GENERALISED CLASSICAL ADJOINT-COMMUTING MAPPINGS ON MATRIX SPACES

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THESIS SUBMITTED IN FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

FACULTY OF SCIENCE
UNIVERSITY OF MALAYA
KUALA LUMPUR

2016

UNIVERSITY OF MALAYA

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Matric No: SHB080010

Name of Degree: Doctor of Philosophy

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GENERALISED CLASSICAL ADJOINT-COMMUTING MAPPINGS ON MATRIX SPACES

Field of Study: Linear Algebra and Matrix Theory

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Abstract

Let m, n be integers with $m, n \geqslant 3$, and let $\mathbb{F}$ and $\mathbb{K}$ be fields. We denote by $\mathcal{M}_n(\mathbb{F})$ the linear space of $n \times n$ matrices over $\mathbb{F}$, $\mathcal{S}_n(\mathbb{F})$ the linear space of $n \times n$ symmetric matrices over $\mathbb{F}$ and $\mathcal{K}_n(\mathbb{F})$ the linear space of $n \times n$ alternate matrices over $\mathbb{F}$. In addition, let $\mathbb{F}$ be a field with an involution $^-$, we denote by $\mathcal{H}_n(\mathbb{F})$ the $\mathbb{F}^-$-linear space of $n \times n$ hermitian matrices over $\mathbb{F}$ and $\mathcal{SH}_n(\mathbb{F})$ the $\mathbb{F}^-$-linear space of $n \times n$ skew-hermitian matrices over $\mathbb{F}$ where $\mathbb{F}^-$ is a fixed field of $\mathbb{F}$. We let adj A be the classical adjoint of a matrix A and I_n be the $n \times n$ identity matrix. In this dissertation, we characterise mappings ψ that satisfy one of the following conditions:

(A1) $\psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ with either $|\mathbb{F}| = 2$ or $|\mathbb{F}| > n + 1$, and

$$\psi(\text{adj }(A + \alpha B)) = \text{adj }(\psi(A) + \alpha\psi(B)) \text{ for all } A, B \in \mathcal{M}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F};$$

(A2) $\psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ where ψ is surjective and

$$\psi(\text{adj }(A - B)) = \text{adj }(\psi(A) - \psi(B)) \text{ for all } A, B \in \mathcal{M}_n(\mathbb{F}).$$

Besides, we also study the structure of ψ on $\mathcal{H}_n(\mathbb{F})$, $\mathcal{S}_n(\mathbb{F})$, $\mathcal{SH}_n(\mathbb{F})$ and $\mathcal{K}_n(\mathbb{F})$. We obtain a complete description of ψ satisfying condition (A1) or (A2) on $\mathcal{M}_n(\mathbb{F})$, $\mathcal{H}_n(\mathbb{F})$ and $\mathcal{S}_n(\mathbb{F})$ if $\psi(I_n) \neq 0$. If $\psi(I_n) = 0$, we prove that such mappings send all rank one matrices to zero. Clearly, $\psi = 0$ when ψ is linear. Some examples of nonlinear mappings ψ satisfying condition (A1) or (A2) with $\psi(I_n) = 0$ are given. In the study of ψ satisfying condition (A1) or (A2) on $\mathcal{K}_n(\mathbb{F})$, we obtain a nice structural result of ψ if $\psi(A) = 0$ for some invertible matrix $A \in \mathcal{K}_n(\mathbb{F})$. Some examples of nonlinear mappings ψ vanishing all invertible matrices are included. In the case of $\mathcal{SH}_n(\mathbb{F})$, some examples of

nonlinear mappings ψ satisfying condition (A1) or (A2) that send all rank one matrices and invertible matrices to zero are given. Otherwise, a nice structural result of ψ is obtained.

Abstrak

Biar m, n integer dengan $m, n \geqslant 3$, dan biar $\mathbb{F}$ dan $\mathbb{K}$ medan. Kami menandakan $\mathcal{M}_n(\mathbb{F})$ sebagai ruang linear matriks $n \times n$ atas $\mathbb{F}$, $\mathcal{S}_n(\mathbb{F})$ sebagai ruang linear matriks symmetri $n \times n$ atas $\mathbb{F}$ dan $\mathcal{K}_n(\mathbb{F})$ sebagai ruang linear matriks selang-seli $n \times n$ atas $\mathbb{F}$. Tambahannya, biar $\mathbb{F}$ satu medan yang mempuyai suatu involusi $^-$ atas $\mathbb{F}$, kami menandakan $\mathcal{H}_n(\mathbb{F})$ sebagai ruang $\mathbb{F}^-$-linear matriks hermitean $n \times n$ atas $\mathbb{F}$ dan $\mathcal{SH}_n(\mathbb{F})$ sebagai ruang $\mathbb{F}^-$-linear matriks hermitean pencong $n \times n$ atas $\mathbb{F}$, di mana $\mathbb{F}^-$ ialah medan tetap bagi $\mathbb{F}$. Biar adj A matrik adjoin A dan I_n matriks identiti $n \times n$. Dalam disertasi ini, kami cirikan pemetaan ψ yang memenuhi salah satu syarat berikut:

(A1) $\psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ dengan $|\mathbb{F}| = 2$ atau $|\mathbb{F}| > n + 1$, dan

$$\psi(\text{adj }(A + \alpha B)) = \text{adj }(\psi(A) + \alpha\psi(B))$$

untuk semua $A, B \in \mathcal{M}_n(\mathbb{F})$ dan $\alpha \in \mathbb{F}$;

(A2) $\psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ di mana ψ adalah surjektif dan

$$\psi(\text{adj }(A - B)) = \text{adj }(\psi(A) - \psi(B)) \text{ untuk semua } A, B \in \mathcal{M}_n(\mathbb{F}).$$

Selain daripada itu, kami juga mengkaji struktur ψ pada $\mathcal{H}_n(\mathbb{F})$, $\mathcal{S}_n(\mathbb{F})$, $\mathcal{SH}_n(\mathbb{F})$ dan $\mathcal{K}_n(\mathbb{F})$. Kami memperolehi pemerihalan lengkap untuk ψ yang mematuhi syarat (A1) atau (A2) pada $\mathcal{M}_n(\mathbb{F})$, $\mathcal{H}_n(\mathbb{F})$ dan $\mathcal{S}_n(\mathbb{F})$ jika $\psi(I_n) \neq 0$. Jika $\psi(I_n) = 0$, kami menunjukkan bahawa pemetaan ψ tersebut memetakan semua matriks yang berpangkat satu kepada kosong. Jelasnya, $\psi = 0$ jika ψ adalah linear. Beberapa contoh pemetaan ψ yang tidak linear, yang mematuhi syarat (A1) atau (A2) dengan $\psi(I_n) = 0$ diberikan. Di dalam pengajian ψ

yang mematuhi syarat (A1) atau (A2) pada $\mathcal{K}_n(\mathbb{F})$, kami memperolehi keputusan yang ψ berstruktur baik jika $\psi(A) \neq 0$ untuk suatu matriks $A \in \mathcal{K}_n(\mathbb{F})$ yang tersongsangkan. Beberapa contoh pemetaan ψ yang tidak linear dan melenyapkan semua matriks yang tersongsangkan diberikan. Untuk kes $\mathcal{SH}_n(\mathbb{F})$, beberapa contoh pemetaan ψ yang tidak linear dan mematuhi syarat (A1) atau (A2) yang memetakan semua matriks yang berpangkat satu dan semua matriks yang tersongsangkan kepada kosong diberikan. Selainnya, struktur ψ yang baik diperolehi.

Acknowledgements

The author would like to thank her supervisor, Assoc. Prof. Dr. Chooi Wai Leong for giving her the opportunity to work on this thesis. She is grateful for his insightful and helpful guidance, and also his patience throughout the period of her PhD research study.

The author would also like to thank the Head of Institute of Mathematical Sciences, University of Malaya and all the staff of the institute for their support and assistance during her postgraduate study.

Last but not least, the author would like to thank Univeriti Tunku Abdul Rahman (UTAR) where she holds the position of Senior Lecturer for partially supporting her PhD study.

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Chapter 1

Introduction

Let A be a square matrix, the *classical adjoint* of A, denoted by adj A, is defined by the transposed matrix of cofactors of the matrix A. More precisely, the (i, j)-entry of adj A of an $n \times n$ matrix A is

$$(\text{adj } A)_{ij} = (-1)^{i+j} \det(A[j|i])$$

where $\det(A[j|i])$ denotes the determinant of the $(n-1) \times (n-1)$ submatrix $A[j|i]$ of A obtained by excluding j-th row and i-th column.

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be vector spaces such that adj $A \in \mathcal{U}_i$ whenever $A \in \mathcal{U}_i$ for $i = 1, 2$. A mapping $\psi : \mathcal{U}_1 \to \mathcal{U}_2$ is said to be *classical adjoint-commuting* if

$$\psi(\text{adj } A) = \text{adj } \psi(A) \text{ for every } A \in \mathcal{U}_1. \tag{1.1}$$

In this dissertation, we mainly study some generalised classical adjoint-commuting mappings. In the next section, we give some notations used in this dissertation. Since the characterisation of classical adjoint-commuting mappings is one of the preserver problems (see [26, 1, 3, 30, 27, 29], we state several types of preserver problems in Section 1.2. Some properties of classical-adjoint which are used in the later part of the dissertation are given in Section 1.4.

1.1 Notations

Unless otherwise stated, the following are some notations used in this dissertation. Let m, n be integers with $m, n \geqslant 2$ and let $\mathbb{F}$ be a field. We denote by

$\mathcal{M}_{m,n}(\mathbb{F})$ the linear space of $m \times n$ matrices over $\mathbb{F}$ ($\mathcal{M}_n(\mathbb{F}) = \mathcal{M}_{n,n}(\mathbb{F})$ in short). For any $A \in \mathcal{M}_n(\mathbb{F})$, A^t denotes the transpose of A and $\mathrm{tr}(A)$ denotes the trace of A. We also denote by $\mathcal{T}_n(\mathbb{F})$ the algebra of all $n \times n$ *upper triangular* matrices over $\mathbb{F}$.

Let $^- : \mathbb{F} \to \mathbb{F}$ be a field involution which is defined by $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{a}\overline{b}$, and $\overline{\overline{a}} = a$ for any $a, b \in \mathbb{F}$. We denote by $\mathbb{F}^- := \{a \in \mathbb{F} : \overline{a} = a\}$ the set of all symmetric elements of $\mathbb{F}$ on the involution $^-$ of $\mathbb{F}$. A matrix $A \in \mathcal{M}_n(\mathbb{F})$ is called a *hermitian matrix on the involution* $^-$ *of* $\mathbb{F}$, or simply *hermitian* if $\overline{A}^t = A$, A is *symmetric* if $A^t = A$, and A is a *skew-hermitian matrix on the involution* $^-$ *of* $\mathbb{F}$, or *skew-hermitian* if $\overline{A}^t = -A$. Here, $\overline{A}$ is the matrix obtained from A by applying $^-$ entrywise. We denote by $\mathcal{H}_n(\mathbb{F})$ the $\mathbb{F}^-$-linear space of $n \times n$ hermitian matrices over $\mathbb{F}$, and $\mathcal{S}_n(\mathbb{F})$ the linear space of $n \times n$ symmetric matrices over $\mathbb{F}$. It is obvious that $\mathcal{H}_n(\mathbb{F}) = \mathcal{S}_n(\mathbb{F})$ when the involution $^-$ of $\mathbb{F}$ is identity, i.e. $\mathbb{F}^- = \mathbb{F}$. We also denote by $\mathcal{SH}_n(\mathbb{F})$ the $\mathbb{F}^-$-linear space of $n \times n$ skew-hermitian matrices over $\mathbb{F}$. A matrix $A \in \mathcal{M}_n(\mathbb{F})$ is *alternate* if $uAu^t = 0$ for every row vector $u \in \mathbb{F}^n$, or equivalently, if $A^t = -A$ with zero diagonal entries. We denote by $\mathcal{K}_n(\mathbb{F})$ the linear space of $n \times n$ alternate matrices over $\mathbb{F}$.

I_n denotes the $n \times n$ identity matrix, E_{ij} denotes the unit square matrix whose (i, j)-th entry is one and whose other entries are zero and 0_n denotes the $n \times n$ zero matrix for any integer $n \geqslant 2$.

1.2 Preserver problems

"Linear Preserver Problems" (LPPs) is one of the active and continuing subjects in matrix theory which concerns the classification of linear operators on spaces of matrices that leave certain functions, subsets, relations, etc invariant. The main objective of this dissertation is to study generalised preserver problems, that is, to classify operators (which are not necessarily linear) on spaces of matrices or operators that leave certain functions, subsets, relations, etc invariant. Here, we give a brief survey of linear preserver problems.

In general, there are several types of linear preserver problems. Here, we shall list four most common types of such problems.

Let T be a linear operator on $\mathcal{M}_n(\mathbb{F})$.

I. *T preserves a (scalar valued, vector-valued or set-valued) function φ on* $\mathcal{M}_n(\mathbb{F})$. Characterise those linear operators T on $\mathcal{M}_n(\mathbb{F})$ that satisfy

$$\varphi(T(A)) = \varphi(A) \quad \text{for all } A \in \mathcal{M}_n(\mathbb{F}).$$

An example of Type I LPP is the classical theorem of G. Frobenius (Proposition 1.2.1) which characterises bijective linear operators on complex matrices $\mathcal{M}_n(\mathbb{C})$ that preserve the determinant (see [6]) in 1897:

Proposition 1.2.1. *Let T be an invertible linear operator on $\mathcal{M}_n(\mathbb{C})$ preserving determinants, i.e.,* $\det T(A) = \det A$ *for every $A \in \mathcal{M}_n(\mathbb{C})$. Then there exist invertible matrices P and Q in $\mathcal{M}_n(\mathbb{C})$ with* $\det(PQ) = 1$ *such that either*

$$T(A) = PAQ \quad \textit{for every } A \in \mathcal{M}_n(\mathbb{C}),$$

or

$$T(A) = PA^tQ \quad \textit{for every } A \in \mathcal{M}_n(\mathbb{C}).$$

II. *T preserves a subset $\mathcal{U}$* of $\mathcal{M}_n(\mathbb{F})$. Characterise those linear operators T on $\mathcal{M}_n(\mathbb{F})$ that satisfy

$$T(\mathcal{U}) \subseteq \mathcal{U} \quad \text{or} \quad T(\mathcal{U}) = \mathcal{U}.$$

In 1959, M. Marcus and R. Purves [21] proved the following proposition (Type II LPP).

Proposition 1.2.2. *Let T be a linear operator on $\mathcal{M}_n(\mathbb{F})$ that preserves the invertible matrices, i.e., $T(A)$ is invertible whenever A is invertible. Then there exist invertible matrices P and Q in $\mathcal{M}_n(\mathbb{F})$ such that either*

$$T(A) = PAQ \quad for\ every\ A \in \mathcal{M}_n(\mathbb{F}),$$

or

$$T(A) = PA^tQ \quad for\ every\ A \in \mathcal{M}_n(\mathbb{F}).$$

III. *T preserves a relation or an equivalence relation $\sim$* on $\mathcal{M}_n(\mathbb{F})$. Characterise those linear operators T on $\mathcal{M}_n(\mathbb{F})$ that satisfy

$$T(A) \sim T(B) \quad \text{whenever} \quad A \sim B$$

or

$$T(A) \sim T(B) \quad \text{if and only if} \quad A \sim B$$

with $A, B \in \mathcal{M}_n(\mathbb{F})$.

The following Type III LPP is proved by F. Hiai [8].

Proposition 1.2.3. *Let T be a linear operator that preserves similarity on $\mathcal{M}_n(\mathbb{F})$, i.e., $T(A)$ is similar to $T(B)$ whenever A is similar to B in $\mathcal{M}_n(\mathbb{F})$. Then there exist $a, b \in \mathbb{F}$ and an invertible matrix $Q \in \mathcal{M}_n(\mathbb{F})$ such that either*

$$T(A) = aQ^{-1}AQ + b(\mathrm{tr}(A))I_n \quad for\ every\ A \in \mathcal{M}_n(\mathbb{F}),$$

or

$$T(A) = aQ^{-1}A^TQ + b(\mathrm{tr}(A))I_n \quad for\ every\ A \in \mathcal{M}_n(\mathbb{F}).$$

IV. *T preserves or commutes with a transformation τ on $\mathcal{M}_n(\mathbb{F})$.* Characterise those linear operators T on $\mathcal{M}_n(\mathbb{F})$ that satisfy

$$\tau(T(A)) = T(\tau(A)) \quad \text{for every } A \in \mathcal{M}_n(\mathbb{F}).$$

The following is an example of Type IV LPP where the classical adjoint-commuting (see Definition 1.4) linear mapping on $n \times n$ complex matrices was studied by Sinkhorn [26] in 1982.

Proposition 1.2.4. *Let T be a linear operator on $\mathcal{M}_n(\mathbb{C})$ such that $T(\text{adj } A) = \text{adj } T(A)$ for every $A \in \mathcal{M}_n(\mathbb{C})$. For $n \geqslant 3$, there exist an invertible complex matrix P, $\lambda \in \mathbb{C}$ with $\lambda^{n-2} = 1$ such that the mapping is of the form*

$$T(A) = \lambda PAP^{-1} \textit{ for every } A \in \mathcal{M}_n(\mathbb{C})$$

or

$$T(A) = \lambda PA^tP^{-1} \textit{ for every } A \in \mathcal{M}_n(\mathbb{C}).$$

Since 1897 much effort has been devoted to the study of linear preserver problems, there have been several excellent survey papers such as [19, 20, 7, 24, 17].

In recent years, many linear preserver results have also been extended to the nonlinear analogues by considering additive preserver problems, multiplicative preserver problems, and even, preserver problems on spaces of matrices without any algebraic assumption. For an extensive expository survey of the subject of these nonlinear preserver problems, see [9, 32] and the reference therein.

1.3 Decomposition of matrices

In this section, some results on decomposition of hermitian matrices and alternate matrices are stated which will be useful in obtaining the main results.

Proposition 1.3.1. *Let $\mathbb{F}$ be a field with an involution $^-$. Then $A \in \mathcal{M}_n(\mathbb{F})$ is a hermitian matrix if and only if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$*

such that

$$A = P\left(\sum_{i=1}^{k} \alpha_i E_{ii}\right)\overline{P}^t \tag{1.2}$$

for some nonzero scalars $\alpha_1, \cdots, \alpha_k \in \mathbb{F}$ *with* $\overline{\alpha}_i = \alpha_i$ *for all* $i = 1, \cdots k,$ *or*

$$A = P(L_1 \oplus \cdots \oplus L_r \oplus 0_{n-2r})P^t \tag{1.3}$$

where $L_1 = \cdots = L_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{F})$ *whenever* A *is alternate and the involution* $^-$ *is identity.*

Proposition 1.3.2. *Let* $A \in \mathcal{M}_n(\mathbb{F})$*. Then the following statements are equivalent.*

1. $A \in \mathcal{K}_n(\mathbb{F})$.
2. $A^t = -A$ *if* char $\mathbb{F} \neq 2$ *and* $A^t = A$ *with zero diagonal elements if* char $\mathbb{F} = 2$.
3. $A^t = -A$ *with zero diagonal elements.*

Proposition 1.3.3. $A \in \mathcal{K}_n(\mathbb{F})$ *if and only if either* $A = 0$ *or there exist an invertible matrix* P *in* $\mathcal{M}_n(\mathbb{F})$ *and an integer* $1 \leqslant k \leqslant \left\lfloor \frac{n}{2} \right\rfloor$ *such that*

$$A = P(J_1 \oplus \cdots \oplus J_k \oplus 0_{n-2k})P^t \tag{1.4}$$

where $J_1 = \cdots = J_k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Here, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Remark 1.3.4. In view of Proposition 1.3.3, any alternate matrices are of even rank.

1.4 Some properties of classical adjoint

The classical adjoint is sometimes called adjugate and is one of the important matrix functions on square matrices. An early history of the notion of classical adjoint is given by Muir in his book, *The Theory of Determinants* [22], where he stated that the present form of the classical adjoint is due to the study of quadratic forms by Gauss in the fifth chapter of Gauss' *Disquisitioned Arithmeticae*, published in 1801.

The main reason to define the classical adjoint is the following well known result.

Proposition 1.4.1. *Let n be an integer with $n \geqslant 2$. If $A \in \mathcal{M}_n(\mathbb{F})$, then*

$$A(\text{adj } A) = (\text{adj } A)A = (\det A)I_n.$$

If $A \in \mathcal{M}_1(\mathbb{F})$, then adj A is defined to be the 1×1 identity matrix. Thus Proposition 1.4.1 also holds for $n = 1$. As a consequence of Proposition 1.4.1,

$$\text{adj } B = (\det B)B^{-1} \text{ if } B \in \mathcal{M}_n(\mathbb{F}) \text{ is invertible.}$$

In addition, the results of the next theorem follow.

Proposition 1.4.2. *Let n be an integer with $n \geqslant 2$ and let $A, B \in \mathcal{M}_n(\mathbb{F})$.*

(a)
$$\text{rank adj } A = \begin{cases} 0 & \textit{if } \text{ rank } A \leqslant n-2, \\ 1 & \textit{if } \text{ rank } A = n-1, \\ n & \textit{if } \text{ rank } A = n. \end{cases}$$

(b) adj $I_n = I_n$.

(c) adj $(\alpha A) = \alpha^{n-1}$adj A *where* $\alpha \in \mathbb{F}$.

(d) adj $(AB) =$ (adj B)(adj A).

(e) adj $A^{-1} = (\text{adj } A)^{-1}$.

(f) adj $A^t = (\text{adj } A)^t$.

(g) $\det(\text{adj } A) = (\det A)^{n-1}$.

(h) adj $(\text{adj } A) = (\det A)^{n-2}A$.

(i) $A^{-1} = (\det A)^{-1}\text{adj } A$.

(j) $(\text{adj } A)^{-1} = (\det A)^{-1}A$.

(k) $P \in \mathcal{M}_n(\mathbb{F})$ *is invertible* $\implies$ adj $(P^{-1}AP) = P^{-1}(\text{adj } A)P$.

(l) $AB = BA \implies (\text{adj } A)B = B(\text{adj } A)$.

In general, adj is not a linear mapping. adj is linear when $n = 2$. adj is also not onto $\mathcal{M}_n(\mathbb{F})$. The following result is proved over $\mathbb{C}$, the set of all complex numbers, in [26].

Proposition 1.4.3. *Let n be an integer with $n \geqslant 2$. If $A \in \mathcal{M}_n(\mathbb{C})$ and* rank $A = n$, 1 *or* 0, *then there exists $B \in \mathcal{M}_n(\mathbb{C})$ such that $A =$* adj B.

Let n be an integer with $n \geqslant 2$ and let k, $n_1, \cdots, n_k$ be a sequence of positive integers satisfying $n_1 + \cdots + n_k = n$. We denote by $\mathcal{T}_{n_1,\cdots,n_k}$, the subalgebra of $\mathcal{M}_n(\mathbb{F})$ consisting of all block matrices (A_{ij}) of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}$$

where $A_{ij} \in \mathcal{M}_{n_i,n_j}(\mathbb{F})$ for all $1 \leqslant i \leqslant j \leqslant k$. $\mathcal{T}_{n_1,\cdots,n_k}$ is said to be a *triangular matrix algebra.* In particular, when $n_i = 1$ for all i, then it forms the algebra of all n-square upper triangular matrices, i.e. $\mathcal{T}_n(\mathbb{F})$. Proposition 1.4.4 is proved by Chooi in [2] and we have proved a similar result on hermitian matrices (see Proposition 1.4.6).

Proposition 1.4.4. *Let n be an integer with $n \geqslant 2$ and let $\mathbb{F}$ be a field. If $A \in \mathcal{T}_{n_1,\cdots,n_k}(\mathbb{F})$ is of rank one, then there exists a rank $n-1$ matrix $B \in \mathcal{T}_{n_1,\cdots,n_k}(\mathbb{F})$ such that $A = \text{adj } B$.*

Corollary 1.4.5. *Let $A \in \mathcal{M}_n(\mathbb{F})$ be of rank one. Then there exists a rank $n-1$ matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that $A = \text{adj } B$.*

Proof. By Proposition 1.4.4, when $k = 1$, $A \in \mathcal{M}_n(\mathbb{F})$. Thus, the result is obtained. $\square$

Proposition 1.4.6. *Let n be an integer with $n \geqslant 2$ and let $\mathbb{F}$ be a field which possesses an involution $^-$ of $\mathbb{F}$. If $A \in \mathcal{H}_n(\mathbb{F})$ is of rank one, then there exists a rank $n-1$ matrix $B \in \mathcal{H}_n(\mathbb{F})$ such that $A = \text{adj } B$.*

Proof. Since $A \in \mathcal{H}_n(\mathbb{F})$ is of rank one, by Proposition 1.3.1, there exist an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ and a nonzero scalar $\alpha \in \mathbb{F}^-$ such that $A = P(\alpha E_{11})P^{-1}$. Let $Q = \text{adj } P$ and $\theta = (\det P\overline{P})^{n-2}$. Obviously, Q is an invertible matrix in $\mathcal{M}_n(\mathbb{F})$ and θ is a nonzero scalar in $\mathbb{F}^-$. Let

$$B = \overline{Q}^t(I_n - E_{11} + (\theta^{-1}\alpha - 1)E_{22})Q \in \mathcal{H}_n(\mathbb{F})$$

which is of rank $n-1$. Then

$$\begin{aligned}
\text{adj } B &= \text{adj } \left(\overline{Q}^t(I_n - E_{11} + (\theta^{-1}\alpha - 1)E_{22})Q\right) \\
&= (\text{adj } Q)\text{adj } (I_n - E_{11} + (\theta^{-1}\alpha - 1)E_{22})(\text{adj } \overline{Q}^t) \\
&= (\text{adj } (\text{adj } P))(\theta^{-1}\alpha E_{11})(\text{adj } (\text{adj } \overline{P}^t) \\
&= P(\theta^{-1}\alpha E_{11})\overline{P}^t.
\end{aligned}$$

$\square$

1.5 Fundamental theorems of geometry of matrices

To conclude this chapter, we state the fundamental theorems of geometry of matrices which are applied in the characterisation of the preserver problems we study in this dissertation. In this section we state the fundamental theorems of geometry of rectangular matrices, hermitian matrices and alternate matrices over arbitrary fields (see [31] or [10] for more details).

Definition 1.5.1. Let m, n be integers and let $\mathbb{F}$ be a field. Let $A, B \in \mathcal{M}_{m,n}(\mathbb{F})$. The *arithmetic distance* between A and B, $d(A, B) = \text{rank }(A - B)$. A and B are said to be *adjacent* if $d(A, B) = 1$.

Theorem 1.5.2 (Fundamental theorem of the geometry of rectangular matrices)**.** *Let m, n be integers with $m, n \geqslant 2$ and let $\mathbb{F}$ be a field. Let $\phi : \mathcal{M}_{m,n}(\mathbb{F}) \to \mathcal{M}_{m,n}(\mathbb{F})$ be a bijective mapping. Assume that for every $A, B \in \mathcal{M}_{m,n}(\mathbb{F})$, A and B are adjacent if and only if $\phi(A)$ and $\phi(B)$ are adjacent. Then one of the following holds:*

$$\phi(A) = PA^\sigma Q + R \quad \textit{for every} \quad A \in \mathcal{M}_{m,n}(\mathbb{F}); \tag{1.5}$$

$$m = n \textit{ and } \phi(A) = P(A^\sigma)^t Q + R \textit{ for all } A \in \mathcal{M}_n(\mathbb{F}) \tag{1.6}$$

where $\sigma : \mathbb{F} \to \mathbb{F}$ is an automorphism, A^σ is a matrix obtained from A by applying σ entrywise, $R \in \mathcal{M}_{m,n}(\mathbb{F})$, $P \in \mathcal{M}_m(\mathbb{F})$ and $Q \in \mathcal{M}_n(\mathbb{F})$ are invertible matrices.

In fact, the theorem stated above holds in the more general case when $\mathbb{F}$ is a division ring. Since in this dissertation, we consider only the case where matrices are over a field, we state the theorem over a field $\mathbb{F}$.

Definition 1.5.3. Let n be an integer with $n \geqslant 2$ and let $\mathbb{F}$ be a field that possesses an involution $^-$ of $\mathbb{F}$. Let $A, B \in \mathcal{H}_n(\mathbb{F})$. The *arithmetic distance* between A and B, $d(A, B) = \text{rank } (A - B)$. A and B are said to be *adjacent* if $d(A, B) = 1$.

Theorem 1.5.4 (Fundamental theorem of the geometry of hermitian matrices)**.** *Let m, n be integers with $m, n \geqslant 3$ and let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively. Let $\phi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a bijective mapping. Assume that for all $A, B \in \mathcal{H}_n(\mathbb{F})$, A and B are adjacent if and only if $\phi(A)$ and $\phi(B)$ are adjacent. Then*

$$\phi(A) = \alpha P A^\sigma \widehat{P}^t + H_0 \quad \textit{for every} \quad A \in \mathcal{H}_n(\mathbb{F}) \tag{1.7}$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a nonzero isomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$, A^σ is the matrix obtained from A by applying σ entrywise, $P \in \mathcal{M}_m(\mathbb{K})$ is an invertible matrix, $H_0 \in \mathcal{H}_m(\mathbb{K})$ and $\alpha \in \mathbb{K}^\wedge$ is nonzero.

Definition 1.5.5. Let n be an integer with $n \geqslant 2$ and let $\mathbb{F}$ be a field. Let $A, B \in \mathcal{K}_n(\mathbb{F})$. The *arithmetic distance* between A and B, $d(A, B) = \frac{1}{2}\text{rank } (A - B)$. A and B are said to be *adjacent* if $d(A, B) = 1$.

Theorem 1.5.6 (Fundamental theorem of the geometry of alternate matrices)**.** *Let n be an integer with $n \geqslant 4$ and let $\mathbb{F}$ be a field. Let $\phi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_n(\mathbb{F})$ be a bijective mapping. Assume that for every $A, B \in \mathcal{K}_n(\mathbb{F})$, A and B are adjacent if and only if $\phi(A)$ and $\phi(B)$ are adjacent. Then ϕ is either of the form*

$$\phi(A) = \alpha P A^\sigma P^t + K_0 \quad \textit{for every} \quad A \in \mathcal{K}_n(\mathbb{F}) \tag{1.8}$$

or when $n = 4$,

$$\phi(A) = \alpha P (A^*)^\sigma P^t + K_0 \quad \textit{for every} \quad A \in \mathcal{K}_4(\mathbb{F}), \tag{1.9}$$

where $\sigma : \mathbb{F} \to \mathbb{F}$ *is an automorphism,* A^{σ} *is the matrix obtained from* A *by applying* σ *entrywise,* $P \in \mathcal{M}_n(\mathbb{F})$ *is invertible,* $\alpha \in \mathbb{F}$ *is a nonzero scalar,* $K_0 \in \mathcal{K}_n(\mathbb{F})$ *and for* $n = 4$,

$$A^{*} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^{*} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}. \tag{1.10}$$

Chapter 2

Preliminary results

2.1 Introduction

There are many applications of the classical adjoint in matrix theory. In particular, it was employed to various studies of generalised invertibility of matrices [25].

Sinkhorn [26] initiated the study of classical adjoint-commuting linear mappings on $n \times n$ complex matrices in 1982. By using continuity argument and Proposition 1.2.1 (Frobenius' classical theorem [6]), he proved, for $n \geqslant 3$, that there exist an invertible complex matrix P, $\lambda \in \mathbb{C}$ with $\lambda^{n-2} = 1$ such that the mapping is either of the form $A \mapsto \lambda PAP^{-1}$ or of the form $A \mapsto \lambda PA^tP^{-1}$ (see Proposition 1.2.4). Since then, classical adjoint-commuting linear mappings and classical adjoint-commuting additive mappings on various matrix spaces have been studied. In 1987, classical adjoint-commuting linear mappings on $\mathcal{M}_n(\mathbb{F})$ with $\mathbb{F}$ any infinite field and $n \geqslant 2$ were studied in [1]. The mappings were also studied on $\mathcal{S}_n(\mathbb{F})$ for any field $\mathbb{F}$ of characteristic not equal to 2 with $n \geqslant 2$. They have also characterised the classical adjoint-commuting linear mappings on $\mathcal{K}_n(\mathbb{F})$ where $\mathbb{F}$ is an infinite field of characteristic not equal to 2 and n is an even positive integer. After that, in 1998, classical adjoint-commuting linear mappings on $\mathcal{T}_n(\mathbb{F})$ with $\mathbb{F}$ a field and $n \geqslant 3$ an integer, were studied in [3]. They proved that the mapping is a bijective classical adjoint-commuting linear mapping on $\mathcal{T}_n(\mathbb{F})$ if and only if there exist an invertible matrix $P \in \mathcal{T}_n(\mathbb{F})$ and a nonzero scalar $\lambda \in \mathbb{F}$ such that the mapping is either of the form $A \mapsto \lambda PAP^{-1}$ or $A \mapsto \lambda PA^{\sim}P^{-1}$ where $A^{\sim}$ is the matrix obtained from $A = (a_{ij})$ by reflecting the diagonal $a_{1n}, a_{2,n-1}, \cdots, a_{n1}$ and $\lambda^{n-1} = \lambda$. Let $n \geqslant 3$, $m \geqslant 2$. In 2010, Chooi [2] proved that $\psi : \mathcal{T}_{n_1,\cdots n_k} \to \mathcal{M}_m(\mathbb{F})$ is a classical adjoint-commuting additive mapping if and only if $\psi = 0$, or $m = n$ and there exist an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$, integers $0 = s_0 < s_1 < \cdots < s_k = k$, and a nonzero field homomorphism σ on $\mathbb{F}$ such that

$$\psi(A) = P\left(\bigoplus_{i=1}^{r} \lambda_1 \Theta_i(A_i)^{\sigma}\right) P^{-1} \text{ for every } A \in \mathcal{T}_{n_1,\cdots,n_k},$$

where $\bigoplus_{i=1}^{r} A_i$ is the $(\epsilon_1, \cdots, \epsilon_r)$-block diagonal matrix induced by A where $\epsilon_i = \delta_{s_i} - \delta_{s_{i-1}}$ with $\delta_{s_i} = n_1 + \cdots + n_{s_i}$, $\delta_k = n$, and $\lambda_1, \cdots, \lambda_r$ are nonzero elements in $\mathbb{F}$ satisfying $\prod_{j=1}^{r} \lambda_j^{\epsilon_j} = \lambda_i^2$ for $i = 1, \cdots, r$ and for each $1 \leqslant i \leqslant r$, $\Theta_i : \mathcal{T}_{n_{(s_{i-1}+1)},\cdots,n_{s_i}} \to \mathcal{M}_{\epsilon_i}(\mathbb{F})$ is a linear mapping defined by $\Theta_i(A_i) = \mu A_i(\alpha) + (a - \mu)A_i(\alpha)^t$ for all $A_i \in \mathcal{T}_{n_1,\cdots,n_k}$. Besides the above-mentioned results, classical adjoint-commuting linear mappings as well as additive mappings on various matrix spaces have been studied in some papers, see [4, 27, 28, 29, 30].

Motivated by their works, we study classical adjoint-commuting mappings ψ between matrix algebras over an arbitrary field by dropping the linearity and the additivity of ψ. Let m, n be integers with $m, n \geqslant 3$ and let $\mathbb{F}$ and $\mathbb{K}$ be fields. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be subspaces of $\mathcal{M}_n(\mathbb{F})$ and $\mathcal{M}_m(\mathbb{K})$, respectively, such that adj $A \in U_i$ whenever $A \in U_i$ for $i = 1, 2$. We investigate the structure of mappings $\psi : \mathcal{U}_1 \to \mathcal{U}_2$ satisfying one of the two conditions:

(A1) $\psi(\text{adj } (A + \alpha B)) = \text{adj } (\psi(A) + \alpha\psi(B))$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$ when $\mathbb{F} = \mathbb{K}$;

(A2) $\psi(\text{adj }(A-B)) = \text{adj }(\psi(A)-\psi(B))$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$.

We notice that if ψ satisfies condition (A1) or (A2), then

$$\psi(0) = \psi(\text{adj }(0-0)) = \text{adj }(\psi(0)-\psi(0)) = 0.$$

This implies

$$\psi(\text{adj }(A)) = \psi(\text{adj }(A-0)) = \text{adj }(\psi(A)-\psi(0)) = \text{adj }(\psi(A)),$$

i.e. ψ is a classical adjoint-commuting mapping (see (1.1)).

2.2 Some requirements

In this section, we give some results established for the construction of the main results. Recall that if we say that $A \in \mathcal{H}_n(\mathbb{F})$, we mean A is a hermitian matrix over a field $\mathbb{F}$ which possesses an involution $^-$.

