ON EIGENVALUES OF CERTAIN CAYLEY GRAPHS

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ABSTRACT

Let S_n be the symmetric group on $[n] = \{1, \ldots, n\}$. The k-point fixing graph, $\mathscr{F}(n,k)$ is defined to be the graph with vertex set S_n and two vertices g and h of $\mathscr{F}(n,k)$ are joined if and only if gh^{-1} fixes exactly k points. $\mathscr{F}(n,k)$ is a Cayley graph on S_n generated by $\mathscr{S}(n,k)$, the union of the conjugacy classes that fixes exactly k points. A subset I of S_n is said to be an independent set in $\mathscr{F}(n,k)$ if and only if any two vertices in I are not adjacent to each other. The problem of finding such a set is called the maximum independent set problem and it is an NP-hard optimization problem. We are going to determine the size of the largest independent set in $\mathscr{F}(n,k)$ for $0 < k << n$ by using the Delsarte-Hoffman Bound. In order to do so, eigenvalues of the adjacency matrix of $\mathscr{F}(n,k)$ are required in finding a bound for the size of a largest independent set in $\mathscr{F}(n,k)$.
To determine the eigenvalues of the adjacency matrix of $\mathscr{F}(n,k)$, we use the representation theory of symmetric groups. In particular, we use the character theory of symmetric groups. We apply the branching rule and results from Foulkes to derive a recurrence formula for eigenvalues of $\mathscr{F}(n,k)$. Then we apply our results and some of the results regarding the eigenvalues and size of largest independent set of $\mathscr{F}(n,0)$ to determine the sign of the eigenvalues of $\mathscr{F}(n,1)$. Then, we determine the smallest eigenvalue of $\mathscr{F}(n,1)$ by techniques in extremal combinatorics. We use the largest and smallest eigenvalues of $\mathscr{F}(n,1)$ and apply the Delsarte-Hoffman Bound to determine a bound for the size of a largest independent set in $\mathscr{F}(n,1)$. When $0 < k << n$, we determine the smallest eigenvalues of $\mathscr{F}(n,k)$ and the Specht module where it occurs. Similarly, we use the largest and smallest eigenvalues of $\mathscr{F}(n,k)$ and apply the Delsarte-Hoffman Bound to determine a bound for the size of a largest independent set in $\mathscr{F}(n,k)$.
We also consider $\mathscr{F}(n,0)$, the derangement graph with generating set D_n, the derangement set. For any fixed positive integer $k \leq n$, the Cayley graph on S_n generated by the subset of D_n consisting of permutations without any i-cycles for all $1 \leq i \leq k$ is a regular subgraph of the derangement graph. We determine the smallest eigenvalue of these subgraphs and show that the set of all largest independent sets in these subgraphs is equal to the set of all the largest independent sets in $\mathscr{F}(n,0)$ provided that $k << n$.

ABSTRAK

Biar S_n sebagai kumpulan simetrik pada $[n] = \{1, ..., n\}$. Graf menetapkan k-titik, $\mathscr{F}(n,k)$ ditakrifkan sebagai graf dengan S_n sebagai set bucu dan dua bucu, g dan h disambungkan jika dan hanya jika gh^{-1} menetapkan setepat-tepatnya k-titik. $\mathscr{F}(n,k)$ adalah satu graf Cayley pada S_n yang dijanakan oleh $\mathscr{S}(n,k)$, di mana S_n adalah kesatuan kelas-kelas konjugasi yang menetapkan setepat-tepatnya k-titik. Suatu subset I bagi S_n dikatakan sebagai satu set berdikari dalam $\mathscr{F}(n,k)$ jika dan hanya jika mana-mana dua bucu dalam I tidak bersebelahan antara satu sama lain. Masalah mencari set seperti ini dipanggil masalah set berdikari maksimum dan merupakan satu masalah pengoptimuman NP-susah. Kita akan menentukan saiz set berdikari yang paling besar dalam $\mathscr{F}(n,k)$ untuk $0 < k << n$ dengan menggunakan Batasan Delsarte-Hoffman. Untuk mencapai tujuan ini, nilai eigen bagi matriks bersebelahan untuk $\mathscr{F}(n,k)$ diperlukan untuk menentukan batasan untuk saiz set berdikari yang paling besar dalam $\mathscr{F}(n,k)$.

Untuk menentukan nilai eigen bagi matriks bersebelahan untuk $\mathscr{F}(n,k)$, kita menggunakan teori perwakilan kumpulan simetri. Khususmya, kita menggunakan teori watak kumpulan simetri. Kita menggunakan Peraturan Bercabang dan keputusan dari Foulkes untuk memperoloehi formula pengulangan untuk nilai eigen $\mathscr{F}(n,k)$. Seterusnya kita menggunakan keputasan kita serta beberapa keputusan mengenai nilai eigen dan saiz set berdikari terbesar dalam $\mathscr{F}(n,0)$ untuk menentukan tanda nilai eigen bagi $\mathscr{F}(n,1)$. Seterusnya, kita menentukan nilai eigen yang terkecil dalam $\mathscr{F}(n,1)$ melalui teknik-teknik kombinatorik ekstrimal. Kita menggunakan nilai eigen yang terbesar dan terkecil dalam $\mathscr{F}(n,1)$ dan menggunakan Batasan Delsarte-Hoffman untuk menentukan batasan untuk saiz set berdikari yang terbesar dalam $\mathscr{F}(n,1)$. Apabila $0 < k << n$, kita menentukan nilai eigen terkecil dalam $\mathscr{F}(n,k)$ dan modul Specht di mana ianya berlaku. Selain itu, kita menggunakan nilai eigen yang terbesar dan terkecil dalam $\mathscr{F}(n,k)$ dan menggunakan Batasan Delsarte-Hoffman untuk menentukan batasan untuk saiz sesuatu set berdikari yang terbesar dalam $\mathscr{F}(n,k)$.

Kita juga mempertimbangkan $\mathscr{F}(n,0)$, graf kekacauan dengan set penjana D_n, iaitu set kekacauan. Bagi mana-mana integer positif yang tetap $k \leq n$, graf Cayley pada S_n yang dijanakan oleh subset D_n yang terdiri daripada pilih atur tanpa apa-apa i-kitaran bagi $1 \leq i \leq k$ adalah subgraf yang biasa dari graf kekacauan itu. Kita menentukan nilai eigen yang terkecil dalam subgraf ini dan menunjukkan bahawa set yang mengandungi semua set berdikari yang terbesar di subgraf ini adalah sama dengan set yang mengandungi semua set berdikari yang terbesar dalam $\mathscr{F}(n,0)$ bagi $k << n$.

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Terry Lau

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CHAPTER 1: INTRODUCTION AND LITERATURE REVIEW

In computer science, several computational problems related to independent sets have been studied. The independent set problem and the clique problem are complimentary. Therefore, many computational results may be applied equally well to either problem. However, the maximum independent set problem is NP-hard and it is also hard to be determined. Therefore, we are interested in other alternatives to determine the size of a maximum independent set.

In 1970, A.J. Hoffman (1970) proved the Delsarte-Hoffman Bound, which gives a bound on the largest independent set of a regular graph. With this bound, we are able to bound the largest independent set by determining the largest and smallest eigenvalues of the graph. In particular, Cayley graph is a special kind of regular graph which is generated by a group and a generating set. By considering some groups with certain algebraic structures, we are able to determine the eigenvalues even though the graph structure is complicated.

Let S_n be the symmetric group on $[n] = \{1, \ldots, n\}$. Consider the Cayley graph on S_n generated by the derangement set, D_n, i.e. the set of elements that fixes no point in $[n]$, such a Cayley graph is well known as the derangement graph. Several results of the derangement graph have been well studied by various people. In particular, Renteln (2007) has proved a recurrence formula for the eigenvalues of partitions in the derangement graph, thus determining the smallest eigenvalue of the derangement graph. By applying Delsarte-Hoffman Bound, the largest independent set in the derangement graph is also determined. Furthermore, Ku and Wales (2010) have developed an Alternating Sign Property for the eigenvalues of the derangement graph.

In this thesis, we consider the k-point fixing graph, namely $\mathscr{F}(n,k)$ which is the Cayley graph on S_n generated by $\mathscr{S}(n,k)$, the union of the conjugacy classes that consists of permutations that fix exactly k points. In particular, the derangement graph is a 0-point fixing graph. We want to determine the size of the largest independent set in $\mathscr{F}(n,k)$ for $0 < k << n$ by using the Delsarte-Hoffman Bound. In order to do so, eigenvalues of the adjacency matrix of $\mathscr{F}(n,k)$ are required in finding a bound for the size of a largest independent set in $\mathscr{F}(n,k)$. Furthermore, for any fixed positive integer $k \leq n$, we consider a regular subgraph of the derangement graph, where this subgraph is a Cayley graph on S_n generated by a subset of D_n consisting of permutations without any i-cycles for all $1 \leq i \leq k$. We want to determine the smallest eigenvalue of these subgraphs and the largest independent set in these subgraphs given $k << n$.

1.1 Definitions & Terminology

In this section we will provide the basic and important definitions for the thesis.

Definition 1.1.1. *We define the following terminologies:*

1. *A* multigraph, Γ *consists of a non-empty finite set of vertices, denoted by* $V(\Gamma)$ *and a finite (possibly empty) set of edges, denoted by* $E(\Gamma)$ *such that each edge in* $E(\Gamma)$ *joins two distinct vertices in* $V(\Gamma)$ *and two distinct vertices in* $V(\Gamma)$ *are joined by a finite (possibly zero) number of edges.*

2. *The* order *of* Γ, *denoted by* $v(\Gamma)$, *is the number of vertices in* $V(\Gamma)$ *while the* size *of* Γ, *denoted by* $e(\Gamma)$, *is the number of edges in* $E(\Gamma)$.

3. *A multigraph* Γ *is called a* simple graph *if any two vertices in* $V(\Gamma)$ *are joined by at most one edge.*

Throughout this thesis, we use the term graph *to represent the term* simple graph.

In this thesis, we are interested in Cayley graphs. They are special cases of regular graphs. It is important for us give the definition of regular graph and we need to use the degree of the graph later.

Definition 1.1.2. *Let* Γ *be a graph with* $V(\Gamma) = \{v_1, \ldots, v_n\}$.

1. *The* degree *of a vertex,* v_i *in* Γ, *denoted by* $d(v_i)$, *is the number of edges incident with* v_i.

2. *If every $v_i \in V(\Gamma)$ has the same degree, we say that Γ is a* regular *graph. In particular, if $d(v_i) = k$ for $i \in \{1, \dots, n\}$, we say that Γ is a k-regular graph. We denote $d(\Gamma) = k$ for a k-regular graph.*

We are interested in identifying independent sets in Cayley graphs. We shall observe that the degrees of a graph are needed in determining the cardinality of an independent set. We first define what is an independent set:

Definition 1.1.3.

1. *An* independent set *is a set of vertices in a graph such that no two of which are adjacent. The size of an independent set is the number of vertices which it contains.*

2. *A maximum independent set is a* largest independent set *for a given graph and its size is the largest independent number, which is denoted by $\alpha(\Gamma)$.*

For every graph with $v(\Gamma) = n$, we are able to determine an $n \times n$ real matrix $A(\Gamma)$ to represent its adjacency. In our context, $A(\Gamma)$ is important as we will study its eigenvalues in determining the largest independent number.

Definition 1.1.4. *Let Γ be a simple graph with $v(\Gamma) = n$. The* adjacency matrix*, $A(\Gamma)$ of a graph Γ is the integer matrix with rows and columns indexed by the vertices of Γ, such that the uv-entry of $A(\Gamma)$ is 1 if u is adjacent to v and 0 otherwise.*

The adjacency matrix of a simple graph Γ, $A(\Gamma)$ is a real symmetric matrix. We know that all eigenvalues of $A(\Gamma)$ are real numbers by the following lemmas:

Lemma 1.1.5. *Let A be a real symmetric matrix. If u and v are eigenvectors of A with different eigenvalues, then u and v are orthogonal.*

Proof. Suppose that $Au = \lambda u$ and $Av = \tau v$, with $\lambda \neq \tau$. Since A is symmetric, $u^T Av = (v^T Au)^T$. The L.H.S of this equation is $\tau u^T v$ whereas the R.H.S is $\lambda u^T v$. Since $\tau \neq \lambda$, then $u^T v = 0$, giving us $u \perp v$. $\square$

Lemma 1.1.6. *The eigenvalues of a real symmetric matrix A are real numbers.*

Proof. Let u be an eigenvector of A with eigenvalue λ. By taking the complex conjugate of the equation $Au = \lambda u$, we obtain $\overline{Au} = A\overline{u} = \overline{\lambda}\overline{u}$, and so $\overline{u}$ is also an eigenvector of A.

By definition an eigenvector is not the $\mathbf{0}$ vector, so $u^T\overline{u} > 0$. By Lemma 1.1.5, u and $\overline{u}$ cannot have different eigenvalues, so $\lambda = \overline{\lambda}$, and the assertion is true. □

In the context of determining the largest independent number using the Delsarte-Hoffman Bound, we are expecting real eigenvalues from a graph so that we can obtain an upper bound for a largest independent set as a real number.
The focus of our thesis would be on properties of Cayley graphs. We need the following definitions before defining Cayley graphs.

Definition 1.1.7. *A* group *is a set, G, together with an operation $\circ$, i.e $(G,\circ)$ which satisfies the following axioms*

1. *Closure: For all $a,b \in G$, $a \circ b \in G$.*
2. *Associativity: For all $a,b,c \in G$, $(a\circ b)\circ c = a\circ(b\circ c)$.*
3. *Identity Element: There exists an element $\mathbb{1} \in G$ such that $\forall a \in G$, $a \circ \mathbb{1} = \mathbb{1} \circ a = a$.*
4. *For each $a \in G$, there exists an element $b \in G$ such that $a \circ b = b \circ a = \mathbb{1}$. Such b is denoted as a^{-1}.*

Definition 1.1.8. *Let G be a finite group and let $S \subseteq G$ be a subset of G such that $1 \notin S$ and $s \in S \Rightarrow s^{-1} \in S$,, the corresponding* Cayley graph*, denoted as $\Gamma(G,S)$ has the following vertex set and edge set*

$$V(\Gamma(G,S)) = G$$

$$E(\Gamma(G,S)) = \{(g,h) \mid \exists s \in S \text{ such that } h^{-1}g = s\}$$

S is called the generating set *for $\Gamma(G,S)$.*

We now define what it means by vertex-transitivity. In particular, a Cayley graph is a vertex-transitive graph and thus it possesses the properties of regularity.

Definition 1.1.9. *A graph Γ is* vertex-transitive *if given any vertices v_1, v_2 of Γ, there is an automorphism $f : V(\Gamma) \to V(\Gamma)$ such that $f(v_1) = v_2$.*

This will mean that the graph properties of any two vertices in a vertex-transitive graph are the same.

Theorem 1.1.10. (Ku and Wong, 2013 [42]) *$\Gamma(G,S)$ is vertex-transitive. In particular, $\Gamma(G,S)$ is a regular graph.*

Theorem 1.1.10 is a well-known result, and it is important as the properties of vertex-transitivity and regularity are required for Theorem 1.4.1 later.
We now state some well known results of the degree of a Cayley graph and its relationship with the largest eigenvalue of the adjacency matrix of the Cayley graph.

Theorem 1.1.11. (Ku and Wong, 2013 [42]) *Let d be the degree of any vertex in $\Gamma(G,S)$, then $d = |S|$. Moreover, the largest eigenvalue of $A(\Gamma(G,S))$ is equal to d.*

1.2 Representation Theory of Symmetric Groups

In this section, we would like to use the Frobenius-Schur-Others Theorem (Theorem 1.2.17) to determine all the eigenvalues of the adjacency matrix of some Cayley graphs. In particular, we are interested in finding the largest and smallest eigenvalues of these graphs.

1.2.1 Introduction and Background

We first introduce the definitions and concepts in group theory:

Definition 1.2.1. *Given two groups $(G,\cdot)$ and $(H,*)$, a* group homomorphism *from $(G,\cdot)$ to $(H,*)$ is a function $\phi : G \to H$ such that for all $u,v \in G$,*

$$\phi(u\cdot v) = \phi(u) * \phi(v)$$

Definition 1.2.2. *A subset S of the domain U of a mapping $T : U \to U$ is an* invariant set *under the mapping when*

$$x \in S \Rightarrow T(x) \in S.$$

In particular, a conjugation invariant subset is the invariant subset under conjugation mapping.

Lemma 1.2.3. *Let $A, B \subset G$ where G is a group. If A, B are inverse-closed and conjugation-invariant subsets of G, then $A \cup B$ is an inverse-closed and conjugation-invariant subset of G.*

Proof. Let $x \in A \cup B$, then $x \in A$ or $x \in B$. Without loss of generality, we assume that $x \in A$.

Since A is inverse-closed, $x^{-1} \in A$, giving us $x^{-1} \in A \cup B$.

Since A is a conjugation-invariant subset of G, for all $g \in G$, $gxg^{-1} \in A$, giving us $gxg^{-1} \in A \cup B$. □

We now introduce some definitions and results in representation theory which are related to this thesis.

