .*

**Proof.** Let  $R$  be a pseudo weakly clean ring and let  $e \in Id(R)$ . Let  $x \in eRe$ . Since  $R$  is pseudo weakly clean, there exist  $g \in Id(R)$  and  $u \in U(R)$  such that  $x - g - u \in (1 - g)Rx$  or  $x + g - u \in (1 - g)Rx$ . Write  $h = 1 - g$  and  $f = 1 - e$ . Since  $x \in eRe$ , it follows that  $xf = 0$ . Hence,  $gf + uf = 0$  or  $gf - uf = 0$ . It follows that

$$\begin{aligned} eu^{-1}gf &= \begin{cases} eu^{-1}(-uf), & \text{if } gf + uf = 0 \\ eu^{-1}(uf), & \text{if } gf - uf = 0 \end{cases} \\ &= 0 \quad (\because ef = 0). \end{aligned}$$

Hence,  $eu^{-1}g \in eRe$ .

For  $x + g - u \in (1 - g)Rx$ , we have  $gf = uf$ ,  $eu^{-1}gf = 0$  and  $eu^{-1}g \in eRe$  from the previous paragraph. Observe that  $eu^{-1}geue = eu^{-1}gue = eu^{-1}gu(1 - f) = eu^{-1}g(u - uf) = eu^{-1}g(u - gf) = eu^{-1}gu - eu^{-1}gf = eu^{-1}gu$ . It follows that  $(eueu^{-1}g)^2 = eu(eu^{-1}geue)u^{-1}g = eu(eu^{-1}gu)u^{-1}g = eueu^{-1}g$ . Hence,  $e' = eueu^{-1}g \in Id(eRe)$ . We also note that  $(hfu^{-1}g)^2 = hfu^{-1}(gh)fu^{-1}g = 0$ . Hence,  $v = (1 + hfu^{-1}g)u$  is invertible in  $R$  where  $v^{-1} = u^{-1}(1 - hfu^{-1}g)$ . We have  $vf = (1 + hfu^{-1}g)uf = uf + hfu^{-1}guf = gf + hfu^{-1}gf = gf +$



$hfu^{-1}uf = gf + hf = f$  which implies that  $v^{-1}f = f$ . Next, we show that  $v' = eve$  is invertible in  $eRe$ . We have  $v'(ev^{-1}e) = (eve)v^{-1}e = ev(1-f)v^{-1}e = e(v-vf)v^{-1}e = e(v-f)v^{-1}e = e$ . Similarly,  $(ev^{-1}e)v' = e$ . It follows that  $v' = eve \in U(eRe)$ . Since  $x + g - u \in (1-g)Rx$ , we then have  $g(x + g - u) = 0$ . It follows that  $gx = -g + gu$ . Then

$$\begin{aligned}
e'(x + e' - v') &= eueu^{-1}gx + eueu^{-1}g - eueu^{-1}g(eve) \\
&= eueu^{-1}(-g + gu) + eueu^{-1}g - eu(eu^{-1}ge)ve \\
&= eueu^{-1}gu - eu(eu^{-1}g)ve \\
&= eueu^{-1}gu - eueu^{-1}g((1 + hfu^{-1}g)u)e \\
&= eueu^{-1}gu - eueu^{-1}gue \quad (\because gh = 0) \\
&= eueu^{-1}gu(1 - e) \\
&= eueu^{-1}guf \\
&= eueu^{-1}gf \quad (\because uf = gf) \\
&= eueu^{-1}uf \quad (\because gf = uf) \\
&= euef \\
&= 0.
\end{aligned}$$

Hence,  $x + e' - v' = (e - e')(x + e' - v') \in (e - e')R$ . Note that  $v' = eve = e(1 + hfu^{-1}g)ue = e(1 + (1 - g)fu^{-1}g)ue = e(1 + fu^{-1}g - gfu^{-1}g)ue = eue - egfu^{-1}gue = eue - eufu^{-1}gue = eue - eufu^{-1}(1 - h)ue = eue + eufu^{-1}hue$ . Then  $e' - v' = eueu^{-1}g - eue - eufu^{-1}hue = eueu^{-1}(g - u) - eufu^{-1}h(ue - ge) = eueu^{-1}(g - u) - eufu^{-1}h(u - g) \in Rx$  (because  $u - g = x - (1 - g)rx$  for some  $r \in R$ ). It follows that  $x + e' - v' \in Rx$ . Hence,  $x + e' - v' = (e - e')(x + e' - v') \in (e - e')Rx$ .

For  $x - g - u \in (1 - g)Rx$ , by taking  $e' = eueu^{-1}g \in Id(eRe)$ ,  $v' = eve$  (where  $v = (1 - hfu^{-1}g)u$  with  $v^{-1} = u^{-1}(1 + hfu^{-1}g)$ ) and by arguments similar to those in the previous paragraph, we obtain  $x - e' - v' \in (e - e')Rx$ . Thus,  $x$  is pseudo weakly clean in  $eRe$ . This completes the proof.  $\square$

**Proposition 5.4.5.** *Let  $R$  be a ring and let  $x \in R$ . If there exists  $f^2 = f \in Rx$  such that  $(1 - f)x(1 - f)$  is pseudo weakly clean in  $(1 - f)R(1 - f)$ , then  $x$  is pseudo weakly clean in  $R$ .*

**Proof.** Let  $x \in R$ . Suppose that there exists  $f^2 = f \in Rx$  such that  $exe$  is pseudo weakly clean in  $eRe$  where  $e = 1 - f$ . Then  $exe = u + g + (e - g)yexe$  or  $exe = u - g + (e - g)yexe$  for some  $u \in U(eRe)$ ,  $g \in Id(eRe)$  and  $y \in eRe$ . Note that  $f + u \in U(R)$  with  $(f + u)^{-1} = f + v$  where  $v \in eRe$  such that  $uv = e = vu$  and  $1 + exf \in U(R)$  with  $(1 + exf)^{-1} = 1 - exf$ . We also note that  $exfu = ex(1 - e)u = 0$  since  $u \in eRe$ . Thus  $f + u + exf = (1 + exf)(f + u) \in U(R)$ . Let  $\alpha = f + u + exf$ . For  $exe = u + g + (e - g)yexe$ , we have the equation

$$gxe = g(exe) = g(u + g + (e - g)yexe) = gu + g + g(e - g)yexe = gu + g.$$

It follows that  $g\alpha = g(u + f + exf) = gu + gf + gexf = gu + gfx = gu + gx(1 - e) = gu + gx - gxe = gu + gx - (gu + g) = gx - g$ . Hence,  $g(x - g - \alpha) = 0$ , which gives us

$$x - g - \alpha = (1 - g)(x - g - \alpha) \in (1 - g)R. \quad (5.5)$$

By the hypothesis,  $xe = x(1 - f) = x - xf \in Rx$ . It follows that  $u + g = exe - (e - g)yexe = (e - (e - g)ye)xe \in Rx$ . Thus,  $\alpha + g = f + u + exf + g = (1 + ex)f + (u + g) \in Rx$ . It follows that

$$x - \alpha - g = x - (\alpha + g) \in Rx. \quad (5.6)$$

By (5.5) and (5.6), we have that  $x - \alpha - g \in (1 - g)Rx$ .

Similarly, for  $exe = u - g + (e - g)yexe$ , we have the equation

$$gxe = g(exe) = g(u - g + (e - g)yexe) = gu - g + g(e - g)yexe = gu - g.$$

Note that  $u - f \in U(R)$  with  $(u - f)^{-1} = v - f$ . Then  $u - f + exf = (1 - exf)(u - f) \in U(R)$ . Let  $\beta = u - f + exf$ . Then  $g\beta = g(u - f + exf) = gu - gf + gexf = gu + gxf = gu + gx(1 - e) = gu + gx - gxe = gu + gx - (gu - g) = gx + g$ . Hence,  $g(x + g - \beta) = 0$ , which gives us

$$x + g - \beta = (1 - g)(x + g - \beta) \in (1 - g)R. \quad (5.7)$$

By the hypothesis, we have  $xe = x(1 - f) = x - xf \in Rx$ . It follows that  $u - g = exe - (e - g)yexe = (e - (e - g)ye)xe \in Rx$ . Thus,  $\beta - g = u - f + exf - g = (ex - 1)f + (u - g) \in Rx$ . It follows that

$$x - \beta + g = x - (\beta - g) \in Rx. \quad (5.8)$$

By (5.7) and (5.8),  $x - \beta + g \in (1 - g)Rx$ . Therefore,  $x$  is pseudo weakly clean in  $R$ . □

A ring  $R$  is called semi-abelian if there exist orthogonal idempotents  $e_1, \dots, e_n \in R$  such that  $1 = e_1 + \dots + e_n$  and each  $e_iRe_i$  is an abelian ring.

**Proposition 5.4.6.** *Let  $R$  be a semi-abelian ring. If  $R$  is pseudo weakly clean, then there exist orthogonal idempotents  $e_1, \dots, e_n \in R$  such that  $1 = e_1 + \dots + e_n$  and each  $e_iRe_i$  is an abelian weakly clean ring.*

**Proof.** Since  $R$  is semi-abelian, there exist orthogonal idempotents  $e_1, \dots, e_n \in R$  such that  $1 = e_1 + \dots + e_n$  and each  $e_iRe_i$  is an abelian ring. Since  $R$  is pseudo weakly clean, it follows by Proposition 5.4.4 that each  $e_iRe_i$  is pseudo weakly

clean. Since  $e_iRe_i$  is abelian and pseudo weakly clean, it follows by Theorem 5.3.1 that  $e_iRe_i$  is weakly clean for each  $i = 1, \dots, n$ .  $\square$

Polynomial rings over pseudo weakly clean rings are not pseudo weakly clean as shown in the following:

**Proposition 5.4.7.** *Let  $R$  be a ring. Then the polynomial ring  $R[x]$  is never pseudo weakly clean.*

**Proof.** We show that the indeterminate  $x \in R[x]$  is not pseudo weakly clean. Suppose that  $x = u + e + (1 - e)rx$  or  $x = u - e + (1 - e)rx$  for some  $u \in U(R[x])$ ,  $e \in Id(R[x])$  and  $r \in R[x]$ . We may write  $e = e_0 + e_1x + \dots + e_nx^n$  and  $u = u_0 + u_1x + \dots + u_nx^n$  where  $(e_n, u_n) \neq (0, 0)$ . Then

$$\begin{aligned} x &= u + e + rx - erx \\ &= (u_0 + u_1x + \dots + u_nx^n) + (e_0 + e_1x + \dots + e_nx^n) + rx \\ &\quad - (e_0 + e_1x + \dots + e_nx^n)rx \end{aligned}$$

or

$$\begin{aligned} x &= u - e + rx - erx \\ &= (u_0 + u_1x + \dots + u_nx^n) - (e_0 + e_1x + \dots + e_nx^n) + rx \\ &\quad - (e_0 + e_1x + \dots + e_nx^n)rx. \end{aligned}$$

It follows that  $e_0 = -u_0$  or  $e_0 = u_0$ ; hence,  $e_0 \in U(R)$ . Since  $e_0$  is also an idempotent in  $R$  (because  $e \in Id(R[x])$ ), we have that  $e_0 = 1$ . Suppose that  $e \neq 1$ . Then  $e$  has the form  $e = 1 + x^m a$ , where  $1 \leq m \leq n$  and  $a = a_0 + a_1x + \dots + a_{n-m}x^{n-m}$  with  $a_0 \neq 0$ . Since  $e^2 = e$ , we thus have that  $1 + x^m(2a) + x^{2m}(a^2) = 1 + x^m a$ . Comparing coefficients of  $x^m$ , we obtain  $2a_0 = a_0$  which gives us