Lemma 2.2.1. *Let $n \geqslant 2$ and let $\mathbb{F}$ be a field which possesses an involution $^-$ of $\mathbb{F}$. If $A \in \mathcal{H}_n(\mathbb{F})$ is a nonzero rank r matrix, then $A = A_1 + \cdots + A_k$ for some rank one matrices $A_1, \cdots, A_k \in \mathcal{H}_n(\mathbb{F})$ with*

$$k = \begin{cases} r+1 & \text{when } A \text{ is alternate and the involution } ^- \text{ is identity,} \\ r & \text{otherwise.} \end{cases}$$

Proof. We consider two cases. First, if A is of Form (1.2) in Proposition 1.3.1, i.e. $A = P(\alpha E_{11} + \cdots + \alpha_r E_{rr})\overline{P}^t$ for some invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ and some nonzero scalars $\alpha_1, \cdots, \alpha_r \in \mathbb{F}^-$, then we choose $A_i = P(\alpha_i E_{ii})\overline{P}^t$ for $i = 1, \cdots, r$. It is obvious that $A_i \in \mathcal{H}_n(\mathbb{F})$ is of rank one, and $A = A_1 + \cdots + A_r$, as claimed. Next, we consider the case where A is alternate and the involution $^-$ of $\mathbb{F}$ is identity, then A is of Form (1.3) in Proposition 1.3.1 i.e. $A = Q(L_1 \oplus \cdots \oplus L_{r/2} \oplus 0_{n-r})Q^t$ for some invertible matrix $Q \in \mathcal{M}_n(\mathbb{F})$, and hence, r is even and $\mathbb{F}$ has characteristic 2. By letting $B = Q(E_{11} + E_{22})Q^t$ which is of rank 2, we have $A + B \in \mathcal{H}_n(\mathbb{F})$ is of odd rank $r - 1$. By Proposition 1.3.1, $A + B$ is of Form (1.2). Thus, there exists an invertible matrix $R \in \mathcal{M}_n(\mathbb{F})$ such that

$$A + B = R(\beta_1 E_{11} + \cdots + \beta_{r-1} E_{r-1,r-1})R^t$$

for some nonzero scalars $\beta_1 \cdots, \beta_{r-1} \in \mathbb{F}^- = \mathbb{F}$. Now, we choose $A_i = R(\beta_i E_{ii})R^t$ for $i = 1, \cdots, r-1$, and $A_r = Q(-E_{11})Q^t$ and $A_{r+1} = Q(-E_{22})Q^t$. Evidently, $A_i \in \mathcal{H}_n(\mathbb{F})$ is of rank one for $i = 1, \cdots, r+1$, and $A = A_1 + \cdots + A_r + A_{r+1}$. We are done. $\square$

Lemma 2.2.2. *Let n be an integer with $n \geqslant 3$ and $\mathcal{R} = \mathcal{M}_n(\mathbb{F})$, $\mathcal{K}_n(\mathbb{F})$, or $\mathcal{H}_n(\mathbb{F})$. If $A, B \in \mathcal{R}$, then the following hold.*

(a) *If A is of rank r, then there exists a rank $n - r$ matrix $X_1 \in \mathcal{R}$ such that* rank $(A + X_1) = n$.

(b) *There exists a matrix $X_2 \in \mathcal{R}$ such that* rank $(A+X_2) =$ rank $(B+X_2) = n$.

(c) *There exists a nonzero matrix $X_3 \in \mathcal{R}$ such that either A or X_3 is of rank n but not both with* rank $(A + X_3) = n$.

Proof.

Case I: We first consider the case where $\mathcal{R} = \mathcal{M}_n(\mathbb{F})$.

(a) If $r = 0$, we choose $X_1 = I_n$. We now suppose A is of rank $r \neq 0$. Then there exist invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{F})$ such that $A = P(E_{11} + \cdots + E_{rr})Q$. By letting $X_1 = P(E_{r+1,r+1} + \cdots + E_{nn})Q$, we have $A + X_1 = PQ$ which is of rank n and it is clear that rank $X_1 = n - r$.

(b) If $A = B$, then we select $X_2 = I_n - A$. Thus, the result holds. We now assume $A \neq B$. Let $C = A - B$ and let rank $C = r \leqslant n$. Then there exist invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{F})$ such that $C = P(E_{11} + \cdots + E_{rr})Q$. Let $X_2 = D - B$, where

$$D = \begin{cases} P((E_{12} + \cdots + E_{r,r+1}) + E_{r+1,1} + (E_{r+2,r+2} + \cdots + E_{nn}))Q & \text{if } r < n, \\ P(E_{11} + (E_{12} + \cdots + E_{n-1,n}) + E_{n1})Q & \text{if } r = n. \end{cases}$$

Then $A + X_2 = C + D$ and $B + X_2 = D$ where both $C + D$ and D are of rank n.

(c) If A is of rank n, then we obtain the result by letting $X_3 = AE_{12}$. We consider rank $A = r < n$. Then there exist invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{F})$ such that $A = P(E_{11} + \cdots + E_{rr})Q$. We choose

$$X_3 = P((E_{12} + \cdots + E_{r,r+1}) + E_{r+1,1} + (E_{r+2,r+2} + \cdots + E_{nn}))Q.$$

It can be shown that rank $X_3 = n$ and $\det(A + X_3) = \det(PQ) \neq 0$ implies $A + X_3$ is of rank n.

Case II: Consider $\mathcal{R} = \mathcal{H}_n(\mathbb{F})$.

Note that, here, $\mathbb{F}$ is a field which possesses an involution $^-$ of $\mathbb{F}$. If a nonzero matrix $A \in \mathcal{H}_n(\mathbb{F})$ is of rank r, then by Proposition 1.3.1, there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ such that either A is of the form:

$$A = P(\alpha_1 E_{11} + \cdots + \alpha_r E_{rr})\overline{P}^t \tag{2.1}$$

for some nonzero scalars $\alpha_1, \cdots \alpha_r \in \mathbb{F}^-$; or if A is alternate and the involution $^-$ of $\mathbb{F}$ is identity, then A can be written in the form:

$$A = P(L_1 \oplus \cdots \oplus L_{r/2})P^t \tag{2.2}$$

where r is even and $\mathbb{F}$ is of characteristic 2, and

$$L_1 = \cdots = L_{r/2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{F}).$$

(a) If $r = 0$, we select $X_1 = I_n$ and if $r = n$, we select $X_1 = 0$. Now, we suppose $1 < r < n$. Then we set

$$X_1 = \begin{cases} P(E_{r+1,r+1} + \cdots + E_{nn})\overline{P}^t & \text{if } A \text{ is of Form (2.1)}, \\ P(E_{r+1,r+1} + \cdots + E_{nn})P^t & \text{if } A \text{ is of Form (2.2)}. \end{cases}$$

In addition, we have $X_1 \in \mathcal{H}_n(\mathbb{F})$ is of rank $n - r$ and rank $(A + X_1) = n$. We are done.

(b) If $A = B$, then we choose $X_2 = I_n - A$. Suppose $A \neq B$. Let $H = A - B$. Then $H \in \mathcal{H}_n(\mathbb{F})$ and $0 <$ rank $H = r \leqslant n$. First, we consider H is of Form (2.1), then we select

$$C = \begin{cases} P(\alpha_1 Z_{12} + \cdots + \alpha_{r-1} Z_{r-1,r} + E_{r+1,r+1} + \cdots + E_{nn})\overline{P}^t & \text{if } r < n, r \text{ is even}, \\ P(\alpha_1 Z_{12} + \cdots + \alpha_{n-1} Z_{n-1,n})\overline{P}^t & \text{if } r = n, r \text{ is even}, \\ P(\alpha_1 Z_{12} + \cdots + \alpha_r Z_{r,r+1} + E_{r+2,r+2} + E_{nn})\overline{P}^t & \text{if } r < n, r \text{ is odd}, \\ P(\alpha_1 Z_{12} + \cdots + \alpha_{n-2} Z_{n-2,n-1} + E_{n-1,n} + E_{n,n-1})\overline{P}^t & \text{if } r = n, r \text{ is odd} \end{cases}$$

where $\alpha Z_{ij} := E_{ij} + E_{ji} - \alpha E_{ii} \in \mathcal{H}_n(\mathbb{F})$ for $1 \leqslant i < j \leqslant n$ and $\alpha \in \mathbb{F}^-$.

Next, we consider H which is alternate and the involution $^-$ of $\mathbb{F}$ is identity, then H is of Form (2.2). Let x be the greatest integer less than or equal to $\frac{n}{2}$, and let y be the smallest integer greater than or equal to $\frac{n}{2}$. Let h be an odd integer satisfying $x - 1 \leqslant h \leqslant x$. We set

$$C = \begin{cases} PT_{1n}P^t & \text{if } r < y + 1, \\ P(T_{1n} - S_h)P^t & \text{if } r \geqslant y + 1, \text{ and } h \neq x \text{ or } h \neq y, \\ P(T_{1,n-1} - S_{h-2} + E_{nn})P^t & \text{if } r \geqslant y + 1 \text{ and } h = x = y \end{cases}$$

where $T_{1k} := E_{1k} + E_{2,k-1} + \cdots + E_{k1}$ for $1 \leqslant k \leqslant n$, and $S_k := (E_{12} + E_{21}) + (E_{34} + E_{43}) + \cdots + (E_{k,k+1} + E_{k+1,k})$ for $1 \leqslant k < n$ with odd integer k. In both cases of H, it can be shown that $C \in \mathcal{H}_n(\mathbb{F})$ is of rank n and rank $(H + C) = n$. By letting $X_2 = D - B$, we have $X_2 \in \mathcal{H}_n(\mathbb{F})$, and $A + X_2 = H + C$ and $B + X_2 = C$. We are done.

(c) If rank $A = n$, then we let

$$X_3 = \begin{cases} P(\alpha_1 E_{11} + E_{12} + E_{21})\overline{P}^t & \text{if } A \text{ is of Form (2.1)}, \\ PE_{11}P^t & \text{if } A \text{ is of Form (2.2)}. \end{cases}$$

In both cases of A, we see that $X_3 \in \mathcal{H}_n(\mathbb{F})$ with rank $X_3 < n$ and rank $(A + X_3) = n$. We now suppose rank $A = r < n$. If $A = 0$, then we choose $X_3 = I_n$. If $A \neq 0$, we first consider the case where A is of Form (2.1). Then by using the same definition of αZ_{ij} as in part (b), we let

$$X_3 = \begin{cases} P(\alpha_1 Z_{12} + \cdots + \alpha_{r-1} Z_{r-1,r} + E_{r+1,r+1} + \cdots + E_{nn})\overline{P}^t & \text{if } r \text{ is even}, \\ P(\alpha_1 Z_{12} + \cdots + \alpha_r Z_{r,r+1} + E_{r+2,r+2} + \cdots + E_{nn})\overline{P}^t & \text{if } r \text{ is odd}. \end{cases}$$

Next, we consider the case where A is of Form (2.2). Then by using the same definitions of x, y and h as in part (b), we let

$$X_3 = \begin{cases} PT_{1n}P^t & \text{if } r < y+1, \\ P(T_{1n} - S_h)P^t & \text{if } r \geqslant y+1, \text{ and } h \neq x \text{ or } h \neq y, \\ P(T_{1,n-1} - S_{h-2} + E_{nn})P^t & \text{if } r \geqslant y+1 \text{ and } h = x = y. \end{cases}$$

In both cases of A, it can be verified that $X_3 \in \mathcal{H}_n(\mathbb{F})$ is of rank n and rank $(A + X_3) = n$.

Case III: We now consider $\mathcal{R} = \mathcal{K}_n(\mathbb{F})$.

By Remark 1.3.4, n is even. Recall from (1.4), if $A \in \mathcal{K}_n(\mathbb{F})$ is of rank r, then $r \geqslant 0$ is necessarily even, and there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ such that

$$A = P(J_1 \oplus \cdots \oplus J_{r/2} \oplus 0_{n-r})P^t. \tag{2.3}$$

where $J_1 = \cdots = J_{r/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{F})$.

(a) By choosing $X_1 = P(0_r \oplus J_{r+1} \oplus \cdots \oplus J_{n/2})P^t \in \mathcal{K}_n(\mathbb{F})$, we have $A + X_1$ is of rank n and it is obvious that rank $X_1 = n - r$.

(b) Suppose that $A = B$. Then from (a), there exists a matrix $X_2 \in \mathcal{K}_n(\mathbb{F})$ such that rank $(A + X_2) = n$. We consider $A \neq B$. Let $H := A - B \in \mathcal{K}_n(\mathbb{F})$ be of rank r with $0 < r \leqslant n$ even. By (2.3), there exists an invertible matrix $Q \in \mathcal{M}_n(\mathbb{F})$ such that $H = Q(J_1 \oplus \cdots \oplus J_{r/2} \oplus 0_{n-r})Q^t$. Let h be the odd integer such that $\frac{n}{2} - 1 \leqslant h \leqslant \frac{n}{2}$. and by letting

$$S = (E_{1n} - E_{2,n-1}) + \cdots + (E_{n-1,2} - E_{n1}) \in \mathcal{K}_n(\mathbb{F}),$$

$$T = J_1 \oplus \cdots \oplus J_{n/4} \oplus 0_{n-2} \in \mathcal{K}_n(\mathbb{F}),$$

$$V = J_1 \oplus \cdots \oplus J_{(n+2)/4} \oplus 0_{(n-2)/2} \in \mathcal{K}_n(\mathbb{F}),$$

$$Z_p = E_{1p} + E_{2,p-1} + \cdots + E_{p1} \in \mathcal{M}_p(\mathbb{F}) \text{ with } p = (n-4)/2,$$

$$Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \in \mathcal{K}_4(\mathbb{F}) \text{ and}$$

$$U = \begin{pmatrix} 0_{(n-4)/2} & 0 & Z_{(n-4)/2} \\ 0 & Z & 0 \\ -Z_{(n-4)/2} & 0 & 0_{(n-4)/2} \end{pmatrix} \in \mathcal{K}_n(\mathbb{F}),$$

we set

$$C = \begin{cases} QSQ^t & \text{if } r < \frac{n}{2} + 1, \\ Q(S-T)Q^t & \text{if } r \geqslant \frac{n}{2} + 1 \text{ and } h = \frac{n}{2} - 1, \\ Q(U-V)Q^t & \text{if } r \geqslant \frac{n}{2} + 1 \text{ and } h = \frac{n}{2}. \end{cases}$$

It can be shown that $C \in \mathcal{K}_n(\mathbb{F})$ is of rank n and rank $(H + C) = n$. Let $X_2 := C - B$. In addition, we have $X_2 \in \mathcal{K}_n(\mathbb{F})$, and $A + X_2 = H + C$ and $B + X_2 = C$ are of rank n. We are done.

(c) If rank $A = n$, then by (2.3), we have $A = P(J_1 \oplus \cdots \oplus J_{n/2})P^t$. We choose

$$X_3 := P(E_{1n} - E_{n1})P^t \in \mathcal{K}_n(\mathbb{F}).$$

It is obvious that rank $X_3 = 2 < n$ and rank $(A + X_3) = n$. Now, we consider rank $A = r < n$. If $A = 0$, then we select $X_3 = J_1 \oplus \cdots \oplus J_{n/2}$. If $A \neq 0$, we let h be the odd integer such that $\frac{n}{2} - 1 \leqslant h \leqslant \frac{n}{2}$. By (2.3), we set

$$X_3 = \begin{cases} PSP^t & \text{if } r < \frac{n}{2} + 1, \\ P(S-T)P^t & \text{if } r \geqslant \frac{n}{2} + 1 \text{ and } h = \frac{n}{2} - 1, \\ P(U-V)P^t & \text{if } r \geqslant \frac{n}{2} + 1 \text{ and } h = \frac{n}{2} \end{cases}$$

where S, T, U, $V \in \mathcal{K}_n(\mathbb{F})$ are as defined in part (b). Then $X_3 \in \mathcal{K}_n(\mathbb{F})$ is of rank n and rank $(A + X_3) = n$. We are done.

$\square$

Lemma 2.2.3. *Let n be an integer with $n \geqslant 3$.*

(a) *Let $\mathbb{F}$ be a field and let $A, B \in \mathcal{M}_n(\mathbb{F})$ or $\mathcal{K}_n(\mathbb{F})$. If $|\mathbb{F}| > n + 1$ and* rank $(A + B) = n$, *then there exists a scalar $\lambda \in \mathbb{F}$ with $\lambda \neq 1$ such that* rank $(A + \lambda B) = n$.

(b) *Let $\mathbb{K}$ be a field which possesses an involution $^\wedge$ of $\mathbb{K}$ and let $A, B \in \mathcal{H}_n(\mathbb{K})$. If $|\mathbb{K}^\wedge| > n + 1$ and* rank $(A + B) = n$, *then there exists a scalar $\lambda \in \mathbb{K}^\wedge$ with $\lambda \neq 1$ such that* rank $(A + \lambda B) = n$.

Proof.

(a) For each $x \in \mathbb{F}$, we let $p(x) = \det(A + xB)$. Then $p(x) \in \mathbb{F}[x]$ is a nonzero polynomial of x over $\mathbb{F}$. First, we let $A, B \in \mathcal{M}_n(\mathbb{F})$. If $B = 0$, the result holds true by choosing $x = 0$. So, we consider $B \neq 0$ and rank $B = r \leqslant n$, then there exist invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{F})$ such that $B = P(E_{11} + \cdots + E_{rr})Q$. So,

$$\begin{aligned} p(x) &= \det(A + xB) \\ &= \det(P(P^{-1}AQ^{-1})Q + P(x(E_{11} + \cdots + E_{rr})Q)) \\ &= \det(PQ)\det(P^{-1}AQ^{-1} + x(E_{11} + \cdots + E_{rr})) \\ &= \eta \det(C + x(E_{11} + \cdots + E_{rr})) \end{aligned}$$

with $C = P^{-1}AQ^{-1}$ and $\eta = \det(PQ)$. Thus, p is a polynomial of degree at most $r \leqslant n$. Since $|\mathbb{F}| > n + 1$, there exists a scalar $\lambda \in \mathbb{F}$ with $\lambda \neq 1$ such that $p(\lambda) \neq 0$. Therefore, rank $(A + \lambda B) = n$.

Next, let $A, B \in \mathcal{K}_n(\mathbb{F})$. If $B = 0$, by choosing $x = 0$, the result is obtained. If $B \neq 0$ and rank $B = r \leqslant n$, then by (2.3), there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ such that $B = P(J_1 \oplus \cdots \oplus J_{r/2} \oplus 0_{n-r})P^t$. Then

$$\begin{aligned} p(x) &= \det(A + xB) \\ &= \det(P(P^{-1}A(P^{-1})^t)P^t + P(x(J_1 \oplus \cdots \oplus J_{r/2} \oplus 0_{n-r})P^t)) \\ &= \det(PP^t)\det(P^{-1}A(P^{-1})^t + x(J_1 \oplus \cdots \oplus J_{r/2} \oplus 0_{n-r})) \\ &= \zeta \det(H + x(J_1 \oplus \cdots \oplus J_{r/2} \oplus 0_{n-r})) \end{aligned}$$

where $\zeta = \det(PP^t) \in \mathbb{F}$ is nonzero and $H = P^{-1}A(P^{-1})^t \in \mathcal{K}_n(\mathbb{F})$. Since $|\mathbb{F}| > n+1$ and p is of degree at most $r \leqslant n$, it follows that there exists a scalar $\lambda \in \mathbb{F}$ with $\lambda \neq 1$ such that $p(\lambda) \neq 0$. Then rank $(A + \lambda B) = n$.

(b) For each $x \in \mathbb{K}^\wedge$, we let $p(x) = \det(A + xB)$. Then we have $p(1) \neq 0$ and $\widehat{p(x)} = \widehat{\det(A + xB)} = \det(A + xB) = p(x)$ as $A + xB \in \mathcal{H}_n(\mathbb{K})$. Thus, p is a nonzero polynomial over $\mathbb{K}^\wedge$. If $B = 0$, then rank $A = n$, and hence the result follows by choosing $x = 0$. Next, we consider $B \neq 0$ and rank $B = r \leqslant n$. If B is of Form (2.1), then

$$\begin{aligned} p(x) &= \det(A + xB) \\ &= \det(P(P^{-1}A(\widehat{P}^{-1})^t)\widehat{P}^t + xP(\alpha_1 E_{11} + \cdots + \alpha_r E_{rr})\widehat{P}^t) \\ &= \det(P\widehat{P}^t)\det(P^{-1}A(\widehat{P}^{-1})^t + x(\alpha_1 E_{11} + \cdots + \alpha_r E_{rr})) \\ &= \zeta \det((S + x(\alpha_1 E_{11} + \cdots + \alpha_r E_{rr})) \end{aligned}$$

where $S = P^{-1}A(\widehat{P}^{-1})^t \in \mathcal{H}_n(\mathbb{K})$ and $0 \neq \zeta = \det(P\widehat{P}^t) \in \mathbb{K}^\wedge$.

If B is of Form (2.2), then

$$\begin{aligned} p(x) &= \det(A + xB) \\ &= \det(P(P^{-1}A(P^{-1})^t)P^t + xP(E_{12} + E_{21} + \cdots + E_{r-1,r} + E_{r,r-1})P^t) \\ &= \det(PP^t)\det(P^{-1}A(P^{-1})^t + x(E_{12} + E_{21} + \cdots + E_{r-1,r} + E_{r,r-1})) \\ &= \eta \det(T + x(E_{12} + E_{21} + \cdots + E_{r-1,r} + E_{r,r-1})) \end{aligned}$$

where $T = P^{-1}A(P^{-1})^t \in \mathcal{H}_n(\mathbb{K})$ and $0 \neq \eta = \det(PP^t) \in \mathbb{K}^\wedge = \mathbb{K}$. It can be shown that for both cases p is a nonzero polynomial of degree at most $r \leqslant n$. Since $|\mathbb{K}^\wedge| > n+1$, there exists a scalar $\lambda \in \mathbb{K}^\wedge$ with $\lambda \neq 1$ such that $p(\lambda) \neq 0$. Therefore, we have rank $(A + \lambda B) = n$.

□

In Lemma 2.2.4 and Lemma 2.2.5, we let m, n be integers with $m, n \geqslant 3$ and let $\psi : \mathcal{R}_1 \to \mathcal{R}_2$ be a mapping satisfying (A2) where $\mathcal{R}_1 = \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{F})$) and $\mathcal{R}_2 = \mathcal{M}_m(\mathbb{K})$ (respectively, $\mathcal{H}_m(\mathbb{K})$). For the case where $\mathcal{R}_1 = \mathcal{H}_n(\mathbb{F})$ and $\mathcal{R}_2 = \mathcal{H}_m(\mathbb{K})$, $\mathbb{F}$ and $\mathbb{K}$ are fields which possess involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively.

Lemma 2.2.4. *Let m, n be integers with $m, n \geqslant 3$. Let $\mathcal{R}_1 = \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{F})$) and $\mathcal{R}_2 = \mathcal{M}_m(\mathbb{K})$ (respectively, $\mathcal{H}_m(\mathbb{K})$). Let $\psi : \mathcal{R}_1 \to \mathcal{R}_2$ be a mapping satisfying* (A2) *and let $A \in \mathcal{R}_1$. Then the following statements hold.*

(a) rank $\psi(A) \leqslant 1$ *if* rank $A = 1$.

(b) rank $\psi(A) \leqslant m - 1$ *if* rank $A = n - 1$.

(c) rank $\psi(A) \leqslant m - 2$ *if* rank $A \leqslant n - 2$.

Proof.

(a) If A is of rank one, then adj $\psi(A) = \psi(\text{adj } A) = 0$ implies rank $\psi(A) \neq m$. By Corollary 1.4.5 (respectively, Proposition 1.4.6), there exists a rank $n-1$ matrix $B \in \mathcal{R}_1$ such that $A = \text{adj } B$. Hence,

$$\text{adj } \psi(B) = \psi(\text{adj } B) = \psi(A) \implies \text{rank } \psi(B) < m$$

as rank $\psi(A) \neq m$. Thus, by $\psi(A) = \text{adj } \psi(B)$ and rank $\psi(B) < m$, we conclude that rank $A \leqslant 1$.

(b) Since rank $A = n-1$, then adj (adj $\psi(A)) = \psi(\text{adj } (\text{adj } A)) = 0$. Therefore, rank $\psi(A) \leqslant m-1$.

(c) If rank $A \leqslant n-2$, then adj $\psi(A) = \psi(\text{adj } A) = \psi(0) = 0$. This implies rank $\psi(A) \leqslant m-2$.

□

Lemma 2.2.5. *Let m, n be integers with $m, n \geqslant 3$ and let $\mathcal{R}_1 = \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{F})$) and $\mathcal{R}_2 = \mathcal{M}_m(\mathbb{K})$ (respectively, $\mathcal{H}_m(\mathbb{K})$). Let $\psi : \mathcal{R}_1 \to \mathcal{R}_2$ be a mapping satisfying* (A2) *and let $A \in \mathcal{R}_1$. Then ψ is injective if and only if*

$$\text{rank } A = n \iff \text{rank } \psi(A) = m.$$

Proof. We first suppose ψ is injective. Let $A \in \mathcal{R}_1$. By Lemma 2.2.4 (b) and (c), if rank $\psi(A) = m$, then rank $A = n$. Conversely, we let rank $A = n$. Suppose rank $\psi(A) < m$. Then $\psi(\text{adj } (\text{adj } A)) = \text{adj } (\text{adj } \psi(A)) = 0$ since $m \geqslant 3$. It follows that adj (adj $A) = 0$ as $\ker \psi = \{0\}$. This contradicts the fact that rank $A = n$. Therefore, rank $\psi(A) = m$.

Next, we prove the necessity. Suppose there exist some matrices $A, B \in \mathcal{R}_1$ such that $\psi(A) = \psi(B)$. We assume rank $(A - B) = r$. Then by Lemma 2.2.2

(a), there exists a rank $n-r$ matrix $C \in \mathcal{R}_1$ such that $A-B+C$ is of rank n. Then rank (adj $(A-B+C)) = n$. So, we have rank (adj $\psi(A-B+C)) =$ rank $(\psi(\text{adj }(A-B+C))) = m$. Thus

$$\begin{aligned}\text{adj } \psi(C) &= \text{adj } (\psi(B-(B-C))) \\ &= \text{adj } (\psi(B)-\psi(B-C)) \\ &= \text{adj } (\psi(A)-\psi(B-C)) \\ &= \text{adj } (\psi(A-B+C))\end{aligned}$$

which is of rank m. Therefore, rank $\psi(C) = m$ implies rank $C = n$. Hence, $r = 0$ implies $A = B$. It follows that ψ is injective. □

Lemma 2.2.6. *Let m, n be integers with $m, n \geqslant 3$, and let $\mathbb{F}$ be a field such that either $|\mathbb{F}| = 2$ or $|\mathbb{F}| > n+1$, $\mathbb{K}$ be a field which possesses an involution $^\wedge$ of $\mathbb{K}$, and $\mathbb{K}^\wedge$ is a fixed field of the involution $^\wedge$ of $\mathbb{K}$ with $|\mathbb{K}^\wedge| = 2$ or $|\mathbb{K}^\wedge| > n+1$. Let ψ be a mapping satisfying* (A1) *from $\mathcal{M}_n(\mathbb{F})$ into $\mathcal{M}_m(\mathbb{F})$ (respectively, from $\mathcal{H}_n(\mathbb{K})$ into $\mathcal{H}_m(\mathbb{K})$). If*

$$\text{rank } (A+\alpha B) = n \iff \text{rank } (\psi(A)+\alpha\psi(B)) = m \tag{2.4}$$

for all $A, B \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$) and $\alpha \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$), then ψ is linear (respectively, additive).

Proof. Let $A, B \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$), and $\alpha \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$) such that rank $(A+\alpha B) = n$. We observe that from (2.4), if we let $B = 0$, then we have

$$\text{rank } A = n \iff \text{rank } \psi(A) = m \tag{2.5}$$

for every $A \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$). By Lemma 2.2.5, ψ is injective and hence we have

$$\text{rank } \psi(A+\alpha B) = \text{rank } (\psi(A)+\alpha\psi(B)) = m$$

as adj $(\psi(A+\alpha B)) = \psi(\text{adj }(A+\alpha B)) = \text{adj }(\psi(A)+\alpha\psi(B))$. Then

$$\psi(A+\alpha B)\text{adj }\psi(A+\alpha B) = (\det\psi(A+\alpha B))I_m,$$

$$(\psi(A)+\alpha\psi(B))\text{adj }(\psi(A)+\alpha\psi(B)) = (\det(\psi(A)+\alpha\psi(B)))I_m.$$

In addition,

$$\frac{\psi(A+\alpha B)}{\det\psi(A+\alpha B)}\text{adj }\psi(A+\alpha B) = I_m = \frac{\psi(A)+\alpha\psi(B)}{\det(\psi(A)+\alpha\psi(B))}\text{adj }\psi(A+\alpha B).$$

By the uniqueness of the inverse of adj $\psi(A+\alpha B)$, we have

$$\psi(A+\alpha B) = \frac{\det\psi(A+\alpha B)}{\det(\psi(A)+\alpha\psi(B))}(\psi(A)+\alpha\psi(B)). \tag{2.6}$$

By repeating similar arguments as for (2.6), we have

$$\psi(A+\alpha B) = \frac{\det\psi(A+\alpha B)}{\det(\psi(A)+\psi(\alpha B))}(\psi(A)+\psi(\alpha B)). \tag{2.7}$$

If $A=0$, then rank $(\alpha B) = n$ and hence by (2.6),

$$\psi(\alpha B) = \frac{\det\psi(\alpha B)}{\det(\alpha\psi(B))}(\alpha\psi(B)). \tag{2.8}$$

Next, we claim that

$$\psi(\alpha A) = \alpha\psi(A) \tag{2.9}$$

for every nonzero scalar $\alpha \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$) and every rank n matrix $A \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$). By Lemma 2.2.2(c), there exists a nonzero singular matrix $C \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$) such that rank $(C+\alpha A) = n$. By Lemma 2.2.5(c) and (2.4), we have

$$\text{rank }\psi(C+\alpha A) = \text{rank }(\psi(C)+\alpha\psi(A)) = \text{rank }(\psi(C)+\psi(\alpha A)) = m.$$

By (2.6) and (2.7), we obtain

$$\frac{\det\psi(C+\alpha A)}{\det(\psi(C)+\alpha\psi(A))}(\psi(C)+\alpha\psi(A)) = \frac{\det\psi(C+\alpha A)}{\det(\psi(C)+\psi(\alpha A))}(\psi(C)+\psi(\alpha A))$$

and hence

$$\frac{\psi(C)+\alpha\psi(A)}{\det(\psi(C)+\alpha\psi(A))} = \frac{\psi(C)+\psi(\alpha A)}{\det(\psi(C)+\psi(\alpha A))} \tag{2.10}$$

We let $\mu_1 = \det(\psi(C)+\alpha\psi(A))$ and $\mu_2 = \det(\psi(C)+\psi(\alpha A))$ be nonzero scalars in $\mathbb{F}$ (respectively, $\mathbb{K}^\wedge$). Then by (2.10), we have

$$\mu_1\psi(\alpha A) - \mu_2\alpha\psi(A) = (\mu_2 - \mu_1)\psi(C). \tag{2.11}$$

Suppose $\mu_1 \neq \mu_2$. Since rank $A = n$, it follows from (2.8) that $\psi(\alpha A)$ and $\psi(A)$ are linearly dependent. So, $\psi(\alpha A) = \gamma\psi(A)$ for some $\gamma \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$) since $\psi(\alpha A), \psi(A) \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$). Thus, we obtain

$$(\mu_1\gamma - \mu_2\alpha)\psi(A) = (\mu_2 - \mu_1)\psi(C).$$

Therefore, $\psi(A)$ and $\psi(C)$ are linearly dependent. In addition, since $\psi(A)$ and $\psi(C)$ are nonzero, we obtain rank $\psi(A) =$ rank $\psi(C)$, a contradiction. Thus, $\mu_1 = \mu_2$ implies $\det(\psi(C)+\alpha\psi(A)) = \det(\psi(C)+\psi(\alpha A))$. Therefore, by (2.10) we have $\psi(C)+\alpha\psi(A) = \psi(C)+\psi(\alpha A)$ and this implies $\psi(\alpha A) = \alpha\psi(A)$.

Now, we want to show that if $A, B \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$) with rank $(A+B) = n$, then

$$A, B \text{ are linearly independent} \implies \psi(A), \psi(B) \text{ are linearly independent.} \tag{2.12}$$

Suppose to the contrary that $\psi(A)$ and $\psi(B)$ are linearly dependent. Then there exists a scalar $\lambda \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$) such that $\psi(B) = \lambda\psi(A)$. Since rank $(A+B) = n$, it follows from (2.4) that rank $(\psi(A)+\psi(B)) = m$. This implies rank $(1+\lambda)\psi(A) = m$ and hence rank $\psi(A) = m$. By Lemma 2.2.5, we have rank $A = n$. Thus, $\psi(B) = \lambda\psi(A) = \psi(\lambda A)$ by (2.9). Since ψ is injective, we obtain $B = \lambda A$ which means A and B are linearly dependent, a contradiction. Therefore, (2.12) is proved.

We next claim that if $A, B \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$) such that rank $(A+B) = n$ with $0 < \text{rank } A < n$ and rank $B = n$, then

$$\psi(A+B) = \psi(A) + \psi(B). \tag{2.13}$$

By substituting $\alpha = 1$ into (2.6), we obtain

$$\frac{\psi(A+B)}{\det \psi(A+B)} = \frac{\psi(A)+\psi(B)}{\det(\psi(A)+\psi(B))}. \tag{2.14}$$

Note that $\psi(A+B)$ and $\psi(A)+\psi(B)$ are in $\mathcal{M}_m(\mathbb{F})$ (respectively, $\mathcal{H}_m(\mathbb{K})$) and hence $\det \psi(A+B)$, $\det(\psi(A)+\psi(B)) \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$). If $|\mathbb{F}| = 2$ (respectively, $|\mathbb{K}^\wedge| = 2$), then $\det \psi(A+B) = 1 = \det(\psi(A)+\psi(B))$. So, we are done. If $|\mathbb{F}| > n+1$ (respectively, $|\mathbb{K}^\wedge| > n+1$), then by Lemma 2.2.3, there exists a nonzero scalar $\alpha_0 \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$) such that rank $(A + (1+\alpha_0)B) = n$. By (2.14), we have

$$\frac{\psi(A+B)+\psi(\alpha_0 B)}{\det(\psi(A+B)+\psi(\alpha_0 B))} = \frac{\psi(A+B+\alpha_0 B)}{\det(\psi(A+B+\alpha_0 B))} = \frac{\psi(A)+\psi(B+\alpha_0 B)}{\det(\psi(A)+\psi(B+\alpha_0 B))}.$$

Since rank $A < n$, we have $1+\alpha_0 \neq 0$, and hence rank $((1+\alpha_0)B) = n$. Thus, by (2.9),

$$\psi(B+\alpha_0 B) = (1+\alpha_0)\psi(B) = \psi(B) + \alpha_0\psi(B) = \psi(B) + \psi(\alpha_0 B).$$

So,

$$\frac{\psi(A+B)+\psi(\alpha_0 B)}{\det(\psi(A+B)+\psi(\alpha_0 B))} = \frac{\psi(A)+\psi(B)+\psi(\alpha_0 B)}{\det(\psi(A)+\psi(B+\alpha_0 B))}. \tag{2.15}$$

Let $\lambda_1 = \det(\psi(A+B)+\psi(\alpha_0 B))$ and $\lambda_2 = \det(\psi(A)+\psi(B+\alpha_0 B))$. It is clear that λ_1 and λ_2 are nonzero scalars in $\mathbb{F}$ (respectively, $\mathbb{K}^\wedge$). In view of (2.14), we see that $\psi(A+B)$ and $\psi(A)+\psi(B)$ are linearly dependent. So, there exists a scalar $\beta \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$) such that $\psi(A)+\psi(B) = \beta\psi(A+B)$. Then by

(2.15), we have

$$(\lambda_1\beta - \lambda_2)\psi(A+B) + (\lambda_2 - \lambda_1)\psi(\alpha_0 B) = 0. \tag{2.16}$$

Since A and B are linearly independent, it follows that $A+B$ and $\alpha_0 B$ are linearly independent. In addition, since rank $((A+B)+\alpha_0 B) = n$, we obtain $\psi(A+B)$ and $\psi(\alpha_0 B)$ are linearly independent by (2.12). From (2.16), we have $\lambda_1 = \lambda_2$ and this implies

$$\psi(A+B) + \psi(\alpha_0 B) = \psi(A) + \psi(B) + \psi(\alpha_0 B)$$

and hence $\psi(A+B) = \psi(A) + \psi(B)$.

Next, we show that ψ is homogenous (respectively, $\mathbb{K}^\wedge$-homogeneous), that is

$$\psi(\alpha A) = \alpha\psi(A) \tag{2.17}$$

for every $A \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$) and $\alpha \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$). It is obvious that (2.17) holds when $\alpha = 0$, $A = 0$ or rank $A = n$. Now, we consider $\alpha \neq 0$ and A is a nonzero singular matrix. By Lemma 2.2.2(c), there exists a rank n matrix $X \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$) such that rank $(\alpha A + X) = n$. This implies rank $(A + \alpha^{-1}X) = n$. It follows from (2.9) and (2.13) that

$$\begin{aligned}
\psi(\alpha A) + \psi(X) &= \psi(\alpha A + X) \\
&= \psi(\alpha(A + \alpha^{-1}X)) \\
&= \alpha\psi(A + \alpha^{-1}X) \\
&= \alpha(\psi(A) + \psi(\alpha^{-1}X)) \\
&= \alpha\psi(A) + \alpha\psi(\alpha^{-1}X) \\
&= \alpha\psi(A) + \psi(X).
\end{aligned}$$

Therefore, $\psi(\alpha A) = \alpha\psi(A)$.

Now, we show that

$$\psi(A+B) = \psi(A) + \psi(B) \tag{2.18}$$

for every $A, B \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$) with rank $(A+B) = n$. It is clear that the claim holds when $|\mathbb{F}| = 2$ (respectively, $|\mathbb{K}^\wedge| = 2$) by (2.14). Consider $|\mathbb{F}| > n+1$ (respectively, $|\mathbb{K}^\wedge| > n+1$). If A and B are linearly dependent, then $B = \gamma A$ for some scalar $\gamma \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$). By (2.17), we have

$$\begin{aligned}
\psi(A+B) &= \psi((1+\gamma)A) \\
&= (1+\gamma)\psi(A) \\
&= \psi(A) + \gamma\psi(A) \\
&= \psi(A) + \psi(\gamma A) \\
&= \psi(A) + \psi(B).
\end{aligned}$$

Consider the case where A and B are linearly independent. By Lemma 2.2.3, there exists $\beta_0 \in \mathbb{F}$ (respectively, $\mathbb{K}^\wedge$) such that rank $(A + (1+\beta_0)B) = n$. By (2.14) and (2.17), we have

$$\frac{\psi(A+B) + \psi(\beta_0 B)}{\det(\psi(A+B) + \psi(\beta_0 B))} = \frac{\psi(A) + \psi(B) + \psi(\beta_0 B)}{\det(\psi(A) + \psi(B) + \psi(\beta_0 B))}. \tag{2.19}$$

Since A and B are linearly independent, $A+B$ and $\beta_0 B$ are also linearly independent and hence $\psi(A+B)$ and $\psi(\beta_0 B)$ are linearly independent by (2.12). By using similar arguments as in the proof of (2.16), it can be shown that $\det(\psi(A+B) + \psi(\beta_0 B)) = \det(\psi(A) + \psi(B) + \psi(\beta_0 B))$. Then by (2.19), we obtain (2.18).