Definition 1.2.4. *Let V be a vector space over the field $\mathbb{F}$. An* automorphism *of V is a linear operator $\phi : V \to V$ where ϕ is an isomorphism.*

Definition 1.2.5. *If V is a vector space over the field $\mathbb{F}$, the* general linear group *of V, written $GL(V)$ is the group of all automorphisms of V.*

Definition 1.2.6. *Let G be a group and V a vector space. A group homomorphism $\rho : G \to GL(V)$ is a* representation *of G and V is a* representation space *of G.*

Definition 1.2.7. *If G is a group and X is a set, then a (left)* group action *of G on X is a binary function,*

$$\psi : G \times X \to X \quad \textit{denoted} \quad \psi((g,x)) = g \cdot x$$

which satisfies the following 2 axioms:

1. *$(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$;*

2. *If $\mathbb{1}$ is the identity element of G, then $\mathbb{1} \cdot x = x$ for all $x \in X$.*

The group G is said to act on *X.*

Definition 1.2.8. *Let G act on a set X, and V be a vector space having basis $\{v_x | x \in X\}$. If $g \in G$, we define $\rho(g)$ to be the linear map $V \to V$ such that $\rho(g)(v_x) = v_{g \cdot x}$, then $\rho : g \mapsto \rho(g)$ defines a representation of G, known as the* permutation representation *of G on X.*

Remark 1.2.9. *The regular representation of G is the permutation representation of G on G by regular left action.*

Definition 1.2.10. *Given two vector spaces V and W, two representations*

$$\rho_1 : G \to GL(V) \quad \text{and} \quad \rho_2 : G \to GL(W)$$

are said to be isomorphic *if there exists a vector space isomorphism*

$$\Phi : V \to W$$

such that for all $g \in G$,

$$\Phi \circ (\rho_1(g)v) = \rho_2(g) \circ \Phi(v)$$

for all $v \in V$. If there is no such isomorphism, then we say V and W are non-isomorphic.

Definition 1.2.11. *A subspace W of V that is invariant under the group action is called a subrepresentation. If V has exactly two* subrepresentations*, namely the zero-dimensional subspace and V itself, then the representation is said to be* irreducible*; if it has a proper subrepresentation of nonzero dimension, the representation is said to be* reducible.

We need to use a special kind of representation, namely character of a representation to evaluate the eigenvalues. We now define character and some related definitions in ring and module theory.

Definition 1.2.12. *A* character*, $\chi = \chi_\rho = \chi_V : G \to \mathbb{C}$ is defined by $\chi(g) = tr(\rho(g))$ for $g \in G$.*

Definition 1.2.13. *An* Abelian *group $(G, \circ)$ is a group which possesses* commutativity*, i.e for all $a, b \in G$*

$$a \circ b = b \circ a.$$

Definition 1.2.14. *A* ring*, R is a set equipped with two associative binary operations, called addition $(+)$ and multiplication $(\times)$, such that*

1. *R is an Abelian group under $+$;*

2. *distributive law holds, i.e*

$$r(s+t) = rs+rt,$$
$$(s+t)r = sr+tr$$

for all $r,s,t \in R$.

Definition 1.2.15. *A* left R-module *M over the ring R consists of an abelian group $(M,+)$ and an operation $R \times M \to M$ such that for all $r,s \in R$, $x,y \in M$,*

1. *$r(x+y) = rx+ry$;*
2. *$(r+s)x = rx+sx$;*
3. *$(rs)x = r(sx)$;*
4. *$\mathbb{1}_R x = x$ if R has multiplicative identity $\mathbb{1}_R$.*

Definition 1.2.16. *For a finite group G, the group module $\mathbb{C}G$ is the complex vector space with basis G and multiplication defined by extending the group multiplication linearly; explicitly,*

$$\left(\sum_{g\in G} x_g g\right)\left(\sum_{h\in G} y_h h\right) = \sum_{g,h\in G} x_g y_h (gh).$$

Identifying a function $f : G \to \mathbb{C}$ with $\sum_{g\in G} f(g)g$, we may consider $\mathbb{C}[G]$ as the group module $\mathbb{C}G$. If Γ is a Cayley graph on G with inverse-closed generating set X, the adjacency matrix of Γ, $A(\Gamma)$ acts on the group module $\mathbb{C}G$ by left multiplication by $\sum_{g\in X} g$.
With the definitions defined, we can study the following theorem in determining eigenvalues of some Cayley graphs.
The following theorem is the result of the work of many people which Frobenius and Schur started.

Theorem 1.2.17. (Frobenius-Schur-others, Ellis, 2012 [17]) *Let G be a finite group; let $X \subset G$ be an inverse-closed, conjugation-invariant subset of G and let Γ be $\Gamma(G,X)$. Let $(\rho_1,V_i),\ldots,(\rho_k,V_k)$ be a complete set of non-isomorphic irreducible representations of G. Let U_i be the sum of all submodules of the group module $\mathbb{C}G$ which are isomorphic to*

V_i. We have

$$\mathbb{C}G = \bigoplus_{i=1}^{k} U_i$$

and each U_i is an eigenspace of A with dimension $dim(V_i)^2$ and eigenvalue

$$\eta_{V_i} = \frac{1}{dim(V_i)} \sum_{g \in X} \chi_i(g)$$

where $\chi_i(g) = tr(\rho_i(g))$ denotes the character of the irreducible representation (ρ_i, V_i).

We want to make use of Theorem 1.2.17 in determining the eigenvalues of Cayley graphs on S_n. Therefore, we will discuss the representation theory of S_n in order to apply Theorem 1.2.17 in the next subsection.

1.2.2 Symmetric Group, Partitions and Specht Module

We provide the perspective of representation theory of the symmetric group via general representation theory. Our objective in this section is to build the modules M^{λ}, the permutation module corresponding to S^{λ}, the Specht Module. First, we introduce the concepts of symmetric group, partitions and Young diagram.

Definition 1.2.18. *The* symmetric group*, S_n on a set $X = \{1, 2, \ldots, n\}$ is the group whose underlying set is the collection of all bijections from X to X and whose group operation is that of function composition*

$$S_n = \{\sigma \mid \sigma : X \to X, \sigma \text{ is a bijection}\}$$

Definition 1.2.19. *A* partition *of n is a non-increasing sequence of integers summing to n, i.e a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$. We write $\lambda \vdash n$.*

Definition 1.2.20. *The* cycle-type *of a permutation $\sigma \in S_n$ is the partition of n obtained by expressing σ as a product of disjoint cycles and listing its cycle-lengths in non-increasing order precisely.*

Therefore, we know that the conjugacy classes of S_n are precisely

$$\{\sigma \in S_n : \text{ cycle-type}(\sigma) = \alpha\}_{\alpha \vdash n}$$

Moreoever, there is an explicit one-to-one correspondence between irreducible representations of S_n (up to isomorphism) and partitions of n, which we now describe.

Definition 1.2.21. *Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a partition of n. The* Young diagram *or* Ferrers diagram *of α is an array of n dots, having k left-justified rows where row i contains α_i dots.*

Definition 1.2.22. *If the array contains the numbers $1, 2, \ldots, n$ in some order in place of the dots, we call it an α-tableau.*

Definition 1.2.23. *Two α-tableaux are* row-equivalent *if for each row, they have the same numbers in that row. If an α-tableau t has rows $R_1, \ldots, R_k \subset [n]$ and columns $C_1, \ldots, C_l \subset [n]$, we let $R_t = S_{R_1} \times \ldots \times S_{R_k}$ be the* row-stabilizer *of t and $C_t = S_{C_1} \times \ldots \times S_{C_l}$ be the* column-stabilizer.

Definition 1.2.24. *An α-tabloid is an α-tableau with unordered row entries. We write $[t]$ for the tabloid produced by a tableau t.*

Now, we have sufficient tools to construct our M^α. Consider the natural left action of S_n on the set X^α of all α-tabloids; let $M^\alpha = \mathbb{C}[X^\alpha]$ be the corresponding permutation module, the complex vector space with basis X^α and S_n action given by extending this action linearly.

Definition 1.2.25. *Given α-tableau t, we define the corresponding α-polytabloid*

$$e_t := \sum_{\pi \in C_t} \varepsilon(\pi)\pi[t]$$

where ε is the character of sign representation, $S^{(1^n)}$.

Definition 1.2.26. *We define the* Specht module *S^α to be the submodule of M^α spanned by the α-polytabloids:*

$$S^\alpha = span\{e_t : t \text{ is an } \alpha\text{-tableau}\}$$

Lemma 1.2.27. (Stanley, 1999 [64]) *The Specht modules S^α are a complete set of pairwise non-isomorphic, irreducible representations of S_n. Hence any irreducible representation ρ of S_n is isomorphic to some S^α. In particular, both the conjugacy classes of S_n and the irreducible characters of S_n are indexed by partitions λ of $[n]$.*

We study the Specht modules, S^α because they are important in applying Theorem 1.2.17 to find the eigenvalues of Cayley graphs on S_n. Notice that Lemma 1.2.27 fulfills the hypothesis for Theorem 1.2.17.

Example 1.2.28. *A few examples of S^α,*

- $S^{(n)} = M^{(n)}$ *is the trivial representation.*
- $S^{(1^n)}$ *is the sign representation and* $M^{(1^n)}$ *is the left-regular representation.*

Definition 1.2.29. *A tableau is* standard *if the numbers are strictly increasing along each row and down each column.*

Proposition 1.2.30. (Ellis, 2012 [17]) *For any partition α of n,*

$$\{e_t : t \text{ is a standard } \alpha\text{-tableau}\}$$

is a basis for the Specht module S^α.

We next define the hook length as there is a relationship between the dimension of a Specht module, S^α and hook length. We require the dimension of Specht Module so that we can apply Theorem 1.2.17 to find the eigenvalues.

Definition 1.2.31. *Let $\lambda \vdash n$. For each cell (i, j) in a Young diagram of a partition α, we define the* hook-length $h_\lambda(a,b)$ *as the size of the set of all the boxes with coordinate (i, j) where $i = a$ and $j \geq b$, or $i \geq a$ and $j = b$.*

Notation 1.2.32. *We use the following notations in this thesis:*

- $[\alpha]$ *- equivalence class of the irreducible representations of S^α.*
- χ_α *- irreducible character of χ_{S^α}.*
- ξ_α *- character of the permutation representation M^α.*
- f^α *- dimension of the Specht module S^α.*

Theorem 1.2.33. (Ellis, 2012 [17]) *Let $H^\alpha = \prod(\text{hook lengths of } [\alpha])$, then the dimension of S^α is*

$$f^\alpha = \frac{n!}{H^\alpha}.$$

Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a partition of n. For each i with $\alpha_i > \alpha_{i+1}$, we define

$$\alpha^{i-} = (\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \dots, \alpha_r).$$

Note that $\alpha^{i-} \vdash (n-1)$. Furthermore, α^{i-} is the partition whose Ferrers diagram is obtained by deleting the box at the end of the ith row of the Ferrers diagram of α. We shall also need the following theorem to find f^{α}:

Theorem 1.2.34. (The Branching Theorem, Sagan, 2001 [62, Lemma 2.8.2 on p. 77] & [62, Theorem 2.8.3]) *For any partition α of n, the restriction $[\alpha] \downarrow S_{n-1}$ is isomorphic to a direct sum of those irreducible representation $[\alpha^{i-}]$ of S_{n-1}, then*

$$[\alpha] \downarrow S_{n-1} = \sum_{i:\alpha_i > \alpha_{i+1}} [\alpha^{i-}]. \tag{1.1}$$

and

$$f^{\alpha} = \sum_{i:\alpha_i > \alpha_{i+1}} f^{\alpha^{i-}}. \tag{1.2}$$

Theorem 1.2.35. (Ellis, 2012 [17]) *The set of α-tabloids form a basis for M^{α}, therefore $\xi_{\alpha}(\sigma)$, the trace of the corresponding permutation representation at σ, is precisely the number of α-tabloids fixed by σ.*

Theorem 1.2.35 is important as it gives us a combinatorial idea to calculate $\xi_{\alpha}(\sigma)$ without looking at the algebra of the corresponding α. We need this to calculate the character values in Theorem 1.2.17. We now study a property about the tensor product which is important in Theorem 1.2.37.

Definition 1.2.36. *If $U \in [\alpha]$ and $V \in [\beta]$, we define $[\alpha] + [\beta]$ to be the equivalence class of $U \oplus V$ and $[\alpha] \otimes [\beta]$ to be the equivalence class of $U \otimes V$; since $\chi_{U \otimes V} = \chi_U \cdot \chi_V$.*

Theorem 1.2.37. (Ellis, 2012 [17]) *For any partition α of n, we have*

$$S^{(1^n)} \otimes S^{\alpha} \cong S^{\alpha'}$$

where α' (or α^T) is the transpose of α, the partition of n with Young diagram obtained by interchanging rows and columns in the Young diagram of α. In particular, $[1^n] \otimes [\alpha] =$

$[\alpha']$ *and* $\chi_{\alpha'} = \varepsilon \cdot \chi_\alpha$ *where* ε *is the character of sign representation.*

Theorem 1.2.37 is important because one can determine the character of a partition by taking the multiplication of its sign character and character of its transpose. The use of Theorem 1.2.37 will be seen in later parts.

Example 1.2.38. *If* $n = 7$*,*

1. $(3,2,2) \vdash 7$.

2. *We sometimes write* $(3,2,2)$ *as* $(3,2^2)$.

3. *The Young diagram of* $(3,2^2)$ *is*

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{array}$$

4. *A* $(3,2^2)$*-tableau*

$$\begin{array}{ccc} 6 & 1 & 7 \\ 5 & 4 & \\ 3 & 2 & \end{array}$$

5. *A* $(3,2^2)$*-tabloid*

$$\begin{array}{l} \{1 \quad 6 \quad 7\} \\ \{4 \quad 5\} \\ \{2 \quad 3\} \end{array}$$

6. *Dimension of* S^α *is*

$$f^\alpha = \frac{n!}{\prod(\textit{hook lengths of } [\alpha])} = \frac{7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1} = 21$$

with Hook lengths of α *are*

$$\begin{array}{ccc} 5 & 4 & 1 \\ 3 & 2 & \\ 2 & 1 & \end{array}$$

7.

$$[1^7] \otimes [3,2,2] = [3,2,2]' = \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{bmatrix}' = \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & & \end{bmatrix}.$$

Before we decompose M^{α}, we need to have the following terminology:

Definition 1.2.39. *Let α, β be partitions of n. A* generalized *α-tableau is produced by replacing each dot in the Young diagram of α with a number between* 1 *and n; if a generalized α-tableau has β_i i's $(1 \leq i \leq n)$ it is said to have content β. A generalized α-tableau is said to be* semistandard *if the numbers are non-decreasing along each row and strictly increasing down each column.*

Definition 1.2.40. *Let α, β be partitions of n. The* Kostka number*, $K_{\alpha,\beta}$ is the number of semistandard generalized α-tableaux with content β.*

With the terminology defined, we now explain how the permutation modules M^{β} decompose into irreducibles.

Theorem 1.2.41. (Young's Rule, Sagan, 2001 [62]) *For any partition β of n, the permutation module M^{β} decomposes into irreducibles as follows:*

$$M^{\beta} \cong \bigoplus_{\alpha \vdash n} K_{\alpha,\beta} S^{\alpha}$$

Example 1.2.42. *$M^{(n-1,1)}$ which corresponds to the natural permutation action of S_n on $[n]$, decomposes as*

$$M^{(n-1,1)} \cong S^{(n-1,1)} \oplus S^{(n)}$$

giving us

$$\xi_{(n-1,1)} = \chi_{(n-1,1)} + 1$$

as $S^{(n)}$ is the trivial representation with dimension 1.

We now return to consider $\Gamma(S_n, X)$ using Theorem 1.2.17. To make use of Theorem 1.2.17, we must make sure the generating set $X \subset G$ is an inverse-closed, conjugation-invariant subset of G. We have the following property about conjugacy classes:

Proposition 1.2.43. *Let C_λ be the conjugacy class of type $\lambda = \langle 1^{m_1} 2^{m_2} \ldots n^{m_n} \rangle$, then C_λ is an inverse-closed and conjugation-invariant subset of S_n. In particular, $\bigcup_\lambda C_\lambda$ is an inverse-closed and conjugation-invariant subset of S_n.*

Proof. Let $\sigma \in C_\lambda$. Then $\sigma = (i_{1,1} \ldots i_{1,a})(i_{2,1} \ldots i_{2,b}) \ldots (i_{j,1} \ldots i_{j,c})$ as a product of disjoint cycles. It follows that its inverse $\sigma^{-1} = (i_{j,1} \ldots i_{j,c})^{-1} \ldots (i_{2,1} \ldots i_{2,b})^{-1} (i_{1,1} \ldots i_{1,a})^{-1}$ is also in C_λ.

By definition of conjugacy classes, for all $\sigma \in C_\lambda$, $\tau\sigma\tau^{-1} \in C_\lambda$ for all $\tau \in S_n$. Therefore C_λ satisfies the desired properties. By Lemma 1.2.3, $\bigcup_\lambda C_\lambda$ is an inverse-closed, conjugation-invariant subset of G. $\square$

With all the tools developed, we are now ready to apply Theorem 1.2.17 to calculate the eigenvalues of the Cayley graphs on S_n. We have the following corollary:

Corollary 1.2.44. *Write U_α for the sum of all copies of S^α in $\mathbb{C}S_n$. We have*

$$\mathbb{C}S_n = \bigoplus_{\alpha \vdash n} U_\alpha$$

and each U_α is an eigenspace of $\Gamma(S_n, X)$, with $dim(U_\alpha) = (f^\alpha)^2$ and corresponding eigenvalue

$$\eta_\alpha = \frac{1}{f^\alpha} \sum_{\sigma \in X} \chi_\alpha(\sigma).$$

Let $S \subseteq S_n$ be closed under conjugation. Since central characters are algebraic integers (Isaacs, 1976 [33, Theorem 3.7 on p. 36]) and the characters of the symmetric group are integers (Isaacs, 1976 [33, 2.12 on p. 31] or Serre, 1977 [63, Corollary 2 on p. 103]), by Theorem 1.2.17, the eigenvalues of $\Gamma(S_n, S)$ are integers.

Corollary 1.2.45. *The eigenvalues of a Cayley graph $\Gamma(S_n, S)$ are integers.*

1.3 *k*-point Fixing Graph

Let $0 \leq k < n$ and $\mathscr{S}(n,k)$ be the set of all $\sigma \in S_n$ such that σ fixes exactly k elements. Note that $\mathscr{S}(n,k)$ is an inverse-closed subset of S_n. The *k-point fixing graph* is defined to be

$$\mathscr{F}(n,k) = \Gamma(S_n, \mathscr{S}(n,k)).$$

That is, two vertices g, h of $\mathscr{F}(n,k)$ are joined if and only if gh^{-1} fixes exactly k points. Note that the 0-point fixing graph is the derangement graph.