$a_0 = 0$ ; a contradiction. It follows that  $e = 1$  and hence,  $x = u + 1$  or  $x = u - 1$ . Therefore,  $1 - x = -u \in U(R[x])$  or  $1 + x = u \in U(R[x])$ . Suppose that the former occurs and let  $(1 - x)^{-1} = b_0 + b_1x + b_2x^2 + \cdots + b_lx^l$ . Then

$$\begin{aligned} 1 &= (1 - x)(1 - x)^{-1} = (1 - x)^{-1} - x(1 - x)^{-1} \\ &= b_0 + (b_1 - b_0)x + (b_2 - b_1)x^2 + \cdots + (b_l - b_{l-1})x^l - b_lx^{l+1}. \end{aligned}$$

By comparing coefficients, we obtain  $b_0 = 1$ ,  $b_1 - b_0 = 0, \dots, b_l - b_{l-1} = 0$  and  $b_l = 0$ ; a contradiction. Now suppose that  $1 + x \in U(R[x])$  and let  $(1 + x)^{-1} = b'_0 + b'_1x + b'_2x^2 + \cdots + b'_lx^l$ . Then

$$\begin{aligned} 1 &= (1 + x)(1 + x)^{-1} = (1 + x)^{-1} + x(1 + x)^{-1} \\ &= b'_0 + (b'_1 + b'_0)x + (b'_2 + b'_1)x^2 + \cdots + (b'_l + b'_{l-1})x^l + b'_lx^{l+1}. \end{aligned}$$

By comparing coefficients, we get  $b'_0 = 1$ ,  $b'_1 + b'_0 = 0, \dots, b'_l + b'_{l-1} = 0$  and  $b'_l = 0$ . Again, we have a contradiction. Hence,  $x \in R[x]$  is not pseudo weakly clean and therefore,  $R[x]$  is not pseudo weakly clean.  $\square$

Formal power series rings over commutative pseudo weakly clean rings are however pseudo weakly clean as shown in the following:

**Proposition 5.4.8.** *Let  $R$  be a commutative ring. Then the formal power series ring  $R[[x]]$  is pseudo weakly clean if and only if  $R$  is pseudo weakly clean.*

**Proof.** Suppose that  $R[[x]]$  is pseudo weakly clean. Then it follows by the isomorphism  $R \cong R[[x]]/(x)$  and Proposition 5.4.1 that  $R$  is pseudo weakly clean. Conversely, suppose that  $R$  is pseudo weakly clean. Since  $R$  is commutative, it follows by Theorem 5.3.1 that  $R$  is weakly clean. Then by [2, Theorem 1.9],  $R[[x]]$  is weakly clean. Hence,  $R[[x]]$  is pseudo weakly clean.  $\square$

In the remainder of this section, we determine some sufficient conditions for a group ring to be pseudo weakly clean. First we recall some basic facts about group rings.

Let  $R$  be a ring and let  $G$  be a group. The augmentation ideal of  $RG$ , denoted by  $\Delta$ , is the ideal of  $RG$  generated by  $\{1 - g \mid g \in G\}$ . The homomorphism  $\delta : RG \rightarrow R$  given by

$$\delta \left( \sum_{g \in G} r_g g \right) = \sum_{g \in G} r_g$$

is called the augmentation mapping of  $RG$ . It is known that the kernel of  $\delta$  is the augmentation ideal  $\Delta$  and that  $RG/\Delta \cong R$ .

The following proposition will be used later.

**Proposition 5.4.9.** *Let  $R$  be a commutative ring. Then  $R$  is clean if and only if  $R$  is pseudo clean.*

**Proof.** ( $\Rightarrow$ ): It is clear by definition that every clean ring is pseudo clean.

( $\Leftarrow$ ): By [59, Proposition 2.3(i)], every pseudo clean ring is exchange. Since commutative exchange rings are clean (by [50, Proposition 1.8(2)]), it follows that commutative pseudo clean rings are clean.  $\square$

Let  $p$  be a prime number. A group  $G$  is called a  $p$ -group if the order of each element in  $G$  is a power of  $p$ .

**Proposition 5.4.10.** *Let  $R$  be a commutative ring and let  $G$  be an abelian  $p$ -group with  $p \in J(R)$ . Then  $RG$  is pseudo clean if and only if  $R$  is pseudo clean.*

**Proof.** ( $\Rightarrow$ ): Since any homomorphic image of a pseudo clean ring is pseudo clean (by [59, Proposition 2.3]) and  $R \cong RG/\Delta$  is a homomorphic image of  $RG$ , it follows that  $R$  is pseudo clean.

( $\Leftarrow$ ): Assume that  $R$  is pseudo clean. It follows by Proposition 5.4.9 that  $R$  is clean. By [62, Theorem 2.3], it follows that  $RG$  is clean (hence, pseudo clean).  $\square$

Since any local ring is clean (hence, pseudo weakly clean), we have the following corollary.

**Corollary 5.4.1.** *Let  $R$  be a commutative local ring with maximal ideal  $M$  and let  $G$  be an abelian  $p$ -group with  $p \in M$ . Then  $RG$  is pseudo weakly clean.*

**Lemma 5.4.1.** *[73, Lemma 2] Let  $R$  be a ring and let  $G$  be a group. If  $G$  is a locally finite  $p$ -group where  $p$  is a prime with  $p \in J(R)$ , then  $\Delta \subseteq J(RG)$ .*

**Proposition 5.4.11.** *Let  $R$  be a ring and let  $p$  be a prime number with  $p \in J(R)$ . Let  $G$  be a locally finite group with  $G = KH$  where  $K$  is a normal  $p$ -subgroup of  $G$  and  $H$  is a subgroup of  $G$ . If  $RH$  is pseudo clean, then  $RG$  is pseudo clean.*

**Proof.** For any  $g \in G$ , since  $G = KH$ , there exist  $k \in K$  and  $h \in H$  such that  $g = kh = (k - 1)h + h \in \sum_{k \in K} (1 - k)RG + RH$ . By Lemma 5.4.1, we have  $\Delta(RK) \subseteq J(RK)$  where  $\Delta(RK)$  is the augmentation ideal of  $RK$ . Since  $G$  and  $G/K \cong H$  are locally finite, so  $J(RK) \subseteq J(RG)$  (by [66, Lemma 4.1]). Hence,  $\Delta(RK) \subseteq J(RG)$  and this implies that  $\sum_{k \in K} (1 - k)RG \subseteq \Delta(RK)(RG) \subseteq J(RG)$ . We thus have

$$RG = J(RG) + RH. \quad (5.9)$$

By [24, Proposition 9],  $RH \cap J(RG) \subseteq J(RH)$ . Since  $RH/(RH \cap J(RG)) \cong RG/J(RG)$  and  $RG/J(RG)$  is semiprimitive, so  $J(RH) \subseteq RH \cap J(RG)$ . Thus, we have  $J(RH) = RH \cap J(RG)$ . Therefore,  $RH/J(RH) \cong RG/J(RG)$ . Since  $RH$  is pseudo clean, we have that  $RH/J(RH)$  is pseudo clean and hence, so is  $RG/J(RG)$  (by [59, Proposition 2.3]). To show that  $RG$  is pseudo clean, it

remains to show that idempotents of  $RG/J(RG)$  can be lifted to idempotents of  $RG$  (by [59, Proposition 2.3]). Let  $x^2 - x \in J(RG)$  where  $x \in RG$ . By (5.9), we may write  $x = y + z$  with  $y \in J(RG)$  and  $z \in RH$ . It follows that  $z^2 - z \in RH \cap J(RG) = J(RH)$ . Since  $RH$  is pseudo clean, there exists  $e^2 = e \in RH \subseteq RG$  such that  $z - e \in J(RH)$  (by [59, Proposition 2.3]). Hence,  $x - e = y + (z - e) \in J(RG) + J(RH) \subseteq J(RG)$ .  $\square$

**Proposition 5.4.12.** *Let  $R$  be a ring and let  $p$  be a prime number with  $p \in J(R)$ . If  $R$  is pseudo clean and  $G$  is a locally finite  $p$ -group, then  $RG$  is pseudo weakly clean.*

**Proof.** Since  $G$  is a locally finite  $p$ -group, we may take  $K = G$  and  $H = \{1\}$  in Proposition 5.4.11. Then since  $RH = R\{1\} \cong R$  is pseudo clean, it follows by Proposition 5.4.11 that  $RG$  is pseudo clean (hence, pseudo weakly clean).  $\square$

## 5.5 Pseudo weakly clean in non-unital rings

In this section, by a non-unital ring we mean an associative ring without identity. Let  $I$  be a non-unital ring and let  $x \in I$ . In [59], Ster introduced the notion of a pseudo clean non-unital ring. An element  $x \in I$  is said to be pseudo clean in  $I$  if  $x = e + p + erx$  for some  $e \in Id(I)$ ,  $r \in I$  and  $p \in Q(I)$  where  $Q(I) = \{p \in I \mid \exists q \in I \text{ such that } p + q + pq = 0 = p + q + qp\}$ . A non-unital ring  $I$  is said to be pseudo clean if each  $x \in I$  is pseudo clean in  $I$ .

**Proposition 5.5.1.** *Let  $R$  be a unital ring and let  $I$  be a non-unital subring of  $R$  such that  $I$  is pseudo clean. If  $x \in I$ , then  $x$  is pseudo weakly clean in  $R$ .*

**Proof.** Since  $x \in I$  and  $I$  is a pseudo clean non-unital ring, we have that  $x = e + p + erx$  for some  $e \in Id(I)$ ,  $p \in Q(I)$  and  $r \in I$ . Note that  $x =$



$e + p + erx = (-1 + e) + (1 + p) + erx$ . By taking  $f = 1 - e$ , we have that  $x = -f + (1 + p) + (1 - f)rx$  with  $f \in Id(R)$  and  $1 + p \in U(R)$  where  $(1 + p)^{-1} = 1 + q$ . Hence,  $x$  is pseudo weakly clean in  $R$ .  $\square$

We now extend the notion of pseudo clean in a non-unital ring to that of pseudo weakly clean. Let  $I$  be a non-unital ring. Let  $Q(I) = \{p \in I \mid \exists p' \in I \text{ such that } p + p' + pp' = 0 = p + p' + p'p\}$  and let  $Q'(I) = \{q \in I \mid \exists q' \in I \text{ such that } q - q' + qq' = 0 = q - q' + q'q\}$ . An element  $x \in I$  is said to be pseudo weakly clean in  $I$  if  $x = e + p + erx$  or  $x = -e + q + erx$  for some  $e \in Id(I)$ ,  $r \in I$ ,  $p \in Q(I)$  and  $q \in Q'(I)$ . A non-unital ring  $I$  is said to be pseudo weakly clean if all of its elements are pseudo weakly clean. Clearly, a non-unital pseudo clean ring is pseudo weakly clean.

We extend Proposition 5.5.1 as follows:

**Proposition 5.5.2.** *Let  $R$  be a unital ring and let  $I$  be a non-unital subring of  $R$ . If  $I$  is pseudo weakly clean, then every  $x \in I$  is pseudo weakly clean in  $R$ .*

**Proof.** Let  $x \in I$ . Since  $I$  is pseudo weakly clean, we have that  $x = e + p + erx$  or  $x = -e + q + erx$  for some  $e \in Id(I)$ ,  $p \in Q(I)$ ,  $q \in Q'(I)$  and  $r \in I$ . For  $x = e + p + erx$ , we have  $x = -(1 - e) + (1 + p) + erx$  where  $1 - e \in Id(R)$ . Since  $p \in Q(I)$ , there exists  $p' \in I$  such that  $p + p' + pp' = 0 = p + p' + p'p$ . Then  $(1 + p)(1 + p') = 1 + p + p' + pp' = 1 = (1 + p')(1 + p)$ , that is,  $1 + p \in U(R)$ . By taking  $f = 1 - e$  and  $u = 1 + p$ , we may write  $x = -f + u + (1 - f)rx$  where  $f \in Id(R)$ ,  $u \in U(R)$  and  $r \in R$ . Hence,  $x$  is pseudo weakly clean in  $R$ . For  $x = -e + q + erx$ , we may write  $x = (1 - e) + (-1 + q) + erx$  where  $1 - e \in Id(R)$ . Since  $q \in Q'(I)$ , there exists  $q' \in I$  such that  $q - q' + qq' = 0 = q - q' + q'q$ . Then  $(-1 + q)(1 + q') = -1 + q - q' + qq' = -1 = (1 + q')(-1 + q)$ , that is,  $-1 + q \in U(R)$ .