Next, we want to show that ψ is additive. Let $A, B \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$). By Lemma 2.2.2(b), there exists a matrix $X \in \mathcal{M}_n(\mathbb{F})$ (respectively,

$\mathcal{H}_n(\mathbb{K})$) such that rank $(A + X) =$ rank $(A + B + X) = n$. By (2.18), we have

$$\psi(A + B) + \psi(X) = \psi(A + B + X) = \psi(A + X) + \psi(B).$$

Since rank $(A + X) = n$, by (2.18) again, we have $\psi(A + X) = \psi(A) + \psi(X)$. Thus, we obtain

$$\psi(A + B) + \psi(X) = \psi(A) + \psi(B) + \psi(X)$$

$$\implies \psi(A + B) = \psi(A) + \psi(B)$$

for all matrices $A, B \in \mathcal{M}_n(\mathbb{F})$ (respectively, $\mathcal{H}_n(\mathbb{K})$). We are done. $\square$

Chapter 3

Classical adjoint-commuting mappings between matrix algebras

3.1 Introduction

In this chapter, we let m, n be integers with $m, n \geqslant 3$ and let $\mathbb{F}$ and $\mathbb{K}$ be fields. We characterise mappings $\psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ that satisfy one of the following conditions (see (A1) and (A2) in Section 2.1):

(AM1) $\psi(\text{adj }(A + \alpha B)) = \text{adj }(\psi(A) + \alpha\psi(B))$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$ when $\mathbb{F} = \mathbb{K}$,

(AM2) $\psi(\text{adj }(A - B)) = \text{adj }(\psi(A) - \psi(B))$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$.

3.2 Some basic properties

Lemma 3.2.1. *Let m, n be integers with $m, n \geqslant 3$ and let $\mathbb{F}$ and $\mathbb{K}$ be fields. Let $\psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ be a mapping satisfying* (AM2). *Then the following statements are equivalent.*

(a) $\psi(I_n) = 0$.

(b) $\psi(A) = 0$ *for every rank one matrix* $A \in \mathcal{M}_n(\mathbb{F})$.

(c) $\text{rank } \psi(A) \leqslant m - 2$ *for every* $A \in \mathcal{M}_n(\mathbb{F})$.

(d) $\psi(\text{adj } A) = 0$ *for every* $A \in \mathcal{M}_n(\mathbb{F})$.

Proof.

(a) $\Longrightarrow$ (b):

Let $1 \leqslant i \leqslant n$. We have

$$\psi(E_{ii}) = \psi(\text{adj } (I_n - E_{ii})) = \text{adj } (\psi(I_n) - \psi(E_{ii})) = \text{adj } (-\psi(E_{ii})) = 0$$

since $m \geqslant 3$ and rank $\psi(E_{ii}) \leqslant 1$. Therefore $\psi(E_{ii}) = 0$ for all $1 \leqslant i \leqslant n$. We next show that $\psi(\alpha E_{ij}) = 0$ for all $1 \leqslant i, j \leqslant n$ and $\alpha \in \mathbb{F}$. The result is clear when $\alpha = 0$. We now suppose $\alpha \neq 0$. Since adj $(I_n - E_{ii} - E_{jj} + \alpha E_{jj}) = \alpha E_{ii}$ with $j \neq i$, we have

$$\begin{aligned}
\psi(\alpha E_{ii}) &= \psi(\text{adj } (I_n - E_{ii} - E_{jj} + \alpha E_{jj})) \\
&= \text{adj } (\psi(I_n + \alpha E_{jj} - E_{ii}) - \psi(E_{jj})) \\
&= \text{adj } (\psi(I_n + \alpha E_{jj}) - \psi(E_{ii})) \\
&= \text{adj } (\psi(I_n) - \psi(-\alpha E_{jj})) \\
&= \text{adj } (-\psi(-\alpha E_{jj})) = 0
\end{aligned}$$

since rank $\psi(-\alpha E_{jj}) \leqslant 1$. For each $1 \leqslant i \neq j \leqslant n$, adj $(I_n - E_{ii} - E_{jj} + (-1)^{i+j}\alpha E_{ij}) = \alpha E_{ij}$. By similar arguments, we obtain

$$\begin{aligned}
\psi(\alpha E_{ij}) &= \psi(\text{adj } (I_n - E_{ii} - E_{jj} + (-1)^{i+j}\alpha E_{ij})) \\
&= \text{adj } (\psi(I_n + (-1)^{i+j}\alpha E_{ij} - E_{ii}) - \psi(E_{jj})) \\
&= \text{adj } (\psi(I_n + (-1)^{i+j}\alpha E_{ij}) - \psi(E_{ii})) \\
&= \text{adj } (\psi(I_n) - \psi(-(-1)^{i+j}\alpha E_{ij})) \\
&= \text{adj } (-\psi(-(-1)^{i+j}\alpha E_{ij})) = 0.
\end{aligned}$$

Hence, $\psi(\alpha E_{ij}) = 0$ for every $1 \leqslant i, j \leqslant n$ and $\alpha \in \mathbb{F}$.

Let $A \in \mathcal{M}_n(\mathbb{F})$ be of rank one. Then by Proposition 1.4.6, there exists a rank $n-1$ matrix $B = (b_{ij}) \in \mathcal{M}_n(\mathbb{F})$ such that $A = \text{adj } B$. Thus, $\psi(A) = \psi(\text{adj } B)$ and hence

$$
\begin{aligned}
\psi(A) &= \text{adj } \psi(B) \\
&= \text{adj } \psi\left(\sum_{i,j=1}^{n} b_{ij}E_{ij}\right) \\
&= \text{adj } \psi\left(\sum_{\substack{i,j=1, \\ (i,j)\neq(1,1)}}^{n} b_{ij}E_{ij} - (-b_{11})E_{11}\right) \\
&= \text{adj } \psi\left(\sum_{\substack{i,j=1, \\ (i,j)\neq(1,1)}}^{n} b_{ij}E_{ij} - \psi(-b_{11}E_{11})\right) \\
&= \text{adj } \psi\left(\sum_{\substack{i,j=1, \\ (i,j)\neq(1,1)}}^{n} b_{ij}E_{ij}\right) \\
&= \text{adj } \psi\left(\sum_{\substack{i,j=1, \\ (i,j)\neq(1,1),(1,2)}}^{n} b_{ij}E_{ij} - \psi(-b_{12}E_{12})\right) \\
&= \text{adj } \psi\left(\sum_{\substack{i,j=1, \\ (i,j)\neq(1,1),(1,2)}}^{n} b_{ij}E_{ij}\right).
\end{aligned}
$$

By repeating similar arguments, we obtain

$$\psi(A) = \text{adj } \psi(b_{nn}E_{nn}) = 0.$$

(b) $\Longrightarrow$ (c):

Let $A \in \mathcal{M}_n(\mathbb{F})$ with rank $A \leqslant n-1$. Then rank adj $A \leqslant 1$. By (b), adj $\psi(A) = \psi(\text{adj } A) = 0$. The result holds. Now we consider $A \in \mathcal{M}_n(\mathbb{F})$ of rank n. Then there exist rank one matrices $A_1, \cdots, A_n \in \mathcal{M}_n(\mathbb{F})$ such that $A = A_1 + \cdots + A_n$. Hence,

$$
\begin{aligned}
\text{adj } \psi(A) &= \text{adj } \psi(A_1 + \cdots + A_n) \\
&= \text{adj } (\psi(A_1 + \cdots + A_{n-1}) - \psi(-A_n)) \\
&= \text{adj } \psi(A_1 + \cdots + A_{n-1}) \\
&= \text{adj } (\psi(A_1 + \cdots + A_{n-2}) - \psi(-A_{n-1})).
\end{aligned}
$$

We continue in this way to obtain

$$\text{adj } \psi(A) = \text{adj } \psi(A_1) = 0.$$

Therefore, rank $\psi(A) \leqslant m-2$.

(c) $\Longrightarrow$ (d): Since rank $\psi(A) \leqslant m-2$, $\psi(\text{adj } A) = \text{adj } \psi(A) = 0$.

(d) $\Longrightarrow$ (a): $\psi(I_n) = \psi(\text{adj } I_n) = 0$. $\square$

Lemma 3.2.2. *Let m, n be integers and let $\mathbb{F}$ and $\mathbb{K}$ be fields. Let $\psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ be a mapping satisfying* (AM2). *Let $A, B \in \mathcal{M}_n(\mathbb{F})$. If $\psi(I_n) \neq 0$, then ψ is injective and*

$$\text{rank } (A-B) = n \iff \text{rank } (\psi(A) - \psi(B)) = m.$$

Proof. Since $\psi(I_n) \neq 0$ and adj $\psi(I_n) = \psi(\text{adj } I_n) = \psi(I_n)$, we have rank $\psi(I_n) = m$. Let $1 \leqslant i \leqslant n$. Then

$$\text{rank adj } \left(\psi(E_{ii}) - \psi\left(\sum_{j=1, j\neq i}^{n} -E_{jj}\right)\right) = \text{rank adj } \psi(I_n) = m.$$

This implies rank $\left(\psi(E_{ii}) - \psi\left(\sum_{j=1,j\neq i}^{n} -E_{jj}\right)\right) = m$. By Lemma 2.2.4,

$$\text{rank } \psi(E_{ii}) \leqslant 1 \text{ and rank } \psi\left(\sum_{j=1, j\neq i}^{n} -E_{jj}\right) \leqslant m-1.$$

These show that rank $\psi(E_{ii}) = 1$.

Next, we show that rank $\psi(\alpha E_{ij}) = 1$ for every nonzero scalar $\alpha \in \mathbb{F}$ and $1 \leqslant i, j \leqslant n$. Suppose there exists a nonzero scalar $\alpha_0 \in \mathbb{F}$ such that $\psi(\alpha_0 E_{ij}) = 0$ for $1 \leqslant i, j \leqslant n$. Since $n \geqslant 3$, then if $i = j$, we can select two distinct integers $1 \leqslant s, t \leqslant n$ with $s, t \neq i$; or if $i \neq j$, we choose an integer $1 \leqslant s \leqslant n$ with $s \neq i, j$, such that

$$E_{ss} = \begin{cases} \text{adj } (I_n - E_{ss} - (1+\alpha_0)E_{ii} - (1+\alpha_0^{-1})E_{tt}) & \text{if } i = j, \\ \text{adj } (I_n - E_{ii} - E_{jj} - E_{ss} + \alpha_0^{-1}E_{ji} - \alpha_0 E_{ij}) & \text{if } i \neq j. \end{cases}$$

Then

$$
\begin{aligned}
\psi(E_{ss}) &= \begin{cases} \psi\left(\text{adj }(I_n - E_{ss} - (1+\alpha_0)E_{ii} - (1+\alpha_0^{-1})E_{tt})\right) & \text{if } i = j, \\ \psi\left(\text{adj }(I_n - E_{ii} - E_{jj} - E_{ss} + \alpha_0^{-1}E_{ji} - \alpha_0 E_{ij})\right) & \text{if } i \neq j \end{cases} \\
&= \begin{cases} \text{adj }\left(\psi(I_n - E_{ss} - E_{ii} - (1+\alpha_0^{-1})E_{tt}) - \psi(\alpha_0 E_{ii})\right) & \text{if } i = j, \\ \text{adj }\left(\psi(I_n - E_{ii} - E_{jj} - E_{ss} + \alpha_0^{-1}E_{ji}) - \psi(\alpha_0 E_{ij})\right) & \text{if } i \neq j \end{cases} \\
&= \begin{cases} \text{adj } \psi(I_n - E_{ss} - E_{ii} - (1+\alpha_0^{-1})E_{tt}) & \text{if } i = j, \\ \text{adj } \psi(I_n - E_{ii} - E_{jj} - E_{ss} + \alpha_0^{-1}E_{ji}) & \text{if } i \neq j \end{cases} \\
&= \begin{cases} \psi\left(\text{adj }(I_n - E_{ss} - E_{ii} - (1+\alpha_0^{-1})E_{tt})\right) & \text{if } i = j, \\ \psi\left(\text{adj }(I_n - E_{ii} - E_{jj} - E_{ss} + \alpha_0^{-1}E_{ji})\right) & \text{if } i \neq j \end{cases} \\
&= \psi(0) = 0,
\end{aligned}
$$

a contradiction. Therefore,

$$\text{rank } \psi(\alpha E_{ij}) = 1 \text{ for every nonzero scalar } \alpha \in \mathbb{F} \text{ and } 1 \leqslant i, j \leqslant n.$$

Let $X \in \mathcal{M}_n(\mathbb{F})$ be of rank one. Then there exist an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ and a nonzero scalar $\lambda \in \mathbb{F}$ such that $X = P(\lambda E_{st})P^{-1}$ for some integers $1 \leqslant s, t \leqslant n$. We define the mapping $\phi_P : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ by

$$\phi_P(A) = \psi(PAP^{-1}) \text{ for every } A \in \mathcal{M}_n(\mathbb{F}).$$

Let $A, B \in \mathcal{M}_n(\mathbb{F})$. We have

$$
\begin{aligned}
\phi_P(\text{adj }(A - B)) &= \psi(P(\text{adj }(A - B)P^{-1}) \\
&= \psi(\text{adj }(P(A - B)P^{-1}) \\
&= \text{adj }(\psi(PAP^{-1}) - \psi(PBP^{-1})) \\
&= \text{adj }(\phi_P(A) - \phi_P(B)).
\end{aligned}
$$

Therefore, ϕ_P satisfies (AM2). Since $\phi_P(I_n) = \psi(PI_nP^{-1}) = \psi(I_n) \neq 0$, we obtain rank $\phi_P(\alpha E_{ij}) = 1$ for all nonzero scalar $\alpha \in \mathbb{F}$ and $1 \leqslant i, j \leqslant n$. Thus, $\psi(X) = \psi(P(\lambda E_{st})P^{-1}) = \phi_P(\lambda E_{st})$ implies

$$\text{rank } \psi(X) = 1 \text{ for every rank one matrix } X \in \mathcal{M}_n(\mathbb{F}). \tag{3.1}$$

Next, let $A, B \in \mathcal{M}_n(\mathbb{F})$ such that $\psi(A) = \psi(B)$. Suppose $A - B \neq 0$. Then there exists a matrix $C \in \mathcal{M}_n(\mathbb{F})$ of rank at most $n - 2$ such that rank $(A - B +$

$C) = n - 1$. Thus, rank adj $(A - B + C) = 1$. It follows that rank $\psi($adj $(A - B + C)) = 1$ by (3.1). On the other hand,

$$\begin{aligned}
\psi(\text{adj } (A - B + C)) &= \text{adj } (\psi(A + C) - \psi(B)) \\
&= \text{adj } (\psi(A + C) - \psi(A)) \\
&= \text{adj } \psi(C) \\
&= 0
\end{aligned}$$

which is a contradiction. This implies $A = B$ and hence ψ is injective.

Let $A, B \in \mathcal{M}_n(\mathbb{F})$. Since ψ is injective, by Lemma 2.2.5,

$$\begin{aligned}
\text{rank } (A - B) = n &\iff \text{rank adj } (A - B) = n \\
&\iff \text{rank } \psi(\text{adj } (A - B)) = m \\
&\iff \text{rank adj } (\psi(A) - \psi(B)) = m \\
&\iff \text{rank } (\psi(A) - \psi(B)) = m.
\end{aligned}$$

□

3.3 Some examples

We should point out that, in order to obtain a nice structural form of ψ which satisfies condition (AM1) or (AM2), the condition of $\psi(I_n) \neq 0$ in Theorem 3.4.1 is indispensable. In Lemma 3.2.1, we proved that ψ sends all rank one matrices to zero if $\psi(I_n) = 0$. Under the condition of (AM1) or (AM2), beside the zero mapping, there are some nonzero classical adjoint-commuting mappings sending rank one matrices to zero. Thus, in this section, we give some examples of such mappings.

Example 3.3.1. Let m, n be integers with $m, n \geqslant 3$ and let $\mathbb{F}$ and $\mathbb{K}$ be fields.

(i) Let $\tau : \mathcal{M}_n(\mathbb{F}) \to \mathbb{K}$ be a nonzero function and let $\psi_1 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$

be the mapping defined by

$$\psi_1(A) = \begin{cases} \tau(A)(E_{11}) & \text{if } A \in \mathcal{M}_n(\mathbb{F}) \text{ is of rank } r \text{ with } 1 < r < n, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let $\mathcal{E} := \{\operatorname{adj} A : A \in \mathcal{M}_n(\mathbb{F}) \text{ is invertible}\}$ and let $\psi_2 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ be the mapping defined by

$$\psi_2(A) = \begin{cases} 0 & \text{if } A \in \mathcal{M}_n(\mathbb{F}) \text{ is of rank 0 or 1, or } A \in \mathcal{E}, \\ E_{11} & \text{otherwise.} \end{cases}$$

Example 3.3.2. Let m, n be integers with $m, n \geqslant 4$. We define the mapping $\psi_3 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ by

$$\psi_3(A) = \begin{cases} \sum_{i=1}^{m-2} E_{ii} & \text{if rank } A = 2, \\ E_{11} + E_{22} & \text{if } A \text{ is of rank } r \text{ with } 2 < r < n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.3.3. Let m, n be integers with $m, n \geqslant 5$. We define the mapping $\psi_4 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ by

$$\psi_4(A) = \begin{cases} E_{11} + E_{22} & \text{if rank } A = r \text{ and } r \text{ is odd,} \\ E_{22} + E_{33} + E_{44} & \text{if rank } A = r \text{ and } r \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily checked that each ψ_i for $i = 1, 2, 3, 4$ is a classical adjoint-commuting mapping satisfying condition (AM1) or (AM2) with $\psi_i(I_n) = 0$. We also observe that these mappings are neither injective nor surjective.

3.4 Characterisation of classical adjoint-commuting mappings between matrix algebras

Theorem 3.4.1. *Let m, n be integers with $m, n \geqslant 3$, and let $\mathbb{F}$ be a field with $|\mathbb{F}| = 2$ or $|\mathbb{F}| > n + 1$. Then $\psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ is a mapping satisfying*

(AM1) *if and only if* $\psi(A) = 0$ *for every rank one matrix* $A \in \mathcal{M}_n(\mathbb{F})$ *and* rank $(\psi(A) + \alpha\psi(B)) \leqslant m - 2$ *for all* $A, B \in \mathcal{M}_n(\mathbb{F})$ *and* $\alpha \in \mathbb{F}$*; or* $m = n$*, and*

$$\psi(A) = \lambda PAP^{-1} \text{ for every } A \in \mathcal{M}_n(\mathbb{F})$$

or

$$\psi(A) = \lambda PA^tP^{-1} \text{ for every } A \in \mathcal{M}_n(\mathbb{F}).$$

where $P \in \mathcal{M}_n(\mathbb{F})$ *is an invertible matrix and* $\lambda \in \mathbb{F}$ *is a scalar with* $\lambda^{n-2} = 1$.

Proof. The sufficiency can be proved easily. We now prove the necessity. We observe that if ψ satisfies (AM1), then ψ satisfies (AM2). Thus, Lemmas 2.2.4, 2.2.5, 3.2.1 and 3.2.2 hold for ψ satisfying (AM1). We first consider the case where $\psi(I_n) = 0$. By Lemma 3.2.1, $\psi(A) = 0$ for every rank one matrix $A \in \mathcal{M}_n(\mathbb{F})$ and $\psi(\text{adj } A) = 0$ for every $A \in \mathcal{M}_n(\mathbb{F})$. Then we obtain adj $(\psi(A) + \alpha\psi(B)) = \psi(\text{adj } (A + \alpha B)) = 0$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$. Thus,

$$\text{rank } (\psi(A) + \alpha\psi(B)) \leqslant m - 2 \text{ for all } A, B \in \mathcal{M}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}.$$

Next, consider $\psi(I_n) \neq 0$. Let $A, B \in \mathcal{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$. Then by Lemma 2.2.5, we have

$$\begin{aligned}
\text{rank } (A + \alpha B) = n &\iff \text{rank adj } (A + \alpha B) = n \\
&\iff \text{rank } \psi(\text{adj } (A + \alpha B)) = m \\
&\iff \text{rank adj } (\psi(A) + \alpha\psi(B)) = m \\
&\iff \text{rank } (\psi(A) + \alpha\psi(B)) = m.
\end{aligned}$$

It follows from Lemma 2.2.6 that ψ is linear. Therefore, by [27, Theorem 3.4] (or [2, Corollary 3.10]), we are done. □

Theorem 3.4.2. *Let* m, n *be integers with* $m, n \geqslant 3$*, and let* $\mathbb{F}$ *and* $\mathbb{K}$ *be fields. Then* $\psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{K})$ *is a surjective mapping satisfying* (AM2) *if and* *only if* $m = n$, $\mathbb{F}$ *and* $\mathbb{K}$ *are isomorphic, and*

$$\psi(A) = \lambda P A^\sigma P^{-1} \text{ for every } A \in \mathcal{M}_n(\mathbb{F})$$

or

$$\psi(A) = \lambda P (A^\sigma)^t P^{-1} \text{ for every } A \in \mathcal{M}_n(\mathbb{F})$$

where $\sigma : \mathbb{F} \to \mathbb{K}$ *is a field isomorphism,* A^σ *is the matrix obtained from* A *by applying* σ *entrywise,* $P \in \mathcal{M}_n(\mathbb{K})$ *is an invertible matrix, and* $\lambda \in \mathbb{K}$ *is a scalar with* $\lambda^{n-2} = 1$.

Proof. The sufficiency part is trivial. We now prove the necessity part. Suppose $\psi(I_n) = 0$. Then by Lemma 3.2.1, rank $\psi(A) \leqslant m - 2$ for every $A \in \mathcal{M}_n(\mathbb{F})$. This implies ψ is not surjective. Therefore, $\psi(I_n) \neq 0$ and hence by Lemma 3.2.2,

$$\text{rank } (A - B) = n \iff \text{rank } (\psi(A) - \psi(B)) = m \text{ for all } A, B \in \mathcal{M}_n(\mathbb{F}).$$

We consider two cases in this proof.

Case I: $|\mathbb{F}| \neq 2$.

By [14, Theorem 3.2] and the fundamental theorem of rectangular matrices (see Theorem 1.5.2), we have $m = n$ and either

$$\psi(A) = P A^\sigma Q + R \text{ for every } A \in \mathcal{M}_n(\mathbb{F})$$

or

$$\psi(A) = P (A^\sigma)^t Q + R \text{ for every } A \in \mathcal{M}_n(\mathbb{F})$$

where $\sigma : \mathbb{F} \to \mathbb{K}$ is an isomorphism, $P, Q \in \mathcal{M}_n(\mathbb{K})$ are invertible matrices, and $R \in \mathcal{M}_n(\mathbb{K})$. For both cases above, $R = 0$ since $\psi(0) = 0$. In addition, since adj $\psi(I_n) = \psi(\text{adj } I_n) = \psi(I_n)$, we have adj $(PQ) = PQ$. Thus,

$$
\begin{aligned}
& \psi(\text{adj } A) = \text{adj } (\psi A) \\
\Longrightarrow\ & P(\text{adj } A)Q = \text{adj } (PAQ) \\
\Longrightarrow\ & PQ(Q^{-1}(\text{adj } A)Q) = (\text{adj } Q)(\text{adj } A)(\text{adj } P) \\
\Longrightarrow\ & PQ(Q^{-1}(\text{adj } A)Q) = \frac{1}{\det Q}(\text{adj } Q)(\text{adj } A)(\det Q)(\text{adj } P) \\
\Longrightarrow\ & PQ(Q^{-1}(\text{adj } A)Q) = Q^{-1}(\text{adj } A)(\frac{\det(PQ)}{\det P})(\text{adj } P) \\
\Longrightarrow\ & PQ(Q^{-1}(\text{adj } A)Q) = Q^{-1}(\text{adj } A)P^{-1}\det(PQ)I_n \\
\Longrightarrow\ & PQ(Q^{-1}(\text{adj } A)Q) = Q^{-1}(\text{adj } A)P^{-1}(\text{adj } (PQ))PQ.
\end{aligned}
$$

Hence, we obtain $PQ(Q^{-1}(\text{adj } A)Q) = (Q^{-1}(\text{adj } A)Q)PQ$ for every $A \in \mathcal{M}_n(\mathbb{F})$. Since $\{Q^{-1}E_{ij}Q : E_{ij} \in \mathcal{M}_n(\mathbb{K})\}$ spans $\mathcal{M}_n(\mathbb{K})$, it follows that PQ commutes with all matrices in $\mathcal{M}_n(\mathbb{K})$. Thus, $PQ = \lambda I_n$ for some nonzero scalar $\lambda \in \mathbb{K}$. Again, since $\psi(\text{adj } I_n) = \text{adj } \psi(I_n)$, we have $PQ = \text{adj } (PQ)$ and hence $\lambda I_n =$ adj (λI_n). Therefore $\lambda^{n-2} = 1$. Consequently, the theorem holds.

Case II: $|\mathbb{F}| = 2$.

Then rank $(A + B) = n$ if and only if rank $(\psi(A) + \psi(B)) = m$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$. Let $A, B \in \mathcal{M}_n(\mathbb{F})$ with rank $(A + B) = n$. Then $\psi(A + B)$ and $\psi(A) + \psi(B)$ are of rank m. Since

$$\psi(A+B)\text{adj } (\psi(A+B)) = \det(\psi(A+B))I_m,$$

$$(\psi(A) + \psi(B))\text{adj } (\psi(A) + \psi(B)) = \det(\psi(A) + \psi(B))I_m$$

and

$$
\begin{aligned}
\text{adj } (\psi(A+B)) &= \psi(\text{adj } (A+B)) \\
&= \psi(\text{adj } (A-B)) \\
&= \text{adj } (\psi(A) - \psi(B)) \\
&= \text{adj } (\psi(A) + \psi(B)),
\end{aligned}
$$

we have

$$\frac{\psi(A+B)}{\det \psi(A+B)} = \frac{\psi(A)+\psi(B)}{\det(\psi(A)+\psi(B))}.$$

As $\det \psi(A+B) = \det(\psi(A)+\psi(B)) = 1$, we obtain $\psi(A+B) = \psi(A)+\psi(B)$ for all $A, B \in \mathcal{M}_n(\mathbb{F})$ with rank $(A+B) = n$. By using similar argument as in the last paragraph of the proof of Lemma 2.2.6, if can be shown that ψ is additive. Therefore, the result follows from [29, Theorem 5.1] and [2, Corollary 3.10]. $\square$

Chapter 4

Classical adjoint-commuting mappings on hermitian and symmetric matrices

4.1 Introduction

Throughout this chapter, unless otherwise stated, we let m, n be integers with $m, n \geqslant 3$ and let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively. We let $\mathbb{F}^- := \{a \in \mathbb{F} : \overline{a} = a\}$ and $\mathbb{K}^\wedge$ the set of all symmetric elements of $\mathbb{F}$ and $\mathbb{K}$, respectively. It can be shown that $\mathbb{F}^-$ is a subfield of $\mathbb{F}$ and we say that $\mathbb{F}^-$ is the *fixed field* on the involution $^-$ of $\mathbb{F}$ whereas $\mathbb{K}^\wedge$ is the fixed field on the involution $^\wedge$ of $\mathbb{K}$. The involution $^-$ of $\mathbb{F}$ is *proper* if $^-$ is not identity and hence there exists $i \in \mathbb{F}$ such that $\overline{i} = -i$ when $\mathbb{F}$ has characteristic $\neq 2$, and $\overline{i} = 1 + i$ when $\mathbb{F}$ has characteristic 2, such that $\mathbb{F} = \mathbb{F}^- \oplus i\mathbb{F}^-$ as an $\mathbb{F}^-$-linear space. See [23] for more details .

In this chapter, we study the structure of $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ that satisfies the following conditions (see (A1) and (A2) in Section 2.1):

(AH1) $\psi(\text{adj }(A + \alpha B)) = \text{adj }(\psi(A) + \alpha\psi(B))$ for all $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$ when $(\mathbb{F}, ^-) = (\mathbb{K}, ^\wedge)$,

(AH2) $\psi(\text{adj }(A - B)) = \text{adj }(\psi(A) - \psi(B))$ for all $A, B \in \mathcal{H}_n(\mathbb{F})$.

4.2 Some basic properties

Let m, n be integers with $m, n \geqslant 3$. Let $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying (AH2). It can be easily shown that

$$\psi(0) = 0 \text{ and } \psi(\text{adj } A) = \text{adj } (\psi(A)) \text{ for every } A \in \mathcal{H}_n(\mathbb{F}).$$

Lemma 4.2.1. *Let n be an integer with $n \geqslant 3$ and let $A \in \mathcal{H}_n(\mathbb{F})$ be a nonzero matrix. Then there exists a matrix $C \in \mathcal{H}_n(\mathbb{F})$ of rank at most $n-2$ such that* rank $(A + C) = n - 1$.

Proof. Suppose rank $A = r$. If $r = n-1$, we choose $C = 0$. We are done. We assume $1 \leqslant r \leqslant n-2$. We choose $C = P(E_{r+1,r+1} + \cdots + E_{n-1,n-1})\overline{P}^t$. It is clear that $C \in \mathcal{H}_n(\mathbb{F})$ and rank $C \leqslant n-2$. It can be shown that rank $(A+C) = n-1$ for both Forms (1.2) and (1.3) of A. If $r = n$, we let

$$C = \begin{cases} P(-\alpha_n E_{nn})\overline{P}^t & \text{if } A \text{ is of Form (1.2)}, \\ P(E_{11} + E_{22})P^t & \text{if } A \text{ is of Form (1.3)}. \end{cases}$$

We note that if A is of Form (1.3), rank $A = n \geqslant 4$. Therefore, $C \in \mathcal{H}_n(\mathbb{F})$ with rank $C \leqslant n-2$ and rank $(A + C) = n - 1$. $\square$

Lemma 4.2.2. *Let m, n be integers with $m, n \geqslant 3$. Let $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying* (AH2). *Let $P \in \mathcal{M}_n(\mathbb{F})$ be a fixed invertible matrix, and let $\phi_P : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be defined by*

$$\phi_P(A) = \psi(PA\overline{P}^t) \textit{ for every } A \in \mathcal{H}_n(\mathbb{F}).$$

If rank $\phi_P(I_n) \neq m$, *then $\phi_P(A) = 0$ for every rank one matrix $A \in \mathcal{H}_n(\mathbb{F})$, and* rank $\phi_P(A) \leqslant m-2$ *for every $A \in \mathcal{H}_n(\mathbb{F})$.*

Proof. Let $A, B \in \mathcal{H}_n(\mathbb{F})$. Then

$$\begin{aligned} \text{adj } \phi_P(A - B) &= \text{adj } \psi(P(A-B)\overline{P}^t) \\ &= \psi(\text{adj } (PA\overline{P}^t - PB\overline{P}^t)) \\ &= \text{adj } (\psi(PA\overline{P}^t) - \psi(PB\overline{P}^t)) \\ &= \text{adj } (\phi_P(A) - \phi_P(B)). \end{aligned}$$

Thus, we obtain

$$\text{adj } \phi_P(A - B) = \text{adj } (\phi_P(A) - \phi_P(B)) \text{ for all } A, B \in \mathcal{H}_n(\mathbb{F}). \tag{4.1}$$

By the definition of ϕ_P, Lemma 2.2.4 (a), (b) and (c) are true for ϕ_P.

Let $\theta := \det(P\overline{P}^t)^{n-2}$, $\vartheta := \theta^{n-1}$ and $Q := \text{adj } P$. It is clear that $\theta, \vartheta \in \mathbb{F}^-$ are nonzero and rank $Q = n$. We now show that

$$\phi_P(\theta I_n) = 0. \tag{4.2}$$

Since adj (adj $(P\overline{P}^t)) = \det(P\overline{P}^t)^{n-2}P\overline{P}^t = \theta P\overline{P}^t$ and rank $\phi_P(I_n) \neq m$, we have

$$\begin{aligned}
\phi_P(\theta I_n) &= \psi(P(\theta I_n)\overline{P}^t) \\
&= \psi(\theta P\overline{P}^t) \\
&= \psi(\text{adj } (\text{adj } (P\overline{P}^t))) \\
&= \text{adj } (\text{adj } \psi(P\overline{P}^t)) \\
&= \text{adj } (\text{adj } \phi_P(I_n)) \\
&= 0.
\end{aligned}$$

Since

$$\begin{aligned}
\psi(\vartheta\overline{Q}^t Q) &= \psi(\theta^{n-1}\text{adj } (P\overline{P}^t)) \\
&= \psi(\text{adj } (\theta P\overline{P}^t)) \\
&= \text{adj } \psi(\theta P\overline{P}^t) \\
&= \text{adj } \phi_P(\theta I_n),
\end{aligned}$$

we obtain

$$\psi(\vartheta\overline{Q}^t Q) = 0 \tag{4.3}$$

by (4.2). Next, we show that

$$\psi(\overline{Q}^t \vartheta E_{ii} Q) = 0 \text{ for every } i = 1, \cdots, n. \tag{4.4}$$

Let $i = 1, \cdots, n$. By $\theta^{n-1}E_{ii} = \text{adj } (\theta(I_n - E_{ii}))$, we have

$$\begin{aligned}
\psi(\overline{Q}^t \vartheta E_{ii} Q) &= \psi(\overline{Q}^t (\theta^{n-1} E_{ii}) Q) \\
&= \psi((\text{adj } \overline{P}^t)(\text{adj } (\theta(I_n - E_{ii})))(\text{adj } P) \\
&= \psi(\text{adj } (P(\theta I_n - \theta E_{ii})\overline{P}^t)) \\
&= \text{adj } \psi(P(\theta I_n - \theta E_{ii})\overline{P}^t) \\
&= \text{adj } \phi_P(\theta I_n - \theta E_{ii}).
\end{aligned}$$

It follows from (4.1), (4.2) that

$$\begin{aligned}
\psi(\overline{Q}^t \vartheta E_{ii} Q) &= \text{adj } (\phi_P(\theta I_n) - \phi(\theta E_{ii})) \\
&= \text{adj } (-\phi_P(\theta E_{ii})) \\
&= 0
\end{aligned}$$

since rank $\phi_P(\theta E_{ii}) \leqslant 1$ by Lemma 2.2.4 (a). The next claim is for $i = 1, \cdots, n$,

$$\phi_P(\alpha E_{ii}) = 0 \text{ for every } \alpha \in \mathbb{F}^-. \tag{4.5}$$

It is clear that the result holds if $\alpha = 0$. We suppose $\alpha \neq 0$. Then

$$\begin{aligned}
\phi_P(\alpha E_{ii}) &= \psi(P(\alpha E_{ii})\overline{P}^t) \\
&= \psi(\theta P(\theta^{-1}\alpha E_{ii})\overline{P}^t) \\
&= \psi((\det P)^{n-2} P(\theta^{-1}\alpha E_{ii})(\det \overline{P})^{n-2}\overline{P}^t).
\end{aligned}$$

Since

$$\text{adj } (\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\alpha E_{jj})) = \theta^{-1}\alpha E_{ii} \text{ with } j \neq i$$

and adj $Q = (\det P)^{n-2} P$, we obtain

$$\begin{aligned}
\phi_P(\alpha E_{ii}) &= \psi((\text{adj } Q)\text{adj } (\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\alpha E_{jj})(\text{adj } \overline{Q}^t)) \\
&= \psi(\text{adj } (\overline{Q}^t(\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\alpha E_{jj})Q)) \\
&= \text{adj } \psi(\overline{Q}^t(\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\alpha E_{jj})Q) \\
&= \text{adj } (\psi(\vartheta\overline{Q}^t Q + \overline{Q}^t(\theta^{-1}\vartheta^{2-n}\alpha E_{jj})Q - \overline{Q}^t\vartheta E_{ii}Q) - \psi(\overline{Q}^t\vartheta E_{jj}Q)).
\end{aligned}$$

Thus, it follows from (4.3) and (4.4) that

$$\begin{aligned}
\phi_P(\alpha E_{ii}) &= \text{adj } (\psi(\vartheta \overline{Q}^t Q + \overline{Q}^t(\theta^{-1}\vartheta^{2-n}\alpha E_{jj})Q - \overline{Q}^t \vartheta E_{ii} Q)) \\
&= \text{adj } (\psi(\vartheta \overline{Q}^t Q + \overline{Q}^t(\theta^{-1}\vartheta^{2-n}\alpha E_{jj})Q) - \psi(\overline{Q}^t \vartheta E_{ii} Q)) \\
&= \text{adj } (\psi(\vartheta \overline{Q}^t Q + \overline{Q}^t(\theta^{-1}\vartheta^{2-n}\alpha E_{jj})Q)) \\
&= \text{adj } (\psi(\vartheta \overline{Q}^t Q) - \psi(-\overline{Q}^t(\theta^{-1}\vartheta^{2-n}\alpha E_{jj})Q)) \\
&= \text{adj } (-\psi(-\overline{Q}^t(\theta^{-1}\vartheta^{2-n}\alpha E_{jj})Q)) \\
&= 0
\end{aligned}$$

as rank $\psi(-\overline{Q}^t(\theta^{-1}\vartheta^{2-n}\alpha E_{jj})Q) \leqslant 1$ and $m \geqslant 3$. This implies

$$\text{adj } \phi_P(A + \alpha_1 E_{11} + \cdots + \alpha_n E_{nn}) = \text{adj } \phi_P(A) \tag{4.6}$$

for every $A \in \mathcal{H}_n(\mathbb{F})$ and $\alpha_1, \cdots, \alpha_n \in \mathbb{F}^-$. Since adj $(I_n - E_{ii} - E_{jj} + \alpha E_{jj}) = \alpha E_{ii}$, we have

$$\begin{aligned}
\psi(\overline{Q}^t(\alpha E_{ii})Q) &= \psi((\text{adj } \overline{P}^t)\text{adj } (I_n - E_{ii} - E_{jj} + \alpha E_{jj})(\text{adj } P)) \\
&= \psi(\text{adj } (P(I_n - E_{ii} - E_{jj} + \alpha E_{jj})\overline{P}^t)) \\
&= \text{adj } \psi(P(I_n - E_{ii} - E_{jj} + \alpha E_{jj})\overline{P}^t) \\
&= \text{adj } \phi_P(I_n - E_{ii} - E_{jj} + \alpha E_{jj}).
\end{aligned}$$