Clearly, $\mathscr{F}(n,k)$ is vertex-transitive, so it is $|\mathscr{S}(n,k)|$-regular and the largest eigenvalue of $\mathscr{F}(n,k)$ is $|\mathscr{S}(n,k)|$. Furthermore, $\mathscr{S}(n,k)$ is closed under conjugation. Therefore, by Corollary 1.2.45, the eigenvalues of the k-point fixing graph are integers. Since $\mathscr{S}(n,k)$ is closed under conjugation, the eigenvalue $\eta_{\chi_\lambda}(k)$ of the k-point fixing graph can be denoted by $\eta_\lambda(k)$. Throughout the thesis, we shall use this notation.

1.4 Literature Review

In this section, we will review some of the results that are related to this thesis.

1.4.1 Delsarte-Hoffman Bound

We are interested in *regular* graphs and their *adjacency* matrices. In particular, we want to determine its *eigenvalues* so that we can apply the Delsarte-Hoffman Bound.

We introduce the following theorem in order to bound the largest independent set of a Cayley graph.

Theorem 1.4.1. (Delsarte-Hoffman Bound, Hoffman, 1970 [29]) *Let Γ be a d-regular graph with n vertices. Let A be the adjacency matrix of Γ . Let $\{\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_n\}$ be an orthonormal system of eigenvectors of A, with corresponding eigenvalues $d=\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n = \tau$ (so that $\mathbf{v}_1 = (1,\dots,1)$ is the all-1's vector). If I is an independent set in Γ, then*

$$|I| \leq \frac{n}{1-\frac{d}{\tau}}.$$

Furthermore, if equality holds, then

$$\mathbf{1}_I \in \mathrm{Span}\left(\{\mathbf{v}_1\} \cup \{\mathbf{v}_i \ : \ \gamma_i = \tau\}\right),$$

where $\mathrm{Span}(\mathbf{b}_1,\dots,\mathbf{b}_m)$ *is the vector space spanned by* $\mathbf{b}_1,\dots,\mathbf{b}_m$.

By Theorem 1.1.11, we can determine the largest eigenvalue of $A(\Gamma(G,S))$ by determining the degree of any vertex in $\Gamma(G,S)$. In order to use Theorem 1.4.1, we need to find the smallest eigenvalue of the graph, which requires the use of Representation Theory of symmetric groups.

1.4.2 0-point Fixing Graph

The 0-point fixing graph, which is well-known as *derangement graph*, is the Cayley graph $\Gamma(S_n, D_n)$ where $D_n = \mathscr{S}(n,0)$ is the set of derangements in S_n. Since D_n is closed under conjugation, by Corollary 1.2.45, the eigenvalues of the derangement graph are integers. We now list out some of the known results that are related to the 0-point fixing graph. Renteln (2007) proved a recurrence formula for eigenvalues of a 0-point fixing graph. To describe the Renteln's recurrence formula for $\mathscr{F}(n,0)$, we require some terminology. To the Ferrers diagram of a partition λ, we assign *xy*-coordinates to each of its boxes by defining the upper-left-most box to be $(1,1)$, with the x axis increasing to the right and the y axis increasing downwards. Then the *hook* of λ is the union of the boxes $(x',1)$ and $(1,y')$ of the Ferrers diagram of λ, where $x' \geq 1$, $y' \geq 1$. Let $\widehat{h}_\lambda$ denote the hook of λ and let h_λ denote the size of $\widehat{h}_\lambda$. Similarly, let $\widehat{c}_\lambda$ and c_λ denote the first column of λ and the size of $\widehat{c}_\lambda$ respectively. Note that c_λ is equal to the number of rows of λ. When λ is clear from the context, we will replace $\widehat{h}_\lambda$, h_λ, $\widehat{c}_\lambda$ and c_λ by $\widehat{h}$, h, $\widehat{c}$ and c respectively. Let $\lambda - \widehat{h} \vdash n - h$ denote the partition obtained from λ by removing its hook. Also, let $\lambda - \widehat{c}$ denote the partition obtained from λ by removing the first column of its Ferrers diagram, i.e. $(\lambda_1, \ldots, \lambda_r) - \widehat{c} = (\lambda_1 - 1, \ldots, \lambda_r - 1) \vdash n - r$.

Theorem 1.4.2. (Renteln's Recurrence Formula, Renteln, 2007 [61, Theorem 6.5]) *For any partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, the eigenvalues of the derangement graph $\mathscr{F}(n,0)$ satisfy the following recurrence:*

$$\eta_\lambda(0) = (-1)^h \eta_{\lambda - \widehat{h}}(0) + (-1)^{h+\lambda_1} h \eta_{\lambda - \widehat{c}}(0)$$

with initial condition $\eta_\emptyset(0) = 1$.

Applying Theorem 1.4.2, Renteln (2007) settled the following conjecture made by Ku and Wong (2007):

Theorem 1.4.3. (Ku and Wong, 2007 [52]) *The smallest eigenvalue of the adjacency matrix of $\mathscr{F}(n,0)$ is given by*

$$\eta_{(n-1,1)} = -\frac{d_n}{n-1}$$

which occurs at the partition $(n-1,1)$.

With the smallest eigenvalue of $\mathscr{F}(n,0)$ determined, we are now able to bound the largest independent set of $\mathscr{F}(n,0)$. Moreover, we are able to determine the exact largest independent number by identifying the existence of an independent set with cardinality of the bound.

Corollary 1.4.4. *The largest independent number of the derangement graph $\mathscr{F}(n,0)$ is $\alpha(\mathscr{F}(n,0)) = (n-1)!$.*

Proof. By Theorems 1.4.1 and 1.1.11, we have

$$\alpha(\mathscr{F}(n,0)) \leq \frac{n!}{1-\frac{d_n}{-\frac{d_n}{n-1}}} = \frac{n!}{1+n-1} = (n-1)!.$$

It suffices for us to verify the existence of an independent set, I with $|I| = (n-1)!$. Let I be an independent set, i.e

$$I = \{g \in S_n : g(1) = 1\}$$

we have $|I| = (n-1)!$, giving us $\alpha(\Gamma) = (n-1)!$. □

Ku and Wong (2013) have proved a recurrence formula for eigenvalues of 0-point fixing graph. To describe the Ku-Wong's recurrence formula for $\mathscr{F}(n,0)$, we need a new terminology. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, let $\widehat{l_\lambda}$ denote the last row of λ and l_λ denote the size of $\widehat{l_\lambda}$. Clearly, we have $l_\lambda = \lambda_r$. Let $\lambda - \widehat{l_\lambda}$ denote the partition obtained from λ by deleting the last row. When λ is clear from the context, we will replace $\widehat{l_\lambda}$, l_λ by $\widehat{l}$ and l respectively.

Theorem 1.4.5. (Ku-Wong's Recurrence Formula, Ku and Wong, 2013 [42, Theorem 1.4]) *For any partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, the eigenvalues of the derangement graph $\mathscr{F}(n,0)$ satisfy the following recurrence:*

$$\eta_\lambda(0) = (-1)^{\lambda_r}\eta_{\lambda-\widehat{l}}(0) + (-1)^{r-1}\lambda_r\eta_{\lambda-\widehat{c}}(0)$$

with initial condition $\eta_{\emptyset}(0) = 1$.

The following theorem is called the Alternating Sign Property (ASP) for $\mathscr{F}(n,0)$, which is proved by Ku and Wales (2010) and Ku and Wong (2013) by using Renteln's Recurrence Formula and Ku-Wong's Recurrence Formula respectively.

Theorem 1.4.6. (Alternating Sign Property for $\mathscr{F}(n,0)$, Ku and Wales, 2010 [41, Theorem 1.2]; Ku and Wong, 2013 [42, Theorem 1.3]) *Let $n \geq 2$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$,*

$$\begin{aligned} \operatorname{sign}(\eta_\lambda(0)) &= (-1)^{|\lambda|-\lambda_1} \\ &= (-1)^{\#\text{cells under the first row of } \lambda} \end{aligned}$$

where $\operatorname{sign}(\eta_\lambda(0))$ *is* 1 *if* $\eta_\lambda(0)$ *is positive or* -1 *if* $\eta_\lambda(0)$ *is negative.*

The following corollary is a consequence of Theorems 1.4.5 and 1.4.6.

Corollary 1.4.7. *For any partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ with $r \geq 2$, the absolute value of the eigenvalues of the derangement graph $\mathscr{F}(n,0)$ satisfy the following recurrence:*

$$|\eta_\lambda(0)| = |\eta_{\lambda-\widehat{l}}(0)| + \lambda_r |\eta_{\lambda-\widehat{c}}(0)|$$

with initial condition $|\eta_{\emptyset}(0)| = 1$.

1.4.3 Intersecting families

Let $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A family $\mathscr{A}$ of subsets of $[n]$ is *t-intersecting* if $|A \cap B| \geq t$ for all $A, B \in \mathscr{A}$. One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

Theorem 1.4.8 (Erdős, Ko, and Rado, 1961 [20]; Frankl, 1978 [21]; Wilson, 1984 [67])**.** *Suppose $\mathscr{A} \subseteq \binom{[n]}{k}$ is t-intersecting and $n > 2k - t$. Then for $n \geq (k-t+1)(t+1)$, we have*

$$|\mathscr{A}| \leq \binom{n-t}{k-t}.$$

Moreover, if $n > (k-t+1)(t+1)$ then equality holds if and only if

$$\mathscr{A} = \left\{ A \in \binom{[n]}{k} \ : \ T \subseteq A \textit{ for some } t\textit{-set } T \right\}.$$

Later, Ahlswede and Khachatrian (1997, [1]) extended the Erdős-Ko-Rado theorem by determining the structure of all t-intersecting set systems of maximum size for all possible n (see also [4, 22, 34, 45, 57, 59, 65] for some related results). There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems (see [2, 5, 6, 7, 8, 11, 12, 14, 16, 19, 23, 27, 28, 30, 31, 32, 36, 37, 52, 43, 44, 46, 47, 48, 49, 50, 51, 53, 55, 58, 66, 68]).

We say that a pair of families $\mathscr{A}, \mathscr{B} \subseteq S_n$ is cross-intersecting if for any $\sigma \in \mathscr{A}$, $\pi \in \mathscr{B}$, there exists an $i \in [n]$ such that $\sigma(i) = \pi(i)$. Recall that $S_{i,j} = \{\pi \in S_n : \pi(i) = j\}$. Leader (2005, [54]) conjectured the following theorem which was later proved by Ellis (2012, [17, Theorem 2.6 and 2.8]).

Theorem 1.4.9. *For $n \geq 4$, if $\mathscr{A}, \mathscr{B} \subseteq S_n$ are cross-intersecting, then*

$$|\mathscr{A}||\mathscr{B}| \leq ((n-1)!)^2.$$

Furthermore, equality holds if and only if $\mathscr{A} = \mathscr{B} = S_{i,j}$ for some $i, j \in [n]$.

CHAPTER 2: RECURRENCE FORMULA FOR $\mathscr{F}(n,k)$ AND ASP FOR $\mathscr{F}(n,1)$

In this chapter, we will derive a recurrence formula for the eigenvalues of $\mathscr{F}(n,k)$ using a result from Foulkes (1978). Then, we will make use of the recurrence formula to determine the Alternating Sign Property for the eigenvalues of $\mathscr{F}(n,1)$. The results of this chapter have been published in Ku, Lau and Wong (2015, [38]).

2.1 Recurrence formula for $\mathscr{F}(n,k)$

For each $\sigma \in S_n$, we denote it's conjugacy class by $\mathrm{Con}_{S_n}(\sigma)$, i.e., $\mathrm{Con}_{S_n}(\sigma) = \{\gamma^{-1}\sigma\gamma \,:\, \gamma \in S_n\}$. Let $\mu \vdash n$ be the partition that represents $\mathrm{Con}_{S_n}(\sigma)$. We shall denote the size of $\mathrm{Con}_{S_n}(\sigma)$ by $N_{S_n}(\mu)$.

Let $A \subseteq S_n$ and $\alpha \in S_n$. The set $\alpha^{-1}A\alpha$ is defined as

$$\alpha^{-1}A\alpha = \{\alpha^{-1}\sigma\alpha \,:\, \sigma \in A\}.$$

Let $0 \le k < n$. Each $\beta \in S_{n-k}$ can be considered as an element $\overline{\beta}$ of S_n by defining $\overline{\beta}(j) = \beta(j)$ for $1 \le j \le n-k$ and $\overline{\beta}(j) = j$ for $n-k+1 \le j \le n$. The $\overline{\beta}$ is called the *extension* of β to S_n. The set of derangements D_{n-k} in S_{n-k} can be considered as a subset of S_n ($\overline{D}_{n-k} = \{\overline{\sigma} \,:\, \sigma \in D_{n-k}\}$). Furthermore, $\bigcup_{\sigma\in S_n} \sigma^{-1}\overline{D}_{n-k}\sigma \subseteq \mathscr{S}(n,k)$.

Let $\gamma \in \mathscr{S}(n,k)$. Then γ fixes exactly k elements, i.e., $\gamma(i_j) = i_j$ for $j = 1,2,\ldots,k$ and $\gamma(a) \neq a$ for $a \in [n] \setminus \{i_1,i_2,\ldots,i_k\} = \{b_1,b_2,\ldots,b_{n-k}\}$. Let $\sigma_0(b_j) = j$ for $1 \le j \le n-k$ and $\sigma_0(i_j) = n-k+j$ for $1 \le j \le k$. Then $\sigma_0 \in S_n$ and $\sigma^{-1}\gamma\sigma \in \overline{D}_{n-k}$. Hence, the following lemma follows.

Lemma 2.1.1.

$$\mathscr{S}(n,k) = \bigcup_{\sigma\in S_n} \sigma^{-1}\overline{D}_{n-k}\sigma.$$

By Lemma 2.1.1, there are σ_{k1}, σ_{k2}, $\ldots$, $\sigma_{ks_k} \in D_{n-k}$ such that

$$\mathscr{S}(n,k) = \bigcup_{i=1}^{s_k} \mathrm{Con}_{S_n}(\overline{\sigma}_{ki}), \tag{2.1}$$

and σ_{ki} is not conjugate to σ_{kj} in S_{n-k} for $i \neq j$. Furthermore,

$$D_{n-k} = \bigcup_{i=1}^{s_k} \mathrm{Con}_{S_{n-k}}(\sigma_{ki}). \tag{2.2}$$

Note that $\chi_\lambda(\sigma) = \chi_\lambda(\beta)$ for all $\sigma \in \mathrm{Con}_{S_n}(\beta)$. Let $\mathrm{Con}_{S_n}(\beta)$ be represented by the partition $\varphi(\beta) \vdash n$. Then by Theorem 1.2.17 and Corollary 1.2.45, the eigenvalues of $\mathscr{F}(n,k)$ are integers given by

$$\eta_\lambda(k) = \frac{1}{f^\lambda} \sum_{i=1}^{s_k} N_{S_n}(\varphi(\overline{\sigma}_{ki}))\chi_\lambda(\varphi(\overline{\sigma}_{ki})), \tag{2.3}$$

where $\chi_\lambda(\varphi(\overline{\sigma}_{ki})) = \chi_\lambda(\overline{\sigma}_{ki})$.

Assume that $0 < k < n$. Note that each $\overline{\sigma}_{ki}$ $(1 \leq i \leq s_k)$ must consist of at least one 1-cycle in its cycle decomposition. Therefore $\varphi(\overline{\sigma}_{ki}) = (\nu_1, \nu_2, \ldots, \nu_r) \vdash n$ and $\nu_r = 1$. Note that $\varphi(\overline{\sigma}_{ki}) - \widehat{l}_{\varphi(\overline{\sigma}_{ki})} = (\nu_1, \nu_2, \ldots, \nu_{r-1}) \vdash (n-1)$. We are now ready to state the following lemma which is a special case of Theorem 3.4 in Foulkes (1978).

Lemma 2.1.2. (Foulkes, 1978 [24, Theorem 3.4]) *If the Ferrers diagrams obtained from λ by removing 1 node from the right hand side from any row of the diagram so that the resulting diagram will still be a partition of $(n-1)$ are those of $\mu_1, \ldots, \mu_q$, then*

$$\chi_\lambda(\varphi(\overline{\sigma}_{ki})) = \sum_{j=1}^{q} \chi_{\mu_j}(\varphi(\overline{\sigma}_{ki}) - \widehat{l}_{\varphi(\overline{\sigma}_{ki})}),$$

for all $1 \leq i \leq s_k$.

Example 2.1.3. *Let $n = 7$ and $\lambda = (3,3,1)$, then*

$$\chi_{(3,3,1)}((6,1)) = \chi_{(3,3)}((6)) + \chi_{(3,2,1)}((6)),$$
$$\chi_{(3,3,1)}((4,2,1)) = \chi_{(3,3)}((4,2)) + \chi_{(3,2,1)}((4,2)),$$
$$\chi_{(3,3,1)}((3,3,1)) = \chi_{(3,3)}((3,3)) + \chi_{(3,2,1)}((3,3)),$$
$$\chi_{(3,3,1)}((2,2,2,1)) = \chi_{(3,3)}((2,2,2)) + \chi_{(3,2,1)}((2,2,2)).$$

We shall need the following lemma in Stanley (1999).