By letting  $f = 1 - e$  and  $v = -1 + q$ , we may write  $x = f + v + (1 - f)rx$  where  $f \in Id(R)$ ,  $v \in U(R)$  and  $r \in R$ . Thus,  $x$  is pseudo clean (hence, pseudo weakly clean) in  $R$ .  $\square$

The converse of Proposition 5.5.2 is not necessarily true. For example, take  $R = \mathbb{Q}$  and  $I = 2\mathbb{Z}$ . Note that any  $x \in I$  is clean in  $R$  (hence, pseudo weakly clean in  $R$ ) but  $I$  itself is not pseudo weakly clean since  $2 \in I$  is not pseudo weakly clean in  $I$ .

In the next proposition, we show that the converse of Proposition 5.5.2 is true when  $I$  is a proper ideal of  $R$ .

**Proposition 5.5.3.** *Let  $R$  be a unital ring and let  $I$  be a proper ideal of  $R$ . If  $x \in I$  is pseudo weakly clean in  $R$ , then  $x$  is pseudo weakly clean in  $I$ .*

**Proof.** Let  $x \in I$  be pseudo weakly clean in  $R$ . Then  $x = e + u + (1 - e)rx$  or  $x = -e + u + (1 - e)rx$  for some  $e \in Id(R)$ ,  $u \in U(R)$  and  $r \in R$ . For  $x = e + u + (1 - e)rx$ , we have  $x = (-1 + e) + (1 + u) + (1 - e)rx$ . By taking  $f = 1 - e$ , we may write  $x = -f + (1 + u) + frx$  with  $f \in Id(R)$ . Since  $x \in I$  and  $I$  is an ideal of  $R$ , we have that  $e + u = x - (1 - e)rx \in I$ . By multiplying  $f$  on the left, it follows that  $fu \in I$  and therefore,  $f \in I$ . We then have  $1 + u = f + (e + u) \in I$  and  $(1 + u) - (-1 - u^{-1}) + (1 + u)(-1 - u^{-1}) = 0 = (1 + u) - (-1 - u^{-1}) + (-1 - u^{-1})(1 + u)$ , that is,  $1 + u \in Q'(I)$ . By letting  $q = 1 + u$ , we may then write  $x = -f + q + f(fr)x$ , where  $f \in Id(I)$ ,  $q \in Q'(I)$  and  $fr \in I$ . It follows that  $x$  is pseudo weakly clean in  $I$ .

For  $x = -e + u + (1 - e)rx$ , we have  $x = (1 - e) + (-1 + u) + (1 - e)rx$ . By taking  $f = 1 - e$  and  $v = -u$ , it follows that  $x = f + (-1 - v) + frx$  with  $f \in Id(R)$ . Since  $x \in I$  and  $I$  is an ideal of  $R$ , we have that  $-e + u = x - (1 - e)rx \in I$ . By

multiplying  $f$  on the left, it follows that  $fu \in I$  and hence,  $f \in I$ . We then have  $-1 - v = -f + (-e + u) \in I$  and  $(-1 - v) + (-1 - v^{-1}) + (-1 - v)(-1 - v^{-1}) = 0 = (-1 - v) + (-1 - v^{-1}) + (-1 - v^{-1})(-1 - v)$ , that is,  $-1 - v \in Q(I)$ . By taking  $p = -1 - v$ , we may then write  $x = f + p + f(fr)x$  where  $f \in Id(I)$ ,  $p \in Q(I)$  and  $fr \in I$ . It follows that  $x$  is pseudo weakly clean in  $I$ .  $\square$

As an application of Propositions 5.5.2 and 5.5.3 we have the following:

**Proposition 5.5.4.** *Let  $I$  be a proper ideal of a unital ring  $R$  and let  $e \in Id(R)$ . If  $I$  is pseudo weakly clean, then  $eIe$  is pseudo weakly clean.*

**Proof.** Suppose that  $I$  is pseudo weakly clean and let  $x \in eIe$ . By Proposition 5.5.2,  $x$  is pseudo weakly clean in  $R$ . It follows by Proposition 5.4.4 that  $x$  is pseudo weakly clean in  $eRe$ . Thus, every  $x \in eIe$  is pseudo weakly clean in  $eRe$ . By Proposition 5.5.3, it follows that  $eIe$  is pseudo weakly clean.  $\square$

# Chapter 6

## Some Results on $g(x)$ -clean Rings

### 6.1 Introduction

Let  $R$  be a ring and let  $g(x)$  be a polynomial in  $Z(R)[x]$ . In [10], an element  $r \in R$  is called  $g(x)$ -clean if  $r = u + s$  for some  $u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ . The ring  $R$  is  $g(x)$ -clean if every element in  $R$  is  $g(x)$ -clean. Note that if  $r \in R$  is  $g(x)$ -clean and  $g(x)$  is a factor of a polynomial  $h(x) \in Z(R)[x]$ , then  $r$  is also  $h(x)$ -clean.

Clearly, if  $g(x) = x^2 - x$ , then  $g(x)$ -clean rings are clean. However, in general,  $g(x)$ -clean rings are not necessarily clean. An example is as follows:

**Example 6.1.1.** Let  $\mathbb{Z}_{(7)} = \{m/n \mid m, n \in \mathbb{Z}, \gcd(7, n) = 1\}$  and let  $C_3$  be the cyclic group of order 3. By [61, Example 2.7], the group ring  $\mathbb{Z}_{(7)}C_3$  is  $(x^6 - 1)$ -clean. However, Han and Nicholson [32] have shown that  $\mathbb{Z}_{(7)}C_3$  is not clean.

Conversely, for a clean ring  $R$ , there may exist a  $g(x) \in Z(R)[x]$  such that  $R$  is not  $g(x)$ -clean as shown in the following example:

**Example 6.1.2.** (see [29, Example 2.3]) Let  $R$  be a Boolean ring containing more than two elements. Let  $c \in R$  where  $0 \neq c \neq 1$  and let  $g(x) = x^2 + (1 +$

$c)x+c = (x+1)(x+c)$ . Since  $R$  is Boolean, so it is clean. Suppose that  $R$  is  $g(x)$ -clean. Then  $c = u + s$  for some  $u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ . Note that  $u = 1$  since  $R$  is Boolean. Therefore,  $s = c + 1$ . However,  $g(c + 1) = c \neq 0$  which contradicts the assumption that  $g(s) = 0$ . Hence, it follows that  $R$  is clean but not  $g(x)$ -clean.

Let  $R$  be a ring and let  $g(x) \in Z(R)[x]$ . An element  $r \in R$  is called weakly  $g(x)$ -clean if  $r = u + s$  or  $r = u - s$  for some  $u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ . We say that  $R$  is weakly  $g(x)$ -clean if every element in  $R$  is weakly  $g(x)$ -clean. Clearly, a  $g(x)$ -clean ring is weakly  $g(x)$ -clean.

In this chapter, we investigate further properties of  $g(x)$ -clean rings for certain types of  $g(x) \in Z(R)[x]$ . In particular, we consider polynomials  $g(x) \in Z(R)[x]$  such that  $R$  is  $g(x)$ -clean if  $R$  is clean and vice versa. In the last section we define  $c$ -topologically boolean rings and show, via set-theoretic topology, that among conditions equivalent to  $R$  being an  $x(x - c)$ -clean ring where  $c \in U(R) \cap Z(R)$  is that  $R$  is right (respectively, left)  $c$ -topologically boolean.

## 6.2 Some properties of $g(x)$ -clean rings

In this section we investigate some properties of  $g(x)$ -clean rings. First, we show that given a ring  $R$ , there exists  $g(x) \in Z(R)[x]$  such that if an element  $z \in R$  is  $g(x)$ -clean in a corner of  $R$ , then  $z$  is  $g(x)$ -clean in  $R$ .

**Proposition 6.2.1.** *Let  $R$  be a ring and let  $g(x) = x(x - 1) \in Z(R)[x]$ . If  $z \in eRe$  is  $g(x)$ -clean in  $eRe$  for some  $e \in Id(R)$ , then  $z$  is  $g(x)$ -clean in  $R$ .*

**Proof.** Suppose that there exists  $e \in Id(R)$  such that an element  $z \in eRe$  is  $g(x)$ -clean in  $eRe$ . That is,  $z = v + s$  where  $v, s \in eRe$  such that  $vw = e = wv$

for some  $w \in eRe$  and  $g(s) = 0$ . Then  $u = v - (1 - e)$  is a unit in  $R$  with  $u^{-1} = w - (1 - e)$  and  $z - u = s + (1 - e)$ . Since  $s(1 - e) = 0 = (1 - e)s$ , we have

$$\begin{aligned}
g(s + (1 - e)) &= (s + (1 - e))((s + (1 - e)) - 1) \\
&= (s + (1 - e))((s - 1) + (1 - e)) \\
&= s(s - 1) \\
&= g(s) \\
&= 0.
\end{aligned}$$

Therefore,  $z = u + (s + (1 - e))$  is  $g(x)$ -clean in  $R$ . □

By Proposition 6.2.1, we readily have the following general result.

**Corollary 6.2.1.** *Let  $R$  be a ring and let  $g(x) = x(x - 1)h(x) \in Z(R)[x]$ . If  $z \in eRe$  is  $x(x - 1)$ -clean in  $eRe$  for some  $e \in Id(R)$ , then  $z$  is  $g(x)$ -clean in  $R$ .*

As a direct consequence of Corollary 6.2.1, we obtain the following known result.

**Corollary 6.2.2.** *Let  $R$  be a ring. If  $z \in eRe$  is clean in  $eRe$  for some  $e \in Id(R)$ , then  $z$  is clean in  $R$ .*

We next obtain polynomials  $g(x) \in Z(R)[x]$  where  $R$  is a weakly clean ring such that  $R$  is also  $g(x)$ -clean.

**Proposition 6.2.2.** *Let  $R$  be a weakly clean ring and let  $g(x) = x(x^2 - 1) \in Z(R)[x]$ . Then  $R$  is  $g(x)h(x)$ -clean for any  $h(x) \in Z(R)[x]$ .*

**Proof.** Let  $r \in R$ . Then  $r = u + e$  or  $r = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . Note that  $g(e) = e(e^2 - 1) = 0$  and  $g(-e) = (-e)((-e)^2 - 1) = -e(e - 1) = 0$ . Thus,  $r$  is  $g(x)$ -clean and hence,  $g(x)h(x)$ -clean for any  $h(x) \in Z(R)[x]$ . □

**Corollary 6.2.3.** *Let  $R$  be a ring and let  $g(x) = x(x^2 - 1) \in Z(R)[x]$ . If  $R$  is clean or local or Boolean, then  $R$  is  $g(x)h(x)$ -clean for any  $h(x) \in Z(R)[x]$ .*

**Proof.** This follows readily by Proposition 6.2.2 and the fact that clean (local, Boolean) rings are weakly clean.  $\square$

Proposition 6.2.2 may be extended as follows:

**Proposition 6.2.3.** *Let  $R$  be a weakly clean ring and let  $g(x) = (x^m - x^n)h(x) \in Z(R)[x]$  where  $m, n$  are positive integers of the same parity. Then  $R$  is  $g(x)$ -clean.*

**Proof.** Let  $r \in R$ . Then  $r = u + e$  or  $r = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . Since  $m, n$  are of the same parity, we have  $g(e) = (e^m - e^n)h(e) = 0$  and  $g(-e) = ((-e)^m - (-e)^n)h(-e) = 0$ . Thus,  $r$  is  $g(x)$ -clean.  $\square$

For a ring  $R$  and polynomial  $g(x) \in Z(R)[x]$ , we say that an element  $r \in R$  is  $(n, g(x))$ -clean if  $r = s + u_1 + \cdots + u_n$  for some  $u_1, \dots, u_n \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ . The ring  $R$  is  $(n, g(x))$ -clean if all of its elements are  $(n, g(x))$ -clean. Clearly, a  $(1, g(x))$ -clean ring is  $g(x)$ -clean.