So, (4.1) and (4.5) imply

$$\begin{aligned}
\psi(\overline{Q}^t(\alpha E_{ii})Q) &= \text{adj } (\phi_P(I_n - E_{ii} - E_{jj}) - \phi_P(-\alpha E_{jj})) \\
&= \text{adj } (\phi_P(I_n - E_{ii} - E_{jj})) \\
&= \text{adj } (\phi_P(I_n - E_{ii}) - \phi_P(E_{jj})) \\
&= \text{adj } (\phi_P(I_n) - \phi_P(E_{ii})) \\
&= \text{adj } \phi_P(I_n).
\end{aligned}$$

Again, by applying (4.1) and (4.5) repeatedly,

$$\begin{aligned}\psi(\overline{Q}^t(\alpha E_{ii})Q) &= \text{adj }(\phi_P(E_{11}+E_{22}+\cdots+E_{n-1,n-1})-\phi(-E_{nn}))\\ &= \text{adj }\phi_P(E_{11}+E_{22}+\cdots+E_{n-1,n-1})\\ &\ \ \vdots\\ &= \text{adj }\phi_P(E_{11}).\end{aligned}$$

Therefore,

$$\psi(\overline{Q}^t(\alpha E_{ii})Q) = 0 \text{ for every } \alpha \in \mathbb{F}^- \text{ and for every } i = 1,\cdots,n. \tag{4.7}$$

It follows that

$$\text{adj }(\psi(A) - \psi(\overline{Q}^t(\alpha_1 E_{11}+\cdots+\alpha_n E_{nn})Q)) = \text{adj }\psi(A) \tag{4.8}$$

for every $A \in \mathcal{H}_n(\mathbb{F})$ and $\alpha_1,\cdots,\alpha_n \in \mathbb{F}^-$. Let i,j,k be distinct integers with $1 \leqslant i,j,k \leqslant n$. Let

$$Y_{ijk} := I_n - E_{ii} - E_{jj} - 2E_{kk}.$$

Let $a \in \mathbb{F}^-$ be nonzero. Then $\overline{a}a \in \mathbb{F}^-$ and

$$\text{adj }(aE_{ij}+\overline{a}E_{ji}+Y_{ijk}) = aE_{ij}+\overline{a}E_{ji}+\overline{a}aY_{ijk}$$

implies

$$\begin{aligned}&\psi(\overline{Q}^t(aE_{ij}+\overline{a}E_{ji}+\overline{a}aY_{ijk})Q)\\ &= \psi((\text{adj }\overline{P}^t)\text{adj }(aE_{ij}+\overline{a}E_{ji}+Y_{ijk})(\text{adj }P))\\ &= \text{adj }\psi(P(aE_{ij}+\overline{a}E_{ji}+Y_{ijk})\overline{P}^t)\\ &= \text{adj }\phi_P(aE_{ij}+\overline{a}E_{ji}+Y_{ijk}).\end{aligned}$$

Then by (4.5), we have

$$\begin{aligned}&\psi(\overline{Q}^t(aE_{ij}+\overline{a}E_{ji}+\overline{a}aY_{ijk})Q)\\ &= \text{adj }\phi_P(aE_{ij}+\overline{a}E_{ji}+Y_{ijk}-E_{ss}-\phi_P(-E_{ss})) \text{ for } s \neq i,j\\ &= \text{adj }\phi_P(aE_{ij}+\overline{a}E_{ji}+Y_{ijk}-E_{ss})\end{aligned}$$

By using similar argument repeatedly, we obtain

$$\begin{aligned}
\psi(\overline{Q}^t(aE_{ij}+\overline{a}E_{ji}+\overline{a}aY_{ijk})Q) &= \text{adj } \phi_P(aE_{ij}+\overline{a}E_{ji}-E_{kk}) \\
&= \text{adj } (\phi_P(aE_{ij}+\overline{a}E_{ji})-\phi_P(E_{kk})) \\
&= \text{adj } \phi_P(aE_{ij}+\overline{a}E_{ji}) \\
&= \text{adj } \psi(P(aE_{ij}+\overline{a}E_{ji})\overline{P}^t) \\
&= \psi(\text{adj } (P(aE_{ij}+\overline{a}E_{ji})\overline{P}^t)) \\
&= \psi((\text{adj } \overline{P}^t)\text{adj } (aE_{ij}+\overline{a}E_{ji})(\text{adj } P)) \\
&= \psi(\overline{Q}^t\mathcal{E}Q)
\end{aligned}$$

where $\mathcal{E}=-a\overline{a}E_{kk}$ if $n=3$, or $\mathcal{E}=0$ if $n>3$. Thus

$$\psi(\overline{Q}^t(aE_{ij}+\overline{a}E_{ji}+\overline{a}aY_{ijk})Q)=0 \tag{4.9}$$

for all distinct integers $1\leqslant i,j,k\leqslant n$ and scalar $a\in\mathbb{F}^-$. Next, we claim that

$$\phi_P(A)=0 \text{ for every rank one matrix } A\in\mathcal{H}_n(\mathbb{F}).$$

Let $A\in\mathcal{H}_n(\mathbb{F})$ be of rank one. Then by Proposition 1.4.6, there exists a matrix $B=(b_{ij})\in\mathcal{H}_n(\mathbb{F})$ of rank $n-1$ such that $\theta^{-1}A=\text{adj } B$. Thus,

$$\begin{aligned}
\phi_P(A) &= \psi(PA\overline{P}^t) \\
&= \psi(\theta P(\theta^{-1}A)\overline{P}^t) \\
&= \psi(\det(P\overline{P}^t)^{n-2})P(\text{adj } B)\overline{P}^t)
\end{aligned}$$

by substituting $\theta=\det(P\overline{P}^t)^{n-2}$. Then we have

$$\begin{aligned}
\phi_P(A) &= \psi((\det P)^{n-2}P(\text{adj } B)(\det\overline{P}^t)^{n-2}\overline{P}^t) \\
&= \psi((\text{adj } Q)(\text{adj } B)(\text{adj } \overline{Q}^t)) \\
&= \psi(\text{adj } (\overline{Q}^tBQ)) \\
&= \text{adj } \psi(\overline{Q}^tBQ).
\end{aligned}$$

Since $B \in \mathcal{H}_n(\mathbb{F})$, $b_{ij} = \overline{b_{ji}}$ for all $1 \leqslant i < j \leqslant n$, and $b_{ii} \in \mathbb{F}^-$ for every $1 \leqslant i \leqslant n$.

Then we obtain

$$\begin{aligned}\phi_P(A) &= \text{adj } \psi\left(\sum_{1\leqslant i<j\leqslant n} \overline{Q}^t(b_{ij}E_{ij} + \overline{b_{ji}}E_{ji})Q + \sum_{i=1}^{n} \overline{Q}^t(b_{ii}E_{ii})Q\right) \\ &= \text{adj } \psi\left(\sum_{1\leqslant i<j\leqslant n} \overline{Q}^t(b_{ij}E_{ij} + \overline{b_{ji}}E_{ji})Q\right)\end{aligned}$$

by (4.8). Thus,

$$\phi_P(A) = \text{adj } \psi\left(\sum_{\substack{1\leqslant i<j\leqslant n,\\ i\neq 1 \text{and} j\neq 2}} \overline{Q}^t[(b_{ji}E_{ji} + \overline{b_{ji}}E_{ij}) + \overline{a}aY_{12k} - (aE_{21} + \overline{a}E_{12} + \overline{a}aY_{12k})]Q\right)$$

and it follows from (4.9) that

$$\phi_P(A) = \text{adj } \psi\left(\sum_{\substack{1\leqslant i<j\leqslant n,\\ i\neq 1 \text{and} j\neq 2}} \overline{Q}^t[(b_{ji}E_{ji} + \overline{b_{ji}}E_{ij}) + \overline{a}aY_{12k}]Q\right).$$

By letting $a = b_{21}$, we obtain

$$\phi_P(A) = \text{adj } \psi\left(\sum_{\substack{1\leqslant i<j\leqslant n,\\ i\neq 1 \text{and} j\neq 2}} \overline{Q}^t(b_{ji}E_{ji} + \overline{b_{ji}}E_{ij})Q + \overline{Q}^t(\overline{b_{21}}b_{21}Y_{12k})Q\right).$$

Thus,

$$\begin{aligned}\phi_P(A) &= \text{adj } \left(\psi\left(\sum_{\substack{1\leqslant i<j\leqslant n,\\ i\neq 1 \text{and} j\neq 2}} \overline{Q}^t(b_{ji}E_{ji} + \overline{b_{ji}}E_{ij})Q\right) - \psi(-\overline{Q}^t(\overline{b_{21}}b_{21}Y_{12k})Q)\right) \\ &= \text{adj } \psi\left(\sum_{\substack{1\leqslant i<j\leqslant n,\\ i\neq 1 \text{and} j\neq 2}} \overline{Q}^t(b_{ji}E_{ji} + \overline{b_{ji}}E_{ij})Q\right)\end{aligned}$$

by (4.8). Continuing using similar arguments, we obtain

$$\begin{aligned}\phi_P(A) &= \text{adj } \psi\left(\sum_{\substack{1\leqslant i<j\leqslant n,\\ i\neq 1 \text{and} j\neq 2,3}} \overline{Q}^t(b_{ji}E_{ji} + \overline{b_{ji}}E_{ij})Q\right) \\ &\vdots \\ &= \text{adj } \psi(\overline{Q}^t(b_{n,n-1}E_{n,n-1} + \overline{b_{n,n-1}}E_{n-1,n})Q).\end{aligned}$$

Let $b = b_{n,n-1}$. Then

$$\begin{aligned}
&\phi_P(A)\\
&= \text{adj } \psi(\overline{Q}^t(\bar{b}bY_{n-1,n,n-2} - ((-b)E_{n,n-1} + \overline{(-b)}E_{n-1,n} + \overline{(-b)}(-b)Y_{n-1,n,n-2}))Q)\\
&= \text{adj } \psi(\overline{Q}^t(\bar{b}bY_{n-1,n,n-2})Q)\\
&= \text{adj } \psi(0 - (-\overline{Q}^t(\bar{b}bY_{n-1,n,n-2})Q))\\
&= \text{adj } (\psi(0) - \psi(-\overline{Q}^t(\bar{b}bY_{n-1,n,n-2})Q))\\
&= \text{adj } \psi(0)\\
&= 0.
\end{aligned}$$

Therefore, $\phi_P(A) = 0$ for every rank one matrix $A \in \mathcal{H}_n(\mathbb{F})$.

It is clear that adj $\phi_P(A) = 0$ if $A = 0$. Let $A \in \mathcal{H}_n(\mathbb{F})$ be of rank r with $1 \leqslant r \leqslant n$. Then by Lemma 2.2.1, there exist rank one matrices $A_1, \cdots, A_s \in \mathcal{H}_n(\mathbb{F})$ with $r \leqslant s \leqslant r+1$ such that $A = A_1 + \cdots + A_s$. It follows from (4.1) that

$$\begin{aligned}
\text{adj } \phi_P(A) &= \text{adj } \phi_P(A_1 + \cdots + A_s)\\
&= \text{adj } (\phi_P(A_1 + \cdots + A_{s-1}) - \phi(-A_s))\\
&= \text{adj } \phi_P(A_1 + \cdots + A_{s-1}).
\end{aligned}$$

By using (4.1) repeatedly, we have

$$\text{adj } \phi_P(A) = \text{adj } \phi_P(A_1) = 0.$$

In conclusion, rank $\phi_P(A) \leqslant m-2$ for every $A \in \mathcal{H}_n(\mathbb{F})$. □

Lemma 4.2.3. *Let n be an integer with $n \geqslant 3$. Let $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_n(\mathbb{K})$ be defined by*

$$\psi(A) = \lambda Q A^\sigma \widehat{Q}^t \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a nonzero field homomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$, $Q \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix and $\lambda \in \mathbb{K}^\wedge$ is a nonzero *scalar. If* adj $\psi(I_n) = \psi(I_n)$*, then there exists a nonzero scalar* $\zeta \in \mathbb{K}^\wedge$ *such that*

$$\widehat{Q}^t Q = \zeta I_n \ \textit{and}\ (\lambda\zeta)^{n-2} = 1.$$

Proof. Since adj $\psi(I_n) = \psi(I_n)$, we obtain adj $(\lambda Q\widehat{Q}^t) = \lambda Q\widehat{Q}^t$ which implies $\lambda^{n-1}(\text{adj }\widehat{Q}^t)(\text{adj }Q) = \lambda Q\widehat{Q}^t$. Then

$$Q\widehat{Q}^t = \lambda^{n-2}(\text{adj }\widehat{Q}^t)(\text{adj }Q)$$

and hence

$$\begin{aligned}(\widehat{Q}^t Q)^2 &= \widehat{Q}^t(Q\widehat{Q}^t)Q = \widehat{Q}^t(\lambda^{n-2}(\text{adj }\widehat{Q}^t)(\text{adj }Q))Q \\ &= \lambda^{n-2}\widehat{Q}^t(\text{adj }\widehat{Q}^t)(\text{adj }Q)Q = \lambda^{n-2}\det(\widehat{Q}^t Q)I_n.\end{aligned}$$

Thus

$$(\widehat{Q}^t Q)^2 = \lambda^{n-2}\det(\widehat{Q}^t Q)I_n. \tag{4.10}$$

Let $1 \leqslant i < j \leqslant n$. Since adj $(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = -(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})$, we obtain

$$\begin{aligned}\text{adj }\psi(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) &= \psi(\text{adj }(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})) \\ &= \psi(-(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})).\end{aligned}$$

It follows that

$$\text{adj }(\lambda Q(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t) = -\lambda Q(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t$$

and hence

$$\begin{aligned}&\lambda^{n-1}(\text{adj }\widehat{Q}^t)\text{adj }(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\text{adj }Q) \\ =\ & -\lambda Q(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t\end{aligned}$$

By computing

$$\lambda^{n-2}(\text{adj }\widehat{Q}^t)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\text{adj }Q) = Q(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t$$

$$\Rightarrow \lambda^{n-2}\widehat{Q}^t(\text{adj }\widehat{Q}^t)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\text{adj }Q)Q = \widehat{Q}^t Q(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t Q$$

$\Rightarrow \lambda^{n-2}\det(\widehat{Q}^tQ)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = \widehat{Q}^tQ(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^tQ.$

By (4.10), we have

$(\widehat{Q}^tQ)(\widehat{Q}^tQ)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = (\widehat{Q}^tQ)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\widehat{Q}^tQ)$

which implies

$$(\widehat{Q}^tQ)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\widehat{Q}^tQ)$$

for all $1 \leqslant i < j \leqslant n$. Hence $\widehat{Q}^tQ = \zeta I_n$ and also $Q\widehat{Q}^t = \zeta I_n$ for some nonzero scalar $\zeta \in \mathbb{K}^{\wedge}$ since $\widehat{Q}^tQ \in \mathcal{H}_n(\mathbb{K})$. Moreover, adj $(\lambda\zeta I_n) =$ adj $(\lambda Q\widehat{Q}^t)$ implies

$$\begin{aligned} \lambda^{n-1}\zeta^{n-1}I_n &= \text{adj } \psi(I_n) = \psi(I_n) \\ &= \lambda Q\widehat{Q}^t = \lambda\zeta I_n. \end{aligned}$$

Therefore, $(\lambda\zeta)^{n-2} = 1$. □

Lemma 4.2.4. *Let m, n be integers with $m, n \geqslant 3$. Let $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying* (AH2)*. Then the following statements are equivalent.*

(a) $\psi(I_n) = 0$.

(b) $\psi(A) = 0$ *for every rank one matrix* $A \in \mathcal{H}_n(\mathbb{F})$.

(c) rank $\psi(A) \leqslant m - 2$ *for every* $A \in \mathcal{H}_n(\mathbb{F})$.

(d) $\psi(\text{adj } A) = 0$ *for every* $A \in \mathcal{H}_n(\mathbb{F})$.

Proof. By letting $P = I_n$ in Lemma 4.2.2, we have $\psi = \phi_P$. Then we obtain (a) $\implies$ (b) $\implies$ (c). $\psi(I_n) = \psi(\text{adj } I_n) = 0$ shows that (d) $\implies$ (a).

We now show (c) $\implies$ (d). Let $A \in \mathcal{H}_n(\mathbb{F})$. Since rank $\psi(A) \leqslant m - 2$, $\psi(\text{adj } A) = \text{adj } (\psi(A)) = 0$. □

Lemma 4.2.5. *Let m, n be integers with $m, n \geqslant 3$. Let $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying* (AH2). *Let $P \in \mathcal{H}_n(\mathbb{F})$ be an arbitrarily fixed invertible matrix. Let $\phi_P : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be defined by*

$$\phi_P(A) = \psi(PA\overline{P}^t) \text{ for every } A \in \mathcal{H}_n(\mathbb{F}). \tag{4.11}$$

If rank $\phi_P(I_n) = m$, *then* rank $\phi_P(\alpha E_{ii}) = 1$ *for all integers $1 \leqslant i \leqslant n$ and nonzero scalars $\alpha \in \mathbb{F}^-$.*

Proof. We let $Q := \text{adj } P$. Then

$$\psi(\overline{Q}^t Q) = \psi(\text{adj } \overline{P}^t \text{adj } P) = \psi(\text{adj } (P\overline{P}^t)) = \text{adj } \psi(P\overline{P}^t) = \text{adj } \phi_P(I_n)$$

which implies rank $\psi(\overline{Q}^t Q) = m$. Thus,

$$\text{rank } (\text{adj } \psi(\overline{Q}^t Q)) = m. \tag{4.12}$$

We claim that

$$\text{rank } \phi(E_{ii}) = 1 \text{ for every } i = 1, \cdots, n.$$

By (4.1),

$$\begin{aligned} \text{rank adj } \left(\phi_P(E_{ii}) - \phi_P \left(\sum_{j=1, j\neq i}^{n} -E_{jj} \right) \right) &= \text{rank adj } \phi_P \left(E_{ii} - \left(\sum_{j=1, j\neq i}^{n} -E_{jj} \right) \right) \\ &= \text{rank adj } \phi_P(I_n) \\ &= m. \end{aligned}$$

This implies rank $\left(\phi_P(E_{ii}) - \phi_P \left(\sum_{j=1, j\neq i}^{n} -E_{jj} \right) \right) = m$ and hence

$$\text{rank } (\phi_P(E_{ii})) + \text{rank } \left(\phi_P \left(\sum_{j=1, j\neq i} -E_{jj} \right) \right) \geqslant m.$$

In addition, by the definition of ϕ_P, (4.13), Lemma 2.2.4(a), (b) and (c) hold for ϕ_P as well. It follows that rank $\phi_P(E_{ii}) \leqslant 1$ and rank $\phi \left(\sum_{j=1, j\neq i}^{n} -E_{jj} \right) \leqslant m-1$. Therefore, rank $\phi_P(E_{ii}) = 1$.

By Lemma 2.2.4(a), rank $\phi_P(\alpha E_{ii}) = $ rank $\psi(\alpha E_{ii}) \leqslant 1$ for every $1 \leqslant i \leqslant n$ and nonzero scalar $\alpha \in \mathbb{F}^-$. Suppose there exist $1 \leqslant i_0 \leqslant n$ and a nonzero scalar $\alpha_0 \in \mathbb{F}^-$ such that $\phi_P(\alpha_0 E_{i_0 i_0}) = 0$. As $n \geqslant 3$, we can choose two distinct integers $1 \leqslant s < t \leqslant n$ with $s, t \neq i_0$ such that

$$-E_{ss} = \text{adj } (I_n - E_{ss} - (1+\alpha_0)E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt}).$$

Then we have

$$\begin{aligned}\psi(\overline{Q}^t(-E_{ss})Q) &= \psi(\text{adj } (P(I_n - E_{ss} - (1+\alpha_0)E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt})\overline{P}^t)) \\ &= \text{adj } \psi(P(I_n - E_{ss} - (1+\alpha_0)E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt})\overline{P}^t)) \\ &= \text{adj } \phi_P(I_n - E_{ss} - (1+\alpha_0)E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt}).\end{aligned}$$

By (4.1),

$$\begin{aligned}\psi(\overline{Q}^t(-E_{ss})Q) &= \text{adj } \phi_P(I_n - E_{ss} - E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt} - (\alpha_0 E_{i_0 i_0})) \\ &= \text{adj } (\phi_P(I_n - E_{ss} - E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt}) - \phi_P(\alpha_0 E_{i_0 i_0}))\end{aligned}$$

and hence

$$\begin{aligned}\psi(\overline{Q}^t(-E_{ss})Q) &= \text{adj } \phi_P(I_n - E_{ss} - E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt}) \\ &= \text{adj } \psi(P(I_n - E_{ss} - E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt})\overline{P}^t) \\ &= \psi(\text{adj } (P(I_n - E_{ss} - E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt})\overline{P}^t)) \\ &= \psi((\text{adj } \overline{P}^t)(\text{adj } (I_n - E_{ss} - E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt}))(\text{adj } P)) \\ &= \psi(\overline{Q}^t(\text{adj } (I_n - E_{ss} - E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt}))Q).\end{aligned}$$

Since adj $(I_n - E_{ss} - E_{i_0 i_0} - (1-\alpha_0^{-1})E_{tt}) = 0$, we obtain $\psi(\overline{Q}^t(-E_{ss})Q) = 0$. Next, we compute

$$\begin{aligned}\text{adj } \psi(\overline{Q}^t Q) &= \text{adj } \psi(\overline{Q}^t((I_n - E_{ss} - E_{tt}) + E_{ss} + E_{tt})Q) \\ &= \text{adj } \psi(\overline{Q}^t((I_n - E_{ss} - E_{tt}) + E_{ss})Q - \overline{Q}^t(-E_{tt})Q) \\ &= \text{adj } (\psi(\overline{Q}^t((I_n - E_{ss} - E_{tt}) + E_{ss})Q) - \psi(\overline{Q}^t(-E_{tt})Q)) \\ &= \text{adj } \psi(\overline{Q}^t((I_n - E_{ss} - E_{tt}) + E_{ss})Q)\end{aligned}$$

and hence

$$\begin{aligned}\text{adj } \psi(\overline{Q}^t Q) &= \text{adj } (\psi(\overline{Q}^t (I_n - E_{ss} - E_{tt})Q) - \psi(\overline{Q}^t(-E_{ss})Q)) \\ &= \text{adj } \psi(\overline{Q}^t(I_n - E_{ss} - E_{tt})Q) \\ &= \psi((\text{adj } Q)(\text{adj } (I_n - E_{ss} - E_{tt}))(\text{adj } \overline{Q}^t)).\end{aligned}$$

Since adj $(I_n - E_{ss} - E_{tt}) = 0$, adj $\psi(\overline{Q}^t Q) = 0$ which contradicts with (4.12). We are done $\square$

Lemma 4.2.6. *Let m, n be integers with $m, n \geqslant 3$. Let $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying* (AH2). *If $\psi(I_n) \neq 0$, then ψ is injective and*

$$\text{rank } (A - B) = n \iff \text{rank } (\psi(A) - \psi(B)) = m$$

for all $A, B \in \mathcal{H}_n(\mathbb{F})$.

Proof. Let $A \in \mathcal{H}_n(\mathbb{F})$ be of rank one. Then by Proposition 1.3.1, there exist an invertible matrix $P \in \mathcal{H}_n(\mathbb{F})$ and a nonzero scalar $a \in \mathbb{F}^-$ such that $A = P(aE_{11})\overline{P}^t$. Let $\phi_P : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be defined by

$$\phi_P(A) = \psi(PA\overline{P}^t) \text{ for every } A \in \mathcal{H}_n(\mathbb{F}). \tag{4.13}$$

Since adj $\psi(I_n) = \psi(I_n)$ and $\psi(I_n) \neq 0$, it follows that rank $\psi(I_n) = m$ and hence rank $\phi_P((\overline{P}^t P)^{-1}) = m$ as $\phi_P((\overline{P}^t P)^{-1}) = \psi(P(\overline{P}^t P)^{-1}\overline{P}^t) = \psi(I_n)$. This implies rank $\phi_P(I_n) = m$ by Lemma 4.2.2; otherwise, rank $\phi_P(A) \leqslant m - 2$ for every $A \in \mathcal{H}_n(\mathbb{F})$ which contradicts with rank $\phi_P((\overline{P}^t P)^{-1}) = m$. Thus, rank adj $\phi_P(I_n) = m$. It follows from Lemma 4.2.5 that rank $\phi_P(aE_{ii}) = 1$ for all integers $1 \leqslant i \leqslant n$ and nonzero scalars $a \in \mathbb{F}^-$. Hence,

$$\text{rank } \psi(A) = \text{rank } \psi(P(aE_{11})\overline{P}^t) = \text{rank } \phi_P(aE_{11}) = 1$$

Let $A, B \in \mathcal{H}_n(\mathbb{F})$ such that $\psi(A) = \psi(B)$. Suppose $A - B \neq 0$. Then by Lemma 4.2.1, there exists a matrix $C \in \mathcal{H}_n(\mathbb{F})$ of rank at most $n - 2$ such that rank $(A - B + C) = n - 1$. Hence, rank adj $(A - B + C) = 1$. This implies rank $\psi(\text{adj } (A - B + C)) = 1$ by Lemma 4.2.5. On the other hand,

$$\begin{aligned}
\psi(\text{adj } (A - B + C)) &= \text{adj } (\psi(A + C) - \psi(B)) \\
&= \text{adj } (\psi(A + C) - \psi(A)) \\
&= \text{adj } \psi(C) \\
&= 0
\end{aligned}$$

which is a contradiction. Therefore $A = B$ implies ψ is injective.

Let $A, B \in \mathcal{H}_n(\mathbb{F})$. As ψ is injective, by Lemma 2.2.5, we have

$$\begin{aligned}
\text{rank } (A - B) = n &\iff \text{rank } \psi(A - B) = m \\
&\iff \text{rank adj } \psi(A - B) = m \\
&\iff \text{rank adj } (\psi(A) - \psi(B)) = m \\
&\iff \text{rank } (\psi(A) - \psi(B)) = m.
\end{aligned}$$

□

4.3 Some examples

If ψ satisfies condition (AH1) or (AH2), we have adj $\psi(I_n) = \psi(I_n)$. Thus, $\psi(I_n)$ is either zero or invertible. If $\psi(I_n) = 0$, ψ sends all rank one matrices to zero by Lemma 4.2.4. By referring to Theorem 4.4.2 and Theorem 4.5.2, the condition $\psi(I_n) \neq 0$ is indispensable as there are some mappings ψ satisfying condition (AH1) or (AH2) which are nonzero and send all rank one matrices to zero. Thus, we give some examples of such mappings in this section.

Let m, n be integers with $m, n \geqslant 3$, and let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively.

Example 4.3.1. Let $\tau : \mathbb{F}^- \to \mathbb{K}^\wedge$ be a nonzero function. We define the mapping $\psi_1 : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ by

$$\psi_1(A) = \begin{cases} \tau(a_{11}) \sum_{i=1}^{m-2} E_{ii} & \text{if } A = (a_{ij}) \in \mathcal{H}_n(\mathbb{F}) \text{ is of rank } r \text{ with } 1 < r < n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.3.2. Let $m, n \geqslant 4$. Let $f : \mathcal{H}_n(\mathbb{F}) \to \mathbb{K}^\wedge$ be a nonzero function and let $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ be a nonzero field homomorphism such that $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$. Let $\psi_2 : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be the mapping defined by

$$\psi_2(A) = \begin{cases} f(A)E_{11} & \text{if rank } A = 2, \\ \sigma(a_{12})E_{12} + \sigma(a_{21})E_{21} & \text{if } A = (a_{ij}) \in \mathcal{H}_n(\mathbb{F}) \text{ is of rank } r, 2 < r < n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.3.3. Let $m, n \geqslant 5$. Let $\tau : \mathbb{F}^- \to \mathbb{K}^\wedge$ be a nonzero function and let $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ be a nonzero field homomorphism such that $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$. Let $A = (a_{ij}) \in \mathcal{H}_n(\mathbb{F})$ and let $\psi_3 : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be defined by

$$\psi_3(A) = \begin{cases} \tau(a_{11})E_{11} + \tau(a_{22})E_{22} & \text{if rank } A = r, 1 < r < n, r \text{ is odd} \\ \sigma(a_{12})E_{12} + \sigma(a_{21})E_{21} + \tau(a_{33})E_{33} & \text{if rank } A = r, 1 < r < n, r \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.3.4. Let $m \geqslant n + 2$ and let $\mathcal{E} = \{\text{adj } A : A \text{ is invertible}\}$. Let $g : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a nonzero mapping and let $\psi_4 : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be defined by

$$\psi_4(A) = \begin{cases} 0 & \text{if } A \in \mathcal{H}_n(\mathbb{F}) \text{ is of rank 0 or 1, or } A \in \mathcal{E}, \\ g(A) \oplus 0_{m-n} & \text{otherwise.} \end{cases}$$

It can be verified that each ψ_i for $i = 1, 2, 3, 4$ satisfies condition (AH1) or (AH2) with $\psi_i(I_n) = 0$. These mappings are neither injective nor surjective.

4.4 Characterisation of classical adjoint-commuting mappings on hermitian matrices

Let m, n be integers with $m, n \geqslant 3$. Let $\mathbb{F}$ be a field which possesses an involution $^-$ of $\mathbb{F}$. We observe that if a mapping $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{F})$ satisfies condition (AH1), then it satisfies condition (AH2). Moreover, if $\psi(I_n) \neq 0$, then ψ is injective by Lemma 4.2.6. By using similar arguments as in the proof of Lemma 4.2.6, it can be shown that

$$\text{rank } (A + \alpha B) = n \iff \text{rank } (\psi(A) + \alpha\psi(B)) = m \tag{4.14}$$

for all $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$.

Theorem 4.4.1. *Let m, n be integers with $m, n \geqslant 3$. Let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively, and $^-$ is proper. Then $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ is a classical adjoint-commuting additive mapping if and only if either $\psi = 0$, or $m = n$ and*

$$\psi(A) = \lambda P A^\sigma \widehat{P}^t \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a nonzero field homomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$, A^σ is the matrix obtained from A by applying σ entrywise, $P \in \mathcal{M}_n(\mathbb{K})$ is invertible with $\widehat{P}^t P = \zeta I_n$, and $\lambda, \zeta \in \mathbb{K}^\wedge$ are scalars with $(\lambda\zeta)^{n-2} = 1$.

Proof. The sufficiency is obvious. We now prove the necessity. Since ψ is additive, it can be easily shown that ψ satisfies (AH2). In addition, $\psi(I_n) = 0$ or rank $\psi(I_n) = m$ as adj $\psi(I_n) = \psi(I_n)$.

Case I: $\psi(I_n) = 0$.

By Lemma 4.2.4, $\psi(A) = 0$ for every rank one matrix $A \in \mathcal{H}_n(\mathbb{F})$. Then it follows from Lemma 2.2.1 and the additivity of ψ that $\psi = 0$.

Case II: rank $\psi(I_n) = m$.

Then by Lemma 4.2.6, ψ is injective. Moreover, rank $\psi(A) \leqslant 1$ for every rank one matrix $A \in \mathcal{H}_n(\mathbb{F})$ by Lemma 2.2.4 (a). This implies that ψ preserves rank one matrices. Next, we suppose $n > m$. Note that

$$m = \text{rank } \psi(I_n) \leqslant \sum_{i=1}^{n} \text{rank } \psi(E_{ii}) = n$$

by the additivity of ψ. By [5, Theorem 2.1], there exist integers $1 \leqslant t_1 < \cdots < t_\ell \leqslant n$, with $m \leqslant \ell < n$ such that rank $\psi(E_{t_1 t_1} + \cdots + E_{t_\ell t_\ell}) = m$. Thus,

$$m = \text{rank adj } \psi(E_{t_1 t_1} + \cdots + E_{t_\ell t_\ell}) = \text{rank } \psi(\text{adj } (E_{t_1 t_1} + \cdots + E_{t_\ell t_\ell})) \leqslant 1$$

as $\ell < n$. This is a contradiction since $m \geqslant 3$. Thus $m = n$. By [23, Main Theorem, p.g.603] and [16, Theorem 2.1 and Remark 2.4], we have

$$\psi(A) = \lambda P A^\sigma \widehat{P}^t \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a nonzero field homomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$, $P \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix and $\lambda \in \mathbb{K}^\wedge$ is a nonzero scalar . Since adj $\psi(I_n) = \psi(I_n)$, it follows from Lemma 4.2.3 that

$$\widehat{P}^t P = \zeta I_n \text{ and } (\lambda\zeta)^{n-2} = 1.$$

$\square$

Theorem 4.4.2. *Let m, n be integers with $m, n \geqslant 3$ and let $\mathbb{F}$ be a field which possesses a proper involution $^-$ of $\mathbb{F}$ such that either $|\mathbb{F}^-| = 2$ or $|\mathbb{F}^-| > n + 1$.*

Then $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{F})$ *is a mapping satisfying* (AH1) *if and only if* $\psi(A) = 0$ *for every rank one matrix* $A \in \mathcal{H}_n(\mathbb{F})$ *and* rank $(\psi(A) + \alpha\psi(B)) \leqslant m - 2$ *for all* $A, B \in \mathcal{H}_n(\mathbb{F})$ *and* $\alpha \in \mathbb{F}^-$*; or* $m = n$ *and*

$$\psi(A) = \lambda P A \overline{P}^t \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $P \in \mathcal{M}_n(\mathbb{F})$ *is invertible with* $\overline{P}^t P = \zeta I_n$ *and* $\lambda, \zeta \in \mathbb{F}^-$ *are scalars with* $(\lambda\zeta)^{n-2} = 1$.

Proof. The sufficiency part is obvious. We now consider the necessity. If $\psi(I_n) = 0$, then by Lemma 4.2.4, $\psi(\text{adj } A) = 0$ for every $A \in \mathcal{H}_n(\mathbb{F})$. By the definition of ψ, this means

$$\text{adj } (\psi(A + \alpha B)) = \psi(\text{adj } (A + \alpha B)) = 0$$

for all $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. Therefore,

$$\text{rank } (\psi(A) + \alpha\psi(B)) \leqslant m - 2$$

for all $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$.

Next, we consider $\psi(I_n) \neq 0$. Then we have (4.14) and hence by Lemma 2.2.6, ψ is additive. In view of Theorem 4.4.1, the result is obtained immediately. □

Theorem 4.4.3. *Let* m, n *be integers with* $m, n \geqslant 3$. *Let* $\mathbb{F}$ *and* $\mathbb{K}$ *be fields which possess involutions* $^-$ *of* $\mathbb{F}$ *and* $^\wedge$ *of* $\mathbb{K}$*, respectively, such that* $|\mathbb{K}^\wedge| = 2$*, or* $|\mathbb{F}^-|, |\mathbb{K}^\wedge| > 3$*, and* $\mathbb{F}$ *and* $\mathbb{K}$ *are not of characteristic* 2 *if* $^-$ *and* $^\wedge$ *are the identity mappings. Then* $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ *is a surjective mapping satisfying* (AH2) *if and only if* $m = n$, $\mathbb{F}$ *and* $\mathbb{K}$ *are isomorphic, and*

$$\psi(A) = \lambda P A^\sigma \widehat{P}^t \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ *is a field isomorphism satisfying* $\widehat{\sigma(a)} = \sigma(\overline{a})$ *for every* $a \in \mathbb{F}$*,* A^σ *is the matrix obtained from* A *by applying* σ *entrywise,* $P \in \mathcal{M}_n(\mathbb{K})$ *is invertible with* $\widehat{P}^t P = \zeta I_n$*, and* $\lambda, \zeta \in \mathbb{K}^\wedge$ *are scalars with* $(\lambda\zeta)^{n-2} = 1$*.*

Proof. The sufficiency part is clear. We now consider the necessity part. If $\psi(I_n) = 0$, then rank $\psi(A) \leqslant m - 2$ for every $A \in \mathcal{H}_n(\mathbb{F})$ by Lemma 4.2.4 which contradicts that ψ is surjective. Thus $\psi(I_n) \neq 0$. Due to Lemma 4.2.6, ψ is injective and hence bijective. In addition,

$$\text{rank } (A - B) = n \iff \text{rank } (\psi(A) - \psi(B)) = m \text{ for all } A, B \in \mathcal{H}_n(\mathbb{F}).$$

Now, we consider two cases

Case I: $|\mathbb{K}^\wedge| = 2$.