Lemma 2.1.4. (Stanley, 1999 [64, (7.18) on p. 299]) *Let $\lambda = (n^{m_n}, \ldots, 2^{m_2}, 1^{m_1}) \vdash n$ and $z_\lambda = \prod_{j=1}^{n}(j^{m_j} m_j!)$, then the size of the conjugacy class represented by λ is*

$$N_{S_n}(\lambda) = \frac{n!}{z_\lambda}.$$

Lemma 2.1.5. *Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash (n-k)$ be a derangement, i.e., $\lambda_r \geq 2$. If*

$$\nu = (\lambda, 1^k) \vdash n, \text{ and } \mu = (\lambda, 1^{k-1}) \vdash (n-1),$$

then

$$N_{S_n}(\nu) = \frac{n}{k} N_{S_{n-1}}(\mu).$$

Proof. The lemma follows from Lemma 2.1.4, by noting that

$$N_{S_n}(\nu) = \frac{n!}{z_\lambda \times 1 \cdot k!}, \text{ and } N_{S_{n-1}}(\mu) = \frac{(n-1)!}{z_\lambda \times 1 \cdot (k-1)!}.$$

□

Theorem 2.1.6. (Recurrence Formula for $\mathscr{F}(n,k)$) *Let $0 < k < n$ and $\lambda \vdash n$. If the Ferrers diagrams obtained from λ by removing 1 node from the right hand side from any row of the diagram so that the resulting diagram will still be a partition of $(n-1)$ are those of $\mu_1, \ldots, \mu_q$, then*

$$\eta_\lambda(k) = \frac{n}{k f^\lambda} \sum_{j=1}^{q} f^{\mu_j} \eta_{\mu_j}(k-1).$$

Proof. Suppose $k = 1$. By equation (2.3),

$$\eta_\lambda(1) = \frac{1}{f^\lambda}\sum_{i=1}^{s_1} N_{S_n}(\varphi(\overline{\sigma}_{1i}))\chi_\lambda(\varphi(\overline{\sigma}_{1i})).$$

Note that $\overline{\sigma}_{1i}$ consists of exactly one 1-cycle and $\varphi(\overline{\sigma}_{1i}) = (\nu_1, \nu_2, \ldots, \nu_r) \vdash n$ with $\nu_r = 1$, $\nu_{r-1} \geq 2$. Therefore $\varphi(\overline{\sigma}_{1i}) - \widehat{l}_{\varphi(\overline{\sigma}_{1i})} = (\nu_1, \nu_2, \ldots, \nu_{r-1}) \vdash (n-1)$ is a derangement. In fact, $\varphi(\overline{\sigma}_{1i}) - \widehat{l}_{\varphi(\overline{\sigma}_{1i})}$ is the partition of $(n-1)$ that represents $\mathrm{Con}_{S_{n-1}}(\sigma_{1i})$. By Lemmas 2.1.2 and 2.1.5,

$$\begin{aligned}
\eta_\lambda(1) &= \frac{1}{f^\lambda}\sum_{i=1}^{s_1} N_{S_n}(\varphi(\overline{\sigma}_{1i}))\left(\sum_{j=1}^{q}\chi_{\mu_j}(\varphi(\overline{\sigma}_{1i}) - \widehat{l}_{\varphi(\overline{\sigma}_{1i})})\right)\\
&= \frac{1}{f^\lambda}\sum_{i=1}^{s_1} nN_{S_{n-1}}(\varphi(\overline{\sigma}_{1i}) - \widehat{l}_{\varphi(\overline{\sigma}_{1i})})\left(\sum_{j=1}^{q}\chi_{\mu_j}(\varphi(\overline{\sigma}_{1i}) - \widehat{l}_{\varphi(\overline{\sigma}_{1i})})\right)\\
&= \frac{n}{f^\lambda}\sum_{j=1}^{q}\left(\sum_{i=1}^{s_1} N_{S_{n-1}}(\varphi(\overline{\sigma}_{1i}) - \widehat{l}_{\varphi(\overline{\sigma}_{1i})})\chi_{\mu_j}(\varphi(\overline{\sigma}_{1i}) - \widehat{l}_{\varphi(\overline{\sigma}_{1i})})\right)\\
&= \frac{n}{f^\lambda}\sum_{j=1}^{q} f^{\mu_j}\eta_{\mu_j}(0),
\end{aligned}$$

where the last equality follows from equations (2.2) and (2.3). Thus, the theorem holds for $k = 1$.

Suppose $k > 1$. (We note here that the proof for $k > 1$ is similar to the proof for $k = 1$. The reason we distinguish them is to make the proof easier to comprehend.)

By equation (2.3),

$$\eta_\lambda(k) = \frac{1}{f^\lambda}\sum_{i=1}^{s_k} N_{S_n}(\varphi(\overline{\sigma}_{ki}))\chi_\lambda(\varphi(\overline{\sigma}_{ki})).$$

Note that $\overline{\sigma}_{ki}$ consists of exactly k 1-cycles and $\varphi(\overline{\sigma}_{ki}) = (\nu_1, \nu_2, \ldots, \nu_r) \vdash n$ with $\nu_j = 1$ for $r-k+1 \leq j \leq r$ and $\nu_{r-k} \geq 2$. Let $\overline{\overline{\sigma}}_{ki}$ be the extension of σ_{ki} to S_{n-1}, i.e., $\overline{\overline{\sigma}}_{ki}(j) = \sigma_{ki}(j)$ for $1 \leq j \leq n-k$ and $\overline{\overline{\sigma}}_{ki}(j) = j$ for $n-k+1 \leq j \leq n-1$. Note that $\varphi(\overline{\sigma}_{ki}) - \widehat{l}_{\varphi(\overline{\sigma}_{ki})} = (\nu_1, \nu_2, \ldots, \nu_{r-1}) \vdash (n-1)$ is the partition of $(n-1)$ that represents $\mathrm{Con}_{S_{n-1}}(\overline{\overline{\sigma}}_{ki})$. Furthermore,

$$\mathscr{S}(n-1, k-1) = \bigcup_{i=1}^{s_k}\mathrm{Con}_{S_{n-1}}(\overline{\overline{\sigma}}_{ki}).$$

Therefore, by Theorem 1.2.17,

$$\eta_{\mu_j}(k-1) = \frac{1}{f^{\mu_j}} \sum_{i=1}^{s_k} N_{S_{n-1}}(\varphi(\overline{\sigma}_{ki}) - \widehat{l}_{\varphi(\overline{\sigma}_{ki})})\chi_{\mu_j}(\varphi(\overline{\sigma}_{ki}) - \widehat{l}_{\varphi(\overline{\sigma}_{ki})}).$$

By Lemmas 2.1.2 and 2.1.5,

$$\begin{aligned}
\eta_\lambda(k) &= \frac{1}{f^\lambda} \sum_{i=1}^{s_k} N_{S_n}(\varphi(\overline{\sigma}_{ki})) \left(\sum_{j=1}^{q} \chi_{\mu_j}(\varphi(\overline{\sigma}_{ki}) - \widehat{l}_{\varphi(\overline{\sigma}_{ki})}) \right) \\
&= \frac{1}{f^\lambda} \sum_{i=1}^{s_k} \frac{n}{k} N_{S_{n-1}}(\varphi(\overline{\sigma}_{ki}) - \widehat{l}_{\varphi(\overline{\sigma}_{ki})}) \left(\sum_{j=1}^{q} \chi_{\mu_j}(\varphi(\overline{\sigma}_{ki}) - \widehat{l}_{\varphi(\overline{\sigma}_{ki})}) \right) \\
&= \frac{n}{kf^\lambda} \sum_{j=1}^{q} \left(\sum_{i=1}^{s_k} N_{S_{n-1}}(\varphi(\overline{\sigma}_{ki}) - \widehat{l}_{\varphi(\overline{\sigma}_{ki})})\chi_{\mu_j}(\varphi(\overline{\sigma}_{1i}) - \widehat{l}_{\varphi(\overline{\sigma}_{1i})}) \right) \\
&= \frac{n}{kf^\lambda} \sum_{j=1}^{q} f^{\mu_j} \eta_{\mu_j}(k-1).
\end{aligned}$$

Hence, the theorem holds for $k > 1$. $\square$

2.2 ASP for $\mathscr{F}(n,1)$

In this section, we want to apply Theorem 2.1.6 to determine whether the Alternating Sign Property for $\mathscr{F}(n,1)$ holds. We first prove some inequalities for the eigenvalues of $\mathscr{F}(n,0)$. Then, we prove the ASP for $\mathscr{F}(n,1)$.

2.2.1 Inequalities for the eigenvalues of $\mathscr{F}(n,0)$

For convenience, if $\lambda = (n)$, we set

$$d_n = \eta_\lambda(0).$$

By Theorem 1.4.5,

$$d_n = (-1)^n + nd_{n-1}, \quad \text{for } n \geq 1, \tag{2.4}$$

where $d_0 = 1$. Note that $d_1 = 0$ and $d_n > 0$ for all $n \neq 1$. Furthermore, for $n \geq 3$,

$$\begin{aligned} d_n &= (-1)^n + nd_{n-1} \\ &\geq nd_{n-1} - 1 \qquad (2.5) \\ &= (n-1)d_{n-1} + d_{n-1} - 1 \geq (n-1)d_{n-1}. \qquad (2.6) \end{aligned}$$

Lemma 2.2.1. *Let $1 \leq p \leq n-1$. If $\lambda = (n-p, 1^p)$ and $\mu = (n-p+1, 1^{p-1})$ are partitions of $[n]$, then*

$$f^\lambda |\eta_\lambda(0)| \leq f^\mu |\eta_\mu(0)|.$$

Furthermore, equality holds if and only if $p = 1$ or $n-p = 1$.

Proof. Note that

$$f^\lambda = \frac{n!}{H^\lambda} = \frac{n!}{n(n-p-1)!p!} \quad \text{and} \quad f^\mu = \frac{n!}{H^\mu} = \frac{n!}{n(n-p)!(p-1)!}.$$

By Theorem 1.4.2 and equation (2.4),

$$\begin{aligned} |\eta_\lambda(0)| &= \left|1 + (-1)^{n-p} nd_{n-p-1}\right|, \\ |\eta_\mu(0)| &= \left|1 + (-1)^{n-p+1} nd_{n-p}\right| \\ &= \left|1 - n + (-1)^{n-p+1} n(n-p)d_{n-p-1}\right|. \end{aligned}$$

Therefore, it is sufficient to show that

$$P_L = (n-p)\left|1 + (-1)^{n-p} nd_{n-p-1}\right| \leq p\left|1 - n + (-1)^{n-p+1} n(n-p)d_{n-p-1}\right| = P_R.$$

Case 1. Suppose n and p are of the same parity (both are even or both are odd). Then

$$\begin{aligned} P_R - P_L &= p(n(n-p)d_{n-p-1} + n - 1) - (n-p)(1 + nd_{n-p-1}) \\ &= n(n-p)(p-1)d_{n-p-1} + (p-1)n \geq 0. \end{aligned}$$

Note that $P_R - P_L = 0$ if and only if $p = 1$.

Case 2. Suppose n and p are of different parity (one even and one odd). Then $d_{n-p-1} \neq 0$, for $n-p \neq 2$. Therefore

$$\begin{aligned} P_R - P_L &= p(1-n+n(n-p)d_{n-p-1}) - (n-p)(nd_{n-p-1}-1) \\ &= n(n-p)(p-1)d_{n-p-1} - (p-1)n \\ &= n(p-1)((n-p)d_{n-p-1}-1) \geq 0. \end{aligned}$$

Note that $P_R - P_L = 0$ if and only if $p = 1$ or $n-p = 1$. $\square$

Lemma 2.2.2. *Let $m \geq q \geq 1$ and $n = m+q$. If $\lambda = (m,q)$ and $\mu = (m+1, q-1)$ are partitions of $[n]$, then*

$$(m-q+1)\,|\eta_\lambda(0)| \leq \left|\eta_\mu(0)\right|.$$

Furthermore, equality holds if and only if $q = 1$ or $m = q = 2$.

Proof. We shall prove by induction on q. Suppose $q = 1$. By Corollary 1.4.7, $|\eta_\lambda(0)| = d_m + d_{m-1}$. By equation (2.4), $\left|\eta_\mu(0)\right| = d_{m+1} = (-1)^{m+1} + (m+1)d_m$. Therefore

$$\begin{aligned} \left|\eta_\mu(0)\right| - m\,|\eta_\lambda(0)| &= (-1)^{m+1} + d_m - md_{m-1} \\ &= (-1)^{m+1} + (-1)^m + md_{m-1} - md_{m-1} = 0. \end{aligned}$$

Suppose $q \geq 2$. Assume that the lemma holds for $q-1$. By Theorem 1.4.2,

$$\eta_\lambda(0) = (-1)^{m+1}d_{q-1} - (m+1)\eta_{(m-1,q-1)}(0).$$

By Theorem 1.4.6, $\operatorname{sign}(\eta_\lambda(0)) = (-1)^q$ and $\operatorname{sign}((m-1,q-1)) = (-1)^{q-1}$. Thus,

$$|\eta_\lambda(0)| = (-1)^{m-q+1}d_{q-1} + (m+1)|\eta_{(m-1,q-1)}(0)|.$$

Similarly, by Theorems 1.4.2 and 1.4.6,

$$|\eta_\mu(0)| = (-1)^{m-q+1}d_{q-2} + (m+2)|\eta_{(m,q-2)}(0)|.$$

By induction, $(m-q+1)|\eta_{(m-1,q-1)}(0)| \leq |\eta_{(m,q-2)}(0)|$.

Therefore

$$
\begin{aligned}
&|\eta_\mu(0)| - (m-q+1)|\eta_\lambda(0)| \\
&\quad \geq (-1)^{m-q}(m-q+1)d_{q-1} + (-1)^{m-q+1}d_{q-2} + |\eta_{(m,q-2)}(0)|.
\end{aligned}
$$

If $q = 2$, then $d_{q-1} = 0$ and $|\eta_\mu(0)| - (m-q+1)|\eta_\lambda(0)| \geq d_m + (-1)^{m-1} \geq 0$. Furthermore, equality holds if and only if $m = q = 2$.

Suppose $q \geq 3$. By Corollary 1.4.7, $|\eta_{(m,q-2)}(0)| = d_m + (q-2)|\eta_{(m-1,q-3)}(0)| > d_m$, where the last inequality follows from $|\eta_{(m-1,q-3)}(0)| \neq 0$. If $m \equiv q \mod 2$, then $|\eta_\mu(0)| - (m-q+1)|\eta_\lambda(0)| > (m-q)d_{q-1} + (d_{q-1} - d_{q-2}) + d_m > 0$. If $m \not\equiv q \mod 2$, then

$$
\begin{aligned}
|\eta_\mu(0)| - (m-q+1)|\eta_\lambda(0)| &> -(m-q+1)d_{q-1} + d_{q-2} + d_m \\
&\geq d_m - (m-q+1)d_{q-1} \\
&\geq (m-1)d_{m-1} - (m-q+1)d_{q-1} \qquad \text{(equation (2.6))} \\
&\geq (q-2)d_{q-1} > 0.
\end{aligned}
$$

This completes the proof of the lemma. $\square$

Lemma 2.2.3. *If $m > q \geq 1$ and $k \geq t \geq 1$, then*

$$(m-q+k+1)|\eta_{(q,q^t)}(0)| \leq k|\eta_{(m+1,q^t)}(0)|.$$

Furthermore, equality holds if and only if $q = 1$, $m = 2$ and $k = t$.

Proof. We shall prove by induction on q. Suppose $q = 1$. Then by Corollary 1.4.7, $|\eta_{(q,q^t)}(0)| = t$ and $|\eta_{(m+1,q^t)}(0)| = td_m + d_{m+1}$. Note that $m \geq 2$. If $m = 2$, then

$$
\begin{aligned}
k|\eta_{(m+1,q^t)}(0)| - (m-q+k+1)|\eta_{(q,q^t)}(0)| &= k(t+2) - (k+2)t \\
&= 2(k-t) \geq 0.
\end{aligned}
$$

Furthermore, equality holds if and only if $k = t$.

If $m = 3$, then

$$\begin{aligned} k|\eta_{(m+1,q^t)}(0)| - (m-q+k+1)|\eta_{(q,q^t)}(0)| &= k(2t+9) - (k+3)t \\ &= kt + 3(3k-t) > 0. \end{aligned}$$

Suppose $m \geq 4$. By equation (2.6), $d_m \geq (m-1)(m-2)d_{m-2} \geq (m-1)(m-2)$. Since

$$\begin{aligned} k(m-1)(m-2) - (m-q+k+1) &= km^2 - (3k+1)m + k \\ &\geq 4km - (3k+1)m + k \\ &= (k-1)m + k > 0, \end{aligned}$$

$k|\eta_{(m+1,q^t)}(0)| - (m-q+k+1)|\eta_{(q,q^t)}(0)| \geq t((k-1)m+k) + d_{m+1} > 0.$

Suppose $q \geq 2$. Assume that

$$(m' - (q-1) + k + 1)|\eta_{(q-1,(q-1)^t)}(0)| \leq k|\eta_{(m'+1,(q-1)^t)}(0)|,$$

for all $m' > q-1$ and $k \geq t \geq 1$.

By Corollary 1.4.7,

$$\begin{aligned} |\eta_{(m+1,q^t)}(0)| &= q|\eta_{(m,(q-1)^t)}(0)| + |\eta_{(m+1,q^{t-1})}(0)| \\ &= q|\eta_{(m,(q-1)^t)}(0)| + q|\eta_{(m,(q-1)^{t-1})}(0)| + |\eta_{(m+1,q^{t-2})}(0)| \\ &\;\;\vdots \\ &= q\left(\sum_{j=1}^{t} |\eta_{(m,(q-1)^j)}(0)|\right) + d_{m+1}. \end{aligned}$$

Similarly,

$$|\eta_{(q,q^t)}(0)| = q\left(\sum_{j=1}^{t} |\eta_{(q-1,(q-1)^j)}(0)|\right) + d_q.$$

By induction, for $1 \leq j \leq t$,

$$(m-q+k+1)|\eta_{(q-1,(q-1)^j)}(0)| \leq k|\eta_{(m,(q-1)^j)}(0)|.$$

By equation (2.6), $d_{m+1} \geq m(m-1)d_{m-1} \geq m(m-1)d_q$. Note that $m \geq 3$ and

$$\begin{aligned} km(m-1)-(m-q+k+1) &= km^2-(k+1)m+q-k-1 \\ &\geq 3km-(k+1)m+q-k-1 \\ &= (2k-1)m+q-k-1 \\ &\geq 3(2k-1)+q-k-1 \\ &= 5k-4+q > 0. \end{aligned}$$

Hence, $(m-q+k+1)|\eta_{(q,q^t)}(0)| \leq k|\eta_{(m+1,q^t)}(0)|$.