In [61, Theorem 2.1], Wang and Chen showed that if  $g(x) = (x - a)(x - b)$  where  $a, b \in Z(R)$  with  $b - a \in U(R)$ , then  $R$  is  $g(x)$ -clean if and only if  $R$  is clean. In [29, Theorem 3.2], Fan and Yang gave another proof of the same result.

In the following, we give an extension to  $n$ -clean rings as follows:

**Theorem 6.2.1.** *Let  $R$  be a ring and let  $g(x) = (x - a)(x - b)h(x) \in Z(R)[x]$  such that  $b - a \in U(R)$ . If  $R$  is  $n$ -clean, then  $R$  is  $(n, g(x))$ -clean ( $n \in \mathbb{N}$ ).*

**Proof.** Let  $r \in R$ . Since  $R$  is  $n$ -clean, then  $(r - a)(b - a)^{-1} = e + u_1 + \cdots + u_n$  for some  $e \in Id(R)$  and  $u_i \in U(R)$  for  $i = 1, \dots, n$ . Thus,  $r = (e(b - a) + a) + u_1(b - a) + \cdots + u_n(b - a)$ , where  $u_i(b - a) \in U(R)$  for  $i = 1, \dots, n$ . Note that

$$g((e(b-a)+a)h(e(b-a)+a)) = e(b-a)(e(b-a) - (b-a))h(e(b-a)+a) = 0 \quad h(e(b-a)+a) = 0.$$

Hence,  $e(b-a) + a$  is a root of  $g(x)$ . It follows that  $R$  is  $(n, g(x))$ -clean.  $\square$

**Corollary 6.2.4.** *Let  $R$  be a ring and let  $g(x) = (x-a)(x-b) \in Z(R)[x]$  such that  $b-a \in U(R)$ . Then  $R$  is  $g(x)$ -clean if and only if  $R$  is  $n$ -clean for all positive integers  $n$ .*

**Proof.** If  $R$  is  $g(x)$ -clean, then by [61, Theorem 2.1] (or [29, Theorem 3.2]), it follows that  $R$  is clean; hence,  $n$ -clean for all positive integers  $n$ . Conversely, if  $R$  is  $n$ -clean for all positive integers  $n$ , then in particular,  $R$  is 1-clean. Hence, by Theorem 6.2.1,  $R$  is  $(1, g(x))$ -clean, that is,  $R$  is  $g(x)$ -clean.  $\square$

### 6.3 Some results on $x(x-c)$ -clean rings

We first obtain some results which generalise parts of Theorem 3.5 in [29].

**Proposition 6.3.1.** *Let  $R$  be a ring which is weakly  $x^k(x-c)$ -clean where  $k$  is a positive integer and  $c \in Z(R)$ . Then  $c \in U(R)$ .*

**Proof.** Let  $g(x) = x^k(x-c)$ . Since  $R$  is weakly  $g(x)$ -clean,  $c = u+s$  or  $c = u-s$  for some  $u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ . For the case  $c = u+s$ , we have that  $s = -u+c$  and hence,  $s^{k+1} = (-u+c)^{k+1} = (-u)^{k+1} + cr$  for some  $r \in R$ . Since  $0 = g(s) = s^k(s-c)$ , we also have  $s^{k+1} = cs^k$ . Thus,  $c(s^k - r) = (-u)^{k+1} \in U(R)$ . This implies that  $c \in U(R)$ . For the case  $c = u-s$ , we have that  $s = u-c$  and hence,  $s^{k+1} = (u-c)^{k+1} = u^{k+1} - cr$  for some  $r \in R$ . Since  $0 = g(s) = s^k(s-c)$ , we also have  $s^{k+1} = cs^k$ . Thus,  $c(s^k + r) = u^{k+1} \in U(R)$  which implies that  $c \in U(R)$ .  $\square$

By using Proposition 6.3.1, we obtain in the following an element-wise characterisation of  $x^k(x-c)$ -clean rings where  $k$  is a positive integer:



**Proposition 6.3.2.** *Let  $R$  be a ring which is  $x(x - c)$ -clean where  $c \in Z(R)$ . Then for any positive integer  $k$ ,  $R$  is  $x^k(x - c)$ -clean and any  $r \in R$  can be expressed as  $r = u + v$  for some  $u, v \in R$  such that  $u \in U(R)$  and*

$$v^k = 1 + \frac{1}{c}s((1 - c)^k - 1)$$

where  $s \in R$  such that  $s(s - c) = 0$ .

**Proof.** Let  $r \in R$ . Since  $R$  is  $x(x - c)$ -clean,  $1 - r = s + w$  for some  $s, w \in R$  such that  $s(s - c) = 0$  and  $w \in U(R)$ . It follows that  $r = (1 - s) + u$  where  $u = -w \in U(R)$ . For any positive integer  $k$ ,  $R$  is clearly also  $x^k(x - c)$ -clean.

The binomial expansion on  $(1 - s)^k$  gives us

$$(1 - s)^k = \binom{k}{0} - \binom{k}{1}s + \binom{k}{2}s^2 - \cdots + (-1)^{k-1} \binom{k}{k-1}s^{k-1} + (-1)^k \binom{k}{k}s^k.$$

Since  $s(s - c) = 0$ , so  $s^{l+1} = cs^l = c^l s$  for all positive integers  $l$ . Hence,

$$(1 - s)^k = \binom{k}{0} - \binom{k}{1}s + \binom{k}{2}cs - \cdots + (-1)^{k-1} \binom{k}{k-1}c^{k-2}s + (-1)^k \binom{k}{k}c^{k-1}s.$$

By Proposition 6.3.1, we have  $c \in U(R)$ . It follows that

$$\begin{aligned} & (1 - s)^k \\ &= 1 + \frac{1}{c}s \left( 1 - \binom{k}{1}c + \binom{k}{2}c^2 + \cdots + (-1)^{k-1} \binom{k}{k-1}c^{k-1} \right. \\ & \quad \left. + (-1)^k \binom{k}{k}c^k - 1 \right) \\ &= 1 + \frac{1}{c}s((1 - c)^k - 1). \end{aligned}$$

This completes the proof. □

In the case  $c = 2$  in Proposition 6.3.2, we have the following:

**Proposition 6.3.3.** *Let  $R$  be a ring which is  $x(x - 2)$ -clean. Then for any positive integer  $k$ ,  $R$  is  $x^k(x - 2)$ -clean and any  $r \in R$  can be expressed as*

$r = u + v$  for some  $u, v \in R$  such that  $u \in U(R)$  and

$$v^k = \begin{cases} 1, & \text{if } k \text{ is even} \\ 1 - s, & \text{if } k \text{ is odd} \end{cases}$$

where  $s \in R$  such that  $s(s - 2) = 0$ .

We now return to the more general polynomial  $g(x) = x(x - c)$  and obtain some conditions equivalent to being  $g(x)$ -clean. First, we prove the following:

**Lemma 6.3.1.** *Let  $R$  be a ring and let  $g(x) = ax^m - bx^n, h(x) = ax^m + bx^n \in Z(R)[x]$  where  $m, n$  are positive integers of different parity. Then  $R$  is  $g(x)$ -clean if and only if  $R$  is  $h(x)$ -clean.*

**Proof.** ( $\Rightarrow$ ): Assume that  $R$  is a  $g(x)$ -clean ring. Then for any  $r \in R$ ,  $-r = u + s$  where  $u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ . It follows that  $r = (-u) + (-s)$ .

Note that

$$\begin{aligned} h(-s) &= a(-s)^m + b(-s)^n \\ &= (-1)^m as^m + (-1)^n bs^n \\ &= \begin{cases} as^m - bs^n, & \text{if } m \text{ is even, } n \text{ is odd} \\ -(as^m - bs^n), & \text{if } m \text{ is odd, } n \text{ is even} \end{cases} \\ &= 0. \end{aligned}$$

It follows that  $r$  is  $h(x)$ -clean.

( $\Leftarrow$ ): Suppose that  $R$  is  $h(x)$ -clean. Then for any  $r \in R$ ,  $-r = u + s$  where  $u \in U(R)$  and  $s \in R$  such that  $h(s) = 0$ . It follows that  $r = (-u) + (-s)$ . Then

since

$$\begin{aligned}
g(-s) &= a(-s)^m - b(-s)^n \\
&= (-1)^m a s^m - (-1)^n b s^n \\
&= \begin{cases} a s^m + b s^n, & \text{if } m \text{ is even, } n \text{ is odd} \\ -(a s^m + b s^n), & \text{if } m \text{ is odd, } n \text{ is even} \end{cases} \\
&= 0,
\end{aligned}$$

we have that  $r$  is  $g(x)$ -clean. □

**Theorem 6.3.1.** *Let  $R$  be a ring and let  $c \in Z(R)$ . Then the following are equivalent:*

- (a)  $R$  is  $x(x - c)$ -clean.
- (b)  $R$  is  $x(x + c)$ -clean.
- (c)  $R$  is  $n$ -clean for all positive integers  $n$  and  $c \in U(R)$ .

**Proof.** (a)  $\Leftrightarrow$  (b): This follows readily by Lemma 6.3.1.

(a)  $\Rightarrow$  (c): Assume (a). By Proposition 6.3.1, we have  $c \in U(R)$ . It follows by Corollary 6.2.4 that  $R$  is  $n$ -clean for all positive integers  $n$ .

(c)  $\Rightarrow$  (a): This follows readily by Theorem 6.2.1 (take  $n = 1$ ). □

As an application of Theorem 6.3.1, consider the group ring  $R = \mathbb{Z}_{(7)}C_3$ . In Example 6.1.1, it is mentioned that  $R$  is not clean. Then since 2 is a unit in  $R$ , it follows by Theorem 6.3.1 that  $R$  is not  $x(x - 2)$ -clean and also not  $x(x + 2)$ -clean.

The set of equivalent conditions in Theorem 6.3.1 can be extended to a larger set, as will be shown in Section 6.4.

It is known by definition that a ring which is  $(x^2 - x)$ -clean is clean; hence,  $n$ -clean for any positive integer  $n$ . We now consider the general case of rings which are  $(x^k - x^l)$ -clean where  $k, l$  are distinct positive integers.

**Proposition 6.3.4.** *If the ring  $R$  is  $(x^k - x^l)$ -clean where  $k, l$  are positive integers with  $k \neq l$ , then  $R$  is 2-clean. In particular,  $R$  is  $n$ -clean for all integers  $n \geq 2$ .*

**Proof.** Let  $r \in R$ . Then  $r = u + s$  where  $u \in U(R)$  and  $s \in R$  such that  $s^k = s^l$ . Assume without loss of generality that  $k > l$ . Then  $s^l = s^{l+1}s^{k-l-1}$  and hence,  $s$  is strongly  $\pi$ -regular. Since strongly  $\pi$ -regular elements are strongly clean, we may write  $s = v + e$  where  $v \in U(R)$  and  $e \in Id(R)$  such that  $ev = ve$ . It follows that  $r = u + v + e$ , that is,  $r$  is 2-clean. Therefore  $R$  is 2-clean and hence,  $n$ -clean for  $n \geq 2$  (by Proposition 1.1.2).  $\square$

Let  $R$  and  $S$  be rings and let  $\theta : Z(R) \rightarrow Z(S)$  be a ring homomorphism with  $\theta(1) = 1$ . For  $g(x) = \sum a_i x^i \in Z(R)[x]$ , let  $\theta'(g(x)) = \sum \theta(a_i) x^i \in Z(S)[x]$ . Then  $\theta$  induces a map  $\theta'$  from  $Z(R)[x]$  to  $Z(S)[x]$ . Clearly, if  $g(x)$  is a polynomial with coefficients in  $\mathbb{Z}$ , then  $\theta'(g(x)) = g(x)$ . We state the following results of Yang [69] which will be used later.