Then rank $(A - B) = n \iff$ rank $(\psi(A) + \psi(B)) = m$ for all $A, B \in \mathcal{H}_n(\mathbb{F})$. Let $A, B \in \mathcal{H}_n(\mathbb{F})$ with rank $(A - B) = n$, then by Lemma 2.2.5,

$$\begin{aligned} \text{rank } \psi(A - B) &= \text{rank } (\psi(A) - \psi(B)) \\ &= \text{rank } (\psi(A) + \psi(B)) \\ &= m. \end{aligned}$$

Thus,

$$\psi(A - B)\text{adj } \psi(A - B) = \det \psi(A - B) I_m$$

and

$$(\psi(A) + \psi(B))\text{adj } (\psi(A) + \psi(B)) = \det(\psi(A) + \psi(B)) I_m.$$

It follows that

$$\frac{\psi(A - B)\text{adj } \psi(A - B)}{\det \psi(A - B)} = \frac{(\psi(A) + \psi(B))\text{adj } (\psi(A) + \psi(B))}{\det(\psi(A) + \psi(B))}.$$

Hence,

$$\frac{\psi(A - B)}{\det \psi(A - B)} = \frac{\psi(A) + \psi(B)}{\det(\psi(A) + \psi(B))}$$

since adj $(\psi(A)+\psi(B)) = \text{adj } (\psi(A)-\psi(B)) = \psi(\text{adj } (A-B)) = \text{adj } \psi(A-B)$. As $\det \psi(A-B) = \det(\psi(A)+\psi(B)) = 1$, we have

$$\psi(A-B) = \psi(A)+\psi(B) \text{ for all } A, B \in \mathcal{H}_n(\mathbb{F}) \text{ if rank } (A-B) = n.$$

By the injectivity of ψ and

$$\psi(-I_n) = \psi(0-I_n) = \psi(0)+\psi(I_n) = \psi(I_n),$$

we have $I_n = -I_n$ and hence $\mathbb{F}$ is of characteristic 2. Thus, $-A = A$ for every $A \in \mathcal{H}_n(\mathbb{F})$. This implies $A-B = A+B$ for all $A, B \in \mathcal{H}_n(\mathbb{F})$. Therefore,

$$\psi(A+B) = \psi(A)+\psi(B) \text{ for all } A, B \in \mathcal{H}_n(\mathbb{F}) \text{ if rank } (A-B) = n. \quad (4.15)$$

Next, we consider the case where rank $(A-B) < n$. By Lemma 2.2.2 (b), there exists a matrix $C \in \mathcal{H}_n(\mathbb{F})$ such that rank $(A+C) =$ rank $(A+B+C) = n$. Then by (4.15), $\psi(A+C) = \psi(A)+\psi(C)$ and

$$\psi(A+B)+\psi(C) = \psi(A+B+C) = \psi(A+C)+\psi(B) = \psi(A)+\psi(C)+\psi(B).$$

This implies

$$\psi(A+B) = \psi(A)+\psi(B) \text{ for all } A, B \in \mathcal{H}_n(\mathbb{F}).$$

by Theorem 4.4.1 and the bijectivity of ψ, the result is proved.

Case II: $|\mathbb{F}^-|, |\mathbb{K}^\wedge| > 3$, and $\mathbb{F}$ and $\mathbb{K}$ are not of characteristic 2 when $^-$ and $^\wedge$ are the identity mappings.

By [14, Theorem 3.6] and the fundamental theorem of the geometry of hermitian matrices, Theorem 1.5.4, we have $m = n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic and

$$\psi(A) = \lambda P A^\sigma \widehat{P}^t + R_0 \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^{-}) \to (\mathbb{K}, ^{\wedge})$ is a field isomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$, A^σ is the matrix obtained from A by applying σ entrywise, $P \in \mathcal{M}_n(\mathbb{K})$ is invertible, $R_0 \in \mathcal{H}_n(\mathbb{K})$ and $\lambda \in \mathbb{K}^\wedge$ is a nonzero scalar. As $\psi(0) = 0$, $R_0 = 0$. We also have adj $\psi(I_n) = \psi(\text{adj } I_n) = \psi(I_n)$. By Lemma 4.2.3, there exists a nonzero scalar $\zeta \in \mathbb{K}^\wedge$ such that

$$\widehat{P}^t P = \zeta I_n \text{ and} (\lambda\zeta)^{n-2} = 1.$$

We are done. □

4.5 Characterisation of classical adjoint-commuting mappings on symmetric matrices

Let $\mathbb{F}$ be a field which possesses an involution $^-$ of $\mathbb{F}$. If $^-$ is identity, then $\mathcal{H}_n(\mathbb{F}) = \mathcal{S}_n(\mathbb{F})$.

Theorem 4.5.1. *Let m, n be integers with $m, n \geqslant 3$ and let $\mathbb{F}$ and $\mathbb{K}$ be fields. Then $\psi : \mathcal{S}_n(\mathbb{F}) \to \mathcal{S}_m(\mathbb{K})$ is a classical adjoint-commuting additive mapping if and only if $\psi = 0$, or $m = n$ and*

$$\psi(A) = \lambda P A^\sigma P^t \textit{ for every } A \in \mathcal{S}_n(\mathbb{F})$$

where $\sigma : \mathbb{F} \to \mathbb{K}$ is a nonzero field homomorphism, A^σ is the matrix obtained from A by applying σ entrywise, $P \in \mathcal{M}_n(\mathbb{K})$ is invertible with $P^t P = \zeta I_n$, and $\lambda, \zeta \in \mathbb{K}$ are scalars with $(\lambda\zeta)^{n-2} = 1$.

Proof. The sufficiency part can be shown easily. We now show the necessity part. By using similar arguments as in Theorem 4.4.1, we can prove that either

$\psi = 0$, or $m = n$, ψ is injective and preserves rank one matrices. In addition, rank adj $\psi(I_n) = n$. By [15, Theorem 2.1], ψ is of the following forms:

(I) $\psi(A) = \lambda P A^\sigma P^t$ for every $A \in \mathcal{S}_n(\mathbb{F})$, or

(II) $\psi(A) = Q\rho(A)Q^t$ for every $A \in \mathcal{S}_n(\mathbb{F})$ if $n = 3$ and $\mathbb{F} = \mathbb{Z}_2 := \{0, 1\}$

where $\sigma : \mathbb{F} \to \mathbb{K}$ is a field homomorphism, $P \in \mathcal{M}_n(\mathbb{F})$ and $Q \in \mathcal{M}_3(\mathbb{F})$ are invertible, $\lambda \in \mathbb{K}$ is nonzero and $\rho : \mathcal{S}_3(\mathbb{Z}_2) \to \mathcal{S}_3(\mathbb{K})$ is an additive mapping preserving rank one matrices with rank $\rho(I_3) = 3$.

Case I.

Since adj $\psi(I_n) = \psi(\text{adj } I_n) = \psi(I_n)$, by Lemma 4.2.3, we have $P^tP = \zeta I_n$ and $(\lambda\zeta)^{n-2} = 1$, as desired.

Case II.

We observe that ρ is nonzero. Since ψ is additive, $\psi(A) = \psi(-A) = -\psi(A)$ for every $A \in \mathcal{S}_3(\mathbb{Z}_2)$. Thus, $\mathbb{K}$ is of characteristic 2. Let $A \in \mathcal{S}_3(\mathbb{Z}_2)$. From adj $(\psi(A)) = \psi(\text{adj } A)$, we have

$$\begin{aligned}
&\text{adj } (Q\rho(A)Q^t) = Q\rho(\text{adj } A)Q^t \\
\Longrightarrow\ &(\text{adj } Q^t)\text{adj } (\rho(A))(\text{adj } Q) = Q\rho(\text{adj } A)Q^t \\
\Longrightarrow\ &Q^{-1}(\text{adj } Q^t)\text{adj } (\rho(A))(\text{adj } Q)(Q^{-1})^t = \rho(\text{adj } A) \\
\Longrightarrow\ &\rho(\text{adj } A) = H\text{adj } (\rho(A))H^t
\end{aligned}$$

where $H = Q^{-1}(\text{adj } Q^t) \in \mathcal{M}_3(\mathbb{K})$. Since ψ satisfies (AH2) and ψ is injective, by Lemma 2.2.5 ,

$$\text{rank } A = 3 \iff \text{rank } \psi(A) = 3 \iff \text{rank } \rho(A) = 3.$$

So, rank $\rho(E_{ii} + E_{jj}) = 2$ for all $1 \leqslant i \neq j \leqslant 3$ as ρ preserves rank one matrices and rank $\rho(I_3) = 3$. Since rank $\rho(E_{11}) = 1$, by Proposition 1.3.1, there exist an invertible matrix $P_1 \in \mathcal{M}_3(\mathbb{K})$ and a nonzero scalar $\alpha_1 \in \mathbb{K}$ such that $\rho(E_{11}) = \alpha_1 P_1 E_{11} P_1^t$. Let

$$\rho(E_{22}) = P_1 \begin{pmatrix} u_1 & V_1 \\ V_1^t & U_1 \end{pmatrix} P_1^t$$

where $u_1 \in \mathbb{K}$, $V_1 \in \mathcal{M}_{1,2}(\mathbb{K})$ and $U_1 \in \mathcal{S}_2(\mathbb{K})$. If $U_1 = 0$, then $V_1 = 0$ since rank $\rho(E_{22}) = 1$ and hence rank $\rho(E_{11} + E_{22}) < 2$, a contradiction. Thus $U_1 \neq 0$. This implies rank $U_1 = 1$ as rank $\rho(E_{22}) = 1$. Again, by Proposition 1.3.1, there exist an invertible matrix $P_2 \in \mathcal{M}_2(\mathbb{K})$ and a nonzero scalar $\alpha_2 \in \mathbb{K}$ such that $U_1 = \alpha_2 P_2 E_{11} P_2^t$. Then we have

$$\begin{aligned}\rho(E_{22}) &= P_1 \begin{pmatrix} u_1 & V_1 \\ V_1^t & P_2 \begin{pmatrix} \alpha_2 & 0 \\ 0 & 0 \end{pmatrix} P_2^t \end{pmatrix} P_1^t \\ &= P_1 \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} u_1 & v_{11} & v_{12} \\ v_{11} & \alpha_2 & 0 \\ v_{12} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P_2^t \end{pmatrix} P_1^t\end{aligned}$$

where $V_1 = (v_{11}\ v_{12})$. As rank $\rho(E_{22}) = 1$, we have $v_{12} = 0$ and $u_1 = v_{11}^2 \alpha_2^{-1}$. Thus,

$$\begin{aligned}&\rho(E_{22}) \\ &= P_1 \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} 1 & v_{11}\alpha_2^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ v_{11}\alpha_2^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P_2^t \end{pmatrix} P_1^t \\ &= \alpha_2 P_3 E_{22} P_3^t\end{aligned}$$

where $P_3 = P_1 \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} 1 & v_{11}\alpha_2^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{K})$ is invertible. Let

$$\rho(E_{33}) = P_3 \begin{pmatrix} U_2 & V_2 \\ V_2^t & \alpha_3 \end{pmatrix} P_3^t$$

with $\alpha_3 \in \mathbb{K}$, $V_2 \in \mathcal{M}_{2,1}(\mathbb{K})$ and $U_2 \in \mathcal{S}_2(\mathbb{K})$. Since rank $\rho(E_{33}) = 1$ and rank $\rho(I_3) = 3$, we have $\alpha_3 \neq 0$ and hence $U_2 = \alpha_3^{-1} V_2 V_2^t$. Thus,

$$\begin{aligned}\rho(E_{33}) &= P_3 \begin{pmatrix} I_2 & \alpha_3^{-1} V_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ \alpha_3^{-1} V_2^t & 1 \end{pmatrix} P_3^t \\ &= \alpha_3 P_4 E_{33} P_4^t\end{aligned}$$

where $P_4 = P_3 \begin{pmatrix} I_2 & \alpha_3^{-1}V_2 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{K})$ is invertible. Then $\rho(E_{ii}) = \alpha_i P_4 E_{ii} P_4^t$ for $i = 1, 2, 3$ and this implies

$$\psi(E_{ii}) = Q\rho(E_{ii})Q^t = \alpha_i Q P_4 E_{ii} (QP_4)^t \text{ for } i = 1, 2, 3.$$

By letting $P = QP_4 \in \mathcal{M}_3(\mathbb{K})$, we obtain

$$\psi(E_{ii}) = \alpha_i P E_{ii} P^t \text{ for } i = 1, 2, 3.$$

Let i, j, k be three distinct integers with $1 \leqslant i, j, k \leqslant 3$. Then

$$\begin{aligned} & E_{ii} = \text{adj } (E_{jj} + E_{kk}) \\ \implies & \psi(E_{ii}) = \psi(\text{adj } (E_{jj} + E_{kk})) \\ \implies & \alpha_i P E_{ii} P^t = \text{adj } (\psi(E_{jj}) + \psi(E_{kk})) \\ \implies & P(\alpha_i E_{ii})P^t = \text{adj } (\alpha_j P E_{jj} P^t + \alpha_k P E_{kk} P^t). \end{aligned}$$

This implies

$$\begin{aligned} P^t P(\alpha_i E_{ii}) &= P^t \text{adj } (P(\alpha_j E_{jj} + \alpha_k E_{kk})P^t)(P^t)^{-1} \\ &= P^t(\text{adj } P^t)\text{adj } (\alpha_j E_{jj} + \alpha_k E_{kk})(\text{adj } P)(P^t)^{-1}. \end{aligned}$$

Since $P^t(\text{adj } P^t) = (\det P^t)I_3$, we obtain

$$\begin{aligned} P^t P(\alpha_i E_{ii}) &= (\det P^t)(\alpha_j \alpha_k E_{ii})(\text{adj } P)(P^t)^{-1} \\ &= (\alpha_j \alpha_k E_{ii})(\text{adj } P)(\det P^t)(P^t)^{-1} \\ &= (\alpha_j \alpha_k E_{ii})(\text{adj } P)(\text{adj } P^t) \\ &= (\alpha_j \alpha_k E_{ii})\text{adj } (P^t P). \end{aligned}$$

Hence, $P^t P = \text{diag } (\zeta_1, \zeta_2, \zeta_3)$ for some nonzero scalars $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{K}$. Thus, we get

$$\psi(E_{ii}) = \alpha_i P E_{ii} P^t = \alpha_i P E_{ii} (P^t P) P^{-1} = \lambda_i P E_{ii} P^{-1} \text{ for } i = 1, 2, 3$$

where $\lambda_i = \alpha_i \zeta_i \in \mathbb{K}$ is nonzero. We let $D_{ij} := E_{ij} + E_{ji} \in \mathcal{S}_3(\mathbb{Z}_2)$. Let $\psi(D_{ij}) = PA_{ij}P^{-1}$ with $A_{ij} = (a_{st}) \in \mathcal{S}_3(\mathbb{K})$. We now show that $\lambda_i = \lambda_j = \lambda_k = 1$

where i, j, k are distinct with $1 \leqslant i, j, k \leqslant 3$ and $A_{ij} = D_{ij}$. As adj $D_{ij} = E_{kk}$, we obtain adj $(\psi(D_{ij})) = \psi(\text{adj } D_{ij}) = \psi(E_{kk})$ is of rank one. This implies rank $\psi(D_{ij}) = 2$. Thus,

$$\psi(D_{ij})\psi(E_{kk}) = \psi(D_{ij})\text{adj } \psi(D_{ij}) = \det \psi(D_{ij}) I_3 = 0$$
$$\implies (PA_{ij}P^{-1})(\lambda_k PE_{kk}P^{-1}) = 0.$$

Therefore, we have $A_{ij}E_{kk} = E_{kk}A_{ij} = 0$ implies $a_{sk} = a_{ks} = 0$ for $s = 1, 2, 3$. Since rank $(D_{ij} + E_{ii} + E_{jj}) = 1$ implies rank $\psi(D_{ij} + E_{ii} + E_{jj}) = 1$, we get rank $(A_{ij} + \lambda_i E_{ii} + \lambda_j E_{jj}) = 1$. Thus,

$$(a_{ii} + \lambda_i)(a_{jj} + \lambda_j) = a_{ij}^2. \tag{4.16}$$

If $a_{ij} = 0$, we obtain $a_{ii} = -\lambda_i$ or $a_{jj} = -\lambda_j$ but not both as $A_{ij} \neq 0$. Suppose $a_{ii} = -\lambda_i$. Then

$$\psi(D_{ij} + E_{ii} + E_{kk}) = P(A_{ij} + \lambda_i E_{ii} + \lambda_k E_{kk})P^{-1} = P(a_{jj}E_{jj} + \lambda_k E_{kk})P^{-1}$$

implies

$$\text{rank } \psi(D_{ij} + E_{ii} + E_{kk}) = \text{rank } (a_{jj}E_{jj} + \lambda_k E_{kk}) < 3.$$

However, rank $\psi(D_{ij}+E_{ii}+E_{kk}) = 3 \iff$ rank $(D_{ij}+E_{ii}+E_{kk}) = 3$ by Lemma 2.2.5. So, that is a contradiction. Therefore $a_{ii} \neq -\lambda_i$. By similar arguments, $a_{jj} \neq -\lambda_j$. These imply $a_{ij} \neq 0$. Since adj $(D_{ij} + E_{kk}) = D_{ij} + E_{kk}$, we have

$$\text{adj } \psi(D_{ij} + E_{kk}) = \psi(\text{adj } (D_{ij} + E_{kk})) = \psi(D_{ij} + E_{kk})$$

and hence

$$\text{adj } (PA_{ij}P^{-1} + \lambda_k PE_{kk}P^{-1}) = PA_{ij}P^{-1} + \lambda_k PE_{kk}P^{-1}$$

which implies that

$$\text{adj } (A_{ij} + \lambda_k E_{kk}) = A_{ij} + \lambda_k E_{kk}.$$

Thus, $-\lambda_k a_{ij} = a_{ij}$, $a_{ii}\lambda_k = a_{jj}$ and

$$a_{ii}a_{jj} - a_{ij}^2 = \lambda_k. \tag{4.17}$$

These imply $\lambda_k = 1$ and $a_{ii} = a_{jj}$ as $\mathbb{K}$ is of characteristic 2 and $a_{ij} \neq 0$. Since adj $(\psi(E_{ii} + E_{kk})) = \psi(\text{adj }(E_{ii} + E_{kk})) = \psi(E_{jj})$, we have

$$\text{adj }(P(\lambda_i E_{ii} + \lambda_k E_{kk})P^{-1}) = P(\lambda_j E_{jj})P^{-1}$$

and hence $\lambda_i\lambda_k = \lambda_j$. Thus, $\lambda_i = \lambda_j$ as $\lambda_k = 1$. In addition, adj $(\psi(E_{ii} + E_{jj})) = \psi(\text{adj }(E_{ii} + E_{jj})) = \psi(E_{kk})$ implies

$$\text{adj }(P(\lambda_i E_{ii} + \lambda_j E_{jj})P^{-1}) = P(\lambda_k E_{kk})P^{-1}$$

and hence $\lambda_i\lambda_j = \lambda_k$. We obtain $\lambda_i = \lambda_j = 1$. Now, consider adj $(\psi(D_{ij}+E_{jj})) = \psi(\text{adj }(D_{ij}+E_{jj})) = \psi(E_{kk})$. We have adj $(P(A_{ij}+E_{jj})P^{-1}) = PE_{kk}P^{-1}$ implies adj $(A_{ij} + E_{jj}) = E_{kk}$. Thus,

$$a_{ii}(a_{jj} + 1) - a_{ij}^2 = 1. \tag{4.18}$$

Equations (4.17) and (4.18) imply $a_{ii} = a_{jj} = 0$ and hence $a_{ij} = 1$ as the characteristic of $\mathbb{K}$ is 2. Therefore $A_{ij} = D_{ij}$. So, $\psi(A) = PAP^{-1}$ for every $A \in \mathcal{S}_3(\mathbb{Z}_2)$. Since $(PAP^{-1})^t = PAP^{-1}$, we have $P^tPA = AP^tP$ for every $A \in \mathcal{S}_3(\mathbb{Z}_2)$. This implies that there exists a nonzero scalar $\zeta \in \mathbb{K}$ such that $P^tP = \zeta^{-1}I_3$. Thus, we conclude that

$$\psi(A) = \zeta PAP^t \text{ for every } A \in \mathcal{S}_3(\mathbb{Z}_2).$$

$\square$

By letting $^-$ and $^\wedge$ be identity, we obtain Theorem 4.5.2 from Theorem 4.4.2 and Theorem 4.5.3 from Theorem 4.4.3.

Theorem 4.5.2. *Let m, n be integers with $m, n \geqslant 3$. Let $\mathbb{F}$ be a field with either $|\mathbb{F}| = 2$ or $|\mathbb{F}| > n+1$. Then $\psi : \mathcal{S}_n(\mathbb{F}) \to \mathcal{S}_m(\mathbb{F})$ is a mapping satisfying* (AH1) *if and only if either $\psi(A) = 0$ for every rank one matrix $A \in \mathcal{S}_n(\mathbb{F})$ and* rank $(\psi(A) + \alpha\psi(B)) \leqslant m-2$ *for all $A, B \in \mathcal{S}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$; or $m = n$ and*

$$\psi(A) = \lambda PAP^t \text{ for every } A \in \mathcal{S}_n(\mathbb{F})$$

where $P \in \mathcal{M}_n(\mathbb{F})$ is invertible with $P^tP = \zeta I_n$ and $\lambda, \zeta \in \mathbb{F}$ are scalars with $(\lambda\zeta)^{n-2} = 1$.

Theorem 4.5.3. *Let m, n be integers with $m, n \geqslant 3$. Let $\mathbb{F}$ and $\mathbb{K}$ be fields with $|\mathbb{K}| = 2$, or $|\mathbb{F}|, |\mathbb{K}| > 3$ and $\mathbb{F}$ and $\mathbb{K}$ are not of characteristic 2. Then $\psi : \mathcal{S}_n(\mathbb{F}) \to \mathcal{S}_m(\mathbb{K})$ is a surjective mapping satisfying* (AH2) *if and only if $m = n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic, and*

$$\psi(A) = \lambda PA^\sigma P^t \text{ for every } A \in \mathcal{S}_n(\mathbb{F})$$

where $\sigma : \mathbb{F} \to \mathbb{K}$ is a field isomorphism, A^σ is the matrix obtained from A by applying σ entrywise, $P \in \mathcal{M}_n(\mathbb{K})$ is invertible with $P^tP = \zeta I_n$, and $\lambda, \zeta \in \mathbb{K}$ are scalars with $(\lambda\zeta)^{n-2} = 1$.

4.6 Characterisation of classical adjoint-commuting mappings on 2×2 hermitian and symmetric matrices

Let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively. We recall that if $^-$ and $^\wedge$ are proper, then there exists $i \in \mathbb{F}$ with $\bar{i} = -i$ when $\mathbb{F}$ has characteristic not 2, and $\bar{i} = 1 + i$ when $\mathbb{F}$ has characteristic 2 such that $\mathbb{F} = \mathbb{F}^- \oplus i\mathbb{F}^-$. Respectively, there exists $j \in \mathbb{K}$ such that $\mathbb{K} = \mathbb{K}^\wedge \oplus j\mathbb{K}^\wedge$. To conclude this chapter, we give a general description of mappings $\psi : \mathcal{H}_2(\mathbb{F}) \to \mathcal{H}_2(\mathbb{K})$ satisfying condition (AH1) or (AH2).

Let $\mathcal{V}_1, \cdots, \mathcal{V}_n$ be $\mathbb{F}^-$-vector spaces over $\mathbb{F}$ and let $\mathcal{W}$ be a $\mathbb{K}^\wedge$-vector space over $\mathbb{K}$. We know that if $f : \mathcal{V}_1 \times \cdots \times \mathcal{V}_n \to \mathcal{W}$ is an additive mapping, then

$$f(v_1, \cdots, v_n) = \phi_1(v_1) + \cdots + \phi_n(v_n) \text{ for all } (v_1, \cdots, v_n) \in \mathcal{V}_1 \times \cdots \times \mathcal{V}_n$$

where $\phi_i : \mathcal{V}_i \to \mathcal{W}$ is an additive mapping with $\phi_i(v_i) = f(0, \cdots 0, v_i, 0 \cdots, 0)$ for every $v_i \in \mathcal{V}_i$ and $i = 1, \cdots, n$. Furthermore, if $(\mathbb{F}, ^-) = (\mathbb{K}, ^\wedge)$ and f is an $\mathbb{F}^-$-linear mapping which means f is an additive and $\mathbb{F}^-$-homogeneous mapping, then every ϕ_i is $\mathbb{F}^-$-linear. Moreover, if $\mathcal{V}_1 = \cdots = \mathcal{V}_n = \mathcal{W} = \mathbb{F}^-$, then every $\phi_i : \mathbb{F}^- \to \mathbb{F}^-$ is a linear mapping. Thus, for every $1 \leqslant i \leqslant n$, there exists a scalar $\beta_i \in \mathbb{F}^-$ such that $\phi_i(a_i) = \beta_i a_i$ for every $a_i \in \mathbb{F}^-$, and hence, we have

$$f(a_1, \cdots, a_n) = \beta_1 a_1 + \cdots + \lambda_n a_n \text{ for all } (a_1, \cdots, a_n) \in \mathcal{M}_{1,n}(\mathbb{F}^-).$$

With these observations, we obtain the following results.

Proposition 4.6.1. *Let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively. Let $\psi : \mathcal{H}_2(\mathbb{F}) \to \mathcal{H}_2(\mathbb{K})$ be a mapping satisfying* (AH2).

(a) *If $^-$ and $^\wedge$ are proper, then*

$$\psi\begin{pmatrix} a & \overline{b+ic} \\ b+ic & d \end{pmatrix} = \begin{pmatrix} g_1(a) + g_2(b) + g_3(c) + g_4(d) & \widehat{\phi(a-d)} + \widehat{\gamma(b,c)} \\ \phi(a-d) + \gamma(b,c) & g_4(a) - g_2(b) - g_3(c) + g_1(d) \end{pmatrix}$$

for all $a, b, c, d \in \mathbb{F}^-$ where $g_1, g_2, g_3, g_4 : \mathbb{F}^- \to \mathbb{K}^\wedge$, $\phi : \mathbb{F}^- \to \mathbb{K}^\wedge \oplus j\mathbb{K}^\wedge$ and $\gamma : \mathbb{F}^- \times \mathbb{F}^- \to \mathbb{K}^\wedge \oplus j\mathbb{K}^\wedge$ are additive with

$$\phi(a) = g_5(a) + jg_6(a) \text{ for every } a \in \mathbb{F}^-,$$

$$\gamma(b, c) = g_7(b) + g_8(c) + j(g_9(b) + g_{10}(c)) \text{ for all } b, c \in \mathbb{F}^-,$$

where $g_5, g_6, g_7, g_8, g_9, g_{10} : \mathbb{F}^- \to \mathbb{K}^\wedge$ are additive mappings.

(b) *If* $^-$ *and* $^\wedge$ *are identity mappings on* $\mathbb{F}$ *and* $\mathbb{K}$, *respectively, then* $\psi : \mathcal{S}_2(\mathbb{F}) \to \mathcal{S}_2(\mathbb{K})$ *satisfies* (AH2) *with*

$$\psi \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} g_1(a) + g_2(b) + g_3(d) & \phi_1(a-d) + \phi_2(b) \\ \phi_1(a-d) + \phi_2(b) & g_3(a) - g_2(b) + g_1(d) \end{pmatrix}$$

for all $a, b, d \in \mathbb{F}$ *where* $g_1, g_2, g_3, \phi_1, \phi_2 : \mathbb{F} \to \mathbb{K}$ *are additive.*

Proof. Let $A, B \in \mathcal{H}_2(\mathbb{F})$. Then

$$\psi(A - B) = \psi(\text{adj adj } (A - B)) = \text{adj adj } (\psi(A) - \psi(B)) = \psi(A) - \psi(B).$$

This implies $\psi(-B) = -\psi(B)$ and hence

$$\psi(A + B) = \psi(A - (-B)) = \psi(A) - (-\psi(B)) = \psi(A) + \psi(B)$$

for all $A, B \in \mathcal{H}_2(\mathbb{F})$. Thus, ψ is a classical adjoint-commuting additive mapping.

(a) Let $a, b, c, d \in \mathbb{F}^-$ and let $g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4 : \mathbb{F}^- \to \mathbb{K}^\wedge$, $\phi, \phi_1 : \mathbb{F}^- \to \mathbb{K}^\wedge \oplus j\mathbb{K}^\wedge$ and $\gamma : \mathbb{F}^- \times \mathbb{F}^- \to \mathbb{K}^\wedge \oplus j\mathbb{K}^\wedge$ be additive mappings such that

$$\psi \begin{pmatrix} a & \overline{b+ic} \\ b+ic & d \end{pmatrix} = \begin{pmatrix} g_1(a) + g_2(b) + g_3(c) + g_4(d) & \widehat{\phi(a)} + \widehat{\gamma(b,c)} + \widehat{\phi_1(d)} \\ \phi(a) + \gamma(b,c) + \phi_1(d) & h_1(a) + h_2(b) + h_3(c) + h_4(d) \end{pmatrix}.$$

Since ψ is a classical adjoint-commuting mapping, we have $h_1 = g_4$, $h_2 = -g_2$, $h_3 = -g_3$, $h_4 = g_1$ and $\phi = -\phi_1$. Thus,

$$\psi \begin{pmatrix} a & \overline{b+ic} \\ b+ic & d \end{pmatrix} = \begin{pmatrix} g_1(a) + g_2(b) + g_3(c) + g_4(d) & \widehat{\phi(a-d)} + \widehat{\gamma(b,c)} \\ \phi(a-d) + \gamma(b,c) & g_4(a) - g_2(b) - g_3(c) + g_1(d) \end{pmatrix}.$$

In addition, by the additivity of ϕ and γ,

$$\phi(a) = g_5(a) + jg_6(a) \text{ for every } a \in \mathbb{F}^-,$$

$$\gamma(b,c) = g_7(b) + g_8(c) + j(g_9(b) + g_{10}(c)) \text{ for all } b, c \in \mathbb{F}^-,$$

where $g_5, g_6, g_7, g_8, g_9, g_{10} : \mathbb{F}^- \to \mathbb{K}^\wedge$ are additive mappings.

(b) Let $a, b, d \in \mathbb{F}$ and let $g_1, g_2, g_3, h_1, h_2, h_3, \phi_1, \phi_2, \phi_3 : \mathbb{F} \to \mathbb{K}$ be additive mappings such that

$$\psi \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} g_1(a) + g_2(b) + g_3(d) & \phi_1(a) + \phi_2(b) + \phi_3(d) \\ \phi_1(a) + \phi_2(b) + \phi_3(d) & h_1(a) + h_2(b) + h_3(d) \end{pmatrix}.$$

Since adj is linear and ψ is a classical adjoint-commuting mapping, we have $h_1 = g_3$, $h_2 = -g_2$, $h_3 = g_1$ and $\phi_3 = -\phi_1$. Thus,

$$\psi \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} g_1(a) + g_2(b) + g_3(d) & \phi_1(a-d) + \phi_2(b) \\ \phi_1(a-d) + \phi_2(b) & g_3(a) - g_2(b) + g_1(d) \end{pmatrix}$$

for all $a, b, d \in \mathbb{F}$.

$\square$

Proposition 4.6.2. *Let $\mathbb{F}$ be a field which possesses an involutions $^-$ of $\mathbb{F}$. Let $\psi : \mathcal{H}_2(\mathbb{F}) \to \mathcal{H}_2(\mathbb{F})$ be a mapping satisfying* (AH1)*.*

(a) *If $^-$ is proper, then*

$$\psi \begin{pmatrix} a & \overline{b+ic} \\ b+ic & d \end{pmatrix} = \begin{pmatrix} \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d & \overline{\phi(a-d)} + \overline{\gamma(b,c)} \\ \phi(a-d) + \gamma(b,c) & \alpha_4 a - \alpha_2 b - \alpha_3 c + \alpha_1 d \end{pmatrix}$$

for all $a, b, c, d \in \mathbb{F}^-$ where

$\phi : \mathbb{F}^- \to \mathbb{F}^- \oplus i\mathbb{F}^-$ and $\gamma : \mathbb{F}^- \times \mathbb{F}^- \to \mathbb{F}^- \oplus i\mathbb{F}^-$ are linear with

$$\phi(a) = (\alpha_5 + i\alpha_6)a \text{ for every } a \in \mathbb{F}^-,$$

$$\gamma(b,c) = (\alpha_7 b + \alpha_8 c) + i(\alpha_9 b + \alpha_{10} c) \text{ for all } b, c \in \mathbb{F}^-,$$

and α_i are some fixed scalars in $\mathbb{F}^-$ for $i = 1, \cdots, 10$.

(b) *If $^-$ is identity, then $\psi : \mathcal{S}_2(\mathbb{F}) \to \mathcal{S}_2(\mathbb{F})$ satisfying* (AH1) *is linear with*

$$\psi \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} \alpha_1 a + \alpha_2 b + \alpha_3 d & \alpha_4(a-d) + \alpha_5 b \\ \alpha_4(a-d) + \alpha_5 b & \alpha_3 a - \alpha_2 b + \alpha_1 d \end{pmatrix}$$

for all $a, b, d \in \mathbb{F}$ where α_1, α_2, α_3, α_4 and α_5 are some fixed scalars in $\mathbb{F}$.

Proof. Let $A, B \in \mathcal{H}_2(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. Then

$$\psi(A + \alpha B) = \psi(\text{adj adj } (A + \alpha B)) = \text{adj adj } (\psi(A) + \alpha\psi(B)) = \psi(A) + \alpha\psi(B).$$

This implies $\psi(\alpha B) = \alpha\psi(B)$ and $\psi(A+B) = \psi(A)+\psi(B)$ for all $A, B \in \mathcal{H}_2(\mathbb{F})$.

Thus, ψ is a classical adjoint-commuting $\mathbb{F}^-$-linear mapping.

(a) Let $a, b, c, d \in \mathbb{F}^-$ and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ be some fixed scalars in $\mathbb{F}^-$, $\phi, \phi_1 : \mathbb{F}^- \to \mathbb{F}^- \oplus i\mathbb{F}^-$ and $\gamma : \mathbb{F}^- \times \mathbb{F}^- \to \mathbb{F}^- \oplus i\mathbb{F}^-$ be linear mappings such that

$$\psi \begin{pmatrix} a & \overline{b + ic} \\ b + ic & d \end{pmatrix} = \begin{pmatrix} \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d & \overline{\phi(a)} + \overline{\gamma(b,c)} + \overline{\phi_1(d)} \\ \phi(a) + \gamma(b,c) + \phi_1(d) & \beta_1 a + \beta_2 b + \beta_3 c + \beta_4 d \end{pmatrix}.$$

Since ψ is a classical adjoint-commuting mapping, we have $\beta_4 = \alpha_1$, $\beta_2 = -\alpha_2$, $\beta_3 = -\alpha_3$, $\beta_4 = \alpha_1$ and $\phi = -\phi_1$. Thus,

$$\psi \begin{pmatrix} a & \overline{b + ic} \\ b + ic & d \end{pmatrix} = \begin{pmatrix} \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d & \overline{\phi(a - d)} + \overline{\gamma(b,c)} \\ \phi(a - d) + \gamma(b,c) & \alpha_4 a - \alpha_2 b - \alpha_3 c + \alpha_1 d \end{pmatrix}.$$

In addition, by the linearity of ϕ and γ,

$$\phi(a) = (\alpha_5 + i\alpha_6)a \text{ for every } a \in \mathbb{F}^-,$$

$$\gamma(b,c) = (\alpha_7 b + \alpha_8 c) + i(\alpha_9 b + \alpha_{10} c) \text{ for all } b, c \in \mathbb{F}^-,$$

and α_i are some fixed scalars in $\mathbb{F}^-$ for $i = 5, \cdots, 10$.

(b) Let $a, b, d \in \mathbb{F}$ and let α_1, α_2, α_3, α_4, α_5, α_6, β_1, β_2, β_3 be some fixed scalars in $\mathbb{F}$,

$$\psi \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} \alpha_1 a + \alpha_2 b + \alpha_3 d & \alpha_4 a + \alpha_5 b + \alpha_6 d \\ \alpha_4 a + \alpha_5 b + \alpha_6 d & \beta_1 a + \beta_2 b + \beta_3 d \end{pmatrix}.$$

Since ψ is a classical adjoint-commuting mapping, we have $\beta_1 = \alpha_3$, $\beta_2 = -\alpha_2$, $\beta_3 = \alpha_1$ and $\alpha_6 = -\alpha_4$. Thus,

$$\psi \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} \alpha_1 a + \alpha_2 b + \alpha_3 d & \alpha_4(a - d) + \alpha_5 b \\ \alpha_4(a - d) + \alpha_5 b & \alpha_3 a - \alpha_2 b + \alpha_1 d \end{pmatrix}.$$

$\square$

Remark 4.6.3. Let $\mathbb{F}$ be a field of characteristic not 2 and let $\psi : \mathcal{S}_2(\mathbb{F}) \to \mathcal{S}_2(\mathbb{F})$ be a mapping satisfying (AH1). Then by using similar arguments as in [1, Theorem 3],

$$\psi(A) = PA(\text{adj } P) \text{ for every } A \in \mathcal{S}_2(\mathbb{F})$$

where $P \in \mathcal{M}_2(\mathbb{F})$ is invertible with adj $P = \pm P^t$.

Chapter 5

Classical adjoint-commuting mappings on skew-hermitian matrices

5.1 Introduction

As in Chapter 4, throughout this chapter, unless otherwise stated, we let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively, and let m, n be integers with $m, n \geqslant 3$. We let $\mathbb{F}^-$ and $\mathbb{K}^\wedge$ be the sets of all symmetric elements of $\mathbb{F}$ and $\mathbb{K}$, respectively. We also let $S\mathbb{F}^- := \{a \in \mathbb{F} : \overline{a} = -a\}$ and $S\mathbb{K}^\wedge := \{a \in \mathbb{K} : \widehat{a} = -a\}$.

We also observe that if n is a positive even integer, then $\mu^n \in \mathbb{F}^-$ and $\eta^n \in \mathbb{K}^\wedge$ for all $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and for all $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$.