This completes the proof of the lemma. □

Lemma 2.2.4. *If $q \geq 1$ and $t \geq 1$, then*

$$|\eta_{(q^t,q-1)}(0)| < |\eta_{(q^t,q)}(0)|.$$

Proof. We shall prove by induction on q. Suppose $q = 1$. Then by Corollary 1.4.7, $|\eta_{(q^t,q)}(0)| = t > t-1 = |\eta_{(q^t,q-1)}(0)|$.

Suppose $q \geq 2$. Assume that the lemma holds for $q-1$. By Corollary 1.4.7,

$$\begin{aligned} |\eta_{(q^t,q)}(0)| &= q|\eta_{((q-1)^t,q-1)}(0)| + |\eta_{(q^t)}(0)|, \\ |\eta_{(q^t,q-1)}(0)| &= (q-1)|\eta_{((q-1)^t,q-2)}(0)| + |\eta_{(q^t)}(0)|. \end{aligned}$$

By induction, $|\eta_{((q-1)^t,q-2)}(0)| < |\eta_{((q-1)^t,q-1)}(0)|$. Hence, $|\eta_{(q^t,q-1)}(0)| < |\eta_{(q^t,q)}(0)|$.

This completes the proof of the lemma. □

Lemma 2.2.5. *Let $m \geq q \geq 1$, $k \geq 2$ and $n = m+kq$. If $\lambda = (m, q^{k-1}, q)$ and $\mu = (m+1, q^{k-1}, q-1)$ are partitions of $[n]$, then*

$$(m-q+1)|\eta_\lambda(0)| \leq k|\eta_\mu(0)|.$$

Furthermore, equality holds if and only if $q = 1 = m$.

Proof. We shall prove by induction on q. Suppose $q = 1$. By Corollary 1.4.7, $|\eta_\lambda(0)| = kd_{m-1} + d_m$ and $|\eta_\mu(0)| = (k-1)d_m + d_{m+1}$. If $m = 1$, then $|\eta_\lambda(0)| = k = k|\eta_\mu(0)|$ and the lemma holds. If $m=2$, then $2|\eta_\lambda(0)|=2<k(k+1)=k|\eta_\mu(0)|$ and the lemma holds. Suppose $m\geq 3$. Then by equation (2.6),

$$\begin{aligned}
&k|\eta_\mu(0)|-(m-q+1)|\eta_\lambda(0)|\\
&=k((k-1)d_m+d_{m+1})-m\,(kd_{m-1}+d_m)\\
&\geq k((k-1)d_m+md_m)-m\,(kd_{m-1}+d_m)\\
&=(k^2+(k-1)m-k)d_m-kmd_{m-1}\\
&\geq d_{m-1}\left((k^2+(k-1)m-k)(m-1)-km\right)\\
&\geq d_{m-1}\left(2(k^2+(k-1)m-k)-km\right)\\
&=d_{m-1}\,(2k(k-1)+(k-2)m)>0.
\end{aligned}$$

Suppose $q\geq 2$. Assume that the lemma holds for $q-1$. By Theorem 1.4.2,

$$\eta_\lambda(0)=(-1)^k(m+k)\eta_{(m-1,(q-1)^{k-1},q-1)}(0)+(-1)^{m+k}\eta_{((q-1)^{k-1},q-1)}(0).$$

By Theorem 1.4.6, $\mathrm{sign}(\eta_\lambda(0))=(-1)^{kq}$, $\mathrm{sign}(\eta_{(m-1,(q-1)^{k-1},q-1)}(0))=(-1)^{k(q-1)}$ and $\mathrm{sign}(\eta_{((q-1)^{k-1},q-1)}(0))=(-1)^{(k-1)(q-1)}$. Therefore,

$$|\eta_\lambda(0)|=(m+k)|\eta_{(m-1,(q-1)^{k-1},q-1)}(0)|+(-1)^{m-q+1}|\eta_{((q-1)^{k-1},q-1)}(0)|.$$

Similarly,

$$|\eta_\mu(0)|=(m+k+1)|\eta_{(m,(q-1)^{k-1},q-2)}(0)|+(-1)^{m-q}|\eta_{((q-1)^{k-1},q-2)}(0)|.$$

By induction,

$$(m-q+1)|\eta_{(m-1,(q-1)^{k-1},q-1)}(0)|\leq k|\eta_{(m,(q-1)^{k-1},q-2)}(0)|.$$

Suppose $m = q$. Then

$$
\begin{aligned}
(m-q+1)|\eta_\lambda(0)| &= (m-q+1)\left((m+k)|\eta_{(m-1,(q-1)^{k-1},q-1)}(0)| - |\eta_{((q-1)^{k-1},q-1)}(0)|\right) \\
&< (m+k)\left((m-q+1)|\eta_{(m-1,(q-1)^{k-1},q-1)}(0)|\right) \\
&< (m+k+1)\left(k|\eta_{(m,(q-1)^{k-1},q-2)}(0)|\right) \\
&\leq k\left((m+k+1)|\eta_{(m,(q-1)^{k-1},q-2)}(0)| + |\eta_{((q-1)^{k-1},q-2)}(0)|\right) \\
&= k|\eta_\mu(0)|.
\end{aligned}
$$

Suppose $m > q$. Note that

$$
\begin{aligned}
(m-q+1)|\eta_\lambda(0)| &\leq (m-q+1)\left((m+k)|\eta_{(m-1,(q-1)^{k-1},q-1)}(0)| + |\eta_{((q-1)^{k-1},q-1)}(0)|\right) \\
&\leq (m+k)\left(k|\eta_{(m,(q-1)^{k-1},q-2)}(0)|\right) + (m-q+1)|\eta_{((q-1)^{k-1},q-1)}(0)|.
\end{aligned}
$$

By Lemma 2.2.4,

$$
\begin{aligned}
k|\eta_\mu(0)| &\geq k\left((m+k+1)|\eta_{(m,(q-1)^{k-1},q-2)}(0)| - |\eta_{((q-1)^{k-1},q-2)}(0)|\right) \\
&> k\left((m+k+1)|\eta_{(m,(q-1)^{k-1},q-2)}(0)| - |\eta_{((q-1)^{k-1},q-1)}(0)|\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&k|\eta_\mu(0)| - (m-q+1)|\eta_\lambda(0)| \\
&\quad \geq k|\eta_{(m,(q-1)^{k-1},q-2)}(0)| - (m-q+k+1)|\eta_{((q-1)^{k-1},q-1)}(0)|.
\end{aligned}
$$

If $q = 2$, then by Lemma 2.2.3, $k|\eta_\mu(0)| - (m-q+1)|\eta_\lambda(0)| \geq 0$. Suppose $q \geq 3$. By Corollary 1.4.7,

$$
\begin{aligned}
|\eta_{(m,(q-1)^{k-1},q-2)}(0)| &= (q-2)|\eta_{(m-1,(q-2)^{k-1},q-3)}(0)| + |\eta_{(m,(q-1)^{k-1})}(0)| \\
&> |\eta_{(m,(q-1)^{k-1})}(0)|.
\end{aligned}
$$

It then follows from Lemma 2.2.3 that

$$
\begin{aligned}
& k|\eta_{\mu}(0)| - (m-q+1)|\eta_{\lambda}(0)| \\
& > k|\eta_{(m,(q-1)^{k-1})}(0)| - (m-q+k+1)|\eta_{((q-1)^{k-1},q-1)}(0)| > 0.
\end{aligned}
$$

This completes the proof of the lemma. $\square$

Lemma 2.2.6. *Let $r \geq 0$, $m \geq q \geq 1$, $k \geq 1$, $n = m + kq + \sum_{j=1}^{r} \alpha_j$, $q > \alpha_1$ and*

$$
\begin{aligned}
\lambda &= (m, q^{k-1}, q, \alpha_1, \ldots, \alpha_r), \\
\mu &= (m+1, q^{k-1}, q-1, \alpha_1, \ldots, \alpha_r),
\end{aligned}
$$

be partitions of $[n]$. Then

$$
(m-q+1)\,|\eta_{\lambda}(0)| \leq k\left|\eta_{\mu}(0)\right|.
$$

Proof. If $r = 0$, then the lemma follows from Lemma 2.2.2 or 2.2.5, depending on whether $k = 1$ or $k \geq 2$. Suppose $r \geq 1$. Then $q \geq 2$, for $q > \alpha_1 \geq 1$. We shall prove by induction on α_1.

Suppose $\alpha_1 = 1$. Then $\alpha_1 = \cdots = \alpha_r = 1$. By Corollary 1.4.7,

$$
\begin{aligned}
\eta_{\lambda}(0) &= |\eta_{(m,q^{k-1},q,\alpha_1,\ldots,\alpha_{r-1})}(0)| + |\eta_{(m-1,(q-1)^{k-1},q-1)}(0)| \\
&= |\eta_{(m,q^{k-1},q,\alpha_1,\ldots,\alpha_{r-2})}(0)| + 2|\eta_{(m-1,(q-1)^{k-1},q-1)}(0)| \\
&\;\;\vdots \\
&= |\eta_{(m,q^{k-1},q)}(0)| + r|\eta_{(m-1,(q-1)^{k-1},q-1)}(0)|.
\end{aligned}
$$

Similarly,

$$
\eta_{\mu}(0) = |\eta_{(m+1,q^{k-1},q-1)}(0)| + r|\eta_{(m,(q-1)^{k-1},q-2)}(0)|.
$$

By Lemma 2.2.2 or 2.2.5,

$$(m-q+1)|\eta_{(m,q^{k-1},q)}(0)| \leq k|\eta_{(m+1,q^{k-1},q-1)}(0)|, \qquad \text{and}$$
$$(m-q+1)|\eta_{(m-1,(q-1)^{k-1},q-1)}(0)| \leq k|\eta_{(m,(q-1)^{k-1},q-2)}(0)|.$$

Hence, $(m-q+1)\left|\eta_{\lambda}(0)\right| \leq k\left|\eta_{\mu}(0)\right|$.

Suppose $\alpha_1 \geq 2$. Assume that the lemma holds for $\alpha_1 - 1$. By Corollary 1.4.7,

$$\begin{aligned}
\eta_{\lambda}(0) &= |\eta_{(m,q^{k-1},q,\alpha_1,\ldots,\alpha_{r-1})}(0)| + \alpha_r|\eta_{(m-1,(q-1)^{k-1},q-1,\alpha_1-1,\ldots,\alpha_r-1)}(0)| \\
&\ \vdots \\
&= |\eta_{(m,q^{k-1},q)}(0)| + \sum_{j=1}^{r} \alpha_j|\eta_{(m-1,(q-1)^{k-1},q-1,\alpha_1-1,\ldots,\alpha_j-1)}(0)|.
\end{aligned}$$

Similarly,

$$\eta_{\mu}(0) = |\eta_{(m+1,q^{k-1},q-1)}(0)| + \sum_{j=1}^{r} \alpha_j|\eta_{(m,(q-1)^{k-1},q-2,\alpha_1-1,\ldots,\alpha_j-1)}(0)|.$$

By Lemma 2.2.2 or 2.2.5,

$$(m-q+1)|\eta_{(m,q^{k-1},q)}(0)| \leq k|\eta_{(m+1,q^{k-1},q-1)}(0)|.$$

By induction, for $1 \leq j \leq r$,

$$(m-q+1)|\eta_{(m-1,(q-1)^{k-1},q-1,\alpha_1-1,\ldots,\alpha_j-1)}(0)| \leq k|\eta_{(m,(q-1)^{k-1},q-2,\alpha_1-1,\ldots,\alpha_j-1)}(0)|.$$

Hence, $(m-q+1)\left|\eta_{\lambda}(0)\right| \leq k\left|\eta_{\mu}(0)\right|$. $\square$

The following lemma is obvious.

Lemma 2.2.7. *If $u \geq v$, then*

$$\left(\frac{u+1}{u}\right)\left(\frac{v-1}{v}\right) < 1.$$

Lemma 2.2.8. *Let $r \geq 0$, $k \geq 1$, $m \geq q \geq 2$, $n = m + kq + \sum_{j=1}^{r} \alpha_j$, $q > \alpha_1$ and*

$$\lambda = (m, q^{k-1}, q, \alpha_1, \ldots, \alpha_r),$$
$$\mu = (m+1, q^{k-1}, q-1, \alpha_1, \ldots, \alpha_r),$$

be partitions of $[n]$. Then

$$\frac{f^\lambda}{f^\mu} < \frac{(m-q+1)}{k}.$$

Proof. Note that $h_\mu(i,j) = h_\lambda(i,j)$ for all i,j except when $i = q$, $j = 1$ or $j = k+1$. Let $c_i = h_\lambda(i,1)$ and $d_i = h_\lambda(i,k+1)$ for $1 \leq i \leq q-1$. Note that $h_\mu(i,1) = c_i + 1$ and $h_\mu(i,k+1) = d_i - 1$ for $1 \leq i \leq q-1$, and $h_\mu(q,1) = h_\lambda(q,1)$. Therefore

$$\begin{aligned}
\frac{f^\lambda}{f^\mu} &= \frac{H^\mu}{H^\lambda} \\
&= \frac{\left(\prod_{i=1}^{q-1}(c_i+1)\right)\left(\prod_{i=1}^{q-1}(d_i-1)\right)(m+1-q)!(k-1)!}{\left(\prod_{i=1}^{q-1} c_i\right)\left(\prod_{i=1}^{q-1} d_i\right)(m-q)!k!} \\
&= \left(\prod_{i=1}^{q-1}\left(\frac{c_i+1}{c_i}\right)\left(\frac{d_i-1}{d_i}\right)\right)\frac{(m+1-q)}{k} \\
&< \frac{(m-q+1)}{k},
\end{aligned}$$

where the last inequality follows from $c_i > d_i$ and Lemma 2.2.7. $\square$

Theorem 2.2.9. *Let $r \geq 0$, $k \geq 1$, $m \geq q \geq 1$, $n = m + kq + \sum_{j=1}^{r} \alpha_j$, $q > \alpha_1$ and*

$$\lambda = (m, q^{k-1}, q, \alpha_1, \ldots, \alpha_r),$$
$$\mu = (m+1, q^{k-1}, q-1, \alpha_1, \ldots, \alpha_r),$$

be partitions of $[n]$. Then

$$f^\lambda \left|\eta_\lambda(0)\right| \leq f^\mu \left|\eta_\mu(0)\right|.$$

Furthermore, equality holds if and only if $\lambda = (1, 1^{n-1})$ or $\lambda = (n-1, 1)$.

Proof. Suppose $q = 1$. Then $r = 0$ and the theorem follows from Lemma 2.2.1. Suppose

$q \geq 2$. By Lemmas 2.2.6 and 2.2.8,

$$\frac{f^{\lambda}}{f^{\mu}}\left|\eta_{\lambda}(0)\right| < \frac{(m-q+1)}{k}\left|\eta_{\lambda}(0)\right| \leq \left|\eta_{\mu}(0)\right|.$$

This completes the proof of the theorem. □

2.2.2 Proof of ASP for $\mathscr{F}(n,1)$

We now prove the Alternating Sign Property for eigenvalues of $\mathscr{F}(n,1)$.

Theorem 2.2.10. (ASP for $\mathscr{F}(n,1)$) *Let $n \geq 2$ and $\lambda = (\lambda_1,\ldots,\lambda_r) \vdash n$.*

(a) $\eta_{\lambda}(1) = 0$ *if and only if* $\lambda = (n-1,1)$ *or* $\lambda = (2,1^{n-2})$.

(b) *If $r = 1$ and $\lambda \neq (2)$, then $\eta_{\lambda}(1) > 0$.*

(c) *If $r \geq 2$ and $\lambda \neq (n-1,1)$ or $(2,1^{n-2})$, then*

$$\begin{aligned}\operatorname{sign}(\eta_{\lambda}(1)) &= (-1)^{|\lambda|-\lambda_1-1}\\ &= (-1)^{(\text{\#cells under the first row of } \lambda)-1}\end{aligned}$$

where $\operatorname{sign}(\eta_{\lambda}(1))$ *is* 1 *if* $\eta_{\lambda}(1)$ *is positive or* -1 *if* $\eta_{\lambda}(1)$ *is negative.*

Proof. Suppose the Ferrers diagrams obtained from λ by removing 1 node from the right hand side from any row of the diagram so that the resulting diagram will still be a partition of $(n-1)$ are those of $\mu_1,\ldots,\mu_s$. Then by Theorem 2.1.6,

$$\eta_{\lambda}(1) = \frac{n}{kf^{\lambda}}\sum_{j=1}^{s} f^{\mu_j}\eta_{\mu_j}(0).$$

Suppose $r = 1$. Then $s = 1$ and $\mu_1 = (\lambda_1 - 1) = (n-1)$. Thus, $\eta_{\lambda}(1) = \frac{n}{kf^{\lambda}} f^{\mu_1}\eta_{\mu_1}(0) \geq 0$ and with equality if and only if $\mu_1 = (1)$, i.e., $\lambda = (2)$.

Suppose $r \geq 2$. If $\lambda_1 = \lambda_2$, then the first part of each μ_j is λ_1. By Theorem 1.4.6, $\operatorname{sign}(\eta_{\mu_j}(0)) = (\sum_{i=2}^{r}\lambda_i) - 1 = |\lambda| - \lambda_1 - 1$. Hence,

$$\eta_{\lambda}(1) = (-1)^{|\lambda|-\lambda_1-1}\frac{n}{kf^{\lambda}}\sum_{j=1}^{s} f^{\mu_j}|\eta_{\mu_j}(0)|.$$

Note that $\eta_\lambda(1) = 0$ if and only if $s = 1$ and $\mu_1 = (1)$, i.e., $\lambda = (1,1)$. For other partitions λ, $|\eta_\lambda(1)| \neq 0$ and $\text{sign}(\eta_\lambda(1)) = |\lambda| - \lambda_1 - 1$.