**Proposition 6.3.5.** *[29, Proposition 2.4] Let  $\theta : R \rightarrow S$  be a ring epimorphism. If  $R$  is  $g(x)$ -clean, then  $S$  is  $\theta'(g(x))$ -clean.*

**Proposition 6.3.6.** *[29, Proposition 2.7] Let  $g(x) \in \mathbb{Z}[x]$  and let  $\{R_i\}$  be a family of rings. Then the direct product  $\prod_{i \in I} R_i$  is  $g(x)$ -clean if and only if every  $R_i$ ,  $i \in I$ , is  $g(x)$ -clean.*

We now consider the case  $c = 2$  again and obtain further results on  $x(x - 2)$ -clean rings.

**Proposition 6.3.7.** *Let  $R$  be a ring. If  $R$  is  $x(x - 2)$ -clean, then  $2 \in U(R)$  and  $R$  is  $(x^k - 1)$ -clean for any even positive integer  $k$ .*

**Proof.** By Proposition 6.3.1, since  $R$  is  $x(x-2)$ -clean, we have that  $2 \in U(R)$ . By Proposition 6.3.3, if  $k$  is an even positive integer, then any  $r \in R$  can be written as  $r = u + v$  for some  $u \in U(R)$  and  $v \in R$  such that  $v^k = 1$ . It is clear that  $v$  is a root of  $x^k - 1$ . Hence,  $r$  is  $(x^k - 1)$ -clean.  $\square$

Let  $R$  be a ring and let  $2 \in U(R)$ . Recall that in the proof of Proposition 4.3.1, we have shown that  $RC_2 \cong R \times R$  (as rings).

**Proposition 6.3.8.** *Let  $R$  be a ring, let  $2 \in U(R)$  and let  $k$  be a positive integer. Then  $R$  is  $x^k(x-2)$ -clean if and only if  $RC_2$  is  $x^k(x-2)$ -clean.*

**Proof.** If  $R$  is  $x^k(x-2)$ -clean, then by Proposition 6.3.6 we have that  $RC_2 \cong R \times R$  is  $x^k(x-2)$ -clean. Conversely, if  $RC_2$  is  $x^k(x-2)$ -clean, then it follows by Proposition 6.3.5 that  $R \times R$  is  $x^k(x-2)$ -clean. Hence, by Proposition 6.3.6,  $R$  is  $x^k(x-2)$ -clean.  $\square$

**Proposition 6.3.9.** *Let  $R$  be a ring which is  $x^k(x-2)$ -clean and let  $G$  be a finite elementary abelian 2-group. Then  $RG$  is  $x^k(x-2)$ -clean.*

**Proof.** Since  $G$  is a finite elementary abelian 2-group, we have that  $G \cong C_2^{(n)} \cong \underbrace{C_2 \times \cdots \times C_2}_{n \text{ times}}$  for some positive integer  $n$ . By Proposition 6.3.1, we have that  $2 \in U(R)$ . Hence, we have the isomorphism  $RC_2 \cong R \times R$ . Since  $2 \in U(RC_2)$ , we thus have  $R(C_2 \times C_2) \cong (RC_2)C_2 \cong RC_2 \times RC_2 \cong R \times R \times R \times R$ . By induction, it may be shown that  $RG$  is isomorphic to the direct product of  $2^n$  copies of  $R$ . Hence,  $RG$  is  $x^k(x-2)$ -clean by Proposition 6.3.6.  $\square$

In the following, we shall give some further properties of  $x(x-c)$ -clean rings, where  $c$  belongs to the centre of the ring.

**Lemma 6.3.2.** *Let  $R$  be a ring, let  $c \in U(R)$  and let all roots of  $g(x) = x(x - c)$  in  $R$  be central. For any  $a, b \in R$ , if  $ab = c$ , then  $ba = c$ .*

**Proof.** Let  $a, b \in R$  such that  $ab = c$ . Since  $c$  is a root of  $g(x)$ , we have that  $c$  is central and therefore,  $ba(ba - c) = baba - c(ba) = b(ab)a - c(ba) = c(ba) - c(ba) = 0$ . Thus,  $ba$  is a root of  $g(x)$  and hence,  $ba$  is also central. Then  $ca = (ab)a = a(ba) = baa$  and it follows that  $c^2 = c(ab) = (ca)b = (baa)b = bac$ . Since  $c \in U(R)$  (by the hypothesis), it follows that  $c = ba$ .  $\square$

By Proposition 6.3.1 and Lemma 6.3.2, we readily have the following:

**Corollary 6.3.1.** *Let  $R$  be a ring which is  $x(x - c)$ -clean and let all roots of  $x(x - c)$  in  $R$  be central. For any  $a, b \in R$ , if  $ab = c$ , then  $ba = c$ .*

**Proposition 6.3.10.** *Let  $R$  be an abelian ring and let  $e \in Id(R)$ . Then the following hold.*

- (a) *Let  $g(x) = x^k(x - c) \in Z(R)[x]$  where  $k$  is a positive integer. Then  $g(se) = 0$  for any root  $s$  of  $g(x)$  in  $R$ .*
- (b) *Let  $\bar{g}(x) = x(x - c) \in Z(R)[x]$ . Then  $\bar{g}(se + t(1 - e)) = 0$  for any roots  $s, t$  of  $\bar{g}(x)$  in  $R$ .*

**Proof.** (a) Let  $s$  be a root of  $g(x)$ . Then  $s^k(s - c) = g(s) = 0$  and it follows that  $s^{k+1} = s^k c$ . Hence,  $g(se) = (se)^k(se - c) = s^{k+1}e^{k+1} - s^k c e^k = s^k c e - s^k c e = 0$ .

(b) Let  $s, t$  be roots of  $\bar{g}(x)$ . Then  $s(s - c) = 0$  and  $t(t - c) = 0$  from which we have  $s^2 = sc$  and  $t^2 = tc$ . Then  $\bar{g}(se + t(1 - e)) = (se + t(1 - e))((se + t(1 - e)) - c) = (se + t(1 - e))^2 - (se + t(1 - e))c = s^2 e + t^2(1 - e) - (se + t(1 - e))c = (se + t(1 - e))c - (se + t(1 - e))c = 0$ .  $\square$

**Proposition 6.3.11.** *Let  $R$  be a ring and let  $g(x) = x(x - c) \in Z(R)[x]$  with  $c \in U(R)$ . Then  $r \in R$  is weakly clean if and only if  $rc$  is weakly  $g(x)$ -clean.*

**Proof.** ( $\Rightarrow$ ): Suppose that  $r \in R$  is weakly clean. Then  $r = u + e$  or  $r = u - e$  for some  $u \in U(R)$  and  $e \in Id(R)$ . Hence,  $rc = (u + e)c = uc + ec$  or  $rc = (u - e)c = uc - ec$ . Note that  $uc \in U(R)$  and  $g(ec) = ec(ec - c) = -ce(1 - e)c = 0$ . Hence,  $rc$  is weakly  $g(x)$ -clean.

( $\Leftarrow$ ): Suppose that  $rc$  is weakly  $g(x)$ -clean. Then  $rc = u + s$  or  $rc = u - s$  for some  $u \in U(R)$  and  $s \in R$  such that  $g(s) = 0$ . It follows that  $r = uc^{-1} + sc^{-1}$  or  $r = uc^{-1} - sc^{-1}$ . By the hypothesis,  $uc^{-1} \in U(R)$ . Note that  $(sc^{-1})^2 = s^2c^{-2} = (sc)(c^{-2}) = sc^{-1}$ . It follows that  $sc^{-1}$  is an idempotent in  $R$ . Hence,  $r$  is weakly clean.  $\square$

**Proposition 6.3.12.** *Let  $R$  be an abelian ring and let  $g(x) = x^k(x - c) \in Z(R)[x]$  where  $k$  is a positive integer and  $c \in U(R)$ . If  $r \in R$  is clean and  $1 + r$  is  $g(x)$ -clean, then  $r + e$  is  $g(x)$ -clean for some  $e \in Id(R)$ .*

**Proof.** Let  $r = u + e$  and let  $1 + r = v + t$  where  $u, v \in U(R)$ ,  $e \in Id(R)$  and  $t \in R$  such that  $g(t) = 0$ . Then  $r + e = (1 + r)e + r(1 - e) = (v + t)e + (u + e)(1 - e) = (ve + u(1 - e)) + te$ . Since  $R$  is abelian, we have  $ve + u(1 - e) \in U(R)$  with  $(ve + u(1 - e))^{-1} = v^{-1}e + u^{-1}(1 - e)$  and  $g(te) = (te)^k(te - c) = t^{k+1}e - ct^k e = ct^k e - ct^k e = 0$ . It follows that  $r + e$  is  $g(x)$ -clean.  $\square$

In the remainder of this section, we shall consider the  $x(x - c)$ -clean condition in the group ring  $RC_2$  where  $R$  is a commutative ring. We first state the following lemma:

**Lemma 6.3.3.** *[46, Lemma 3.2] Let  $R$  be a commutative ring. Then the element*

$a + bg \in RC_2$  where  $a, b \in R$  and  $g \in C_2$  is invertible if and only if  $a + b, a - b \in U(R)$ .

**Lemma 6.3.4.** *Let  $R$  be a commutative ring. Then the element  $a + bg \in RC_2$  where  $a, b \in R$  and  $g \in C_2$  is a root of  $h(x) = x(x - c) \in R[x]$  if and only if  $a^2 + b^2 = ca$  and  $2ab = cb$ . Moreover, in this case  $a + b$  and  $a - b$  are roots of  $h(x)$ .*

**Proof.** Suppose that  $a + bg \in RC_2$  is a root of  $h(x)$ . Then  $(a + bg)((a + bg) - c) = 0$ . It follows that  $(a + bg)^2 = c(a + bg)$ . Hence,  $a^2 + b^2 + 2abg = ca + cbg$  which gives us the equations  $a^2 + b^2 = ca$  and  $2ab = cb$ . Conversely, let  $a + bg \in RC_2$  such that  $a^2 + b^2 = ca$  and  $2ab = cb$ . Then  $h(a + bg) = (a + bg)((a + bg) - c) = (a + bg)^2 - c(a + bg) = (a^2 + b^2 + 2abg) - (ca + cbg) = (a^2 + b^2 - ca) + (2ab - cb)g = 0$ . For the final assertion, we add the equations  $a^2 + b^2 = ca$  and  $2ab = cb$  to get  $a^2 + b^2 + 2ab = c(a + b)$  which then gives us  $(a + b)^2 = c(a + b)$ . It follows that  $h(a + b) = 0$ . By subtracting the equation  $2ab = cb$  from  $a^2 + b^2 = ca$ , we have that  $a^2 + b^2 - 2ab = c(a - b)$ . Thus,  $(a - b)^2 = c(a - b)$  and hence,  $h(a - b) = 0$ .  $\square$

**Proposition 6.3.13.** *Let  $R$  be a commutative ring, let  $r \in R$  and let  $2 \in U(R)$ . Let  $h(x) = x(x - c) \in R[x]$  and let  $C_2 = \langle g \mid g^2 = 1 \rangle$ . If the element  $r + rg \in RC_2$  can be written as the sum of a unit in  $RC_2$  and a root of  $h(x)$  in  $R$ , then  $r$  is  $n$ -weakly clean for any integer  $n \geq 2$ .*

**Proof.** Suppose that  $r + rg \in RC_2$  can be written as  $r + rg = (a + bg) + s$  for some  $a + bg \in U(RC_2)$  and  $s \in R$  such that  $s(s - c) = 0$ . Then  $a + s = r$  and  $b = r$ . Since  $a + bg \in U(RC_2)$ , we have by Lemma 6.3.3 that  $a + b$  and  $a - b$  are invertible. It follows that  $a + r$  and  $a - r$  are invertible. Since  $s = r - a = -(a - r)$



is invertible and  $s(s - c) = 0$ , we thus have that  $s = c$  and hence,  $c \in U(R)$ . Thus,  $2r - c = (r - c) + r = a + r \in U(R)$  which implies that  $r = 2^{-1}(c + u)$  for some  $u \in U(R)$ . That is,  $r$  can be written as a sum of 2 units in  $R$ . By Proposition 4.2.1,  $r$  is  $n$ -weakly clean for any integer  $n \geq 2$ .  $\square$

## 6.4 More equivalent conditions for $x(x - c)$ -clean rings

Let  $R$  be a ring. A proper right (left) ideal  $P$  of  $R$  is said to be prime if  $aRb \subseteq P$  with  $a, b \in R$  implies that  $a \in P$  or  $b \in P$ . The ring  $R$  is said to be a right (respectively, left)  $pm$ -ring if every prime right (respectively, left) ideal of  $R$  is contained in a unique maximal right (respectively, left) ideal of  $R$ .