Remark 5.1.1. Let $A \in \mathcal{H}_n(\mathbb{F})$ and let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$. If n is an even integer, then $\mu^{n-2}\text{adj } A \in \mathcal{H}_n(\mathbb{F})$.

In this chapter, we study the structure of $\psi : \mathcal{SH}_n(\mathbb{F}) \to \mathcal{SH}_m(\mathbb{K})$ that satisfies the following conditions (see (A1) and (A2) in Section 2.1):

(AS1) $\psi(\text{adj } (A + \alpha B)) = \text{adj } (\psi(A) + \alpha\psi(B))$ for all matrices $A, B \in \mathcal{SH}_n(\mathbb{F})$ and any scalar $\alpha \in \mathbb{F}^-$ when $(\mathbb{F}, ^-) = (\mathbb{K}, ^\wedge)$,

(AS2) $\psi(\text{adj } (A - B)) = \text{adj } (\psi(A) - \psi(B))$ for all matrices $A, B \in \mathcal{SH}_n(\mathbb{F})$.

5.2 Some basic properties

Let m, n be even integers with $m, n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars and let $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying

$$\varphi(\mu^{n-2}\text{adj } (X-Y)) = \eta^{m-2}\text{adj } (\varphi(X)-\varphi(Y)) \text{ for all } X,Y \in \mathcal{H}_n(\mathbb{F}). \qquad \text{(H)}$$

Lemma 5.2.1. *Let m,n be even integers with $m,n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars and let $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying* (H)*. Let $A,B \in \mathcal{H}_n(\mathbb{F})$. Then the following statements hold.*

(a) $\varphi(\mu^{n-2}\text{adj } A) = \eta^{m-2}\text{adj } \varphi(A)$.

(b) $\text{adj } \varphi(A-B) = \text{adj } (\varphi(A)-\varphi(B))$.

Proof.

(a) It is obvious that $\varphi(0)=0$. Thus, we have

$$\begin{aligned}\varphi(\mu^{n-2}\text{adj } A) &= \varphi(\mu^{n-2}\text{adj } (A-0)) \\ &= \eta^{m-2}\text{adj } (\varphi(A)-\varphi(0)) \\ &= \eta^{m-2}\text{adj } \varphi(A).\end{aligned}$$

(b) By (a) and (H), we obtain

$$\begin{aligned}\eta^{m-2}\text{adj } (\varphi(A-B)) &= \varphi(\mu^{n-2}\text{adj } (A-B)) \\ &= \eta^{m-2}\text{adj } (\varphi(A)-\varphi(B)).\end{aligned}$$

□

Lemma 5.2.2. *Let m,n be even integers with $m,n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars, and let $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying* (H)*. Let $A,B \in \mathcal{H}_n(\mathbb{F})$. Then the following statements hold.*

(a) $\text{rank } \varphi(A) \leqslant 1$ *if* $\text{rank } A = 1$.

(b) rank $\varphi(A) \leqslant m-1$ *if* rank $A = n-1$.

(c) rank $\varphi(A) \leqslant m-2$ *if* rank $A \leqslant n-2$.

Proof.

(a) Let $A \in \mathcal{H}_n(\mathbb{F})$ be of rank one. Then by Proposition 1.4.6, there exists a rank $n-1$ matrix $B \in \mathcal{H}_n(\mathbb{F})$ such that adj $B = \frac{1}{\mu^{n-2}}A$. This implies

$$\varphi(A) = \varphi(\mu^{n-2}\text{adj } B) = \eta^{m-2}\text{adj } \varphi(B).$$

Since

$$\eta^{m-2}\text{adj } \varphi(A) = \varphi(\mu^{n-2}\text{adj } A) = \varphi(0) = 0,$$

we have rank $\varphi(A) < m$ which implies rank $\varphi(B) < m$. Hence

$$\text{rank } \varphi(A) = \text{rank } (\eta^{m-2}\text{adj } \varphi(B)) \leqslant 1.$$

(b) Let $A \in \mathcal{H}_n(\mathbb{F})$ be of rank $n-1$. Then rank $\varphi(\mu^{n-2}\text{adj } A) \leqslant 1$ by (a). Thus we obtain adj $\varphi(\mu^{m-2}\text{adj } A) = 0$. On the other hand,

$$\begin{aligned}\text{adj } \varphi(\mu^{n-2}\text{adj } A) &= \text{adj } (\eta^{m-2}\text{adj } \varphi(A)) \\ &= (\eta^{m-2})^{m-1}\text{adj } (\text{adj } \varphi(A)).\end{aligned}$$

This implies adj (adj $\varphi(A)) = 0$. Therefore rank $\varphi(A) \leqslant m-1$.

(c) If rank $A \leqslant n-2$, then $\eta^{m-2}\text{adj } \varphi(A) = \varphi(\mu^{n-2}\text{adj } A) = \varphi(0) = 0$. Therefore, rank $\varphi(A) \leqslant m-2$.

□

Lemma 5.2.3. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars, and let $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying* (H). *Let $A \in \mathcal{H}_n(\mathbb{F})$. Then φ is injective if and only if*

$$\text{rank } A = n \iff \text{rank } \varphi(A) = m.$$

Proof. Since $\varphi(0) = 0$, we have $\ker\varphi = \{0\}$ by the injectivity of φ. In addition, by Lemma 5.2.2 (b) and (c), we observe that rank $\varphi(A) = m$ implies rank $A = n$. If rank $A = n$ and assume that rank $\varphi(A) < m$, then

$$\begin{aligned}\eta^{-(m-2)}\varphi(\mu^{n-2}\text{adj }(\mu^{n-2}\text{adj } A)) &= \eta^{-(m-2)}\eta^{m-2}\text{adj } \varphi(\mu^{n-2}\text{adj } A)\\ &= \text{adj } (\eta^{m-2}\text{adj } \varphi(A))\\ &= (\eta^{m-2})^{m-1}\text{adj } (\text{adj } \varphi(A)) = 0.\end{aligned}$$

This implies $\mu^{n-2}\text{adj } (\mu^{n-2}\text{adj } A) = 0$ as $\ker\varphi = \{0\}$. This contradicts the assumption that rank $A = n$. Therefore, rank $\varphi(A) = m$.

Conversely, suppose $\varphi(A) = \varphi(B)$ for some $A, B \in \mathcal{H}_n(\mathbb{F})$. We suppose rank $(A - B) = r$. By Lemma 2.2.2 (a), there exists a rank $n - r$ matrix $C \in \mathcal{H}_n(\mathbb{F})$ such that rank $(A - B + C) = n$. Then rank $\varphi(A - B + C) = m$. In addition, we have

$$\begin{aligned}\text{adj } \varphi(C) &= \text{adj } (\varphi(B - B + C))\\ &= \text{adj } (\varphi(B) - \varphi(B - C))\\ &= \text{adj } (\varphi(A) - \varphi(B - C))\\ &= \text{adj } (\varphi(A - B + C))\end{aligned}$$

by (b). Thus, rank $\varphi(C) = m$ implies rank $C = m$ and hence $r = 0$. We obtain $A = B$. Therefore, φ is injective. $\square$

Lemma 5.2.4. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars and let $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying* (H). *Suppose $P \in \mathcal{M}_n(\mathbb{F})$ is invertible and let $\phi_P : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be defined by*

$$\phi_P(A) = \varphi(PA\overline{P}^t) \text{ for every } A \in \mathcal{H}_n(\mathbb{F}).$$

Then the following statements hold.

(a) *If* rank $\phi_P(I_n) \neq m$, *then* rank $\phi_P(A) \leqslant m-2$ *for every* $A \in \mathcal{H}_n(\mathbb{F})$ *and* $\phi_P(A) = 0$ *for every rank one matrix* $A \in \mathcal{H}_n(\mathbb{F})$.

(b) *If* rank $\phi_P(I_n) = m$, *then* rank $\phi_P(aE_{ii}) = 1$ *for all integers* $1 \leqslant i \leqslant n$ *and nonzero scalar* $a \in \mathbb{F}^-$.

Proof. Let $A, B \in \mathcal{H}_n(\mathbb{F})$. Then

$$\begin{aligned} \text{adj } \phi_P(A-B) &= \text{adj } \varphi(P(A-B)\overline{P}^t) \\ &= \text{adj } \varphi(PA\overline{P}^t - PB\overline{P}^t) \\ &= \text{adj } (\varphi(PA\overline{P}^t) - \varphi(PB\overline{P}^t)) \\ &= \text{adj } (\phi_P(A) - \phi_P(B)). \end{aligned}$$

Thus,

$$\text{adj } \phi_P(A-B) = \text{adj } (\phi_P(A) - \phi_P(B)) \text{ for all } A, B \in \mathcal{H}_n(\mathbb{F}). \tag{5.1}$$

By definition of ϕ_P, Lemma 5.2.2 (a), (b) and (c) are true for ϕ_P.

We let $\theta := \mu^{n(n-2)} \det(P\overline{P})^{n-2}$, $\vartheta := \mu^{n-2}\theta^{n-1}$ and $H := \text{adj } P$. It is obvious that $\theta, \vartheta \in \mathbb{F}^-$ are nonzero and rank $H = n$.

(a) We observe that

$$\begin{aligned} \mu^{n-2}\text{adj } (\mu^{n-2}\text{adj } (P\overline{P}^t)) &= \mu^{n-2}(\mu^{n-2})^{n-1}\text{adj } (\text{adj } (P\overline{P}^t)) \\ &= \mu^{n(n-2)} \det(P\overline{P}^t)^{n-2} P\overline{P}^t \\ &= \theta P\overline{P}^t. \end{aligned}$$

This implies

$$\begin{aligned} \phi_P(\theta I_n) &= \varphi(\theta P\overline{P}^t) \\ &= \varphi(\mu^{n-2}\text{adj } (\mu^{n-2}\text{adj } (P\overline{P}^t))) \\ &= \eta^{m-2}\text{adj } \varphi(\mu^{n-2}\text{adj } (P\overline{P}^t)) \\ &= \eta^{m-2}\text{adj } (\eta^{m-2}\text{adj } \varphi(P\overline{P}^t)) \\ &= \eta^{m-2}\text{adj } (\eta^{m-2}\text{adj } \phi_P(I_n)). \end{aligned}$$

Therefore,

$$\phi_P(\theta I_n) = 0 \tag{5.2}$$

as rank $\phi_P(I_n) < m$. Hence, we obtain

$$\varphi(\vartheta \overline{H}^t H) = 0 \tag{5.3}$$

since

$$\begin{aligned}
\varphi(\vartheta \overline{H}^t H) &= \varphi(\mu^{n-2}\theta^{n-1}\text{adj }(P\overline{P}^t)) \\
&= \varphi(\mu^{n-2}\text{adj }(\theta P\overline{P}^t)) \\
&= \eta^{m-2}\text{adj }\varphi(\theta P\overline{P}^t) \\
&= \eta^{m-2}\text{adj }\phi_P(\theta I_n).
\end{aligned}$$

We next claim that

$$\varphi(\overline{H}^t \vartheta E_{ii} H) = 0 \text{ for } i = 1, \cdots, n. \tag{5.4}$$

Let $i = 1, \cdots, n$. We compute

$$\begin{aligned}
\varphi(\overline{H}^t \vartheta E_{ii} H) &= \varphi(\overline{H}^t(\mu^{n-2}\theta^{n-1}E_{ii})H) \\
&= \varphi(\mu^{n-2}\overline{H}^t(\theta^{n-1}E_{ii})H).
\end{aligned}$$

Since $\theta^{n-1}E_{ii} = \text{adj }(\theta(I_n - E_{ii}))$, we obtain

$$\begin{aligned}
\varphi(\overline{H}^t \vartheta E_{ii} H) &= \varphi(\mu^{n-2}(\text{adj }\overline{P}^t)\text{adj }(\theta(I_n - E_{ii}))(\text{adj }P)) \\
&= \varphi(\mu^{n-2}\text{adj }(P\theta(I_n - E_{ii})\overline{P}^t)).
\end{aligned}$$

By (5.2) and Lemma 5.2.1 (a),

$$\begin{aligned}
\varphi(\overline{H}^t \vartheta E_{ii} H) &= \eta^{m-2}\text{adj }\varphi(P\theta(I_n - E_{ii})\overline{P}^t) \\
&= \eta^{m-2}\text{adj }\phi_P(\theta(I_n - E_{ii})) \\
&= \eta^{m-2}\text{adj }(\phi_P(\theta I_n) - \phi_P(\theta E_{ii})) \\
&= \eta^{m-2}\text{adj }(-\phi_P(\theta E_{ii})) \\
&= 0
\end{aligned}$$

since rank $\phi_P(\theta E_{ii}) \leqslant 1$ by Lemma 5.2.2 (a). Our next claim is for every $i = 1, \cdots, n$,

$$\phi_P(\alpha E_{ii}) = 0 \text{ for every } \alpha \in \mathbb{F}^-. \tag{5.5}$$

It is clear that the result holds when $\alpha = 0$. We assume $\alpha \neq 0$. Let $\gamma = \mu^{(n-2)(n-1)}\alpha \in \mathbb{F}^-$. Then

$$\begin{aligned}
\phi_P(\alpha E_{ii}) &= \varphi(P(\alpha E_{ii})\overline{P}^t) \\
&= \varphi(P((\mu^{-1})^{(n-2)(n-1)}\gamma)E_{ii}\overline{P}^t) \\
&= \varphi((\mu^{-1})^{(n-2)(n-1)}\theta P(\theta^{-1}\gamma)E_{ii}\overline{P}^t) \\
&= \varphi((\mu^{-1})^{(n-2)(n-1)}\mu^{n(n-2)}\det(P\overline{P})^{n-2}P(\theta^{-1}\gamma)E_{ii}\overline{P}^t) \\
&= \varphi(\mu^{n-2}(\det P)^{n-2}P(\theta^{-1}\gamma)E_{ii}(\det \overline{P})^{n-2}\overline{P}^t).
\end{aligned}$$

Note that adj $(\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\gamma E_{jj}) = \theta^{-1}\gamma E_{ii}$ with $i \neq j$, and adj $H = (\det P)^{n-2}P$. Thus, we have

$$\begin{aligned}
&\phi_P(\alpha E_{ii}) \\
&= \varphi(\mu^{n-2}(\text{adj } H)(\text{adj } (\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\gamma E_{jj}))(\text{adj } \overline{H}^t)) \\
&= \varphi(\mu^{n-2}(\text{adj } (\overline{H}^t(\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\gamma E_{jj})H))) \\
&= \eta^{m-2}\text{adj } (\varphi(\overline{H}^t(\vartheta I_n - \vartheta E_{ii} - \vartheta E_{jj} + \theta^{-1}\vartheta^{2-n}\gamma E_{jj})H))
\end{aligned}$$

and hence by (5.3) and (5.4), we have

$$\begin{aligned}
\phi_P(\alpha E_{ii}) &= \eta^{m-2}\text{adj } (\varphi(\overline{H}^t(\vartheta I_n - \vartheta E_{ii} + \theta^{-1}\vartheta^{2-n}\gamma E_{jj})H) - \varphi(\overline{H}^t(\vartheta E_{jj})H)) \\
&= \eta^{m-2}\text{adj } (\varphi(\overline{H}^t(\vartheta I_n - \vartheta E_{ii} + \theta^{-1}\vartheta^{2-n}\gamma E_{jj})H)) \\
&= \eta^{m-2}\text{adj } (\varphi(\overline{H}^t(\vartheta I_n + \theta^{-1}\vartheta^{2-n}\gamma E_{jj})H) - \varphi(\overline{H}^t(\vartheta E_{ii})H)) \\
&= \eta^{m-2}\text{adj } \varphi(\vartheta\overline{H}^t H + \overline{H}^t(\theta^{-1}\vartheta^{2-n}\gamma E_{jj})H) \\
&= \eta^{m-2}\text{adj } (\varphi(\vartheta\overline{H}^t H) - \varphi(-\overline{H}^t(\theta^{-1}\vartheta^{2-n}\gamma E_{jj})H)) \\
&= \eta^{m-2}\text{adj } (-\varphi(-\overline{H}^t(\theta^{-1}\vartheta^{2-n}\gamma E_{jj})H)) \\
&= 0
\end{aligned}$$

since rank $\varphi(-\overline{H}^t(\theta^{-1}\vartheta^{2-n}\gamma E_{jj})H) \leqslant 1$. It follows that

$$\text{adj } \phi_p(A + \alpha_1 E_{11} + \cdots + \alpha_n E_{nn}) = \text{adj } \phi_P(A) \tag{5.6}$$

for every $A \in \mathcal{H}_n(\mathbb{F})$ and for all scalars $\alpha_1, \cdots, \alpha_n \in \mathbb{F}^-$. We next claim that for each $1 \leqslant i \leqslant n$,

$$\varphi(\overline{H}^t(\alpha E_{ii} H)) = 0 \text{ for every } \alpha \in \mathbb{F}^-. \tag{5.7}$$

Since adj $(I_n - E_{ii} - E_{jj} + \beta E_{jj}) = \beta E_{ii}$ where $i \neq j$ and $\beta = (\mu^{-1})^{n-2}\alpha \in \mathbb{F}^-$ as well as (5.1) and (5.6), we obtain

$$\begin{aligned}
\varphi(\overline{H}^t(\alpha E_{ii})H) &= \varphi((\text{adj } \overline{P}^t)(\mu^{n-2}\beta E_{ii})(\text{adj } P)) \\
&= \varphi(\mu^{n-2}(\text{adj } \overline{P}^t)(\beta E_{ii})(\text{adj } P)) \\
&= \varphi(\mu^{n-2}(\text{adj } \overline{P}^t)(\text{adj } (I_n - E_{ii} - E_{jj} + \beta E_{jj}))(\text{adj } P)) \\
&= \varphi(\mu^{n-2}\text{adj } (P(I_n - E_{ii} - E_{jj} + \beta E_{jj})\overline{P}^t))) \\
&= \eta^{m-2}\text{adj } (\varphi(P(I_n - E_{ii} - E_{jj} + \beta E_{jj})\overline{P}^t)) \\
&= \eta^{m-2}\text{adj } \phi_P(I_n - E_{ii} - E_{jj} + \beta E_{jj}) \\
&= \eta^{m-2}\text{adj } \phi_P(\beta E_{jj}) \\
&= 0.
\end{aligned}$$

Then by Lemma 5.2.1, Lemma 5.2.2 and (5.7),

$$\text{adj } \varphi(A + \overline{H}^t(\alpha_1 E_{11} + \cdots + \alpha_n E_{nn})H) = \text{adj } \varphi(A) \tag{5.8}$$

for every $A \in \mathcal{H}_n(\mathbb{F})$ and for all scalars $\alpha_1, \cdots, \alpha_n \in \mathbb{F}^-$. Let i, j and k be distinct integers with $1 \leqslant i, j, k \leqslant n$. Let $Y_{ijk} := I_n - E_{ii} - E_{jj} - 2E_{kk}$. Let $a \in \mathbb{F}^-$ be a nonzero scalar. Then $\overline{a}a \in \mathbb{F}^-$ and adj $(aE_{ij} + \overline{a}E_{ji} + Y_{ijk}) = aE_{ij} + \overline{a}E_{ji} + \overline{a}aY_{ijk}$. Thus, we obtain

$$
\begin{aligned}
&\varphi(\mu^{n-2}\overline{H}^t(aE_{ij}+\overline{a}E_{ji}+\overline{a}aY_{ijk})H)\\
&=\varphi(\mu^{n-2}(\text{adj }\overline{P}^t)\text{adj }(aE_{ij}+\overline{a}E_{ji}+Y_{ijk})(\text{adj }P))\\
&=\varphi(\mu^{n-2}\text{adj }(P(aE_{ij}+\overline{a}E_{ji}+Y_{ijk})\overline{P}^t))\\
&=\eta^{m-2}\text{adj }\varphi(P(aE_{ij}+\overline{a}E_{ji}+Y_{ijk})\overline{P}^t)\\
&=\eta^{m-2}\text{adj }\phi_P(aE_{ij}+\overline{a}E_{ji}+Y_{ijk})\\
&=\eta^{m-2}\text{adj }\phi_P(aE_{ij}+\overline{a}E_{ji})
\end{aligned}
$$

by Lemma 5.2.1 (a) and (5.6). Since rank $\phi_P(aE_{ij}+\overline{a}E_{ji})\leqslant m-2$,

$$
\varphi(\mu^{n-1}\overline{H}^t(aE_{ij}+\overline{a}E_{ji}+\overline{a}aY_{ijk})H)=0 \tag{5.9}
$$

for all distinct integers $1\leqslant i,j,k\leqslant n$ and scalar $a\in\mathbb{F}^-$.

We now claim that ϕ_P sends all rank one matrices to zero. Let $A\in\mathcal{H}_n(\mathbb{F})$ be of rank one. Then by Proposition 1.4.6, there exists a rank $n-1$ matrix $B=(b_{ij})\in\mathcal{H}_n(\mathbb{F})$ such that $\theta^{-1}A=\text{adj }B$. Hence, we obtain

$$
\begin{aligned}
\phi_P(A)&=\varphi(PA\overline{P}^t)\\
&=\varphi(\theta P(\theta^{-1}A)\overline{P}^t)\\
&=\varphi(\mu^{(n-2)n}\det(P\overline{P})^{n-2}P(\text{adj }B)\overline{P}^t)\\
&=\varphi(\mu^{(n-2)n}((\det P)^{n-2}P)(\text{adj }B)((\det\overline{P})^{n-2}\overline{P}^t))\\
&=\varphi(\mu^{(n-2)n}(\text{adj }H)(\text{adj }B)(\text{adj }\overline{H}^t))\\
&=\varphi(\mu^{n-2}\text{adj }(\mu^{n-2}\overline{H}^tBH))
\end{aligned}
$$

and hence $\phi_P(A)=\eta^{m-2}\text{adj }\varphi(\mu^{n-2}\overline{H}^tBH)$ by Lemma 5.2.1 (a). It follows from (5.8), (5.9) and Lemma 5.2.1 (b) that

$$
\begin{aligned}
&\text{adj }\varphi(\mu^{n-2}\overline{H}^tBH)\\
&=\text{adj }\varphi\left(\sum_{1\leqslant i<j\leqslant n}\mu^{n-2}\overline{H}^t(b_{ji}E_{ji}+\overline{b_{ji}}E_{ij})H+\sum_{i=1}^{n}\overline{H}^t(\mu^{n-2}b_{ii}E_{ii})H\right)
\end{aligned}
$$

which implies

$$\begin{aligned}
&\text{adj } \varphi(\mu^{n-2}\overline{H}^t BH) \\
&= \text{adj } \varphi\left(\sum_{1\leqslant i<j\leqslant n} \mu^{n-2}\overline{H}^t(b_{ji}E_{ji} + \overline{b_{ji}}E_{ij})H\right) \\
&= \text{adj } \varphi\left(\sum_{1\leqslant i<j\leqslant n} \mu^{n-2}\overline{H}^t(b_{ji}E_{ji} + \overline{b_{ji}}E_{ij})H + \mu^{n-2}\overline{H}^t(\overline{b_{21}}b_{21}Y_{12k})H\right) \\
&= \text{adj } \varphi\left(\sum_{\substack{1\leqslant i<j\leqslant n \\ i\neq 1 \text{ and } j\neq 2}} \mu^{n-2}\overline{H}^t(b_{ji}E_{ji} + \overline{b_{ji}}E_{ij})H\right).
\end{aligned}$$

We continue in this way to obtain

$$\text{adj } \varphi(\mu^{n-2}\overline{H}^t BH) = \text{adj } \varphi\left(\mu^{n-2}\overline{H}^t(b_{n,n-1}E_{n,n-1} + \overline{b_{n,n-1}}E_{n-1,n})H\right) = 0$$

as rank $\varphi\left(\mu^{n-2}\overline{H}^t(b_{n,n-1}E_{n,n-1} + \overline{b_{n,n-1}}E_{n-1,n})H\right) \leqslant m-2$. Therefore,

$$\phi_P(A) = 0 \quad \text{for every rank one matrix } A \in \mathcal{H}_n(\mathbb{F}).$$

Let $A = 0$. It is clear that adj $\phi_P(A) = 0$. Let $A \in \mathcal{H}_n(\mathbb{F})$ be of rank r with $1 \leqslant r \leqslant n$. Then by Lemma 2.2.1, there exist rank one matrices $A_1, \cdots, A_k \in \mathcal{H}_n(\mathbb{F})$ with $r \leqslant k \leqslant r+1$ such that $A = A_1 + \cdots + A_k$. By (5.1), we obtain

$$\begin{aligned}
\text{adj } \phi_P(A) =& \text{adj } \phi_P(A_1 + \cdots + A_k) \\
&= \text{adj } (\phi_P(A_1 + \cdots + A_{k-1}) - \phi_P(-A_k)) \\
&= \text{adj } (\phi_P(A_1 + \cdots + A_{k-1})).
\end{aligned}$$

By applying (5.1) repeatedly, we have

$$\text{adj } \phi_P(A) = \text{adj } \phi_P(A_1) = 0.$$

This implies rank $\phi_P(A) \leqslant m-2$ for all matrices $A \in \mathcal{H}_n(\mathbb{F})$, as desired.

(b) We have

$$\begin{aligned}
\varphi(\mu^{n-2}\overline{H}^t H) &= \varphi(\mu^{n-2}\text{adj } (P\overline{P}^t)) \\
&= \eta^{m-2}\text{adj } \varphi(P\overline{P}^t) \\
&= \eta^{m-2}\text{adj } \phi_P(I_n)
\end{aligned}$$

which implies rank $\varphi(\mu^{n-2}\overline{H}^t H) = m$ as rank $\phi_P(I_n) = m$. Suppose there exist an integer i_0 with $1 \leqslant i_0 \leqslant n$ and a nonzero scalar $a_0 \in \mathbb{F}^-$ such that $\phi_P(a_0 E_{i_0 i_0}) = 0$. Let s, t be two distinct integers with $1 \leqslant s, t \leqslant n$ and $s, t \neq i_0$. Since adj $(I_n - E_{ss} - (1 + a_0)E_{i_0 i_0} - (1 - a_0^{-1})E_{tt}) = -E_{ss}$, it follows from (5.1) and Lemma 5.2.2 (c) that

$$
\begin{aligned}
&\varphi(\mu^{n-2}\overline{H}^t(-E_{ss})H) \\
&= \varphi(\mu^{n-2}(\text{adj } \overline{P}^t)\text{adj } (I_n - E_{ss} - (1 + a_0)E_{i_0 i_0} - (1 - a_0^{-1})E_{tt})(\text{adj } P)) \\
&= \varphi(\mu^{n-2}\text{adj } (P(I_n - E_{ss} - (1 + a_0)E_{i_0 i_0} - (1 - a_0^{-1})E_{tt})\overline{P}^t)) \\
&= \eta^{m-2}\text{adj } \varphi(P(I_n - E_{ss} - (1 + a_0)E_{i_0 i_0} - (1 - a_0^{-1})E_{tt})\overline{P}^t) \\
&= \eta^{m-2}\text{adj } \phi_P(I_n - E_{ss} - (1 + a_0)E_{i_0 i_0} - (1 - a_0^{-1})E_{tt}) \\
&= \eta^{m-2}\text{adj } (\phi_P(I_n - E_{ss} - E_{i_0 i_0} - (1 - a_0^{-1})E_{tt}) - \phi_P(a_0 E_{i_0 i_0})) \\
&= \eta^{m-2}\text{adj } \phi_P(I_n - E_{ss} - E_{i_0 i_0} - (1 - a_0^{-1})E_{tt}) \\
&= 0
\end{aligned}
$$

as rank $(I_n - E_{ss} - E_{i_0 i_0} - (1 - a_0^{-1})E_{tt})) = n - 2$. By Lemma 5.2.1 (b) and Lemma 5.2.2 (b), we obtain

$$
\begin{aligned}
\text{adj } \varphi(\mu^{n-2}\overline{H}^t H) &= \text{adj } \varphi(\mu^{n-2}\overline{H}^t(I_n - E_{ss} + E_{ss})H) \\
&= \text{adj } (\varphi(\mu^{n-2}\overline{H}^t(I_n - E_{ss})H) - \varphi(\mu^{n-1}\overline{H}^t(-E_{ss})H)) \\
&= \text{adj } \varphi(\mu^{n-2}\overline{H}^t(I_n - E_{ss})H).
\end{aligned}
$$

Since rank $(\mu^{n-2}\overline{H}^t(I_n - E_{ss})H) \neq n$, it follows that rank $\varphi(\mu^{n-2}\overline{H}^t(I_n - E_{ss})H) \neq m$ and hence rank $\varphi(\mu^{n-2}\overline{H}^t H) \neq m$, a contradiction. Thus, $\phi_P(aE_{ii}) \neq 0$ for all nonzero $a \in \mathbb{F}^-$. Therefore,

$$\text{rank } \phi_P(aE_{ii}) = 1 \text{ for every integer } 1 \leqslant i \leqslant n \text{ and nonzero scalar } a \in \mathbb{F}^-$$

by Lemma 5.2.2 (a).

$\square$

Lemma 5.2.5. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars, and let $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ be a mapping satisfying* (H). *If* $\operatorname{rank} \varphi(I_n) = m$*, then φ is injective and*

$$\operatorname{rank}\,(A - B) = n \iff \operatorname{rank}\,(\varphi(A) - \varphi(B)) = m$$

for all $A, B \in \mathcal{H}_n(\mathbb{F})$.

Proof. Let $A \in \mathcal{H}_n(\mathbb{F})$ be of rank one. Then by Proposition 1.3.1, there exist an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ and a nonzero scalar $\alpha \in \mathbb{F}^-$ such that $A = P(\alpha E_{11})\overline{P}^t$. We define the mapping $\phi_P : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ by

$$\phi_P(A) = \varphi(PA\overline{P}^t) \text{ for every } A \in \mathcal{H}_n(\mathbb{F}).$$

Since $\operatorname{rank} \varphi(I_n) = m$ and $\phi_P(P^{-1}\overline{P^{-1}}^t) = \varphi(P(P^{-1}\overline{P^{-1}}^t)\overline{P}^t) = \varphi(I_n)$, we have $\operatorname{rank} \phi_P(P^{-1}\overline{P^{-1}}^t) = m$. Suppose $\operatorname{rank} \phi_P(I_n) \neq m$. Then by Lemma 5.2.4 (a), $\operatorname{rank} \phi_P(A) \leqslant m - 2$ for every $A \in \mathcal{H}_n(\mathbb{F})$. This contradicts that $\operatorname{rank} \phi_P(P^{-1}\overline{P^{-1}}^t) = m$. Thus, $\operatorname{rank} \phi_P(I_n) = m$ and hence it follows from Lemma 5.2.4 (b) that $\operatorname{rank} \phi_P(aE_{ii}) = 1$ for all integers $1 \leqslant i \leqslant n$ and nonzero scalars $a \in \mathbb{F}^-$. So, $\operatorname{rank} \varphi(A) = \operatorname{rank} \varphi(P(\alpha E_{11})\overline{P}^t) = \operatorname{rank} \phi_P(\alpha E_{11}) = 1$. Therefore φ preserves rank one matrices.

Let $X, Y \in \mathcal{H}_n(\mathbb{F})$ such that $\varphi(X) = \varphi(Y)$. Suppose $X - Y \neq 0$. Then by Lemma 4.2.1, there exists a matrix $Z \in \mathcal{H}_n(\mathbb{F})$ of rank at most $n - 2$ such that $\operatorname{rank}\,(X - Y + Z) = n - 1$. Hence,

$$\operatorname{rank} \operatorname{adj}\,(X - Y + Z) = 1 \Rightarrow \operatorname{rank} \varphi(\mu^{n-2}\operatorname{adj}\,(X - Y + Z)) = 1.$$

However,

$$\begin{aligned}\varphi(\mu^{n-2}\text{adj }(X-Y+Z)) &= \eta^{m-2}\text{adj }\varphi(X-Y+Z)\\ &= \eta^{m-2}\text{adj }(\varphi(X+Z)-\varphi(Y))\\ &= \eta^{m-2}\text{adj }(\varphi(X+Z)-\varphi(X))\\ &= \eta^{m-2}\text{adj }(\varphi(X+Z-X))\\ &= \eta^{m-2}\text{adj }(\varphi(Z))\\ &= 0.\end{aligned}$$

This is a contradiction. Therefore, $X = Y$ implies φ is injective.

Let $A, B \in \mathcal{H}_n(\mathbb{F})$. By the injectivity of φ, in view of Lemma 5.2.1 (a), (b) and Lemma 5.2.3, we obtain

$$\begin{aligned}\text{rank }(A-B) = n &\iff \text{rank }\varphi(\mu^{n-2}\text{adj }(A-B)) = m\\ &\iff \text{rank }\eta^{m-2}\text{adj }(\varphi(A)-\varphi(B)) = m\\ &\iff \text{rank }(\varphi(A)-\varphi(B)) = m,\end{aligned}$$

we are done. $\square$

Lemma 5.2.6. *Let n be an even integer with $n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be any fixed but arbitrarily chosen nonzero scalars, and let $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_n(\mathbb{K})$ be defined by*

$$\varphi(A) = \lambda Q A^\sigma \widehat{Q}^t \textit{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a nonzero field homomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$, $Q \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix and $\lambda \in \mathbb{K}^\wedge$ is a nonzero scalar. If $\eta^{n-2}\text{adj }\varphi(I_n) = \varphi(\mu^{n-2}I_n)$, then there exists a nonzero scalar $\zeta \in \mathbb{K}^\wedge$ such that

$$\widehat{Q}^t Q = \zeta I_n \textit{ and } (\eta\lambda\zeta\sigma(\mu)^{-1})^{n-2} = 1.$$

Proof. $\eta^{n-2}\text{adj }\varphi(I_n) = \varphi(\mu^{n-2}I_n)$ implies

$$\eta^{n-2}\text{adj }(\lambda Q\widehat{Q}^t) = \lambda Q(\mu^{n-2}I_n)^\sigma \widehat{Q}^t = \lambda\sigma(\mu)^{n-2}Q\widehat{Q}^t$$

and hence

$$\eta^{n-2}\lambda^{n-1}(\text{adj }\widehat{Q}^t)(\text{adj }Q) = \lambda\sigma(\mu)^{n-2}Q\widehat{Q}^t$$

$$\implies Q\widehat{Q}^t = (\lambda\eta\sigma(\mu)^{-1})^{n-2}(\text{adj }\widehat{Q}^t)(\text{adj }Q).$$

Let $\xi := (\lambda\eta\sigma(\mu)^{-1})^{n-2} \in \mathbb{K}^\wedge$. Then

$$(\widehat{Q}^tQ)^2 = \widehat{Q}^t(Q\widehat{Q}^t)Q = \widehat{Q}^t(\xi(\text{adj }\widehat{Q}^t)(\text{adj }Q))Q$$

$$= \xi\widehat{Q}^t(\text{adj }\widehat{Q}^t)(\text{adj }Q)Q = \xi\det(\widehat{Q}^tQ)I_n.$$

Thus,

$$(\widehat{Q}^tQ)^2 = \xi\det(\widehat{Q}^tQ)I_n. \tag{5.10}$$

Let $1 \leqslant i < j \leqslant n$. Since adj $(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = -(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})$, we obtain

$$\eta^{n-2}\text{adj }\varphi(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = \varphi(\mu^{n-2}\text{adj }(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})$$

$$= \varphi(-\mu^{n-2}(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})).$$

This implies

$$\eta^{n-2}\text{adj }(\lambda Q(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t) = -\lambda Q\sigma(\mu)^{n-2}(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t$$

and hence

$$\eta^{n-2}(\text{adj }\lambda\widehat{Q}^t)\text{adj }(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\text{adj }Q)$$

$$= -\lambda Q\sigma(\mu)^{n-2}(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t.$$

By computing

$$(\lambda\eta\sigma(\mu^{-1})^{n-2}(\text{adj }\widehat{Q}^t)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\text{adj }Q) = Q(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^t$$

$$\Rightarrow \xi\widehat{Q}^t(\text{adj }\widehat{Q}^t)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\text{adj }Q)Q = \widehat{Q}^tQ(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^tQ$$

$$\Rightarrow \xi\det(\widehat{Q}^tQ)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = \widehat{Q}^tQ(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})\widehat{Q}^tQ$$

$$\Rightarrow (\widehat{Q}^tQ)(\widehat{Q}^tQ)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = (\widehat{Q}^tQ)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\widehat{Q}^tQ),$$

we obtain

$$(\widehat{Q}^tQ)(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) = (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(\widehat{Q}^tQ)$$

for all $1 \leqslant i < j \leqslant n$. Then $\widehat{Q}^t Q = \zeta I_n$ and also $Q\widehat{Q}^t = \zeta I_n$ for some nonzero scalar $\zeta \in \mathbb{K}^\wedge$. In addition, $\eta^{n-2}\text{adj }(\lambda\zeta I_n) = \eta^{n-2}\text{adj }(\lambda Q\widehat{Q}^t)$ and hence

$$\begin{aligned}\eta^{n-2}\lambda^{n-1}\zeta^{n-1}I_n &= \eta^{n-2}\text{adj } \varphi(I_n)\\ &= \varphi(\mu^{n-2}I_n)\\ &= \lambda\sigma(\mu)^{n-2}Q\widehat{Q}^t\\ &= \lambda\sigma(\mu)^{n-2}\zeta I_n.\end{aligned}$$

It follows that $(\eta\lambda\zeta\sigma(\mu)^{-1})^{n-2} = 1$. □

Proposition 5.2.7. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ and $\eta \in \mathbb{K}^\wedge \cup S\mathbb{K}^\wedge$ be any fixed nonzero scalars. Then $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ is an additive mapping satisfying*

$$\varphi(\mu^{n-2}\text{adj } A) = \eta^{m-2}\text{adj } \varphi(A) \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

if and only if either $\varphi = 0$, or $m = n$ and

$$\varphi(A) = \lambda P A^\sigma \widehat{P}^t \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a nonzero field homomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, A^σ is the matrix obtained from A by applying σ entrywise, $P \in \mathcal{M}_n(\mathbb{K})$ is invertible with $\widehat{P}^t P = \zeta I_n$, and $\lambda, \zeta \in \mathbb{K}^\wedge$ are scalars with $(\lambda\zeta\eta\sigma(\mu)^{-1})^{n-2} = 1$.