Suppose $\lambda_1 = m+1 > \lambda_2 = q$. Note that we may write

$$\lambda = (m+1, q^{k-1}, q, \alpha_1, \ldots, \alpha_r),$$

where $r \geq 0$, $k \geq 1$, $m \geq q \geq 1$, and $q > \alpha_1$. Let

$$\begin{aligned}
\mu_1 &= (m, q^{k-1}, q, \alpha_1, \ldots, \alpha_r), \\
\mu_2 &= (m+1, q^{k-1}, q-1, \alpha_1, \ldots, \alpha_r).
\end{aligned}$$

By Theorem 1.4.6, $\text{sign}(\eta_{\mu_1}(0)) = |\lambda| - \lambda_1$ and $\text{sign}(\eta_{\mu_j}(0)) = |\lambda| - \lambda_1 - 1$ for $j \geq 2$. This implies that

$$\eta_\lambda(1) = (-1)^{|\lambda|-\lambda_1-1} \frac{n}{kf^\lambda} \left(f^{\mu_2}|\eta_{\mu_2}(0))| - f^{\mu_1}|\eta_{\mu_1}(0))| + \sum_{j=3}^{s} f^{\mu_j}|\eta_{\mu_j}(0)| \right).$$

By Theorem 2.2.9, $f^{\mu_2}|\eta_{\mu_2}(0)| - f^{\mu_1}|\eta_{\mu_1}(0)| \geq 0$. Furthermore, equality holds if and only if

$$\mu_1 = (1, 1^{n-2}) \text{ or } \mu_1 = (n-2, 1),$$

i.e., $\lambda = (2, 1^{n-2})$ or $(n-1, 1)$. Note also that when this happens, $s = 2$. Therefore $\eta_\lambda(1) = 0$. For other partitions λ, $f^{\mu_2}|\eta_{\mu_2}(0)| - f^{\mu_1}|\eta_{\mu_1}(0)| > 0$. Hence, $|\eta_\lambda(1)| \neq 0$ and $\text{sign}(\eta_\lambda(1)) = |\lambda| - \lambda_1 - 1$.

This completes the proof of the theorem. $\square$

CHAPTER 3: SMALLEST EIGENVALUE AND BOUNDING A LARGEST INDEPENDENT SET IN $\mathscr{F}(n,1)$

In this chapter, we will determine the smallest eigenvalue of $\mathscr{F}(n,1)$ by applying the Recurrence Formula for $\mathscr{F}(n,k)$ (Theorem 2.1.6). Then we will determine a bound for a largest independent set in $\mathscr{F}(n,1)$. The results of this chapter have been published in Ku, Lau and Wong (2016, [40]).

3.1 Some Eigenvalues of $\mathscr{F}(n,1)$

Lemma 3.1.1. (Ellis, 2012 [17, Lemma 2.4]) *For $n \geq 9$, the only Specht modules S^{λ} of dimension $f^{\lambda} < \binom{n-1}{2} - 1$ are as follows:*

(a) $S^{(n)}$ *(the trivial representation), dimension* 1*;*

(b) $S^{(1^n)}$ *(the sign representation), dimension* 1*;*

(c) $S^{(n-1,1)}$*, dimension* $n-1$*;*

(d) $S^{(2,1^{n-2})}$ $(\cong S^{(1^n)} \otimes S^{(n-1,1)})$*, dimension* $n-1$.

Lemma 3.1.2. *For $n \geq 13$, the only Specht modules S^{λ} of dimension $\binom{n-1}{2} - 1 \leq f^{\lambda} < \frac{1}{6}n(n-1)(n-5)$ are as follows:*

(a) $S^{(n-2,2)}$*, dimension* $\binom{n-1}{2} - 1$*;*

(b) $S^{(2^2,1^{n-4})}$*, dimension* $\binom{n-1}{2} - 1$*;*

(c) $S^{(n-2,1^2)}$*, dimension* $\binom{n-1}{2}$*;*

(d) $S^{(3,1^{n-3})}$*, dimension* $\binom{n-1}{2}$.

Proof. By Lemma 3.1.1, it is sufficient to prove the following statement:

(∗) For $n \geq 13$, $f^{\lambda} < \frac{1}{6}n(n-1)(n-5)$ if and only if

$$\lambda \in \left\{(n),(n-1,1),(1^n),(2,1^{n-2}),(n-2,2),(2^2,1^{n-4}),(n-2,1^2),(3,1^{n-3})\right\}.$$

By direct calculation using Theorem 1.2.33, (∗) can be verified for $n = 13, 14$. We proceed by induction. Assume that (∗) holds for $n-2$, $n-1$; we will prove it for n. Let α be a partition of n such that $f^{\alpha} < \frac{1}{6}n(n-1)(n-5)$. Consider the restriction $[\alpha] \downarrow S_{n-1}$, which has the same dimension. First suppose $[\alpha] \downarrow S_{n-1}$ is reducible. If it has one of the eight irreducible representations (in (∗)) as a constituent, then by (1.2), the possibilities of α are as follows:

Table 3.1: The possibilities of α

constituent	possibilities of α
$[n-1]$	(n), $(n-1,1)$
$[1^{n-1}]$	(1^n), $(2,1^{n-1})$
$[n-2,1]$	$(n-1,1)$, $(n-2,2)$, $(n-2,1^2)$
$[2,1^{n-3}]$	$(2,1^{n-2})$, $(2^2,1^{n-4})$, $(3,1^{n-3})$
$[n-3,2]$	$(n-2,2)$, $(n-3,3)$, $(n-3,2,1)$
$[2^2,1^{n-5}]$	$(3,2,1^{n-5})$, $(2^3,1^{n-6})$, $(2^2,1^{n-4})$
$[n-3,1^2]$	$(n-2,1^2)$, $(n-3,2,1)$, $(n-3,1^3)$
$[3,1^{n-4}]$	$(4,1^{n-4})$, $(3,2,1^{n-5})$, $(3,1^{n-3})$

By Lemma 1.2.33, the new irreducible representations above all have dimension $\geq \frac{1}{6}n(n-1)(n-5)$:

Table 3.2: α and dimension of α

α	f^{α}
$(n-3,3)$, $(2^3,1^{n-6})$	$\frac{1}{6}n(n-1)(n-5)$
$(n-3,2,1)$, $(3,2,1^{n-5})$	$\frac{1}{3}n(n-2)(n-4)$
$(n-3,1^3)$, $(4,1^{n-4})$	$\frac{1}{6}(n-1)(n-2)(n-3)$

Hence, (∗) holds, provided that $[\alpha] \downarrow S_{n-1}$ has one of the eight irreducible representations in (∗) as a constituent.

Suppose that the irreducible constituents of $[\alpha] \downarrow S_{n-1}$ do not include any of the eight irreducible representations in (∗). By induction hypothesis for $n-1$, each irreducible constituent has dimension $\geq \frac{1}{6}(n-1)(n-2)(n-6)$. Note that $2\left(\frac{1}{6}(n-1)(n-2)(n-6)\right) > \frac{1}{6}n(n-1)(n-5)$ for $n \geq 15$. Thus, $[\alpha] \downarrow S_{n-1}$ has exactly one irreducible constituent, i.e., it is irreducible. Therefore, $[\alpha] = [s^t]$ for some $s, t \in \mathbb{N}$ with $st = n$, i.e., it has a square

Young diagram. Furthermore, $t \geq 2$.

Now consider

$$[\alpha] \downarrow S_{n-2} = [s^{t-1}, s-2] + [s^{t-2}, s-1, s-1].$$

Note that for $n \geq 15$, neither of these two irreducible constituents are any of the eight irreducible representations in $(*)$. By induction hypothesis for $n-2$, each has dimension $\geq \frac{1}{6}(n-2)(n-3)(n-7)$, but

$$2\left(\frac{1}{6}(n-2)(n-3)(n-7)\right) > \frac{1}{6}n(n-1)(n-5),$$

for $n \geq 15$, contradicting $\dim([\alpha] \downarrow S_{n-2}) < \frac{1}{6}n(n-1)(n-5)$.

This completes the proof of the lemma. □

For convenience, if $\lambda = (n)$, we set

$$d_n = \eta_\lambda(0).$$

By Theorem 1.2.17, $d_n = |D_n|$.

Lemma 3.1.3. *For $n \geq 5$, the followings are the eigenvalues of $\mathscr{F}(n,1)$ for α with dimension $f^\alpha < \frac{1}{6}n(n-1)(n-5)$:*

Table 3.3: Eigenvalues of $\mathscr{F}(n,1)$ for α with small dimension

λ	$\eta_\lambda(1)$
(n)	nd_{n-1}
$(n-1,1)$	0
(1^n)	$(-1)^n n(n-2)$
$(2,1^{n-2})$	0
$(n-2,2)$	$-\frac{1}{(n-3)}\left[d_{n-1}+(-1)^n(n-2)\right]$
$(n-2,1^2)$	$-\frac{n}{(n-1)(n-2)}\left[d_{n-1}+(-1)^{n-1}(n-2)\right]$
$(2^2,1^{n-4})$	$(-1)^{n-1}(n-2)^2$
$(3,1^{n-3})$	$(-1)^n n(n-4)$

Proof. These eigenvalues can be evaluated by using Theorem 1.4.5 (or Theorem 1.4.2) and Theorem 2.1.6.

(a) $\eta_{(n)}(1) = \frac{n}{f^{(n)}}\left[f^{(n-1)}\eta_{(n-1)}(0)\right] = \frac{n}{1}\left[1 \cdot d_{n-1}\right] = nd_{n-1}$.

(b) By Theorem 1.4.5, $\eta_{(n-2,1)}(0) = -\eta_{(n-2)}(0) - \eta_{(n-3)}(0) = -d_{n-2} - d_{n-3}$. Since

$$d_{n-2} = (-1)^{n-2} + (n-2)d_{n-3} \quad \Longleftrightarrow \quad d_{n-3} = \frac{d_{n-2} + (-1)^{n-1}}{n-2}$$

we have

$$\eta_{(n-2,1)}(0) = -\frac{(-1)^{n-1} + (n-1)d_{n-2}}{n-2} = -\frac{d_{n-1}}{n-2} \quad \text{(Theorem 1.4.5)}.$$

Therefore,

$$\begin{aligned}
\eta_{(n-1,1)}(1) &= \frac{n}{f^{(n-1,1)}} \left[f^{(n-2,1)} \eta_{(n-2,1)}(0) + f^{(n-1)} \eta_{(n-1)}(0) \right] \\
&= \frac{n}{n-1} \left[(n-2) \cdot \left(-\frac{d_{n-1}}{n-2} \right) + 1 \cdot d_{n-1} \right] \\
&= 0.
\end{aligned}$$

(c) By Theorem 1.4.2, $\eta_{(1^{n-1})}(0) = (-1)^{n-1} + (-1)^n(n-1) = (-1)^{n-2}(n-2)$. Therefore,

$$\eta_{(1^n)}(1) = \frac{n}{f^{(1^n)}} \left[f^{(1^{n-1})} \eta_{(1^{n-1})}(0) \right] = \frac{n}{1} \left[1 \cdot (-1)^{n-2}(n-2) \right] = (-1)^n n(n-2).$$

(d) By Theorem 1.4.2, $\eta_{(2,1^{n-3})}(0) = (-1)^{n-1} + (-1)^{n+1}(n-1)\eta_{(1)}(0) = (-1)^{n-1}$. Therefore,

$$\begin{aligned}
\eta_{(2,1^{n-2})}(1) &= \frac{n}{f^{(2,1^{n-2})}} \left[f^{(2,1^{n-3})} \eta_{(2,1^{n-3})}(0) + f^{(1^{n-1})} \eta_{(1^{n-1})}(0) \right] \\
&= \frac{n}{n-1} \left[(n-2) \cdot (-1)^{n-1} + 1 \cdot (-1)^{n-2}(n-2) \right] \\
&= 0.
\end{aligned}$$

(e) By Theorem 1.4.5, $\eta_{(n-3,2)}(0) = \eta_{(n-3)}(0) - 2\eta_{(n-4,1)}(0) = d_{n-3} - 2\left(-\frac{d_{n-3}}{n-4} \right) = \frac{n-2}{n-4} d_{n-3}$. Therefore,

$$
\begin{aligned}
\eta_{(n-2,2)}(1) &= \frac{n}{f^{(n-2,2)}}\left[f^{(n-3,2)}\eta_{(n-3,2)}(0)+f^{(n-2,1)}\eta_{(n-2,1)}(0)\right]\\
&= \frac{2n}{n(n-3)}\left[\frac{(n-1)(n-4)}{2}\cdot\frac{n-2}{n-4}d_{n-3}+(n-2)\cdot\left(-\frac{d_{n-1}}{n-2}\right)\right]\\
&= \frac{2}{(n-3)}\left[-d_{n-1}+\frac{(n-1)(n-2)}{2}d_{n-3}\right]\\
&= \frac{2}{(n-3)}\left[-d_{n-1}+\frac{(n-1)(n-2)}{2}\left(\frac{d_{n-2}+(-1)^{n-1}}{n-2}\right)\right]\\
&= \frac{2}{(n-3)}\left[-d_{n-1}+\frac{(n-1)}{2}d_{n-2}-\frac{(n-1)}{2}(-1)^{n}\right]\\
&= \frac{2}{(n-3)}\left[-d_{n-1}+\frac{1}{2}\left(d_{n-1}+(-1)^{n}\right)-\frac{(n-1)}{2}(-1)^{n}\right]\\
&= -\frac{1}{(n-3)}\left[d_{n-1}+(-1)^{n}(n-2)\right].
\end{aligned}
$$

(f) By Theorem 1.4.5, $\eta_{(n-3,1^2)}(0)=-\eta_{(n-3,1)}(0)+\eta_{(n-4)}(0)=-\left(-\frac{d_{n-2}}{n-3}\right)+d_{n-4}$. Therefore,

$$
\begin{aligned}
\eta_{(n-2,1^2)}(1) &= \frac{n}{f^{(n-2,1^2)}}\left[f^{(n-3,1^2)}\eta_{(n-3,1^2)}(0)+f^{(n-2,1)}\eta_{(n-2,1)}(0)\right]\\
&= \frac{2n}{(n-1)(n-2)}\left[\frac{(n-2)(n-3)}{2}\cdot\left(d_{n-4}-\left(-\frac{d_{n-2}}{n-3}\right)\right)+(n-2)\cdot\left(-\frac{d_{n-1}}{n-2}\right)\right]\\
&= \frac{2n}{(n-1)(n-2)}\left[-d_{n-1}+\frac{n-2}{2}d_{n-2}+\frac{(n-2)(n-3)}{2}d_{n-4}\right]\\
&= \frac{2n}{(n-1)(n-2)}\left[-d_{n-1}+\frac{n-2}{2}d_{n-2}+\frac{n-2}{2}\left(d_{n-3}+(-1)^{n}\right)\right]\\
&= \frac{2n}{(n-1)(n-2)}\left[-d_{n-1}+\frac{n-2}{2}d_{n-2}+\frac{n-2}{2}d_{n-3}+\frac{n-2}{2}(-1)^{n}\right]\\
&= \frac{2n}{(n-1)(n-2)}\left[-d_{n-1}+\frac{n-2}{2}d_{n-2}+\frac{1}{2}\left(d_{n-2}+(-1)^{n-1}\right)+\frac{n-2}{2}(-1)^{n}\right]\\
&= \frac{2n}{(n-1)(n-2)}\left[-d_{n-1}+\frac{n-1}{2}d_{n-2}+\frac{n-3}{2}(-1)^{n}\right]\\
&= \frac{2n}{(n-1)(n-2)}\left[-d_{n-1}+\frac{1}{2}\left(d_{n-1}+(-1)^{n}\right)+\frac{n-3}{2}(-1)^{n}\right]\\
&= \frac{2n}{(n-1)(n-2)}\left[-\frac{1}{2}d_{n-1}+\frac{(-1)^{n}}{2}(n-2)\right]\\
&= -\frac{n}{(n-1)(n-2)}\left[d_{n-1}+(-1)^{n-1}(n-2)\right]
\end{aligned}
$$

(g) By Theorem 1.4.2, $\eta_{(2^2,1^{n-5})}(0) = (-1)^{n-2}\eta_{(1)}(0) + (-1)^n(n-2)\eta_{(1,1)}(0) = (-1)^n(n-2)\eta_{(1,1)}(0) = (-1)^{n-1}(n-2)$. Therefore,

$$\begin{aligned}
\eta_{(2^2,1^{n-4})}(1) &= \frac{n}{f^{(2^2,1^{n-4})}}\left[f^{(2,1^{n-3})}\eta_{(2,1^{n-3})}(0) + f^{(2^2,1^{n-5})}\eta_{(2^2,1^{n-5})}(0)\right] \\
&= \frac{2n}{n(n-3)}\left[(n-2)\cdot(-1)^{n-1} + \frac{(n-1)(n-4)}{2}\cdot(-1)^{n-1}(n-2)\right] \\
&= (-1)^{n-1}(n-2)^2.
\end{aligned}$$

(h) By Theorem 1.4.2, $\eta_{(3,1^{n-4})}(0) = (-1)^{n-1} + (-1)^n(n-1)\eta_{(2)}(0) = (-1)^{n-1} + (-1)^n(n-1) = (-1)^n(n-2)$. Therefore

$$\begin{aligned}
\eta_{(3,1^{n-3})}(1) &= \frac{n}{f^{(3,1^{n-3})}}\left[f^{(2,1^{n-3})}\eta_{(2,1^{n-3})}(0) + f^{(3,1^{n-4})}\eta_{(3,1^{n-4})}(0)\right] \\
&= \frac{2n}{(n-1)(n-2)}\left[(n-2)\cdot(-1)^{n-1} + \frac{(n-2)(n-3)}{2}\cdot(-1)^n(n-2)\right] \\
&= (-1)^n n(n-4).
\end{aligned}$$

□

Lemma 3.1.4. (Ellis, 2012 [17, Lemma 2.5]) *Let H be a graph on N vertices whose adjacency matrix A has eigenvalues $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_N$, then*

$$\sum_{i=1}^{N} \eta_i^2 = 2e(H),$$

where $e(H)$ is the number of edges in H.

Lemma 3.1.5. $|\mathscr{S}(n,1)| = nd_{n-1}$.