Given a ring  $R$ , let  $Spec_r(R)$  be the set of all proper right ideals of  $R$  which are prime. It has been shown in [72, Corollary 2.8] that if  $R$  is not a right quasi-duo ring, then  $Spec_r(R)$  is a topological space with the weak Zariski topology but not with the Zariski topology. For a right ideal  $I$  of  $R$ , let  $\mathcal{U}_r(I) = \{P \in Spec_r(R) \mid P \not\supseteq I\}$  and  $\mathcal{V}_r(I) = Spec_r(R) \setminus \mathcal{U}_r(I)$ . Let  $\tau = \{\mathcal{U}_r(I) \mid I \text{ is a right ideal of } R\}$ . Then  $\tau$  contains the empty set and  $Spec_r(R)$ . In general,  $\tau$  is just a subbase of the weak Zariski topology on  $Spec_r(R)$ . For any element  $a \in R$ , let  $\mathcal{U}_r(a) = \mathcal{U}_r(aR)$  and  $\mathcal{V}_r(a) = \mathcal{V}_r(aR)$ . Then  $\mathcal{U}_r(a) = \{P \in Spec_r(R) \mid a \notin P\}$  and  $\mathcal{V}_r(a) = \{P \in Spec_r(R) \mid a \in P\}$ . The left prime spectrum  $Spec_l(R)$  and the weak Zariski topology associated with it are defined analogously. Let  $Max_r(R)$  (respectively,  $Max_l(R)$ ) be the set of all maximal right (respectively, left) ideals of  $R$ . Since maximal right (respectively, left) ideals are prime right (respectively, left) ideals (see [57]),  $Max_r(R)$  (respectively,  $Max_l(R)$ ) inherits the weak Zariski topology on  $Spec_r(R)$  (respectively,  $Spec_l(R)$ ). Let  $U_r(I) = Max_r(R) \cap \mathcal{U}_r(I)$

and  $V_r(I) = \text{Max}_r(R) \cap \mathcal{V}_r(I)$  for any right ideal  $I$  of  $R$ . Then, in particular,  $U_r(a) = \text{Max}_r(R) \cap \mathcal{U}_r(a)$  and  $V_r(a) = \text{Max}_r(R) \cap \mathcal{V}_r(a)$  for any  $a \in R$ . A clopen set in a topological space is a set which is both open and closed. A topological space is said to be zero-dimensional if it has a base consisting of clopen sets.

In [21], a ring  $R$  (not necessarily commutative) is said to be right (respectively, left) topologically boolean, or a right (respectively, left) tb-ring for short, if for every pair of distinct maximal right (respectively, left) ideals of  $R$ , there is an idempotent in exactly one of them. Now let  $g_c(x) = x(x-c) \in Z(R)[x]$ . Here, we define a ring  $R$  to be right (respectively, left)  $c$ -topologically boolean, or a right (respectively, left)  $c$ -tb ring for short, if for every pair of distinct maximal right (respectively, left) ideals of  $R$ , there is a root of  $g_c(x)$  in exactly one of them. We say that  $R$  is a  $c$ -tb ring if it is both right and left  $c$ -tb. Clearly, when  $c = 1$ , a right (respectively, left)  $c$ -tb ring is just a right (respectively, left) tb-ring.

In this section we show that among conditions equivalent to  $R$  being an  $x(x-c)$ -clean ring where  $c \in U(R) \cap Z(R)$  is that  $R$  is right (left)  $c$ -tb. We begin with the following lemmas.

**Lemma 6.4.1.** *Let  $R$  be a ring, let  $g(x) = x(x-c) \in Z(R)[x]$  where  $c \in U(R)$  and let  $s \in R$  be a central root of  $g(x)$ . Let  $N$  be a maximal right ideal of  $R$ . If  $s \notin N$ , then  $c-s \in N$ .*

**Proof.** Since  $g(s) = 0$ , we have that  $s(s-c) = 0 \in P$  for any prime right ideal  $P$  of  $R$ . Then since  $s$  is central, it follows that every prime right ideal of  $R$  contains either  $s$  or  $s-c$ . Now since  $c = s + (c-s)$  and  $c \in U(R)$ , we have that  $1 = sc^{-1} + (c-s)c^{-1}$ . Hence, every prime right ideal of  $R$  contains either  $s$  or  $c-s$  but not both. Since maximal right ideals are prime right ideals (by [57]),

it follows that if  $s \notin N$ , then  $c - s \in N$ . □

**Lemma 6.4.2.** *Let  $R$  be a ring and let  $g(x) = x(x - c) \in Z(R)[x]$  where  $c \in U(R)$ . Let  $s, t \in R$  be central roots of  $g(x)$ . Then  $c^{-1}st$ ,  $s + t - c^{-1}st$  and  $c - s$  are also roots of  $g(x)$ .*

**Proof.** We first note that since  $s(s - c) = 0$  and  $t(t - c) = 0$ , we thus have  $s = c^{-1}s^2$  and  $t = c^{-1}t^2$ . Then

$$\begin{aligned}
 g(c^{-1}st) &= c^{-1}st(c^{-1}st - c) = c^{-2}(st)^2 - st \\
 &= c^{-2}(st)^2 - c^{-1}s^2t \quad (\because s = c^{-1}s^2) \\
 &= c^{-2}(s^2t)(t - c) \\
 &= 0.
 \end{aligned}$$

We also have that

$$\begin{aligned}
 g(s + t - c^{-1}st) &= (s + t - c^{-1}st)(s + t - c^{-1}st - c) \\
 &= s(s - c) + s(t - c^{-1}st) + t(s - c^{-1}st) + t(t - c) \\
 &\quad - c^{-1}st(s - c^{-1}st) - c^{-1}st(t - c) \\
 &= 0.
 \end{aligned}$$

Finally, we note that  $g(c - s) = (c - s)((c - s) - c) = (s - c)s = g(s) = 0$ . □

Let  $R$  be a ring and let  $g(x) = x(x - c) \in Z(R)[x]$  where  $c \in U(R)$ . Let  $\xi = \{U_r(s) \mid s \in R \text{ is a central root of } g(x) = x(x - c)\}$ . By Lemma 6.4.2 and the following lemma, we may deduce that  $\xi$  is closed under intersection and union.

**Lemma 6.4.3.** *Let  $R$  be a ring and let  $g(x) = x(x - c) \in Z(R)[x]$  with  $c \in U(R)$  such that every root of  $g(x)$  is central in  $R$ . If  $s, t \in R$  are roots of  $g(x)$ , then the following hold.*

(a)  $U_r(s) \cap U_r(t) = U_r(c^{-1}st)$ .

(b)  $U_r(s) \cup U_r(t) = U_r(s + t - c^{-1}st)$ .

(c)  $U_r(s) = V_r(c - s)$ . In particular, every set in  $\xi$  is clopen.

**Proof.** (a) Let  $P \in \mathcal{U}_r(s) \cap \mathcal{U}_r(t)$ . Then  $P \in \text{Spec}_r(R)$  with  $s, t \notin P$ . Note that  $c \notin P$ ; otherwise  $1 = cc^{-1} \in P$ , a contradiction. Since  $c, s, t$  are central in  $R$  and  $P$  is a prime right ideal of  $R$ , it follows that  $c^{-1}st \notin P$ . Hence,  $P \in \mathcal{U}_r(c^{-1}st)$  and therefore,  $\mathcal{U}_r(s) \cap \mathcal{U}_r(t) \subseteq \mathcal{U}_r(c^{-1}st)$ . Conversely, suppose that  $P \in \mathcal{U}_r(c^{-1}st)$ . If  $s$  or  $t$  belongs to  $P$ , then since  $s, t$  are central in  $R$  and  $P$  is a right ideal of  $R$ , it follows that  $c^{-1}st \in P$ ; a contradiction. Thus  $s$  and  $t$  do not belong to  $P$ , that is,  $P \in \mathcal{U}_r(s) \cap \mathcal{U}_r(t)$ . Hence,  $\mathcal{U}_r(c^{-1}st) \subseteq \mathcal{U}_r(s) \cap \mathcal{U}_r(t)$ . The equality  $\mathcal{U}_r(s) \cap \mathcal{U}_r(t) = \mathcal{U}_r(c^{-1}st)$  thus follows. Then  $U_r(s) \cap U_r(t) = \mathcal{U}_r(s) \cap \mathcal{U}_r(t) \cap \text{Max}_r(R) = \mathcal{U}_r(c^{-1}st) \cap \text{Max}_r(R) = U_r(c^{-1}st)$ .

(b) Let  $P \in \mathcal{U}_r(s) \cup \mathcal{U}_r(t)$ . Then  $s \notin P$  or  $t \notin P$ . Without loss of generality, suppose that  $s \notin P$ . Since  $s(s - c) = 0 \in P$  and  $s \notin P$  with  $s$  central in  $R$ , it follows that  $s - c \in P$ . Then  $(1 - c^{-1}s)t = -c^{-1}(s - c)t \in P$ . If  $s + (1 - c^{-1}s)t \in P$ , then it will follow that  $s \in P$ ; a contradiction. Thus,  $s + (1 - c^{-1}s)t \notin P$  and hence,  $P \in \mathcal{U}_r(s + (1 - c^{-1}s)t)$ . The inclusion  $\mathcal{U}_r(s) \cup \mathcal{U}_r(t) \subseteq \mathcal{U}_r(s + (1 - c^{-1}s)t)$  therefore holds. For the reverse inclusion, suppose that  $P \in \mathcal{U}_r(s + (1 - c^{-1}s)t)$ . Then  $s + (1 - c^{-1}s)t \notin P$ . If  $s$  and  $t$  both belong to  $P$ , then  $s + (1 - c^{-1}s)t \in P$ ; a contradiction. Hence, either  $s \notin P$  or  $t \notin P$ , that is,  $P \in \mathcal{U}_r(s)$  or  $P \in \mathcal{U}_r(t)$ . Therefore,  $P \in \mathcal{U}_r(s) \cup \mathcal{U}_r(t)$  and the inclusion  $\mathcal{U}_r(s + (1 - c^{-1}s)t) \subseteq \mathcal{U}_r(s) \cup \mathcal{U}_r(t)$  follows. Hence,  $\mathcal{U}_r(s) \cup \mathcal{U}_r(t) = \mathcal{U}_r(s + (1 - c^{-1}s)t)$ . It follows that  $U_r(s) \cup U_r(t) = (\mathcal{U}_r(s) \cap \text{Max}_r(R)) \cup (\mathcal{U}_r(t) \cap \text{Max}_r(R)) = (\mathcal{U}_r(s) \cup \mathcal{U}_r(t)) \cap \text{Max}_r(R) = \mathcal{U}_r(s + (1 - c^{-1}s)t) \cap \text{Max}_r(R) = U_r(s + (1 - c^{-1}s)t)$ .