Proof. The sufficiency part is clear. We now consider the necessity part. Let $A, B \in \mathcal{H}_n(\mathbb{F})$. Since φ is additive, $0 = \varphi(0) = \varphi(A - A) = \varphi(A) + \varphi(-A)$ implies $\varphi(-A) = -\varphi(A)$. Thus

$$\begin{aligned}\varphi(\mu^{n-2}\text{adj }(A - B)) &= \eta^{m-2}\text{adj } \varphi(A - B)\\ &= \eta^{m-2}\text{adj }(\varphi(A) - \varphi(B))\end{aligned}$$

for all $A, B \in \mathcal{H}_n(\mathbb{F})$ and hence (H) is satisfied. We continue the proof by considering two cases.

Case I: rank $\varphi(I_n) \neq m$.

From Lemma 5.2.4 (a), by letting $P = I_n$, $\varphi(A) = \phi_P(A) = 0$ for all rank one matrices $A \in \mathcal{H}_n(\mathbb{F})$. By the additivity of φ, $\varphi = 0$.

Case II: rank $\varphi(I_n) = m$.

By Lemma 5.2.5, φ is injective and by Lemma 5.2.2 (a), φ preserves rank one matrices. Suppose $n > m$. Since

$$m = \text{rank } \varphi(I_n) = \text{rank } (\varphi(E_{11}) + \cdots + \varphi(E_{nn})) \leqslant \sum_{i=1}^{n} \text{rank } \varphi(E_{ii}) = n,$$

we have rank $\varphi(I_n) < n$. By [5, Theorem 2.1], there exist integers $1 \leqslant t_1 < \cdots < t_\ell \leqslant n$, with $m \leqslant \ell < n$ such that rank $\varphi(E_{t_1t_1} + \cdots + E_{t_\ell t_\ell}) = m$. Thus,

$$\begin{aligned} m &= \text{rank } (\eta^{m-2}\text{adj } \varphi(E_{t_1t_1} + \cdots + E_{t_\ell t_\ell})) \\ &= \text{rank } (\varphi(\mu^{n-2}\text{adj } (E_{t_1t_1} + \cdots + E_{t_\ell t_\ell})) \leqslant 1, \end{aligned}$$

a contradiction.

Hence, $m = n$. By [23, Main Theorem, p.g.603] and [16, Theorem 2.1 and Remark 2.4], we have

$$\varphi(A) = \lambda Q A^\sigma \widehat{Q}^t \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a nonzero field homomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for every $a \in \mathbb{F}$, $Q \in \mathcal{M}_n(\mathbb{K})$ is an invertible matrix and $\lambda \in \mathbb{K}^\wedge$ is a nonzero scalar. In view of Lemma 5.2.1 (a), we have $\eta^{n-2}\text{adj } \varphi(I_n) = \varphi(\mu^{n-2} I_n)$ and hence by Lemma 5.2.6, we obtain

$$\widehat{Q}^t Q = \zeta I_n \text{ and } (\eta\lambda\zeta\sigma(\mu)^{-1})^{n-2} = 1.$$

We are done. $\square$

Let m, n be even integers with $m, n \geqslant 4$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ be a fixed but arbitrarily chosen nonzero scalar, and let $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{F})$ be a mapping satisfying

$$\varphi(\mu^{n-2}\text{adj } (X + \alpha Y)) = \mu^{m-2}\text{adj } (\varphi(X) + \alpha\varphi(Y)) \tag{5.11}$$

for all $X, Y \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. Then φ satisfies (H) for $(\mathbb{K}, ^\wedge) = (\mathbb{F}, ^-)$ and $\eta = \mu$ and so Lemmas 5.2.1, 5.2.2, 5.2.3, 5.2.4 and 5.2.5 are true for φ. In particular,

$$\begin{aligned} \mu^{n-2}\text{adj } \varphi(X + \alpha Y) &= \varphi(\mu^{n-2}\text{adj } (X + \alpha Y)) \\ &= \mu^{n-2}\text{adj } (\varphi(X) + \alpha\varphi(Y)). \end{aligned}$$

Thus, we have

$$\text{adj } \varphi(X + \alpha Y) = \text{adj } (\varphi(X) + \alpha\varphi(Y))$$

for all $X, Y \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. Furthermore, if rank $\varphi(I_n) = m$, then by Lemma 5.2.5, φ is injective. Let $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. It follows from Lemma 5.2.3, we have

$$\begin{aligned} \text{rank } (A + \alpha B) = n &\iff \text{rank } \varphi(\mu^{n-2}\text{adj } (A + \alpha B)) = m \\ &\iff \text{rank } \mu^{n-2}\text{adj } (\varphi(A) + \alpha\varphi(B)) = m \\ &\iff \text{rank } (\varphi(A) + \alpha\varphi(B) = m. \end{aligned}$$

Therefore, by following the arguments of the analogous proof in Lemma 2.2.6, we have the following lemma.

Lemma 5.2.8. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mathbb{F}$ be a field which possesses a proper involution $^-$ of $\mathbb{F}$ such that $|\mathbb{F}^-| = 2$ or $|\mathbb{F}^-| > n + 1$. Let $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{F})$ be a mapping satisfying* (H)*. If* rank $\varphi(I_n) = m$*, then φ is additive and*

$$\varphi(\alpha A) = \alpha\varphi(A) \textit{ for every } A \in \mathcal{H}_n(\mathbb{F}) \textit{ and scalar } \alpha \in \mathbb{F}^-.$$

Proposition 5.2.9. *Let m, n be even integers with $m, n \geqslant 4$, and $\mathbb{F}$ be a field which possesses a proper involution $^-$ of $\mathbb{F}$ such that either $|\mathbb{F}^-| = 2$ or $|\mathbb{F}^-| > n+1$. Let $\mu \in \mathbb{F}^- \cup S\mathbb{F}^-$ be a fixed but arbitrarily chosen nonzero scalar. Then $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{F})$ is a mapping satisfying*

$$\varphi(\mu^{n-2}\text{adj }(A + \alpha B)) = \mu^{m-2}\text{adj }(\varphi(A) + \alpha\varphi(B))$$

for all $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$ if and only if $\varphi(A) = 0$ for every rank one matrix $A \in \mathcal{H}_n(\mathbb{F})$ and rank $(\varphi(A) + \alpha\varphi(B)) \leqslant m - 2$ *for all $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$; or $m = n$ and*

$$\varphi(A) = \lambda P A^\sigma \overline{P}^t \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : \mathbb{F} \to \mathbb{F}$ is a field isomorphism satisfying $\overline{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$ and $\sigma(a) = a$ for all $a \in \mathbb{F}^-$, A^σ is the matrix obtained from A by applying σ entrywise, $P \in \mathcal{M}_n(\mathbb{F})$ is invertible with $\overline{P}^t P = \zeta I_n$ and $\lambda, \zeta \in \mathbb{F}^-$ are scalars with $(\lambda\zeta\mu\sigma(\mu)^{-1})^{n-2} = 1$.

Proof. The sufficiency is clear. Now, we prove the necessity. First, we suppose $\varphi(I_n) \neq m$. Then by letting P in Lemma 5.2.1 (a) be I_n, we have $\varphi(A) = 0$ for every rank one matrix $A \in \mathcal{H}_n(\mathbb{F})$, and rank $\varphi(A) \leqslant m-2$ for every $A \in \mathcal{H}_n(\mathbb{F})$.

Next, we suppose rank $\varphi(I_n) = m$. Since $\varphi(0) = 0$, we have

$$\begin{aligned}\varphi(\mu^{n-2}\text{adj }A) &= \varphi(\mu^{n-2}\text{adj }(A + \alpha(0))) \\ &= \mu^{n-2}\text{adj }(\varphi(A) + \alpha\varphi(0)) \\ &= \mu^{n-2}\text{adj }\varphi(A).\end{aligned}$$

Thus, by Lemma 5.2.8 and Proposition 5.2.7, we obtain $m = n$ and

$$\varphi(A) = \lambda Q A^\sigma \overline{Q}^t \text{ for every } A \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : \mathbb{F} \to \mathbb{F}$ is a nonzero field homomorphism satisfying $\overline{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, $Q \in \mathcal{M}_n(\mathbb{F})$ is invertible with $\overline{Q}^t Q = \zeta I_n$, and λ, $\zeta \in \mathbb{F}^-$ are scalars with $(\lambda\zeta\mu\sigma(\mu)^{-1})^{n-2} = 1$. By Lemma 5.2.8, $\varphi(aI_n) = a\varphi(I_n)$ for all $a \in \mathbb{F}^-$ and hence $\lambda Q\sigma(a)\overline{Q}^t = a\lambda Q\overline{Q}^t$. Thus $\sigma(a) = a$ for every $a \in \mathbb{F}^-$. In addition, since $^-$ is proper, there exists a scalar $i \in \mathbb{F}$ with $\bar{i} = -i$ when char $\mathbb{F} \neq 2$, and $\bar{i} = 1 + i$ when char $\mathbb{F} = 2$, such that $\mathbb{F} = \mathbb{F}^- \oplus i\mathbb{F}^-$. So, when char $\mathbb{F} \neq 2$,

$$\overline{\sigma(i)} = \sigma(\bar{i}) = \sigma(-i) = -\sigma(i)$$

and when char $\mathbb{F} = 2$,

$$\overline{\sigma(i)} = \sigma(\bar{i}) = \sigma(1 + i) = 1 + \sigma(i).$$

Thus, we have $\mathbb{F} = \mathbb{F}^- \oplus \sigma(i)\mathbb{F}^-$. Let $\gamma \in \mathbb{F}$. Then there exist scalars $\beta_1, \beta_2 \in \mathbb{F}^-$ such that $\gamma = \beta_1 + \sigma(i)\beta_2$. Let $\delta = \beta_1 + i\beta_2 \in \mathbb{F}$. Thus, we have

$$\sigma(\delta) = \sigma(\beta_1 + i\beta_2) = \sigma(\beta_1) + \sigma(i)\sigma(\beta_2) = \beta_1 + \sigma(i)\beta_2 = \gamma.$$

This shows that σ is surjective and so it is an isomorphism. $\square$

5.3 Some examples

In this section, we give a few examples of nonlinear mappings ψ that satisfy condition (AS1) or (AS2) that send all rank one matrices and invertible matrices to zero. Under the condition of (AS1) or (AS2), nice structural results are obtained if there exists an invertible matrix $X \in \mathcal{SH}_n(\mathbb{F})$ such that $\psi(X)$ is invertible.

Let m, n be even integers with $m, n \geqslant 4$, and let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess proper involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively.

Example 5.3.1. Let $\alpha \in S\mathbb{K}^{\wedge}$ be nonzero scalar and we define the mapping $\psi_1 : \mathcal{SH}_n(\mathbb{F}) \to \mathcal{SH}_m(\mathbb{K})$ by

$$\psi_1(A) = \begin{cases} \alpha \sum_{i=1}^{m-2} E_{ii} & \text{if } A \in \mathcal{SH}_n(\mathbb{F}) \text{ is of rank } r \text{ with } 1 < r < n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.3.2. Let $\beta \in S\mathbb{K}^{\wedge}$ be a nonzero scalar and let $\tau : (\mathbb{F}, ^{-}) \to (\mathbb{K}, ^{\wedge})$ be a field isomorphism such that $\widehat{\tau(a)} = \tau(\overline{a})$ for every $a \in \mathbb{F}$. Let the mapping $\psi_2 : \mathcal{SH}_n(\mathbb{F}) \to \mathcal{SH}_m(\mathbb{K})$ be defined by

$$\psi_2(A) = \begin{cases} \beta E_{11} & \text{if } A \in \mathcal{SH}_n(\mathbb{F}) \text{ is of rank } 2, \\ \tau(a_{12})E_{12} + \tau(a_{21})E_{21} & \text{if } A = (a_{ij}) \in \mathcal{SH}_n(\mathbb{F}) \text{ is of rank } r, 2 < r < n, \\ 0 & \text{otherwise.} \end{cases}$$

We observe that ψ_1 and ψ_2 are mappings that satisfy condition (AS1) or (AS2). Both mappings send rank one matrices and invertible matrices to zero. The mappings are neither injective nor surjective.

5.4 Characterisation of classical adjoint-commuting mappings on skew-hermitian matrices

Let $\mathbb{F}$ be a field which possesses an involution $^{-}$ of $\mathbb{F}$ and let $\mu \in S\mathbb{F}^{-}$ be nonzero. If $A \in \mathcal{SH}_n(\mathbb{F})$, then $(\overline{\mu A})^t = \overline{\mu}\overline{A}^t = -\mu(-A) = \mu A$. It follows that $\mu A \in \mathcal{H}_n(\mathbb{F})$. Conversely, if $\mu A \in \mathcal{H}_n(\mathbb{F})$, then $\mu A = (\overline{\mu A})^t = \overline{\mu}\overline{A}^t = -\mu\overline{A}^t$ and hence $A = -\overline{A}^t$. Thus, $A \in \mathcal{SH}_n(\mathbb{F})$. Therefore, we have shown that

$$A \in \mathcal{SH}_n(\mathbb{F}) \iff \mu A \in \mathcal{H}_n(\mathbb{F}) \tag{5.12}$$

for any fixed nonzero scalar $\mu \in S\mathbb{F}^{-}$. Similarly, we can show that

$$A \in \mathcal{H}_n(\mathbb{F}) \iff \mu A \in \mathcal{SH}_n(\mathbb{F}) \tag{5.13}$$

for any fixed nonzero scalar $\mu \in S\mathbb{F}^-$. Then by (5.12) and (5.13), we have

$$\mathcal{SH}_n(\mathbb{F}) = \mu\mathcal{H}_n(\mathbb{F}) := \{\mu A : A \in \mathcal{H}_n(\mathbb{F})\} \tag{5.14}$$

and

$$\mathcal{H}_n(\mathbb{F}) = \mu\mathcal{SH}_n(\mathbb{F}) := \{\mu A : A \in \mathcal{SH}_n(\mathbb{F})\} \tag{5.15}$$

for any fixed nonzero scalar $\mu \in S\mathbb{F}^-$.

Lemma 5.4.1. *Let m, n be even integers with $m, n \geqslant 4$, and let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess involution $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively. Let $\mu \in S\mathbb{F}^-$ and $\eta \in S\mathbb{K}^\wedge$ be fixed but arbitrarily chosen nonzero scalars. Let $\psi : \mathcal{SH}_n(\mathbb{F}) \to \mathcal{SH}_m(\mathbb{K})$ be a mapping. If $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ is defined by*

$$\varphi(X) = \eta^{-1}\psi(\mu X) \text{ for every } X \in \mathcal{H}_n(\mathbb{F}),$$

then the following statements hold:

(a) $\psi(\text{adj }(A - B)) = \text{adj }(\psi(A) - \psi(B))$ *for all $A, B \in \mathcal{SH}_n(\mathbb{F})$ if and only if* $\varphi(\mu^{n-2}\text{adj }(X - Y)) = \eta^{m-2}\text{adj }(\varphi(X) - \varphi(Y))$ *for all $X, Y \in \mathcal{H}_n(\mathbb{F})$.*

(b) *If $(\mathbb{K}, ^\wedge) = (\mathbb{F}, ^-)$ and $\mu = \eta$, then*

$\psi(\text{adj }(A+\alpha B)) = \text{adj }(\psi(A)+\alpha\psi(B))$ *for all $A, B \in \mathcal{SH}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$ if and only if*

$$\varphi(\mu^{n-2}\text{adj }(X + \alpha Y)) = \eta^{m-2}\text{adj }(\varphi(X) + \alpha\varphi(Y))$$

for all $X, Y \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$.

Proof.

(a) Let $X, Y \in \mathcal{H}_n(\mathbb{F})$. By the definition of φ and (5.14), we have

$$\eta^{m-2}\text{adj }(\varphi(X) - \varphi(Y)) = \eta^{m-2}\text{adj }(\eta^{-1}\psi(\mu X) - \eta^{-1}\psi(\mu Y))$$

and hence

$$\begin{aligned}\eta^{m-2}\text{adj }(\varphi(X)-\varphi(Y)) &= \eta^{m-2}\eta^{-(m-1)}\text{adj }(\psi(\mu X)-\psi(\mu Y))\\ &= \eta^{-1}\psi(\text{adj }\mu(X-Y))\\ &= \eta^{-1}\psi(\mu^{n-1}\text{adj }(X-Y))\\ &= \varphi(\mu^{n-2}\text{adj }(X-Y)).\end{aligned}$$

Conversely, we let $A, B \in \mathcal{SH}_n(\mathbb{F})$. By the definition of φ and (5.15), we obtain

$$\begin{aligned}\text{adj }(\psi(A)-\psi(B)) &= \text{adj }(\eta\varphi(\mu^{-1}A)-\eta\varphi(\mu^{-1}B))\\ &= \eta^{m-1}\text{adj }(\varphi(\mu^{-1}A)-\eta\varphi(\mu^{-1}B))\\ &= \eta(\eta^{m-2}\text{adj }(\varphi(\mu^{-1}A)-\eta\varphi(\mu^{-1}B)))\\ &= \eta(\varphi(\mu^{n-2}\text{adj }(\mu^{-1}(A-B))))\\ &= \eta(\varphi(\mu^{-1}\text{adj }(A-B)))\\ &= \psi(\text{adj }(A-B)).\end{aligned}$$

(b) This part can be proved by using similar arguments as in part (a).

□

Theorem 5.4.2. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mathbb{F}$ and $\mathbb{K}$ be fields which posses proper involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively. Then $\psi : \mathcal{SH}_n(\mathbb{F}) \to \mathcal{SH}_m(\mathbb{K})$ is a classical adjoint-commuting additive mapping if and only if either $\psi = 0$, or $m = n$ and*

$$\psi(A) = \lambda P A^\sigma \widehat{P}^t \text{ for every } A \in \mathcal{SH}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a nonzero field homomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, $P \in \mathcal{M}_n(\mathbb{F})$ is invertible with $\widehat{P}^t P = \zeta I_n$, and $\lambda, \zeta \in \mathbb{K}^\wedge$ are scalars with $(\lambda\zeta)^{n-2} = 1$.

Proof. The sufficiency part is clear. Now, we consider the necessity part. Since ψ is additive, we have $\psi(\text{adj }(A-B)) = \text{adj }(\psi(A)-\psi(B))$ for all $A, B \in \mathcal{SH}_n(\mathbb{F})$. Let $\mu \in S\mathbb{F}^-$ and $\eta \in S\mathbb{K}^\wedge$ be fixed nonzero scalars. By (5.14), we define $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ by

$$\varphi(X) = \eta^{-1}\psi(\mu X) \text{ for every } X \in \mathcal{H}_n(\mathbb{F}). \tag{5.16}$$

In view of Lemma 5.4.1 (a) and $\psi(0) = 0$, we have $\varphi(\mu^{n-2}\text{adj } X) = \eta^{m-2}\text{adj } \varphi(X)$ for every $X \in \mathcal{H}_n(\mathbb{F})$. We now show that φ is additive. Let $X, Y \in \mathcal{H}_n(\mathbb{F})$. Then

$$\begin{aligned}\varphi(X+Y) &= \eta^{-1}\psi(\mu(X+Y)) \\ &= \eta^{-1}(\psi(\mu X) + \psi(\mu Y)) \\ &= \eta^{-1}\psi(\mu X) + \eta^{-1}\psi(\mu Y) \\ &= \varphi(X) + \varphi(Y).\end{aligned}$$

By Proposition 5.2.7, we have either $\varphi = 0$, or $m = n$ and

$$\varphi(X) = \gamma P X^\sigma \widehat{P}^t \text{ for every } X \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^-) \to (\mathbb{K}, ^\wedge)$ is a nonzero field homomorphism with $\widehat{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, $P \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix with $\widehat{P}^t P = \zeta I_n$, $\gamma, \zeta \in \mathbb{K}^\wedge$ are scalars with $(\eta\gamma\zeta\sigma(a)^{-1})^{n-2} = 1$. By (5.16), we obtain

$$\psi(\mu X) = \eta\varphi(X) = \eta\gamma P X^\sigma \widehat{P}^t = \eta\gamma\sigma(\mu)^{-1} P(\mu X)^\sigma \widehat{P}^t \text{ for every } X \in \mathcal{H}_n(\mathbb{F}).$$

Let $\lambda := \eta\gamma\sigma(\mu)^{-1}$. We observe that $\lambda \in \mathbb{K}^\wedge$ since $\eta, \sigma(\mu)^{-1} \in S\mathbb{K}^\wedge$ and $\gamma \in \mathbb{K}^\wedge$. Therefore, by (5.14)

$$\psi(A) = \lambda P A^\sigma \widehat{P}^t \text{ for every } A \in \mathcal{SH}_n(\mathbb{F})$$

with $\widehat{P}^t P = \zeta I_n$ and $(\lambda\zeta)^{n-2} = 1$. $\square$

Theorem 5.4.3. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mathbb{F}$ be a field which possesses a proper involution $^-$ of $\mathbb{F}$ such that either $|\mathbb{F}^-| = 2$ or $|\mathbb{F}^-| > n+1$. Then $\psi : \mathcal{SH}_n(\mathbb{F}) \to \mathcal{SH}_m(\mathbb{F})$ is a mapping satisfying* (AS1) *if and only if either $\psi(A) = 0$ for every rank one matrix $A \in \mathcal{SH}_n(\mathbb{F})$ and* rank $(\psi(A) + \alpha\psi(B)) \leqslant m-2$ *for all $A, B \in \mathcal{SH}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$; or $m = n$ and*

$$\psi(A) = \lambda P A^\sigma \overline{P}^t \text{ for every } A \in \mathcal{SH}_n(\mathbb{F})$$

where $\sigma : \mathbb{F} \to \mathbb{F}$ is a field isomorphism satisfying $\overline{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$ and $\sigma(a) = a$ for all $a \in \mathbb{F}^-$, $P \in \mathcal{M}_n(\mathbb{F})$ is invertible with $\overline{P}^t P = \zeta I_n$, and $\lambda, \zeta \in \mathbb{F}^-$ are scalars with $(\lambda\zeta)^{n-2} = 1$.

Proof. The sufficiency part can be shown easily. We now prove the necessity part. Let $\mu \in S\mathbb{F}^-$ be a fixed nonzero scalar and $\varphi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{F})$ be the mapping defined by

$$\varphi(X) = \mu^{-1}\psi(\mu X) \text{ for every } X \in \mathcal{H}_n(\mathbb{F}). \tag{5.17}$$

By the definition of ψ and Lemma 5.4.1, φ satisfies (5.11). By Proposition 5.2.9, we have either

(I) $\varphi(X) = 0$ for every rank one matrix $X \in \mathcal{H}_n(\mathbb{F})$, and rank $\varphi(X) \leqslant m-2$ for every $X \in \mathcal{H}_n(\mathbb{F})$; or

(II) $m = n$ and $\varphi(X) = \gamma P X^\sigma \overline{P}^t$ for every $X \in \mathcal{H}_n(\mathbb{F})$, where $\sigma : \mathbb{F} \to \mathbb{F}$ is a field isomorphism satisfying $\overline{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$ and $\sigma(a) = a$ for all $a \in \mathbb{F}^-$, $P \in \mathcal{M}_n(\mathbb{F})$ is invertible with $\overline{P}^t P = \zeta I_n$, and $\gamma, \zeta \in \mathbb{F}^-$ are scalars with $(\gamma\zeta\mu\sigma(\mu)^{-1})^{n-2} = 1$.

Let $A \in \mathcal{SH}_n(\mathbb{F})$. Then by (5.17),

$$\psi(A) = \psi(\mu(\mu^{-1}A)) = \mu\varphi(\mu^{-1}A).$$

If Case (I) is true, then

$$\psi(A)=\mu\varphi(\mu^{-1}A)=0 \text{ for every rank one matrix } A\in\mathcal{SH}_n(\mathbb{F})$$

and

$$\operatorname{rank}\psi(A)=\operatorname{rank}\varphi(\mu^{-1}A)\leqslant m-2 \text{ for every } A\in\mathcal{SH}_n(\mathbb{F}).$$

If Case (II) is true, then we have

$$\psi(A)=\mu\varphi(\mu^{-1}A)=\mu\gamma P(\mu^{-1}A)^{\sigma}\overline{P}^t=(\mu\gamma\sigma(\mu)^{-1})PA^{\sigma}\overline{P}^t$$

Thus, we obtain

$$\psi(A)=\lambda PA^{\sigma}\overline{P}^t \text{ for every } A\in\mathcal{SH}_n(\mathbb{F}),$$

where $\lambda=\mu\gamma\sigma(\mu)^{-1}\in\mathbb{F}^-$ and $\zeta\in\mathbb{F}^-$ with $(\lambda\zeta)^{n-2}=1$, and $\overline{P}^tP=\zeta I_n$. □

Theorem 5.4.4. *Let m,n be even integers with $m,n\geqslant 4$. Let $\mathbb{F}$ and $\mathbb{K}$ be fields which possess proper involutions $^-$ of $\mathbb{F}$ and $^\wedge$ of $\mathbb{K}$, respectively, such that either $|\mathbb{K}^\wedge|=2$, or $|\mathbb{F}^-|,|\mathbb{K}^\wedge|>3$. Then $\psi:\mathcal{SH}_n(\mathbb{F})\to\mathcal{SH}_m(\mathbb{K})$ is a surjective mapping satisfying* (AS2) *if and only if $m=n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic, and*

$$\psi(A)=\lambda PA^{\sigma}\widehat{P}^t \text{ for every } A\in\mathcal{SH}_n(\mathbb{F})$$

where $\sigma:(\mathbb{F},^-)\to(\mathbb{K},^\wedge)$ is a field isomorphism satisfying $\widehat{\sigma(a)}=\sigma(\overline{a})$ for every $a\in\mathbb{F}$, $P\in\mathcal{M}_n(\mathbb{K})$ is invertible with $\widehat{P}^tP=\zeta I_n$ and $\lambda,\zeta\in\mathbb{K}^\wedge$ are scalars with $(\lambda\zeta)^{n-2}=1$.

Proof. The sufficiency part is obvious. We now consider the necessity part. Let $\mu\in S\mathbb{F}^-$ and $\eta\in S\mathbb{K}^\wedge$ be fixed nonzero scalars and $\varphi:\mathcal{H}_n(\mathbb{F})\to\mathcal{H}_m(\mathbb{K})$ be defined by

$$\varphi(X)=\eta^{-1}\psi(\mu X) \text{ for every } X\in\mathcal{H}_n(\mathbb{F}). \tag{5.18}$$

By Lemma 5.4.1 (a), φ satisfies (H). Let $K \in \mathcal{H}_m(\mathbb{K})$. Then $\eta K \in \mathcal{SH}_m(\mathbb{K})$ by (5.12). Since ψ is surjective, there is a matrix $H \in \mathcal{SH}_n(\mathbb{F})$ such that $\psi(H) = \eta K$. This implies $\mu^{-1}H \in \mathcal{H}_n(\mathbb{F})$ and

$$\varphi(\mu^{-1}H) = \eta^{-1}\psi(H) = K.$$

Thus, φ is surjective.

We suppose rank $\varphi(I_n) \neq m$. Hence, by letting P in Lemma 5.2.4 (a) be I_n, we have rank $\varphi(X) \leqslant m-2$ for every $X \in \mathcal{H}_n(\mathbb{F})$. This contradicts the surjectivity of φ. Thus, rank $\varphi(I_n) = m$. By Lemma 5.2.5, φ is bijective and satisfies

$$\text{rank } (X - Y) = n \iff \text{rank } (\varphi(X) - \varphi(Y)) = m \text{ for all } X, Y \in \mathcal{H}_n(\mathbb{F}).$$

Next, we consider two cases.

Case I: $|\mathbb{K}^\wedge| = 2$.

Then $-1 = 1$. Thus

$$\text{rank } (X - Y) = n \iff \text{rank } (\varphi(X) + \varphi(Y)) = m \text{ for all } X, Y \in \mathcal{H}_n(\mathbb{F}).$$

We now show that φ is additive. Let $X, Y \in \mathcal{H}_n(\mathbb{F})$. If rank $(X+Y) = n$, then by Lemma 5.2.3,

$$\text{rank } \varphi(X+Y) = \text{rank } \varphi(X - (-Y)) = \text{rank } (\varphi(X) + \varphi(-Y)) = m.$$

Thus,

$$\varphi(X+Y)\text{adj } \varphi(X+Y) = \det \varphi(X+Y) I_m$$

and

$$(\varphi(X) + \varphi(-Y))\text{adj } (\varphi(X) + \varphi(-Y)) = \det(\varphi(X) + \varphi(-Y)) I_m.$$

This implies

$$\frac{\varphi(X+Y)\text{adj } \varphi(X+Y)}{\det \varphi(X+Y)} = I_m = \frac{(\varphi(X)+\varphi(-Y))\text{adj } (\varphi(X)+\varphi(-Y))}{\det(\varphi(X)+\varphi(-Y))}.$$

It follows from Lemma 5.2.1 (b) that

$$\text{adj } \varphi(X+Y) = \text{adj } \varphi(X-(-Y)) = \text{adj } (\varphi(X)-\varphi(-Y)) = \text{adj } (\varphi(X)+\varphi(-Y))$$

and hence

$$\frac{\varphi(X+Y)}{\det \varphi(X+Y)} = \frac{\varphi(X)+\varphi(-Y)}{\det(\varphi(X)+\varphi(-Y))}.$$

As $\det \varphi(X+Y) = \det(\varphi(X)+\varphi(-Y)) = 1$, we have

$$\varphi(X+Y) = \varphi(X)+\varphi(-Y) \text{ for all } X, Y \in \mathcal{H}_n(\mathbb{F}) \text{ with rank } X+Y = n.$$

Since φ is injective and

$$\varphi(-I_n) = \varphi(0-I_n) = \varphi(0)+\varphi(I_n) = \varphi(I_n),$$

we obtain $I_n = -I_n$ and hence $\mathbb{F}$ is of characteristic 2. Thus $\varphi(-Y) = \varphi(Y)$ for every $Y \in \mathcal{H}_n(\mathbb{F})$. Therefore,

$$\varphi(X+Y) = \varphi(X)+\varphi(Y) \text{ for all } X, Y \in \mathcal{H}_n(\mathbb{F}) \text{ with rank } X+Y = n. \quad (5.19)$$

We next consider the case where rank $(X+Y) < n$. There exists a matrix $Z \in \mathcal{H}_n(\mathbb{F})$ such that rank $(X+Z) =$ rank $(X+Y+Z) = n$ by Lemma 2.2.2 (b). Then by (5.19), $\varphi(X+Z) = \varphi(X)+\varphi(Z)$ and

$$\varphi(X+Y)+\varphi(Z) = \varphi(X+Y+Z) = \varphi(X+Z)+\varphi(Y) = \varphi(X)+\varphi(Z)+\varphi(Y).$$

Thus,

$$\varphi(X+Y) = \varphi(X)+\varphi(Y) \text{ for all } X, Y \in \mathcal{H}_n(\mathbb{F}).$$

Therefore, by Proposition 5.2.7 and the bijectivity of φ, we have $m = n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic, and

$$\varphi(X) = \gamma P X^{\sigma} \widehat{P}^{t} \text{ for every } X \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^{-}) \to (\mathbb{K}, ^{\wedge})$ is a nonzero field isomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, $P \in \mathcal{M}_n(\mathbb{F})$ is invertible with $\widehat{P}^{t} P = \zeta I_n$, and $\gamma, \zeta \in \mathbb{K}^{\wedge}$ are scalars with $(\gamma\zeta\eta\sigma(\mu)^{-1})^{n-2} = 1$.

Case II: $|\mathbb{F}^{-}|, |\mathbb{K}^{\wedge}| > 3$.

As $\varphi(0) = 0$, [14, Theorem 3.6] and the fundamental theorem of the geometry of hermitian matrices, Theorem 1.5.4, give $m = n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic, and

$$\varphi(X) = \gamma P X^{\sigma} \widehat{P}^{t} \text{ for every } X \in \mathcal{H}_n(\mathbb{F})$$

where $\sigma : (\mathbb{F}, ^{-}) \to (\mathbb{K}, ^{\wedge})$ is a nonzero field isomorphism satisfying $\widehat{\sigma(a)} = \sigma(\overline{a})$ for all $a \in \mathbb{F}$, $P \in \mathcal{M}_n(\mathbb{F})$ is invertible and $\gamma \in \mathbb{K}^{\wedge}$ is nonzero. By Lemma 5.2.1 (a), $\eta^{n-2} \text{adj } \varphi(I_n) = \varphi(\mu^{n-2} I_n)$. It follows from Lemma 5.2.6 that there exists a nonzero scalar $\zeta \in \mathbb{K}^{\wedge}$ such that

$$\widehat{P}^{t} P = \zeta I_n \text{ and } (\gamma\zeta\eta\sigma(\mu)^{-1})^{n-2} = 1.$$

For both cases, by (5.18), we have

$$\psi(\mu X) = \eta\varphi(X) = \eta\gamma P X^{\sigma} \widehat{P}^{t} = \gamma\eta\sigma(\mu)^{-1} P (\mu X)^{\sigma} \widehat{P}^{t} = \lambda P (\mu X)^{\sigma} \widehat{P}^{t}$$

for every $X \in \mathcal{H}_n(\mathbb{F})$, where $\lambda := \gamma\eta\sigma(\mu)^{-1} \in \mathbb{K}^{\wedge}$, $\widehat{P}^{t} P = \zeta I_n$ and $(\lambda\zeta)^{n-2} = 1$. Therefore by (5.14),

$$\psi(A) = \lambda P A^{\sigma} \widehat{P}^{t} \text{ for every } A \in \mathcal{SH}_n(\mathbb{F}).$$

$\square$

Chapter 6

Classical adjoint-commuting mappings on alternate matrices

6.1 Introduction

Let n be an integer with $n \geqslant 2$ and let $\mathbb{F}$ be a field. A matrix $A \in \mathcal{M}_n(\mathbb{F})$ is *alternate* if $uAu^t = 0$ for every row vector $u \in \mathbb{F}^n$, or equivalently, if $A^t = -A$ with zero diagonal entries. We denote by $\mathcal{K}_n(\mathbb{F})$ the linear space of all $n \times n$ alternate matrices over $\mathbb{F}$.

We recall from Proposition 1.3.3 that $A \in \mathcal{K}_n(\mathbb{F})$ if and only if $A = 0$ or there exist an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ and an integer $1 \leqslant r \leqslant \left\lfloor \frac{n}{2} \right\rfloor$ such that

$$A = P(J_1 \oplus \cdots \oplus J_r \oplus 0_{n-2r})P^t \tag{6.1}$$

where $J_1 = \cdots = J_r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let

$$J_n := J_1 \oplus \cdots \oplus J_{n/2} \in \mathcal{K}_n(\mathbb{F}).$$

If n is even, J_n is invertible and adj $J_n = -J_n$.

Lemma 6.1.1. *Let n be an even integer. If $A \in \mathcal{K}_n(\mathbb{F})$, then* adj $A \in \mathcal{K}_n(\mathbb{F})$ *and*

$$\text{rank adj } A = \begin{cases} 0 & \textit{if } \text{rank } A \neq n, \\ n & \textit{if } \text{rank } A = n. \end{cases} \tag{6.2}$$

Proof. Let $A \in \mathcal{K}_n(\mathbb{F})$. Then every (i, i)-cofactor of A is zero. This implies that the diagonal entries of adj A are all zero. In addition,

$$(\text{adj } A)^t = \text{adj } (A^t) = \text{adj } (-A) = (-1)^{n-1}\text{adj } A = -\text{adj } A$$

since n is even. Thus, adj $A \in \mathcal{K}_n(\mathbb{F})$.

If rank $A = n$, then it is clear that rank adj $A = n$. If rank $A \neq n$, then rank $A \leqslant n-2$ since rank A is even by (6.1). Hence, adj $A = 0$. $\square$

Remark 6.1.2. Let n be an odd integer and let $A \in \mathcal{K}_n(\mathbb{F})$. Then adj $A \notin \mathcal{K}_n(\mathbb{F})$ since $(\text{adj } A)^t = \text{adj } (A^t) = \text{adj } (-A) = (-1)^{n-1}\text{adj } A = \text{adj } A$.

Remark 6.1.3. Let q be an integer with $q \geqslant 2$. Let $\mathbb{F}$ be a field and $\mathbb{F}[x]$ be the ring of polynomials in the indeterminate x over $\mathbb{F}$. If $\mathbb{F}$ is algebraically closed, then

$$x^q - c \in \mathbb{F}[x] \text{ has a root in } \mathbb{F} \text{ for every } c \in \mathbb{F}. \tag{6.3}$$

In addition, we also observe that

- if $\mathbb{F} = \mathbb{F}_p$ is a Galois field of p elements with $p = 2$ or $p^r = kq$ for some positive integers r and k, then condition (6.3) holds in $\mathbb{F}_p$ since $c^p = c$ for every $c \in \mathbb{F}_p$;
- if q is odd and $\mathbb{F}$ is the real field $\mathbb{R}$, then it follows by the intermediate value theorem that condition (6.3) holds in $\mathbb{R}$.

Proposition 6.1.4. *Let n be an integer with $n \geqslant 2$, and let $\mathbb{F}$ be a field. Then $\mathbb{F}$ satisfies condition* (6.3) *for $q = n-1$ if and only if for every rank n matrix $A \in \mathcal{M}_n(\mathbb{F})$, there exists a rank n matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that $A = \text{adj } B$.*

Proof. Let $A \in \mathcal{M}_n(\mathbb{F})$ be of rank n. Let $d := (\det A)^{n-2}$. Then $d \neq 0$ and there exists a nonzero scalar $d_0 \in \mathbb{F}$ such that $d_0^{n-1} = d^{-1}$. Thus

$$A = d^{-1}(dA) = d_0^{n-1}\text{adj } (\text{adj } A) = \text{adj } (d_0\text{adj } A) = \text{adj } B$$

where $B = d_0\text{adj } A \in \mathcal{M}_n(\mathbb{F})$ and rank $B = n$.