Proof. Let

$$A_i = \{\alpha \in S_n \ : \ \alpha(i) = i \ \text{ and } \ \alpha(j) \neq j \ \ \forall j \in [n] \setminus \{i\}\}.$$

Note that the restriction of A_i on $[n] \setminus \{i\}$ is the derangement of $[n] \setminus \{i\}$. Thus, $|A_i| = d_{n-1}$. Since $\mathscr{S}(n,1) = \bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, we have $|\mathscr{S}(n,1)| = nd_{n-1}$. □

Lemma 3.1.6. *Let n be positive integer such that $n \geq 6$, and $\lambda \vdash n$. If the dimension of the Specht module S^{λ}, $f^{\lambda} \geq \frac{1}{6}n(n-1)(n-5)$, then*

$$|\eta_{\lambda}(1)| \leq 6\sqrt{\frac{d_{n-1}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)}}.$$

Proof. By Theorem 1.2.17, Lemmas 3.1.4 and 3.1.5,

$$\sum_{\lambda \vdash n} \left(f^{\lambda}\eta_{\lambda}(1)\right)^2 = 2e\left(\mathscr{F}(n,1)\right) = n!|\mathscr{S}(n,1)| = n!\left(nd_{n-1}\right).$$

This implies that

$$|\eta_{\lambda}(1)| \leq \frac{\sqrt{n!\left(nd_{n-1}\right)}}{f^{\lambda}} \leq \frac{6\sqrt{n!\left(nd_{n-1}\right)}}{n(n-1)(n-5)} = 6\sqrt{\frac{d_{n-1}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)}}.$$

□

3.2 Smallest Eigenvalue of $\mathscr{F}(n,1)$

We now prove some preliminary results to determine the smallest eigenvalue of $\mathscr{F}(n,1)$.

Lemma 3.2.1. *For $n \geq 4$, $d_n > \frac{n!}{3}$.*

Proof. By Theorem 1.4.5, $d_n = (-1)^n + nd_{n-1}$. For $n = 4$, the lemma holds. Suppose $n \geq 5$. Assume that the lemma holds for $n-1$, i.e., $d_{n-1} > \frac{(n-1)!}{3}$. Since both sides are integers, $d_{n-1} \geq \frac{(n-1)!}{3} + 1$. Therefore,

$$d_n \geq -1 + nd_{n-1} \geq -1 + n\left(\frac{(n-1)!}{3} + 1\right) \geq \frac{n!}{3} + (n-1) > \frac{n!}{3}.$$

Hence, the lemma follows. □

Lemma 3.2.2. *Let $n \geq 14$ be a positive integer and $\lambda \vdash n$. If the dimension f^{λ} of the Specht module S^{λ} is at least $\frac{1}{6}n(n-1)(n-5)$, then*

$$|\eta_{\lambda}(1)| < \left|\eta_{(n-2,2)}(1)\right|.$$

Proof. By Lemma 3.1.6,

$$|\eta_\lambda(1)| \leq 6\sqrt{\frac{d_{n-1}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)}}.$$

By Lemma 3.1.3, $\left|\eta_{(n-2,2)}(1)\right| = \frac{1}{(n-3)}\left[d_{n-1}+(-1)^n(n-2)\right] \geq \frac{1}{(n-3)}\left[d_{n-1}-(n-2)\right]$. So, it is sufficient to show that

$$\begin{aligned}
&6\sqrt{\frac{d_{n-1}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)}} < \frac{1}{(n-3)}\left[d_{n-1}-(n-2)\right] \\
\Leftrightarrow\ & \frac{36d_{n-1}(n-2)(n-3)(n-4)(n-6)!}{(n-1)(n-5)} < \frac{1}{(n-3)^2}\left[d_{n-1}-(n-2)\right]^2 \\
\Leftrightarrow\ & 36d_{n-1}(n-2)(n-3)^3(n-4)(n-6)! < (n-1)(n-5)\left[d_{n-1}^2-2(n-2)d_{n-1}+(n-2)^2\right] \\
\Leftrightarrow\ & 36d_{n-1}(n-2)(n-3)^3(n-4)(n-6)! \\
&\qquad +2d_{n-1}(n-1)(n-2)(n-5) < (n-1)(n-5)\left[d_{n-1}^2+(n-2)^2\right]. \qquad (3.1)
\end{aligned}$$

Note that for $n \geq 7$,

$$\begin{aligned}
(n-2)(n-3)^3(n-4)(n-6)! &> (n-2)(n-5)(n-3)^3(n-6)! \\
&> 16(n-2)(n-5)(n-3) \\
&> 2(n-1)(n-2)(n-5).
\end{aligned}$$

Therefore, equation (3.1) follows provided that

$$37d_{n-1}(n-2)(n-3)^3(n-4)(n-6)! < (n-1)(n-5)\left[d_{n-1}^2+(n-2)^2\right]. \qquad (3.2)$$

By Lemma 3.2.1,

$$\begin{aligned}
(n-1)(n-5)\left[d_{n-1}^2+(n-2)^2\right] &> (n-1)(n-5)\left[d_{n-1}\left(\frac{(n-1)!}{3}\right)+(n-2)^2\right] \\
&> (n-1)(n-5)d_{n-1}\frac{(n-1)!}{3}.
\end{aligned}$$

So, equation (3.2) follows provided that

$$(n-1)(n-5)(n-1)! > 111(n-2)(n-3)^3(n-4)(n-6)!,$$

which is equivalent to

$$(n-1)^2(n-5)^2 > 111(n-3)^2. \tag{3.3}$$

Finally, note that equation (3.3) holds for $n = 14, 15$ and for $n \geq 16$,

$$(n-1)^2(n-5)^2 > (n-3)^2(n-5)^2 \geq 11^2(n-3)^2 > 111(n-3)^2.$$

This completes the proof of the lemma. $\square$

Lemma 3.2.3. *Let $n \geq 7$ be a positive integer. Then*

$$|\eta_\lambda(1)| < \left|\eta_{(n-2,2)}(1)\right|,$$

for $\lambda \in \left\{(n-2,1^2),(1^n),(2^2,1^{n-4}),(3,1^{n-3})\right\}$.

Proof. By Lemma 3.1.3, it is sufficient to show the following two equations hold:

$$\frac{1}{(n-3)}\left[d_{n-1}-(n-2)\right] > n(n-2); \tag{3.4}$$
$$\frac{1}{(n-3)}\left[d_{n-1}-(n-2)\right] > \frac{n}{(n-1)(n-2)}\left[d_{n-1}+(n-2)\right]. \tag{3.5}$$

Note that equation (3.4) is equivalent to

$$d_{n-1} > n(n-2)(n-3)+(n-2), \tag{3.6}$$

and equation (3.6) holds provided that

$$d_{n-1} > n(n-1)(n-2). \tag{3.7}$$

Next, equation (3.5) is equivalent to

$$2d_{n-1} > n(n-2)(n-3)+(n-1)(n-2)^2. \tag{3.8}$$

Note that equation (3.8) holds provided that equation (3.7) holds. Thus, it is sufficient to show that equation (3.7) holds.

By Lemma 3.2.1, for $n \geq 7$,

$$d_{n-1} > \frac{(n-1)!}{3} \geq 2(n-1)(n-2)(n-3) > n(n-1)(n-2).$$

This completes the proof of the lemma. $\square$

Theorem 3.2.4. *For $n \geq 7$, the smallest eigenvalue of $\mathscr{F}(n,1)$ is equal to*

$$\eta_{(n-2,2)}(1) = -\frac{1}{(n-3)}\left(d_{n-1} + (-1)^n(n-2)\right)$$

where $d_n = |D_n|$. Furthermore, $\eta_\lambda(1) = -\frac{1}{(n-3)}\left(d_{n-1} + (-1)^n(n-2)\right)$ if and only if $\lambda = (n-2,2)$.

Proof. It follows from Lemmas 3.1.1, 3.1.2, 3.1.3, 3.2.2 and 3.2.3 that for $n \geq 14$,

$$-\frac{1}{(n-3)}\left[d_{n-1} + (-1)^n(n-2)\right]$$

is the smallest eigenvalue of $\mathscr{F}(n,1)$. Furthermore, $\eta_\lambda(1) = -\frac{1}{(n-3)}\left[d_{n-1} + (-1)^n(n-2)\right]$ if and only if $\lambda = (n-2,2)$. For $7 \leq n \leq 13$, the assertion can be verified by checking all the the eigenvalues in Appendix A. This completes the proof of Theorem 3.2.4. $\square$

Corollary 3.2.5. *The size of a largest independent set in $\mathscr{F}(n,1)$ is at most*

$$\frac{n![d_{n-1} + (-1)^n(n-2)]}{(n^2-3n+1)d_{n-1} + (-1)^n(n-2)}.$$

Proof. By Lemma 3.1.5, Theorems 3.2.4 and 1.4.1, if I is an independent set in $\mathscr{F}(n,1)$, then

$$\begin{aligned}
|I| &\leq \frac{n!}{1 - \frac{nd_{n-1}}{-\frac{1}{(n-3)}[d_{n-1}+(-1)^n(n-2)]}} \\
&= \frac{[d_{n-1} + (-1)^n(n-2)]n!}{d_{n-1} + (-1)^n(n-2) + n(n-3)d_{n-1}} \\
&= \frac{n![d_{n-1} + (-1)^n(n-2)]}{(n^2-3n+1)d_{n-1} + (-1)^n(n-2)}.
\end{aligned}$$

$\square$

CHAPTER 4: SMALLEST EIGENVALUE AND BOUNDING A LARGEST INDEPENDENT SET IN $\mathscr{F}(n,k)$ FOR $k << n$

Ellis (2014, [18]) proved that for sufficiently large n, if $\mathscr{A}$ is a family of permutations of $\{1,2,\ldots,n\}$ with no two permutations agreeing in exactly one point, then $|\mathscr{A}| \leq (n-2)!$, with equality if and only if $\mathscr{A}$ is a coset of the stabiliser of two points. Such a family can also be viewed as an independent set in the Cayley graph of the symmetric group generated by permutations with exactly one fixed point.

In this chapter, we will determine the smallest eigenvalue of $\mathscr{F}(n,k)$ for $k << n$ and the partition where it occurs. Then we will determine a bound for a largest independent set in $\mathscr{F}(n,k)$, thus settling the size of a largest family of permutations such that no two of its elements agree in exactly k points is $O((n-t_0)!)$ for n sufficiently large in terms of k.

4.1 Dimension of Specht Module S^{λ}

Lemma 4.1.1. *Let n,t be positive integers such that $\lambda = (n-t, \lambda_2, \ldots, \lambda_r) \vdash n$ with $\sum_{i=2}^{r} \lambda_i = t$, and $\beta = (\lambda_2, \ldots, \lambda_r) \vdash t$. Let u be the number of columns in the Ferrers diagram of β and a_i be the number of boxes in the ith column. Then*

$$f^{\lambda} = f^{\beta}\left(\frac{n!}{t!}\right)\frac{1}{(n-t-u)!\prod_{i=1}^{u}(n-t+a_i-i+1)}.$$

Proof. By Theorem 1.2.33, $f^{\beta} = \frac{t!}{\prod h_{\beta}(a,b)}$. Now,

$$\prod h_{\lambda}(a,b) = \frac{\prod h_{\beta}(a,b)}{\prod_j h_{\lambda}(1,j)}$$

and

$$\prod_j h_{\lambda}(1,j) = (n-t-u)!\prod_{i=1}^{u}(n-t+a_i-i+1).$$

Hence, the lemma holds. □

The following lemma is obvious.

Lemma 4.1.2. *If a, u, v are positive integers with $u \geq v$, then*

$$\binom{v+a}{v}\binom{u}{u+a} \geq 1.$$

Lemma 4.1.3. *Let n, t be positive integers such that $\lambda = (n-t, \lambda_2, \ldots, \lambda_r) \vdash n$ with $\sum_{i=2}^{r} \lambda_i = t$, and $\beta = (\lambda_2, \ldots, \lambda_r) \vdash t$. Let u be the number of columns in the Ferrers diagram of β and a_i be the number of boxes in the ith column. Let $\mu = (n-t, t) \vdash n$. Then the following hold.*

(a)

$$\frac{f^{\lambda}}{f^{\mu} f^{\beta}} = \binom{n-t+1}{n-2t+1}\left(\frac{\prod_{i=1}^{u}(n-t-i+1)}{\prod_{i=1}^{u}(n-t+a_i-i+1)}\right) \geq 1.$$

(b) *Let*

$$(n-t+1)\prod_{i=1}^{u}(n-t-i+1) - (n-2t+1)\prod_{i=1}^{u}(n-t+a_i-i+1) = \sum_{i=0}^{l} c_i n^i.$$

Then $l \leq u-1 \leq t-1$ and $c_l \geq 1$. Furthermore, for $0 \leq i \leq l$,

$$|c_i| \leq 2(4t)^{t+1}.$$

(c) *There exists a positive integer $n_1 = n_1(t)$ such that for all $n \geq n_1$,*

$$\sum_{i=0}^{l} c_i n^i \geq 1.$$

Proof. (a) By Lemma 4.1.1,

$$f^{\lambda} = f^{\beta}\binom{n!}{t!}\frac{1}{(n-t-u)!\prod_{i=1}^{u}(n-t+a_i-i+1)},$$
$$f^{\mu} = \binom{n!}{t!}\frac{1}{(n-2t)!\prod_{i=1}^{t}(n-t-i+2)}.$$

Therefore,

$$\begin{aligned}
\frac{f^{\lambda}}{f^{\mu}f^{\beta}} &= \frac{(n-2t)!\prod_{i=1}^{t}(n-t-i+2)}{(n-t-u)!\prod_{i=1}^{u}(n-t+a_i-i+1)} \\
&= \left(\frac{(n-2t)!\prod_{i=u+1}^{t}(n-t-i+2)}{(n-t-u)!}\right)\left(\frac{\prod_{i=1}^{u}(n-t-i+2)}{\prod_{i=1}^{u}(n-t+a_i-i+1)}\right) \\
&= \left(\frac{n-t-u+1}{n-2t+1}\right)\left(\frac{(n-t+1)\prod_{i=2}^{u}(n-t-i+2)}{\prod_{i=1}^{u}(n-t+a_i-i+1)}\right) \\
&= \left(\frac{n-t+1}{n-2t+1}\right)\left(\frac{\prod_{i=1}^{u}(n-t-i+1)}{\prod_{i=1}^{u}(n-t+a_i-i+1)}\right) \\
&= \left(\frac{n-t+1}{n-2t+1}\right)\left(\prod_{i=0}^{u-1}\left(\frac{n-t-u+i+1}{n-t+a_{u-i}-u+i+1}\right)\right).
\end{aligned} \tag{4.1}$$

Note that $n-2t+1+\sum_{j=0}^{u-1}a_{u-j}=n-t+1$. So,

$$\frac{n-t+1}{n-2t+1}=\frac{n-2t+1+a_u}{n-2t+1}\left(\prod_{i=1}^{u-1}\frac{n-2t+1+\sum_{j=0}^{i}a_{u-j}}{n-2t+1+\sum_{j=0}^{i-1}a_{u-j}}\right).$$

Since $n-t-u+1 \geq n-2t+1$, by Lemma 4.1.2,

$$\left(\frac{n-2t+1+a_u}{n-2t+1}\right)\left(\frac{n-t-u+1}{n-t+a_u-u+1}\right) \geq 1.$$

Note that $a_1 \geq a_2 \geq \cdots \geq a_u$. Let $1 \leq i \leq u-1$ be fixed. Then $\sum_{j=0}^{i-1}a_{u-j}=t-\sum_{j=i}^{u-1}a_{u-j} \leq t-u+i$ where the last inequality follows from the fact that $a_{u-j} \geq 1$ for $i \leq j \leq u-1$. So, $n-2t+1+\sum_{j=0}^{i-1}a_{u-j} \leq n-t-u+i+1$. By Lemma 4.1.2,

$$\left(\frac{n-2t+1+\sum_{j=0}^{i}a_{u-j}}{n-2t+1+\sum_{j=0}^{i-1}a_{u-j}}\right)\left(\frac{n-t-u+i+1}{n-t+a_{u-i}-u+i+1}\right) \geq 1.$$

Thus, $\frac{f^{\lambda}}{f^{\mu}f^{\beta}} \geq 1$.

(b) The coefficient of n^{u+1} on the left side of the equation is zero and the coefficient of n^u is $-t+1+\sum_{i=1}^{u}(-t-i+1)-(-2t+1+\sum_{i=1}^{u}(-t+a_i-i+1))=0$. This implies that $l \leq u-1 \leq t-1$. Now, if $c_l<0$, then for sufficiently large n,

$$(n-t+1)\prod_{i=1}^{u}(n-t-i+1)-(n-2t+1)\prod_{i=1}^{u}(n-t+a_i-i+1)<0.$$

From equation (4.1), we obtain $\frac{f^\lambda}{f^\mu f^\beta} < 1$, contradicting part (a) of this lemma. Hence, $c_l \geq 1$ for c_l is an integer.

Note that $|-t+1| < t$ and $|-t-i+1| < t+i \leq t+u \leq 2t$ for $1 \leq i \leq u$. So, the absolute value of the coefficient of n^j in $(n-t+1)\prod_{i=1}^{u}(n-t-i+1)$ is at most $\binom{u+1}{j}(2t)^{u+1-j} \leq 2^{u+1}(2t)^{u+1} = 2^{2u+2}t^{u+1}$.

Similarly, $|-2t+1| < 2t$ and $|-t+a_i-i+1| < t-a_i+i < t+i \leq t+u \leq 2t$ for $1 \leq i \leq u$ imply that the absolute value of the coefficient of n^j in $(n-2t+1)\prod_{i=1}^{u}(n-t+a_i-i+1)$ is at most $2^{2u+2}t^{u+1}$. Therefore, $|c_i| \leq 2(2^{2u+2}t^{u+1}) = 2(4t)^{t+1}$ for $u \leq t$.