(c) By using Lemma 6.4.1, we have  $U_r(s) = \text{Max}_r(R) \setminus U_r(c - s) = V_r(c - s)$ . It follows that every set in  $\xi$  is clopen.  $\square$

Next, we extend Proposition 2.4 in [21] as follows:

**Proposition 6.4.1.** *Let  $R$  be an  $x(x - c)$ -clean ring with  $c \in Z(R)$  such that every root of  $x(x - c)$  is central in  $R$ . Then  $R$  is a right  $c$ -tb ring.*

**Proof.** By Proposition 6.3.1,  $c \in U(R)$ . Let  $M$  and  $N$  be distinct maximal right ideals of  $R$ . Then there exists  $a \in M \setminus N$  and  $N + aR = R$ . Hence,  $1 - ar \in N$  for some  $r \in R$ . Since  $N$  is a right ideal of  $R$ ,  $c - arc = (1 - ar)c \in N$ . Let  $y = arc$ . Then  $c - y \in N$  and  $y \in M \setminus N$ . Since  $R$  is  $x(x - c)$ -clean, there exist a unit  $u \in R$  and a root  $s \in R$  of  $x(x - c)$  such that  $y = u + s$ . If  $s \in M$ , then  $u = y - s \in M$  from which it follows that  $M = R$ ; a contradiction since  $M$  is a maximal right ideal of  $R$ . Thus,  $s \notin M$ . If  $s \notin N$ , then  $c - s \in N$  (by Lemma 6.4.1) and hence,  $u = y - s = (y - c) + (c - s) \in N$ . It follows that  $N = R$  which is also not possible since  $N$  is a maximal right ideal of  $R$ . We thus have that  $s$  is a root of  $x(x - c)$  belonging to  $N$  only. Hence,  $R$  is a right  $c$ -tb ring.  $\square$

In [21], the following lemma was proven.

**Lemma 6.4.4.** [21, Lemma 2.1] *Let  $R$  be a ring. Then  $\text{Max}_r(R)$  is a compact  $T_1$ -space.*

**Proposition 6.4.2.** *Let  $R$  be a ring and let  $g(x) = x(x - c) \in Z(R)[x]$  with  $c \in U(R)$  such that every root of  $g(x)$  in  $R$  is central. If  $R$  is a right  $c$ -tb ring, then  $\xi$  forms a base for the weak Zariski topology on  $\text{Max}_r(R)$ . In particular,  $\text{Max}_r(R)$  is a compact, zero-dimensional Hausdorff space.*

**Proof.** Note that if  $M_1$  and  $M_2$  are two distinct maximal right ideals of  $R$ , then since  $R$  is a right  $c$ -tb ring, there exists a root  $s \in R$  of  $g(x)$  such that

$s \notin M_1, s \in M_2$  (that is,  $M_1 \in U_r(s), M_2 \notin U_r(s)$ ). The points in  $Max_r(R)$  can therefore be separated by disjoint clopen sets belonging to  $\xi$ . Hence,  $Max_r(R)$  is Hausdorff. By Lemma 6.4.4, we have that  $Max_r(R)$  is compact.

To show that  $\xi$  forms a base for the weak Zariski topology on  $Max_r(R)$ , let  $K \subseteq Max_r(R)$  be a closed subset and take  $M \notin K$ . For each  $N \in K$ , since  $N \neq M$ , there exists a clopen set  $U_r(s_N) \in \xi$  separating  $M$  and  $N$ , say  $N \in U_r(s_N)$ . The collection  $\{U_r(s_N) \mid N \in K\}$  is therefore an open cover of the set  $K$ . Since  $K$  is compact, it has a finite subcover, that is,  $K$  is contained in a finite cover of sets of the form  $U_r(s_N)$  with  $N \in K$ . By Lemma 6.4.3, there exists a clopen set  $C \in \xi$  separating  $M$  from  $K$ . Hence,  $\xi$  forms a base for the weak Zariski topology on  $Max_r(R)$ . Since every set in  $\xi$  is clopen (by Lemma 6.4.3), it follows that  $Max_r(R)$  is zero-dimensional.  $\square$

**Proposition 6.4.3.** *Let  $R$  be a ring and let  $g(x) = x(x - c) \in Z(R)[x]$  with  $c \in U(R)$  such that every root of  $g(x)$  in  $R$  is central. If  $\xi$  forms a base for the weak Zariski topology on  $Max_r(R)$ , then for any  $a \in R$ , there exists a root  $s$  of  $g(x)$  such that  $s \notin M$  for every  $M \in V_r(a)$  and  $s \in N$  for every  $N \in V_r(a - c)$ .*

**Proof.** Consider the disjoint closed sets  $V_r(a)$  and  $V_r(a - c)$ . Since  $\xi$  forms a base for the weak Zariski topology on  $Max_r(R)$  and  $Max_r(R)$  is compact, there is a clopen set  $U_r(s) \in \xi$  separating the sets  $V_r(a)$  and  $V_r(a - c)$ . Without loss of generality, assume that  $V_r(a) \subseteq U_r(s)$  and  $V_r(a - c) \subseteq V_r(s)$ . Then it follows that  $s \notin M$  for every  $M \in V_r(a)$  and  $s \in N$  for every  $N \in V_r(a - c)$ .  $\square$

**Proposition 6.4.4.** *Let  $R$  be a ring and let  $g(x) = x(x - c) \in Z(R)[x]$  with  $c \in U(R)$  such that every root of  $g(x)$  in  $R$  is central. If for every  $a \in R$  there*

exists a root  $s \in Z(R)$  of  $g(x)$  such that  $V_r(a) \subseteq U_r(s)$  and  $V_r(a - c) \subseteq V_r(s)$ , then  $R$  is  $g(x)$ -clean.

**Proof.** Let  $a \in R$ . By the hypothesis, there exists a root  $s \in Z(R)$  of  $g(x)$  such that  $V_r(a) \subseteq U_r(s)$  and  $V_r(a - c) \subseteq V_r(s)$ . We claim that  $a - s$  is a unit. Let  $M$  be a maximal right ideal of  $R$ . Note that if  $a \in M$ , then  $a - s \notin M$ , since  $s \notin M$ . Next, suppose that  $a \notin M$ . If  $a - s \in M$ , then  $s \notin M$ , and hence,  $c - s \in M$  (by Lemma 6.4.1). Then since  $(a - c) + (c - s) = a - s \in M$ , it follows that  $a - c \in M$  and hence,  $s \in M$  (because  $V_r(a - c) \subseteq V_r(s)$ ); a contradiction. Thus,  $a - s \notin M$ . We have therefore shown that  $a - s \notin M$  for any maximal right ideal  $M$  of  $R$ . Hence,  $a - s$  has a right inverse, that is,  $(a - s)v = 1$  for some  $v \in R$ . Then  $(a - s)(vc) = c$  and by Lemma 6.3.2, we have that  $(vc)(a - s) = c$ . Since  $c \in U(R) \cap Z(R)$ , we can conclude that  $a - s$  is a unit in  $R$ . Hence,  $a$  is the sum of a unit and a root of  $g(x)$  in  $R$ . Since  $a$  is arbitrary in  $R$ , it follows that  $R$  is  $g(x)$ -clean.  $\square$

We are now ready for the main result.

**Theorem 6.4.1.** *Let  $R$  be a ring and let  $x(x - c) \in Z(R)[x]$  with  $c \in U(R)$ . If every root of  $x(x - c)$  is central in  $R$ , then the following conditions are equivalent:*

- (a)  $R$  is  $x(x - c)$ -clean.
- (b)  $R$  is  $x(x + c)$ -clean.
- (c)  $R$  is  $n$ -clean for all positive integers  $n$ .
- (d)  $R$  is a right  $c$ -tb ring.
- (e) The collection  $\xi = \{U_r(s) \mid s \in R \text{ is a root of } x(x - c)\}$  forms a base for the weak Zariski topology on  $\text{Max}_r(R)$ .

(f) For every  $a \in R$ , there exists a root  $s \in Z(R)$  of  $x(x - c)$  such that  $V_r(a) \subseteq U_r(s)$  and  $V_r(a - c) \subseteq V_r(s)$ .

(g)  $R$  is a left  $c$ -tb ring.

(h) The collection  $\xi = \{U_l(s) \mid s \in R \text{ is a root of } x(x - c)\}$  forms a base for the weak Zariski topology on  $Max_l(R)$ .

**Proof.** By Theorem 6.3.1, it follows readily that (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (c). By Proposition 6.4.1, we readily have (a)  $\Rightarrow$  (d). The implications (d)  $\Rightarrow$  (e)  $\Rightarrow$  (f) follow by Propositions 6.4.2 and 6.4.3, respectively. The implication (f)  $\Rightarrow$  (a) is straightforward by using Proposition 6.4.4. For (a)  $\Leftrightarrow$  (g)  $\Leftrightarrow$  (h), this follows from the left analogue of the arguments for (a)  $\Rightarrow$  (d)  $\Rightarrow$  (e)  $\Rightarrow$  (f)  $\Rightarrow$  (a).  $\square$

**Remark.** Characterisations of clean-related rings in terms of their topological properties have also recently been obtained in [14]. In that paper, a ring  $R$  is said to be feckly clean provided that for any  $a \in R$ , there exist an element  $e \in R$  and an element  $u \in R$  satisfying  $RuR = R$  such that  $a = e + u$  and  $eR(1 - e) \subseteq J(R)$ . Among others, the authors obtained a characterisation of feckly clean rings in terms of the topological space of all prime ideals containing the Jacobson radical of  $R$ .



# Chapter 7

## Summary of Properties

For the convenience of the reader, we summarise in this final chapter some properties of the rings covered in this dissertation. The properties considered in this chapter have been shown to hold (or not to hold) in the previous chapters or in the available literature.

### 7.1 Corners

**Question 7.1.1.** *Let  $R$  be a clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean) ring and let  $e \in Id(R)$ . Is the corner ring  $eRe$  also clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean)?*

	Yes	Not always	uncertain
Clean		✓	
Weakly clean	✓*		
Pseudo weakly clean	✓		
Uniquely weakly clean			✓
$n$ -weakly clean ( $n \geq 2$ )		✓	

\* For abelian rings

**Question 7.1.2.** *Let  $R$  be a ring and let  $e \in Id(R)$  such that the corner ring  $eRe$  is clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,*

$n$ -weakly clean). Is the ring  $R$  also clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean)?

	Yes	Not always	uncertain
Clean	✓		
Weakly clean	✓		
Pseudo weakly clean	✓		
Uniquely weakly clean			✓
$n$ -weakly clean ( $n \geq 2$ )	✓		

## 7.2 Direct products

**Question 7.2.1.** Let  $R = \prod_{i \in I} R_i$  where each  $R_i$  is a ring. Suppose that  $R$  is clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean). Is each  $R_i$  also clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean)?

	Yes	Not always	uncertain
Clean	✓		
Weakly clean	✓		
Pseudo weakly clean	✓		
Uniquely weakly clean	✓		
$n$ -weakly clean ( $n \geq 2$ )	✓		

**Question 7.2.2.** Let  $R = \prod_{i \in I} R_i$  where each  $R_i$  is a ring. Suppose that each  $R_i$  ( $i \in I$ ) is clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean). Is  $R$  clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean)?