For the sufficiency, we let $c \in \mathbb{F}$ and we show that there exists a scalar $b \in \mathbb{F}$ such that $b^{n-1} - c = 0$. The result is clear if $c = 0$. We suppose $c \neq 0$. Then there exists an invertible matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that adj $B = cI_n$. Thus

$$(\det B)^{n-2}B = \text{adj (adj } B) = \text{adj } (cI_n) = c^{n-1}I_n.$$

This implies $B = bI_n$ for some scalar $b \in \mathbb{F}$.

$$b^{n-1}I_n = \text{adj } B = cI_n \implies b^{n-1} = c.$$

Therefore, $\mathbb{F}$ satisfies condition (6.3) when $q = n - 1$. □

Following from the result above, we obtain the following Lemma.

Lemma 6.1.5. *Let n be an even positive integer and let $\mathbb{F}$ be a field. Then $\mathbb{F}$ satisfies condition* (6.3) *for $q = n - 1$ if and only if for every rank n matrix $A \in \mathcal{K}_n(\mathbb{F})$, there exists a rank n matrix $B \in \mathcal{K}_n(\mathbb{F})$ such that $A =$ adj B.*

Proof. Let $A \in \mathcal{K}_n(\mathbb{F})$ be of rank n. By Proposition 6.1.4, there exists a rank n matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that $A =$ adj B. This implies

$$\text{adj } A = \text{adj adj } B = (\det B)^{n-2}B \in \mathcal{K}_n(\mathbb{F})$$

by Lemma 6.1.1. Thus, $((\det B)^{n-2}B)^t = -(\det B)^{n-2}B$ and the diagonal entries of $(\det B)^{n-2}B$ are zero. It follows that $B^t = -B$ and the diagonal entries of B are zero. That is, $B \in \mathcal{K}_n(\mathbb{F})$.

Next, we let $c \in \mathbb{F}$. We now show that there exists $b_0 \in \mathbb{F}$ such that $b_0^{n-1} = c$. The result is clear if $c = 0$. We suppose $c \neq 0$. Then $cJ_n =$ adj B for some rank n matrix $B \in \mathcal{K}_n(\mathbb{F})$. As $(\det B)^{n-2}B = \text{adj (adj } B) = \text{adj } (cJ_n) = -c^{n-1}J_n$, we have $B = -b_0J_n$ for some scalar $b_0 \in \mathbb{F}$. Thus $b_0^{n-1}J_n = \text{adj } (-b_0J_n) = \text{adj } B = cJ_n$. Therefore, $b_0^{n-1} = c$. This implies $\mathbb{F}$ satisfies condition (6.3) for $q = n - 1$. We are done. □

6.2 Some basic properties

Here and subsequently, we let m, n be even integers with $m, n \geqslant 4$. Let $\mathbb{F}$ and $\mathbb{K}$ be fields. We study the structure of $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ that satisfies one of the following conditions (see (A1) and (A2) in Section 2.1):

(AK1) $\psi(\text{adj }(A + \alpha B)) = \text{adj }(\psi(A) + \alpha\psi(B))$ for all matrices $A, B \in \mathcal{K}_n(\mathbb{F})$ and any scalar $\alpha \in \mathbb{F}$ when $\mathbb{F} = \mathbb{K}$,

(AK2) $\psi(\text{adj }(A - B)) = \text{adj }(\psi(A) - \psi(B))$ for all matrices $A, B \in \mathcal{K}_n(\mathbb{F})$.

We consider only even integers m, n as adj $A \notin \mathcal{K}_n(\mathbb{F})$ if $A \in \mathcal{K}_n(\mathbb{F})$ and n is odd by Remark 6.1.2.

Let m, n be even integers with $m, n \geqslant 4$. Let $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ be a mapping satisfying (AK2). It can be shown that

$$\psi(0) = 0 \text{ and } \psi(\text{adj } A) = \text{adj }(\psi(A)) \text{ for every } A \in \mathcal{K}_n(\mathbb{F}).$$

Lemma 6.2.1. *Let m, n be even integers with $m, n \geqslant 4$. Let $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ be a mapping satisfying* (AK2). *Let $A \in \mathcal{K}_n(\mathbb{F})$. Then the following statements hold.*

(a) *If $\mathbb{F}$ satisfies condition* (6.3) *for $q = n - 1$, then*

$$\text{rank } A = n \implies \text{rank } \psi(A) = 0 \text{ or } m.$$

(b) rank $\psi(A) \leqslant m - 2$ *if* rank $A \leqslant n - 2$.

Proof.

(a) If rank $A = n$, then by Lemma 6.1.5, there exists a rank n matrix $B \in \mathcal{K}_n(\mathbb{F})$ such that $A = \text{adj } B$. Thus $\psi(A) = \psi(\text{adj } B) = \text{adj } \psi(B)$. If rank $\psi(B) = m$, then rank $\psi(A) = m$. If rank $\psi(B) \neq m$, then $\psi(A) = 0$.

(b) If rank $A \leqslant n-2$, then adj $A = 0$. Thus,

$$\text{adj } \psi(A) = \psi(\text{adj } A) = \psi(0) = 0.$$

This implies rank $\psi(A) \leqslant m-2$.

$\square$

Lemma 6.2.2. *Let m, n be even integers with $m, n \geqslant 4$. Let $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ be a mapping satisfying* (AK2)*. Let $A \in \mathcal{K}_n(\mathbb{F})$. Then ψ is injective if and only if*

$$\text{rank } A = n \iff \text{rank } \psi(A) = m.$$

Proof. By Lemma 6.2.1(b), if rank $\psi(A) = m$, then rank $A = n$. Let A be of rank n. Suppose rank $\psi(A) < m$. Then $\psi(\text{adj (adj } A)) = \text{adj (adj } (\psi(A))) = 0$ since $m \geqslant 4$. This implies adj (adj A) $= 0$ by the injectivity of ψ. This contradicts that rank $A = n$. Thus, by Lemma 6.2.1(a), rank $\psi(A) = m$.

Conversely, suppose $\psi(A) = \psi(B)$ for some $A, B \in \mathcal{K}_n(\mathbb{F})$. Let rank $(A - B) = r$. By Lemma 2.2.2(a), there exists a rank $n-r$ matrix $C \in \mathcal{K}_n(\mathbb{F})$ such that rank $(A - B + C) = n$. Then rank adj $(A - B + C) = n$ and hence rank adj $\psi(A - B + C) = \text{rank } \psi(\text{adj } (A - B + C) = m$. By using (AK2), we have

$$\begin{aligned} \text{adj } \psi(C) &= \text{adj } \psi(B - (B - C)) \\ &= \text{adj } (\psi(B) - \psi(B - C)) \\ &= \text{adj } (\psi(A) - \psi(B - C)) \\ &= \text{adj } (\psi(A - B + C)). \end{aligned}$$

Thus, rank $\psi(C) = m$ and this implies $r = 0$. It follows that $A = B$. Therefore ψ is injective. $\square$

Lemma 6.2.3. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mathbb{F}$ be a field satisfying condition* (6.3) *for $q = n - 1$. Let $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ be a mapping satisfying* (AK2). *Let $P \in \mathcal{M}_n(\mathbb{F})$ be invertible and let $\phi_P : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ be defined by*

$$\phi_P(A) = \psi(PAP^t) \text{ for every } A \in \mathcal{K}_n(\mathbb{F}). \tag{6.4}$$

If $\phi_P(J_n) = 0$, then $\phi_P(A) = 0$ for every invertible matrix $A \in \mathcal{K}_n(\mathbb{F})$.

Proof. Let $A, B \in \mathcal{K}_n(\mathbb{F})$ be invertible matrices with rank $(A - B) < n$. Then adj $(P(A - B)P^t) = 0$ implies $\psi(\text{adj } (P(A - B)P^t)) = 0$. By the definition of ϕ_p and (AK2),

$$\begin{aligned}
\text{adj } (\phi_P(A) - \phi_P(B)) &= \text{adj } (\psi(PAP^t) - \psi(PBP^t)) \\
&= \psi(\text{adj } (PAP^t - PBP^t)) \\
&= \psi(\text{adj } (P(A - B)P^t)) \\
&= 0.
\end{aligned}$$

If $\phi_P(A) = 0$, then adj $\phi_P(B) = 0$. This implies rank $\psi(PBP^t) = \text{rank } \phi_P(B) < m$. By Lemma 6.2.1(a), $\phi_P(B) = \psi(PBP^t) = 0$. Therefore

$$\phi_P(A) = 0 \implies \phi_P(B) = 0 \tag{6.5}$$

if $A, B \in \mathcal{K}_n(\mathbb{F})$ are invertible with rank $(A - B) < n$. Let

$$\mathcal{B} := \{J \oplus S \mid S \in \mathcal{K}_{n-2}(\mathbb{F}) \text{ and rank } S = n - 2\} \subseteq \mathcal{K}_n(\mathbb{F})$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{K}_2(\mathbb{F})$. Let $B \in \mathcal{B}$. Then rank $B = n$ and rank $(J_n - B) < n$. Thus, if rank $\phi_P(J_n) = 0$, then by (6.5)

$$\phi_P(B) = 0 \text{ for every } B \in \mathcal{B}. \tag{6.6}$$

Let $A \in \mathcal{K}_n(\mathbb{F})$ be an invertible matrix. Then A can be written in the form:

$$A = \begin{pmatrix} \alpha J & A_1 \\ -A_1^t & C \end{pmatrix} \in \mathcal{K}_n(\mathbb{F}) \tag{6.7}$$

where $\alpha \in \mathbb{F}$, $A_1 = (a_{ij}) \in \mathcal{M}_{2,n-2}(\mathbb{F})$ and $C \in \mathcal{K}_{n-2}(\mathbb{F})$. We now consider two cases.

Case I: $n = 4$.

Then $C = cJ$ for some scalar $c \in \mathbb{F}$. If $a_{21} = a_{22} = 0$, then $\alpha \neq 0$ and $c \neq 0$. Let $B_1 = J \oplus C \in \mathcal{B}$. Then rank $(A - B_1) < 4$. We obtain $\phi_P(A) = 0$ by (6.5) and (6.6).

Next, we suppose $C \neq 0$. We let

$$B_2 = \begin{pmatrix} aJ & \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -a_{11} & 0 \\ -a_{12} & 0 \end{pmatrix} & C \end{pmatrix} \in \mathcal{K}_4(\mathbb{F}) \text{ where } a = \begin{cases} \alpha & \text{if } \alpha \neq 0, \\ 1 & \text{if } \alpha = 0. \end{cases}$$

Thus, B_2 is invertible in both cases. Since rank $(B_1 - B_2) < 4$ when $\alpha \neq 0$, $\phi_P(B_1) = 0$ implies $\phi_P(B_2) = 0$. When $\alpha = 0$, rank $(J_n - B_2) < 4$ implies $\phi_P(B_2) = 0$. Thus, $\phi_P(A) = 0$ by (6.5) since rank $(A - B_2) < 4$.

Now, we suppose $C = 0$. Then A_1 is invertible. If $\alpha \neq 0$, we select

$$B_3 = \begin{pmatrix} \alpha J & \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -a_{11} & 0 \\ -a_{12} & 0 \end{pmatrix} & J \end{pmatrix} \in \mathcal{K}_4(\mathbb{F}).$$

It can be easily seen that B_3 is invertible and rank $(J_n - B_3) < 4$. Thus, $\phi_P(B_3) = 0$. Since rank $(A - B_3) < 4$, $\phi_P(A) = 0$ by (6.5). If $\alpha = 0$, we choose

$$B_4 = \begin{pmatrix} J & A_1 \\ -A_1^t & 0 \end{pmatrix} \in \mathcal{K}_4(\mathbb{F}).$$

It is obvious that B_4 is invertible and $\phi_P(B_4) = 0$. As rank $(A - B_4) < 4$, we have $\phi_P(A) = 0$ by (6.5).

Case II: $n \geqslant 6$.

Let $A \in \mathcal{K}_n(\mathbb{F})$ be invertible of form (6.7). If C is invertible, then we choose

$H_1 = J \oplus C \in \mathcal{K}_n(\mathbb{F})$. It is clear that $H_1 \in \mathcal{B}$ and rank $(A - H_1) < n$. Thus, $\phi_P(A) = 0$ by (6.5). We now suppose C is not invertible. We observe that

$$\operatorname{rank}\left(\begin{pmatrix} \alpha J & A_1 \\ -A_1^t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}\right) = \operatorname{rank} A = n \text{ and } \operatorname{rank} \begin{pmatrix} \alpha J & A_1 \\ -A_1^t & 0 \end{pmatrix} \leqslant 4.$$

Thus rank $C \geqslant n-4$. Since C is not invertible, rank $C = n-4$. By (6.1), there exists an invertible matrix $P \in \mathcal{M}_{n-1}(\mathbb{F})$ such that

$$C = P(J_1 \oplus \cdots \oplus J_{(n-4)/2} \oplus 0_2)P^t \tag{6.8}$$

where $J_i = J$ for $i = 1, \cdots, (n-4)/2$.

If $n \geqslant 8$, we choose $H_2 = J \oplus P(J_1 \oplus \cdots \oplus J_{(n-4)/2} \oplus J)P^t \in \mathcal{K}_n(\mathbb{F})$. It can be easily shown that $H_2 \in \mathcal{B}$. Thus, $\phi_P(H_2) = 0$ by (6.6). Since rank $(A - H_2) < n$, we obtain $\phi_P(A) = 0$ by (6.5).

Next, we suppose $n = 6$. We denote by $\mathcal{G}$ the set of 6×6 invertible alternate matrices of the form

$$G = \begin{pmatrix} aJ & U \\ -U^t & V \end{pmatrix} \in \mathcal{K}_6(\mathbb{F})$$

where $a \in \mathbb{F}$ is nonzero, $U = (u_{ij}) \in \mathcal{M}_{2,4}(\mathbb{F})$ with $u_{2j} = 0$ for $j = 1, \cdots, 4$ and $V \in \mathcal{K}_4(\mathbb{F})$ is invertible. We choose $H_3 = J \oplus V \in \mathcal{K}_6(\mathbb{F})$. As $V \in \mathcal{K}_4(\mathbb{F})$ is invertible, we have $H_3 \in \mathcal{B}$ and hence $\phi_P(H_3) = 0$ by (6.6). We observe that rank $(G - H_3) < 6$. It follows from (6.5) that

$$\phi_P(G) = 0 \text{ for every } G \in \mathcal{G}. \tag{6.9}$$

Let $A \in \mathcal{K}_6(\mathbb{F})$ be an invertible matrix of form (6.7) with singular C. By (6.8), $C = P(J \oplus 0_2)P^t \in \mathcal{K}_4(\mathbb{F})$. We select

$$H_4 = \begin{pmatrix} J & \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -a_{11} & 0 \\ -a_{12} & 0 \\ -a_{13} & 0 \\ -a_{14} & 0 \end{pmatrix} & P(J \oplus J)P^t \end{pmatrix} \in \mathcal{K}_6(\mathbb{F}).$$

Then $H_4 \in \mathcal{G}$ and

$$\text{rank }(A - H_4) = \text{rank} \begin{pmatrix} (\alpha-1)J & \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \\ \begin{pmatrix} 0 & -a_{21} \\ 0 & -a_{22} \\ 0 & -a_{23} \\ 0 & -a_{24} \end{pmatrix} & P(0_2 \oplus J)P^t \end{pmatrix} \leqslant 4.$$

By (6.9) and (6.5), $\phi_P(A) = 0$.

All the cases show that

$$\phi_P(A) = 0 \text{ for every invertible matrix } A \in \mathcal{K}_n(\mathbb{F})$$

if $\phi_P(J_n) = 0$. We are done. □

Lemma 6.2.4. *Let m, n be even integers with $m, n \geqslant 4$ and let $\mathbb{F}$ and $\mathbb{K}$ be fields with $\mathbb{F}$ satisfying condition* (6.3) *for $q = n - 1$. Let $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ be a mapping satisfying condition* (AK2)*. Then the following statements hold.*

(a) *$\psi(J_n) = 0$ if and only if* rank *$\psi(A) \leqslant m - 2$ for every $A \in \mathcal{K}_n(\mathbb{F})$.*

(b) *$\psi(J_n) \neq 0$ if and only if ψ is injective.*

Proof.

(a) Let $A \in \mathcal{K}_n(\mathbb{F})$. If rank $A \leqslant n - 2$, then rank $\psi(A) \leqslant m - 2$ by Lemma 6.2.1(b). Next, we suppose rank $A = n$. Since $\psi(J_n) = 0$, by letting P in Lemma 6.2.3 be I_n , we have $\psi(A) = \phi_P(A) = 0$.

Conversely, if rank $\psi(A) \leqslant m - 2$ for every $A \in \mathcal{K}_n(\mathbb{F})$, then rank $\psi(J_n) \leqslant m - 2$. This implies $\psi(J_n) = 0$ since rank $J_n = n$ and Lemma 6.2.1 (a).

(b) If ψ is injective and $\psi(0) = 0$, $\psi(J_n) \neq 0$. Conversely, we suppose $\psi(J_n) \neq 0$. If rank $\psi(A) = m$, then by Lemma 6.2.1 (b), rank $A = n$. Next, we suppose rank $A = n$. By (6.1), there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ such that $A = PJ_nP^t$. Let $\phi_P : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ be defined by

$$\phi_P(X) = \psi(PXP^t) \text{ for every } X \in \mathcal{K}_n(\mathbb{F}).$$

Thus, $\phi_P(P^{-1}J_n(P^{-1})^t) = \psi(P(P^{-1}J_n(P^{-1})^t)P^t) = \psi(J_n) \neq 0$. If rank $\psi(A) \neq m$, then $\psi(A) = 0$ by Lemma 6.2.1 (a). This implies $\phi_P(J_n) = \psi(PJ_nP^t) = \psi(A) = 0$. Then $\phi_P(X) = 0$ for every invertible matrix $X \in \mathcal{K}_n(\mathbb{F})$. In particular, $\phi_P(P^{-1}J_n(P^{-1})^t) = 0$ which is a contradiction. Therefore,

$$\text{rank } A = n \iff \text{rank } \psi(A) = m.$$

It follows that ψ is injective by Lemma 6.2.2.

□

6.3 Some examples

Let m, n be even integers with $m, n \geqslant 4$ and let $\mathbb{F}$ be a field satisfying condition (6.3) for $q = n - 1$. If ψ satisfies condition (AK1) or (AK2) and $\psi(J_n) = 0$, we have $\psi(A) = 0$ for every invertible matrix $A \in \mathcal{K}_n(\mathbb{F})$ by Lemma 6.2.1 and Lemma 6.2.4. In this section, we give some examples of such mappings that send all invertible matrices to zero.

Example 6.3.1. Let m, n be even integers with $m, n \geqslant 4$ and let $\mathbb{F}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $\tau : \mathcal{K}_n(\mathbb{F}) \to \mathbb{F}$ be a nonzero function and let $\psi_1 : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{F})$ be the mapping defined by

$$\psi_1(A) = \begin{cases} \tau(A)(E_{12} - E_{21}) & \text{if } A \in \mathcal{K}_n(\mathbb{F}) \text{ is of rank } r \text{ with } 2 \leqslant r \leqslant n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.3.2. Let m, n be even integers with $m, n \geqslant 4$ and let $\mathbb{F}$ be a field with $n-1$ elements. Let $f : \mathbb{F} \to \mathbb{F}$ and $g : \mathbb{F} \to \mathbb{F}$ be nonzero functions. Let $A = (a_{ij}) \in \mathcal{K}_n(\mathbb{F})$ and let $\psi_2 : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{F})$ be the mapping defined by

$$\psi_2(A) = \begin{cases} \sum_{i=1}^{\frac{m}{2}-1} f(a_{12})(E_{2i-1,2i} - E_{2i,2i-1}) & \text{if } A \in \mathcal{K}_n(\mathbb{F}) \text{ is of rank two,} \\ g(a_{12})(E_{12} - E_{21}) & \text{if } A \in \mathcal{K}_n(\mathbb{F}) \text{ is of rank } r, 2 < r < n, \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily verified that ψ_1 and ψ_2 are both classical adjoint-commuting mappings satisfying condition (AK1) or (AK2) and send all invertible matrices to zero.

6.4 Characterisation of classical adjoint-commuting mappings on alternate matrices

Let $A \in \mathcal{K}_4(\mathbb{F})$. Here, we note that $A^* \in \mathcal{K}_4(\mathbb{F})$ is defined as in (1.10). That is,

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}. \tag{6.10}$$

Then adj $A^* = (\text{adj } A)^*$ for every $A \in \mathcal{K}_4(\mathbb{F})$.

Let k, n be even integers with $4 \leqslant k \leqslant n$ and let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$. Let S be a subset of $\mathcal{K}_n(\mathbb{F})$ and we define

$$S^{\perp_k} := \{B \in \mathcal{K}_n(\mathbb{F}) \mid \text{rank } (A - B) \leqslant k \text{ for every } B \in S\}$$

and $S^{\perp_k\perp_k} := (S^{\perp_k})^{\perp_k}$ if $S^{\perp_k}$ is nonempty. Let $A, B \in \mathcal{K}_n(\mathbb{F})$. A and B are said to be *adjacent* if rank $(A - B) = 2$ (see Definition 1.5.5). The following lemma was proved in [18, Lemmas 3.2 and 3.3].

Lemma 6.4.1. *Let k, m be even integers with $4 \leqslant k \leqslant m$, and let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$. Let $A, B \in \mathcal{K}_n(\mathbb{F})$ such that* rank $(A - B) \leqslant k$. *Then A, B are adjacent if and only if $|\{A, B\}^{\perp_k\perp_k}| \geqslant 3$.*

Definition 6.4.2. $\varphi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ is called an *adjacency preserving mapping in both directions* if

$$\text{rank } (A - B) = 2 \iff \text{rank } (\varphi(A) - \varphi(B)) = 2 \text{ for all } A, B \in \mathcal{K}_n(\mathbb{F}).$$

We state the following proposition without proof. The details of the proposition can be found in [11, 18, 12, 13].

Proposition 6.4.3. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mathbb{F}$ and $\mathbb{K}$ be fields with at least three elements. If $\varphi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ is a surjective mapping satisfying*

$$\text{rank } (A - B) = n \iff \text{rank } (\varphi(A) - \varphi(B)) = m \tag{6.11}$$

for all $A, B \in \mathcal{K}_n(\mathbb{F})$, then φ is a bijective adjacency preserving mapping in both directions, $m = n$, and $\mathbb{F}$ and $\mathbb{K}$ are isomorphic.

Theorem 6.4.4. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mathbb{K}$ be a field with $|\mathbb{K}| \geqslant 3$, and let $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$ such that $x^{n-1} - c \in \mathbb{F}[x]$ has a root for every $c \in \mathbb{F}$. Then $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ is a surjective mapping satisfying* (AK2) *if and only if $m = n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic, and either*

$$\psi(A) = \lambda P A^\sigma P^t \textit{ for every } A \in \mathcal{K}_n(\mathbb{F})$$

or when $n = 4$,

$$\psi(A) = \lambda P (A^*)^\sigma P^t \textit{ for every } A \in \mathcal{K}_4(\mathbb{F})$$

where $\sigma : \mathbb{F} \to \mathbb{K}$ is a field isomorphism, A^σ is the matrix obtained from A by applying σ entrywise, $P \in \mathcal{M}_n(\mathbb{K})$ is invertible with $P^t P = \zeta I_n$, $\lambda, \zeta \in \mathbb{K}$ are nonzero scalars with $(\lambda\zeta)^{n-2} = 1$, and

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Proof. The sufficiency part can be shown easily. We now consider the necessity part. Suppose $\psi(J_n) = 0$. Then rank $\psi(A) \leqslant m - 2$ for every $A \in \mathcal{K}_n(\mathbb{F})$ by Lemma 6.2.4 (a). This contradicts that ψ is surjective. Thus $\psi(J_n) \neq 0$ and hence ψ is injective by Lemma 6.2.4 (b). It follows from Lemma 6.2.2 that

$$\begin{aligned}\text{rank } (A - B) = n &\iff \text{rank } \psi(A - B) = m\\ &\iff \text{rank adj } (\psi(A - B)) = m\\ &\iff \text{rank } \psi(\text{adj } (A - B)) = m\\ &\iff \text{rank adj } (\psi(A) - \psi(B)) = m\\ &\iff \text{rank } (\psi(A) - \psi(B)) = m.\end{aligned}$$

Then it follows by Proposition 6.4.3 that ψ is a bijective adjacency preserving mapping in both directions, $m = n$, and $\mathbb{F}$ and $\mathbb{K}$ are isomorphic. Since $\psi(0) = 0$ and by Theorem 1.5.6, the fundamental theorem of geometry of alternate matrices, either

$$\psi(A) = \lambda P A^\sigma P^t \text{ for every } A \in \mathcal{K}_n(\mathbb{F}) \tag{6.12}$$

or when $n = 4$,

$$\psi(A) = \lambda P (A^*)^\sigma P^t \text{ for every } A \in \mathcal{K}_4(\mathbb{F}), \tag{6.13}$$

where $\sigma : \mathbb{F} \to \mathbb{K}$ is a field isomorphism, $\lambda \in \mathbb{K}$ is a nonzero scalar and $P \in \mathcal{M}_n(\mathbb{K})$ is invertible.

Next, we want to show that there exists a nonzero scalar $\zeta \in \mathbb{K}$ such that

$$P^t P = \zeta I_n \text{ and } (\lambda\zeta)^{n-2} = 1. \tag{6.14}$$

We first consider case (6.12). We have

$$\begin{aligned}\lambda P \text{adj } (A^\sigma - B^\sigma) P^t &= \lambda P \text{adj } (A - B)^\sigma P^t\\ &= \psi(\text{adj } (A - B))\\ &= \text{adj } (\psi(A) - \psi(B))\end{aligned}$$

and hence

$$\begin{aligned}\lambda P\text{adj }(A^\sigma - B^\sigma)P^t &= \text{adj }(\lambda P A^\sigma P^t - \lambda P B^\sigma P^t)\\ &= \text{adj }(\lambda P(A^\sigma - B^\sigma)P^t)\\ &= \lambda^{n-1}(\text{adj }P^t)\text{adj }(A^\sigma - B^\sigma)(\text{adj }P)\end{aligned}$$

for all $A, B \in \mathcal{K}_n(\mathbb{F})$. Thus,

$$\begin{aligned}\text{adj }(A^\sigma - B^\sigma) &= \lambda^{n-2}P^{-1}(\text{adj }P^t)\text{adj }(A^\sigma - B^\sigma)(\text{adj }P)(P^t)^{-1}\\ &= \lambda^{n-2}P^{-1}(\det P^t)(P^t)^{-1}\text{adj }(A^\sigma - B^\sigma)(\det P)P^{-1}(P^t)^{-1}\\ &= \lambda^{n-2}(\det P^tP)(P^tP)^{-1}\text{adj }(A^\sigma - B^\sigma)(P^tP)^{-1}.\end{aligned}$$

It follows that

$$\text{adj }(A^\sigma - B^\sigma) = \lambda^{n-2}(\det Q)Q^{-1}\text{adj }(A^\sigma - B^\sigma)Q^{-1}$$

where $Q = P^tP$ is invertible and $Q^t = Q$. Thus, we have

$$H = \lambda^{n-2}(\det Q)Q^{-1}HQ^{-1}$$

for every invertible matrix $H \in \mathcal{K}_n(\mathbb{F})$. Let $1 \leqslant i \neq j \leqslant n$. Then $J_n + \lambda(E_{ij} - E_{ji}) \in \mathcal{K}_n(\mathbb{F})$ is invertible. Hence,

$$J_n + \lambda(E_{ij} - E_{ji}) = \lambda^{n-2}(\det Q)Q^{-1}(J_n + \lambda(E_{ij} - E_{ji}))Q^{-1}.$$

Since $J_n \in \mathcal{K}_n(\mathbb{F})$ is invertible, we have $J_n = \lambda^{n-2}(\det Q)Q^{-1}J_nQ^{-1}$. Thus,

$$J_n + \lambda(E_{ij} - E_{ji}) = \lambda^{n-2}(\det Q)Q^{-1}J_nQ^{-1} + \lambda^{n-2}(\det Q)Q^{-1}\lambda(E_{ij} - E_{ji})Q^{-1}$$

$$\implies \lambda(E_{ij} - E_{ji}) = \lambda^{n-2}(\det Q)Q^{-1}\lambda(E_{ij} - E_{ji})Q^{-1}.$$

It follows that

$$Q(E_{ij} - E_{ji}) = \lambda^{n-2}(E_{ij} - E_{ji})\text{adj }Q \text{ for all } 1 \leqslant i \neq j \leqslant n. \tag{6.15}$$

Let $Q = (q_{ij})$. By (6.15) and $Q^t = Q$, we have

$$q_{ij} = 0 \text{ and } q_{ii}q_{jj} - q_{ij}^2 = \lambda^{n-2}(\det Q) \text{ for all } 1 \leqslant i \neq j \leqslant n. \tag{6.16}$$

Thus, we have $q_{ii}q_{jj} = \lambda^{n-2}(\det Q)$ for all $1 \leqslant i \neq j \leqslant n$ and hence $q_{ii} = \zeta$ for some nonzero $\zeta \in \mathbb{F}$ for every $i = 1, \cdots, n$. This implies $P^tP = Q = \zeta I_n$. Then by (6.16), $\zeta^2 = \lambda^{n-2}\zeta^n$ leads to $(\lambda\zeta)^{n-2} = 1$.

Since adj $A^* = (\text{adj } A)^*$ for every $A \in \mathcal{K}_4(\mathbb{F})$, case (6.13) can be shown by using similar arguments. We are done. $\square$

The following corollary is a consequence of Theorem 6.4.4

Corollary 6.4.5. *Let m, n be even integers with $m, n \geqslant 4$. Let $\mathbb{K}$ be a field with $|\mathbb{K}| \geqslant 3$, and $\mathbb{F}$ be a field with $|\mathbb{F}| \geqslant 3$ such that $x^{n-1} - c \in \mathbb{F}[x]$ has a root for every $c \in \mathbb{F}$. Then $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ is a surjective classical adjoint-commuting additive mapping if and only if $m = n$, $\mathbb{F}$ and $\mathbb{K}$ are isomorphic, and either*

$$\psi(A) = \lambda P A^\sigma P^t \text{ for every } A \in \mathcal{K}_n(\mathbb{F})$$

or when $n = 4$,

$$\psi(A) = \lambda P (A^*)^\sigma P^t \text{ for every } A \in \mathcal{K}_4(\mathbb{F})$$

where $\sigma : \mathbb{F} \to \mathbb{K}$ is a field isomorphism, A^σ is the matrix obtained from A by applying σ entrywise, $P \in \mathcal{M}_n(\mathbb{K})$ is an invertible matrix with $P^t P = \zeta I_n$, $\lambda, \zeta \in \mathbb{K}$ are nonzero scalars with $(\lambda\zeta)^{n-2} = 1$ and

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Proof. Since ψ is a surjective classical adjoint commuting additive mapping,

$$\begin{aligned} \psi(\text{adj } (A - B)) &= \psi(\text{adj } (A + (-B))) \\ &= \text{adj } (\psi(A + (-B))) \\ &= \text{adj } (\psi(A) + \psi(-B)). \end{aligned}$$

In addition, $0 = \psi(0) = \psi(B - B) = \psi(B) + \psi(-B)$ implies $\psi(-B) = -\psi(B)$. Thus, $\psi(\text{adj } (A - B)) = \text{adj } (\psi(A) - \psi(B))$ which is (AK2). Therefore, the result is obtained from Theorem 6.4.4. $\square$

By using an analogous proof of Lemma 2.2.6, it can be shown that ψ is linear. The result is formulated in the following lemma.

Lemma 6.4.6. *Let m, n be even integers with $m, n \geqslant 4$, and let $\mathbb{F}$ be a field with $|\mathbb{F}| = 2$ or $|\mathbb{F}| > n+1$ satisfying condition* (6.3) *for $q = n-1$. Let $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{K})$ be a mapping satisfying* (AK1). *If*

$$\text{rank } (A + \alpha B) = n \iff \text{rank } (\psi(A) + \alpha\psi(B)) = m$$

for all $A, B \in \mathcal{K}_n(\mathbb{F})$, then ψ is linear.

Theorem 6.4.7. *Let n be an even integer with $n \geqslant 4$. Let $\mathbb{F}$ be a field with $|\mathbb{F}| > n+1$ such that $x^{n-1} - c \in \mathbb{F}[x]$ has a root for every $c \in \mathbb{F}$. Then $\psi : \mathcal{K}_n(\mathbb{F}) \to \mathcal{K}_m(\mathbb{F})$ is a mapping satisfying* (AK1) *if and only if either $\psi(A) = 0$ for every invertible $A \in \mathcal{K}_n(\mathbb{F})$, and* rank $(\psi(A) + \alpha\psi(B)) \leqslant n-2$ *for every $A, B \in \mathcal{K}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$; or either*

$$\psi(A) = \lambda P A P^t \text{ for every } A \in \mathcal{K}_n(\mathbb{F})$$

or when $n = 4$,

$$\psi(A) = \lambda P A^* P^t \text{ for every } A \in \mathcal{K}_4(\mathbb{F})$$

where $P \in \mathcal{M}_n(\mathbb{F})$ is an invertible matrix with $P^tP = \zeta I_n$, $\lambda, \zeta \in \mathbb{F}$ are nonzero scalars with $(\lambda\zeta)^{n-2} = 1$ and

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{23} \\ -a_{12} & 0 & a_{14} & a_{24} \\ -a_{13} & -a_{14} & 0 & a_{34} \\ -a_{23} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Proof. The sufficiency can be shown easily. We now proceed to the necessity. Since ψ satisfies (AK1), ψ also satisfies (AK2).

We first consider the case where $\psi(J_n) = 0$. Then by Lemma 6.2.4 (a), we have rank $\psi(A) \leqslant n-2$ for every $A \in \mathcal{K}_n(\mathbb{F})$. This implies

$$\text{rank } \psi(A + \alpha B) \leqslant n-2 \text{ for all } A, B \in \mathcal{K}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}.$$

Let $A \in \mathcal{K}_n(\mathbb{F})$ be of rank n. Then there exists a rank n matrix $B \in \mathcal{K}_n(\mathbb{F})$ such that $A = \text{adj } B$ by Lemma 6.1.5. Thus, $\psi(A) = \psi(\text{adj } B) = \text{adj } \psi(B) = 0$ since rank $B \leqslant n-2$. Therefore $\psi(A) = 0$ for every invertible matrix $A \in \mathcal{K}_n(\mathbb{F})$.

Next, we consider $\psi(J_n) \neq 0$. Then by Lemma 6.2.4 (b), ψ is injective and hence it follows from Lemma 6.2.2 that

$$\begin{aligned} \text{rank } (A+\alpha B) = n &\iff \text{rank adj } (A+\alpha B) = n \\ &\iff \text{rank } \psi(\text{adj } (A+\alpha B)) = m \\ &\iff \text{rank adj } (\psi(A)+\alpha\psi(B)) = m \\ &\iff \text{rank } (\psi(A)+\alpha\psi(B)) = m. \end{aligned}$$

Thus, by Lemma 6.4.6, ψ is linear. This implies ψ is surjective. The result follows from Corollary 6.4.5 and the homogeneity of ψ. $\square$

Chapter 7

Conclusion

As a conclusion of the thesis, in this research, we study the classical adjoint-commuting mappings on various types of matrices such as full matrices, hermitian matrices, symmetric matrices, skew-hermitian matrices and alternate matrices. We obtained a number of characterisations of these mappings, such as:

(i) characterisations of classical adjoint-commuting mappings between matrix algebras in Theorems 3.4.1 and 3.4.2;

(ii) characterisations of classical adjoint-commuting mappings on hermitian matrices in Theorems 4.4.1, 4.4.2 and 4.4.3;

(iii) characterisations of classical adjoint-commuting mappings on symmetric matrices in Theorems 4.5.1, 4.5.2 and 4.5.3;

(iv) characterisations of classical adjoint-commuting mappings on skew-hermitian matrices in Theorems 5.4.2, 5.4.3 and 5.4.4;

(v) characterisations of classical adjoint-commuting mappings on alternate matrices in Theorems 6.4.4, 6.4.7 and Corollary 6.4.5.

On the other hand, we have also identified some open problems for future investigation. In our study, we apply Lemma 2.2.3 in the proofs of Theorems 3.4.1, 4.4.2, 4.5.2, 5.4.3. Since Lemma 2.2.3 does not include the case where $|\mathbb{F}| = 3$, this causes that the theorems are not proven for the case where $|\mathbb{F}| = 3$. These theorems can be improved by including the omitted case which we have yet to find a solution. In addition, Theorem 6.4.4 is not proven for fields with exactly two elements. This is another open problem that this thesis has not solved. Furthermore, the research can be continued by considering other matrix spaces such as upper triangular matrices, strictly upper triangular matrices etc.

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List of Publications

1. W.L. Chooi and W.S. Ng, On classical adjoint-commuting mappings between matrix algebras. *Linear Algebra and its Applications*, 432: 2589–2599, 2010. (ISI-Cited Publication)

2. W.L. Chooi and W.S. Ng. Classical adjoint-commuting mappings on hermitian and symmetric matrices, *Linear Algebra and its Applications* 435: 202– 223, 2011. (ISI-Cited Publication)

3. Wai Leong Chooi and Wei Shean Ng, Classical adjoint commuting mappings on alternate matrices and skew-hermitian matrices. *Operators and Matrices*, 8(2): 485–512, 2014. (ISI-Cited Publication)