	Yes	Not always
Clean	✓	
Weakly clean	✓*	
Pseudo weakly clean	✓**	
Uniquely weakly clean	✓	
$n$ -weakly clean ( $n \geq 2$ )	✓***	

**Remarks.**

- \* provided at most one  $R_i$  is not a weakly clean ring
- \*\* provided at most one  $R_i$  is not a pseudo weakly clean ring
- \*\*\* provided at most one  $R_i$  is not an  $n$ -weakly clean ring

### 7.3 Centres

**Question 7.3.1.** *Let  $R$  be a clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean) ring. Is the centre of  $R$  also clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean)?*

	Yes	Not always	uncertain
Clean		✓	
Weakly clean		✓	
Pseudo weakly clean			✓
Uniquely weakly clean	✓		
$n$ -weakly clean ( $n \geq 2$ )			✓

### 7.4 Homomorphic Images

**Question 7.4.1.** *Let  $R$  be a clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean) ring. Are homomorphic images of  $R$  also clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean)?*

	Yes	No	Uncertain
Clean	✓		
Weakly clean	✓		
Pseudo weakly clean	✓		
Uniquely weakly clean			✓
$n$ -weakly clean ( $n \geq 2$ )	✓		

## 7.5 Polynomial rings

**Question 7.5.1.** *Is the polynomial ring over a clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean) ring also clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean)?*

	Yes	No	uncertain
Clean		✓	
Weakly clean		✓	
Pseudo weakly clean		✓	
Uniquely weakly clean		✓	
$n$ -weakly clean ( $n \geq 2$ )		✓	

## 7.6 Power series rings

**Question 7.6.1.** *Is the formal power series ring over a clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean) ring also clean (respectively, weakly clean, pseudo weakly clean, uniquely weakly clean,  $n$ -weakly clean)?*

	Yes	No	Uncertain
Clean	✓		
Weakly clean	✓		
Pseudo weakly clean	✓*		
Uniquely weakly clean	✓		
$n$ -weakly clean ( $n \geq 2$ )	✓		

\* Shown for commutative rings

# Bibliography

- [1] M.S. Ahn, Weakly clean rings and almost clean rings, Ph.D. thesis, The University of Iowa, 2003.
- [2] M.S. Ahn and D.D. Anderson, Weakly clean rings and almost clean rings, Rocky Mountain J. Math. **36** (2006), 783–797.
- [3] D.D. Anderson and V.P. Camillo, Commutative rings whose elements are a sum of a unit and an idempotent, Comm. Algebra **30** (2002), 3327–3336.
- [4] N. Ashrafi and Z. Ahmadi, Weakly  $g(x)$ -clean rings, Iranian J. of Mathematical Sciences and Informatics **7(2)** (2012), 83–91.
- [5] G.F. Birkenmeier, H.E. Heatherly, E.K. Lee, Completely prime ideals and associated radicals, in: S.K. Jain, S.T. Rizvi (Eds.), Proc. Biennial Ohio StateDenison Conference, 1992, World Scientific, Singapore, 1993, 102-129.
- [6] W.D. Burgess and R. Raphael, On commutative clean rings and  $pm$ -rings, Rings, modules and representations, Contemp. Math., Amer. Math. Soc., Providence, RI. **480** (2009), 35–55.
- [7] W.D. Burgess and R. Raphael, On embedding rings in clean rings, Comm. Algebra **41** (2013), 552–564.

- [8] V.P. Camillo and D. Khurana, A characterization of unit regular rings, *Comm. Algebra* **29** (2001), 2293–2295.
- [9] V.P. Camillo, D. Khurana, T.Y. Lam, W.K. Nicholson and Y. Zhou, Continuous modules are clean, *J. Algebra* **304** (2006), 94–111.
- [10] V.P. Camillo, J.J. Simón, The Nicholson-Varadarajan theorem on clean linear transformations, *Glasg. Math. J.* **44** (2002), 365–369.
- [11] V.P. Camillo and H.P. Yu, Exchange rings, units and idempotents, *Comm. Algebra* **22** (1994), 4737–4749.
- [12] V.P. Camillo and H.P. Yu, Stable range one for rings with many idempotents, *Trans. Amer. Math. Soc.* **347** (1995), 3141–3147.
- [13] H. Chen, Exchange rings with stable range one, *Czechoslovak Math. J.* **57(132)** (2007), 579–590.
- [14] H. Chen, H. Kose and Y. Kurtulmaz, On feckly clean rings, *J. Algebra and its Applications* **14** (2015), 1550046 (15 pages).
- [15] H. Chen, B. Ungor and S. Halicioglu, Very clean matrices over local rings, arXiv:1406.1240 [math.RA].
- [16] J. Chen, W.K. Nicholson and Y. Zhou, Group rings in which every element is uniquely the sum of a unit and an idempotent, *J. Algebra* **306** (2006), 453–460.
- [17] J. Chen and Z. Wang, Some progress on clean rings, *Ring theory 2007*, World Sci. Publ., Hackensack, NJ, (2009), 38–53.

- [18] J. Chen, Z. Wang and Y. Zhou, Rings in which elements are uniquely the sum of an idempotent and a unit that commute, *J. Pure Appl. Algebra* **213** (2009), 215–223.
- [19] W. Chen, A question on strongly clean rings, *Comm. Algebra* **34** (2006), 2347–2350.
- [20] W. Chen and S. Cui, On clean rings and clean elements, *Southeast Asian Bull. Math.* **32** (2008), 855–861.
- [21] A.Y.M. Chin, Clean elements in abelian rings, *Proc. Indian Acad. Sci. Math. Sci.* **119** (2009), 145–148.
- [22] A.Y.M. Chin and K.T. Qua, A note on weakly clean rings, *Acta Math. Hungar.* **132** (2011), 113–116.
- [23] A.Y.M. Chin and K.T. Qua, A note on  $n$ -clean group rings, *Publ. Math. Debrecen*, **78** (2011), 569–574.
- [24] I.G. Connell, On the group ring, *Canad. J. Math.* **15** (1963), 650–685.
- [25] M. Contessa, On pm-rings, *Comm. Algebra* **10** (1982), 93–108.
- [26] P.V. Danchev, On weakly exchange rings, *J. Math. Univ. Tokushima.* **48** (2014) 17–22.
- [27] P.V. Danchev and W.Wm. McGovern, Commutative weakly nil clean unital rings, *J. Algebra* **425** (2015) 410–422.
- [28] A.J. Diesl, Nil clean rings, *J. Algebra* **383** (2013) 197–211.

- [29] L. Fan and X. Yang, On rings whose elements are the sum of a unit and a root of a fixed polynomial, *Comm. Algebra* **36** (2008), 269–278.
- [30] L. Fan and X. Yang, On strongly  $g(x)$ -clean rings, arXiv:0803.3353 [math.RA].
- [31] K.R. Goodearl, Von Neumann regular rings, Kreiger Publ. Co. 2nd edition (1991).
- [32] J. Han and W.K. Nicholson, Extensions of clean rings, *Comm. Algebra* **29** (2001), 2589–2595.
- [33] A.H. Handam,  $(n, g(n))$ -clean rings, *International Mathematical Forum* **4** (2009), 1007–1011.
- [34] M. Henriksen, Two classes of rings generated by their units. *J. Algebra* **31** (1974), 182–193.
- [35] V.A. Hiremath and S. Hegde, On strongly clean rings, *International J. of Algebra* **5** (2011), 31–36.
- [36] C.Y. Hong, N.K. Kim and Y. Lee, Exchange rings and their extensions, *J. Pure Appl. Algebra* **179** (2003), 117–126.
- [37] I. Kaplansky, Elementary divisors and modules, *Trans, Amer. Math. Soc.* **66** (1949), 464–491.
- [38] O.H. Kegel and B.A.F. Wehrfritz, Locally finite groups, North-Holland, Amsterdam (1973).
- [39] D. Khurana and T.Y. Lam, Clean matrices and unit-regular matrices, *J. Algebra* **280** (2004), 683–698.



- [40] T.Y. Lam, Lectures on modules and rings, Springer-Verlag, New York (1999).
- [41] T.K. Lee and Y. Zhou, A class of exchange rings, *Glasg. Math. J.* **50** (2008), 509–522.
- [42] G. Marks, On 2-primal Ore extensions, *Comm. Algebra* **29** (2001), 2113–2123
- [43] G. Marks, Reversible and symmetric rings, *J. Pure Appl. Algebra* **174** (2002) 311–318.
- [44] G. Marks, A taxonomy of 2-primal rings, *J. Algebra* **266** (2003) 494–520.
- [45] W.Wm. McGovern, Clean semiprime  $f$ -rings with bounded inversion, *Comm. Algebra* **31** (2003), 3295–3304.
- [46] W.Wm. McGovern, A characterization of commutative clean rings, *Int. J. Math. Game Theory Algebra* **15** (2006), 403–413.
- [47] W.Wm. McGovern, Neat rings, *J. Pure Appl. Algebra* **205** (2006), 243–265.
- [48] A.I. Nicholas, Clean rings and clean group rings, Ph.D. thesis, Bowling Green State University, 2013.
- [49] W.K. Nicholson, Local group rings, *Canad. Math. Bull.* **15** (1972), 137–138.
- [50] W.K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.* **229** (1977), 269–278.
- [51] W.K. Nicholson, Strongly clean rings and Fitting’s lemma, *Comm. Algebra* **27** (1999), 3583–3592.

- [52] W.K. Nicholson and M.F. Yousif, *Quasi-Frobenius Rings*, Cambridge Tracts in Math., Cambridge University Press, 2003.
- [53] W.K. Nicholson and Y. Zhou, Rings in which elements are uniquely the sum of an idempotent and a unit, *Glasg. Math. J.* **46** (2004), 227–236.
- [54] W.K. Nicholson and Y. Zhou, Clean rings: a survey. Chen, J.L. (ed) et al., *Advances in ring theory*. World Sci. Publ., Hackensack, NJ. (2005), 181–198.
- [55] E. Sánchez Campos, On strongly clean rings, Unpublished manuscript, 2002.
- [56] G. Shin, Prime ideals and sheaf representations of a pseudo symmetric ring, *Trans. Amer. Math. Soc.* **184** (1973), 43–60.
- [57] S.H. Sun, Rings in which every prime ideal is contained in a unique maximal right ideal, *J. Pure Appl. Algebra* **78** (1992), 183–194.
- [58] J. Ster, Corner rings of a clean ring need not be clean, *Comm. Algebra*, **40** (2012), 1595–1604.
- [59] J. Ster, Weakly clean rings, *J. Algebra*, **401** (2014), 1–12.
- [60] Z. Wang and J. Chen, On two open problems about strongly clean rings, *Bull. Austral. Math. Soc.* **70** (2004), 279–282.
- [61] Z. Wang and J. Chen, A note on clean rings, *Algebra Colloq.* **14** (2007), 537–540.
- [62] X. Wang and H. You, Cleanness of the group ring of an abelian  $p$ -group over a commutative ring, *Algebra Colloq.* **19** (2012), 539–544.

- [63] J. Wei, Weakly-abel rings and weakly exchange rings, *Acta Math. Hungar.* **137** (2012), 254-262.
- [64] J. Wei, Generalized weakly symmetric rings, *J. Pure Appl. Algebra.* **218** (2014), 1591–1603.
- [65] J. Wei, C. Qu, On left  $NQD$  rings, *Southeast Asian Bull. Math.* **38** (2014), 297–310.
- [66] S.M. Woods, Some results on semiperfect group rings, *Canadian J. Math.* **26** (1974), 121–129.
- [67] G. Xiao and W. Tong,  $n$ -clean rings and weakly unit stable range rings, *Comm. Algebra* **33** (2005), 1501–1517.
- [68] G. Xiao and W. Tong,  $n$ -clean rings, *Algebra Colloq.* **13** (2006), 599–606.
- [69] X. Yang, Strongly clean rings and  $g(x)$ -clean rings, Ph.D. thesis, Memorial University of Newfoundland, 2007.
- [70] X. Yang and Y. Zhou, Some families of strongly clean rings, *Linear Algebra Appl.* **425** (2007), 119–129.
- [71] X. Yang, A survey of strongly clean rings, *Acta Appl. Math.* **108** (2009), 157–173.
- [72] G. Zhang, W. Tong and F. Wang, Spectra of maximal 1-sided ideals and primitive ideals, *Comm. Algebra* **34** (2006), 2879–2896.
- [73] Y. Zhou, On clean group rings, *Advances in Ring Theory* (2010), 335–345.