(c) By part (b) of this lemma, $|c_i| \leq 2(4t)^{t+1}$ for $0 \leq i \leq l-1$, $c_l \geq 1$ and $l < t$. If $l = 0$, then $\sum_{i=0}^{l} c_i n^i = c_l \geq 1$. If $l \geq 1$, then

$$\begin{aligned}\sum_{i=0}^{l} c_i n^i &\geq n^l\left(c_l - \sum_{i=0}^{l-1}|c_i| n^{i-l}\right)\\ &\geq n^l\left(c_l - \frac{2(4t)^{t+1}l}{n}\right)\\ &> n^l\left(1 - \frac{2(4t)^{t+1}t}{n}\right)\\ &\geq \frac{n^l}{2} > 1,\end{aligned}$$

provided that $n \geq 4(4t)^{t+1}t$. □

Lemma 4.1.4. *Let n,t be positive integers with $n \geq 4t$, $\lambda = (n-t,\lambda_2,\ldots,\lambda_r) \vdash n$ with $\sum_{i=2}^{r}\lambda_i = t$, and $\beta = (\lambda_2,\ldots,\lambda_r) \vdash t$. Then*

$$f^\beta\left(\frac{n^t}{2^{2t-1}t!}\right) < f^\lambda < f^\beta\left(\frac{2^t n^t}{t!}\right).$$

Proof. Let u be the number of columns in the Ferrers diagram of β and a_i be the number of boxes in the ith column. By Lemma 4.1.1,

$$\begin{aligned}f^\lambda &= f^\beta\left(\frac{n!}{t!}\right)\frac{1}{(n-t-u)!\prod_{i=1}^{u}(n-t+a_i-i+1)}\\ &= f^\beta\left(\frac{n(n-1)(n-2)\ldots(n-t-u+1)}{t!}\right)\frac{1}{\prod_{i=1}^{u}(n-t+a_i-i+1)}\\ &= f^\beta\left(\frac{n^t}{t!}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{t+u-1}{n}\right)\frac{1}{\prod_{i=1}^{u}\left(1-\frac{t-a_i+i-1}{n}\right)}.\end{aligned}$$

Note that for $1 \le i \le u$, $t - a_i + i - 1 \le t - 1 + u - 1 = t - 2 + u \le 2t - 2$. So, for $n \ge 2(2t-2)$, $1 - \frac{t-a_i+i-1}{n} \ge \frac{1}{2}$. Therefore,

$$\begin{aligned} f^\lambda &\le f^\beta \left(\frac{n^t}{t!}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{t+u-1}{n}\right)2^u \\ &< f^\beta \left(\frac{2^t n^t}{t!}\right). \end{aligned}$$

For $1 \le j \le t+u-1$, we have $j \le t+u-1 \le 2t-1$. So, for $n \ge 2(2t-1)$, $1 - \frac{j}{n} \ge \frac{1}{2}$. Since $\prod_{i=1}^{u}\left(1 - \frac{t-a_i+i-1}{n}\right) < 1$,

$$f^\lambda > f^\beta \left(\frac{n^t}{2^{t+u-1}t!}\right) \ge f^\beta \left(\frac{n^t}{2^{2t-1}t!}\right).$$

$\square$

Lemma 4.1.5. *Let n,t be positive integers with $n \ge (4t+4)^2$, and $\lambda_1, \lambda_2, \ldots, \lambda_r$ be positive integers with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r$, $\lambda_1 \ge \sqrt{n}$, $\sum_{i=2}^{r} \lambda_i = 2(t+1)$. Then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash (\lambda_1 + 2(t+1))$ and*

$$f^\lambda > \frac{n^{t+1}}{(2t+2)!2^{4t+4}}.$$

Proof. Clearly, $\lambda \vdash (\lambda_1 + 2(t+1))$. Let $\beta = (\lambda_2, \ldots, \lambda_r) \vdash 2(t+1)$, u be the number of columns in the Ferrers diagram of β and a_i be the number of boxes in the ith column. By Theorem 1.2.33,

$$\begin{aligned} f^\lambda &= \frac{(\lambda_1 + 2(t+1))!}{\left(\prod h_\beta(a,b)\right)\left(\prod_{i=1}^{\lambda_1} h_\lambda(i,1)\right)} \\ &= f^\beta \frac{(\lambda_1 + 2(t+1))!}{(2t+2)!\left(\prod_{i=1}^{\lambda_1} h_\lambda(i,1)\right)} \\ &\ge \frac{(\lambda_1 + 2(t+1))!}{(2t+2)!\left(\prod_{i=1}^{\lambda_1} h_\lambda(i,1)\right)}, \end{aligned}$$

for $f^\beta \ge 1$. Note that $h_\lambda(i,1) = \lambda_1 + a_i - i + 1$ for $1 \le i \le u$ and $h_\lambda(i,1) = \lambda_1 - i + 1$ for

$u+1 \leq i \leq \lambda_1$. Therefore,

$$\begin{aligned}\prod_{i=1}^{\lambda_1} h_\lambda(i,1) &= (\lambda_1 - u)! \prod_{i=1}^{u} (\lambda_1 + a_i - i + 1) \\ &= (\lambda_1 - u)! \lambda_1^u \left(\prod_{i=1}^{u} \left(1 + \frac{a_i - i + 1}{\lambda_1} \right) \right).\end{aligned}$$

Note that $a_i - i + 1 \leq 2t + 2 < 4t + 4 \leq \sqrt{n} \leq \lambda_1$. So, $\prod_{i=1}^{u} \left(1 + \frac{a_i - i + 1}{\lambda_1}\right) < 2^u$ and

$$\prod_{i=1}^{\lambda_1} h_\lambda(i,1) < (\lambda_1 - u)! \lambda_1^u 2^u.$$

Therefore,

$$\begin{aligned} f^\lambda &> \frac{(\lambda_1 + 2(t+1))!}{(2t+2)!(\lambda_1 - u)! \lambda_1^u 2^u} \\ &= \frac{\left(\prod_{i=1}^{2t+2} (\lambda_1 + 2(t+1) - i + 1)\right) \left(\prod_{i=1}^{u} (\lambda_1 - i + 1)\right)}{(2t+2)! \lambda_1^u 2^u} \\ &= \lambda_1^{2t+2} \frac{\left(\prod_{i=1}^{2t+2} \left(1 + \frac{2(t+1) - i + 1}{\lambda_1}\right)\right) \left(\prod_{i=1}^{u} \left(1 - \frac{i-1}{\lambda_1}\right)\right)}{(2t+2)! 2^u} \\ &> \lambda_1^{2t+2} \frac{\prod_{i=1}^{u} \left(1 - \frac{i-1}{\lambda_1}\right)}{(2t+2)! 2^u}. \end{aligned}$$

From $u \leq 2t+2$, we have $2(i-1) < 2u \leq 4t+4 \leq \sqrt{n} \leq \lambda_1$ for $1 \leq i \leq u$, and $2^{2u} \leq 2^{4t+4}$. Therefore,

$$\begin{aligned} f^\lambda &> (\sqrt{n})^{2t+2} \frac{\frac{1}{2^u}}{(2t+2)! 2^u} \\ &= (\sqrt{n})^{2t+2} \frac{1}{(2t+2)! 2^{2u}} \\ &\geq n^{t+1} \frac{1}{(2t+2)! 2^{4t+4}}. \end{aligned}$$

□

4.2 Eigenvalues with small dimension

Lemma 4.2.1. *Let $\lambda \vdash n$. Then $f^\lambda = f^{\lambda^T}$.*

Theorem 4.2.2. *Let n, k be integers with $n > k \geq 0$, and $\lambda = (n) \vdash n$. Then*

(a) $\eta_\lambda(k) = \binom{n}{k} d_{n-k}$,

(b) $\eta_{\lambda^T}(k) = \binom{n}{k}(-1)^{n-k-1}(n-k-1)$.

Proof. Let $\mu = (n-1) \vdash (n-1)$. By Theorem 1.2.33 and Lemma 4.2.1, $f^\lambda = f^{\lambda^T} = f^\mu = f^{\mu^T} = 1$.

(a) We shall prove by induction on k. When $k = 0$, $\eta_\lambda(0) = d_n$. Suppose $k > 0$. Assume that it holds for $k-1$. By Theorem 2.1.6 and the induction hypothesis,

$$\begin{aligned}
\eta_\lambda(k) &= \frac{n}{kf^\lambda}\left(f^\mu \eta_\mu(k-1)\right) \\
&= \frac{n}{k}\eta_\mu(k-1) \\
&= \frac{n}{k}\binom{n-1}{k-1} d_{(n-1)-(k-1)} \\
&= \binom{n}{k} d_{n-k}.
\end{aligned}$$

(b) We shall prove by induction on k. When $k = 0$, $\eta_{\lambda^T}(0) = (-1)^{n-1}(n-1)$ (Theorem 1.4.2). Suppose $k > 0$. Assume that it holds for $k-1$. By Theorem 2.1.6 and the induction hypothesis,

$$\begin{aligned}
\eta_{\lambda^T}(k) &= \frac{n}{kf^{\lambda^T}}\left(f^{\mu^T}\eta_{\mu^T}(k-1)\right) \\
&= \frac{n}{k}\eta_{\mu^T}(k-1) \\
&= \frac{n}{k}\binom{n-1}{k-1}(-1)^{(n-1)-(k-1)-1}((n-1)-(k-1)-1) \\
&= \binom{n}{k}(-1)^{n-k-1}(n-k-1).
\end{aligned}$$

□

A function $h(n)$ with $n \geq 1$ is said to be a *positive function* if $h(n) > 0$ for all n. Given a function $f(n)$ and a positive function $h(n)$, we write $f(n) = O(h(n))$ if there is a constant A that does not depend on n such that

$$|f(n)| \leq Ah(n), \forall n \geq 1.$$

If A depends on the variable k, we write $f(n) = O_k(h(n))$.

Lemma 4.2.3. *If n,t are integers with $n > t$, then*

$$n(n-1)\dots(n-t)d_{n-t-1} = d_n + O_t(n^t).$$

Proof. By applying equation (2.4) repeatedly,

$$\begin{aligned}
d_n &= nd_{n-1} + O(1) \\
&= n\left((-1)^{n-1} + (n-1)d_{n-2}\right) + O(1) \\
&= n(n-1)d_{n-2} + O(2n) \\
&= n(n-1)\left((-1)^{n-2} + (n-2)d_{n-3}\right) + O(2n) \\
&= n(n-1)(n-2)d_{n-3} + O(3n^2) \\
&\quad \vdots \\
&= n(n-1)(n-2)\dots(n-t)d_{n-t-1} + O((t+1)n^t) \\
&= n(n-1)(n-2)\dots(n-t)d_{n-t-1} + O_t(n^t).
\end{aligned}$$

$\square$

Lemma 4.2.4. *Let n,t be positive integers such that $\lambda = (n-t, \lambda_2, \dots, \lambda_r) \vdash n$, $\sum_{i=2}^{r} \lambda_i = t$, and $\beta = (\lambda_2, \dots, \lambda_r) \vdash t$. Let u be the number of columns in the Ferrers diagram of β and a_i be the number of boxes in the ith column. Then*

$$\eta_\lambda(0) = (-1)^t \left(\prod_{i=1}^{u}(n-t+a_i-i+1)\right) d_{n-t-u} + O_t(n^{u-1}).$$

Proof. We shall prove by induction on u. Suppose $u = 1$. Then $\lambda = (n-t, \overbrace{1, \dots, 1}^{t})$ and $\lambda - \widehat{c} = (n-t-1)$. By Theorem 1.4.2,

$$\eta_\lambda(0) = (-1)^n + (-1)^t n\eta_{(n-t-1)}(0) = (-1)^t nd_{n-t-1} + O(1).$$

Thus, the lemma holds.

Suppose $u > 1$. Assume that the lemma holds for $u-1$. Note that $\lambda - \widehat{h}_\lambda = \beta - \widehat{c}_\beta =$

$(\lambda_2-1,\ldots,\lambda_r-1)$. Therefore $\eta_{\lambda-\widehat{h}_\lambda}(0)=O_t(1)$. By Theorem 1.4.2,

$$\begin{aligned}\eta_\lambda(0)&=(-1)^{n-t+a_1}\eta_{\lambda-\widehat{h}_\lambda}(0)+(-1)^{a_1}(n-t+a_1)\eta_{\lambda-\widehat{c}_\lambda}(0)\\&=(-1)^{a_1}(n-t+a_1)\eta_{\lambda-\widehat{c}_\lambda}(0)+O_t(1).\end{aligned}\tag{4.2}$$

The number of columns in the Ferrers diagram of $\lambda-\widehat{c}_\lambda$ is $u-1$ and a_{i+1} is the number of boxes in the ith column. So, by the induction hypothesis,

$$\begin{aligned}\eta_{\lambda-\widehat{c}_\lambda}(0)&=(-1)^{t-a_1}\left(\prod_{i=1}^{u-1}((n-t-1)+a_{i+1}-i+1)\right)d_{(n-t-1)-(u-1)}+O_t(n^{u-2})\\&=(-1)^{t-a_1}\left(\prod_{i=2}^{u}(n-t+a_i-i+1)\right)d_{n-t-u}+O_t(n^{u-2}).\end{aligned}\tag{4.3}$$

Substituting equation (4.3) into equation (4.2), we obtain

$$\eta_\lambda(0)=(-1)^t\left(\prod_{i=1}^{u}(n-t+a_i-i+1)\right)d_{n-t-u}+O_t(n^{u-1}).$$

Hence, the lemma follows. □

Lemma 4.2.5. *Let n,t be positive integers such that $\lambda=(n-t,\lambda_2,\ldots,\lambda_r)\vdash n$ with $\sum_{i=2}^r\lambda_i=t$, and $\beta=(\lambda_2,\ldots,\lambda_r)\vdash t$. Then*

$$f^\lambda\eta_\lambda(0)=\frac{(-1)^t}{t!}f^\beta d_n+O_t(n^{2t-1}).$$

Proof. Let u be the number of columns in the Ferrers diagram of β and a_i be the number of boxes in the ith column. By Lemma 4.2.4,

$$\eta_\lambda(0)=(-1)^t\left(\prod_{i=1}^{u}(n-t+a_i-i+1)\right)d_{n-t-u}+O_t(n^{u-1}).$$

By Lemma 4.1.1,

$$f^\lambda=f^\beta\left(\frac{n!}{t!}\right)\frac{1}{(n-t-u)!\prod_{i=1}^u(n-t+a_i-i+1)}.$$

Therefore,

$$f^{\lambda} = f^{\beta}\left(\frac{n(n-1)\dots(n-t-u+1)}{t!\,(\prod_{i=1}^{u}(n-t+a_i-i+1))}\right) = O_t(n^t),$$

and

$$f^{\lambda}\eta_{\lambda}(0) = \frac{(-1)^t}{t!}f^{\beta}n(n-1)\dots(n-t-u+1)d_{n-t-u} + O_t(n^{t+u-1}).$$

Note that $u \le t$. So, $O_t(n^{t+u-1}) = O_t(n^{2t-1})$. By Lemma 4.2.3,

$$n(n-1)\dots(n-t-u+1)d_{n-t-u} = d_n + O_t(n^{t+u-1}) = d_n + O_t(n^{2t-1}).$$

Hence,

$$f^{\lambda}\eta_{\lambda}(0) = \frac{(-1)^t}{t!}f^{\beta}d_n + O_t(n^{2t-1}).$$

□

If $\beta = (\lambda_1, \dots, \lambda_r) \vdash m$, $\alpha = (\alpha_1, \dots, \alpha_{r'}) \vdash n$ and $\lambda_r \ge \alpha_1$, then we define

$$(\beta, \alpha) = (\lambda_1, \dots, \lambda_r, \alpha_1, \dots, \alpha_{r'}).$$

Note that (β, α) is a partition of $(m+n)$.

When $0 \le r \le k$, the binomial coefficient $\binom{k}{r} = \frac{k!}{r!(k-r)!}$. When $r > k$, we set $\binom{k}{r} = 0$. Note that for all $r \ge 0$,

$$\binom{k}{r} + \binom{k}{r+1} = \binom{k+1}{r+1}. \tag{4.4}$$

We shall need this equality in the proof of the next theorem.

Theorem 4.2.6. *Let n, k, t be integers with $k \ge 0$, $t > 0$ and $n > k+2t$, $\lambda = (n-t, \lambda_2, \dots, \lambda_r) \vdash n$ with $\sum_{i=2}^{r}\lambda_i = t$, and $\beta = (\lambda_2, \dots, \lambda_r) \vdash t$. Then*

$$f^{\lambda}\eta_{\lambda}(k) = f^{\beta}\binom{n}{k}\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)d_{n-k} + O_t(n^{2t-1+k}).$$

Proof. Let $T = \{i \,:\, i \ge 2 \text{ and } \lambda_i > \lambda_{i+1}\} = \{t_1, \dots, t_{q-1}\}$, $\mu_j = \lambda^{t_{j-1}-}$ for $2 \le j \le q$, and $\mu_1 = (n-t-1, \lambda_2, \dots, \lambda_r)$. Since $\sum_{i=2}^{r}\lambda_i = t$ and $|T| = q-1$, we have $q-1 < t$, i.e., $q \le t$. Note that for $2 \le j \le q$, $\beta^{(t_{j-1}-1)-}$ is the partition whose Ferrers diagram is obtained by deleting the box at the end of the $(t_{j-1}-1)$th row of that of β. Therefore, $\mu_j = (n-t, \beta^{(t_{j-1}-1)-})$ for $2 \le j \le q$.

We shall prove the theorem by induction on k. Suppose $k = 0$. Then the theorem follows from Lemma 4.2.5, by noting that

$$\sum_{r=0}^{t} \binom{0}{r} \frac{(-1)^{t-r}}{(t-r)!} = \frac{(-1)^t}{t!}.$$

Suppose $k > 0$. Assume that the theorem is true for $k-1$. By Theorem 2.1.6,

$$\eta_\lambda(k) = \frac{n}{kf^\lambda} \sum_{j=1}^{q} f^{\mu_j} \eta_{\mu_j}(k-1).$$

Note that $(n-1) > (k-1) + 2t$. Therefore, by the induction hypothesis,

$$\begin{aligned} f^{\mu_1}\eta_{\mu_1}(k-1) =& f^\beta \binom{n-1}{k-1} \left(\sum_{r=0}^{t} \binom{k-1}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) d_{(n-1)-(k-1)} + O_t(n^{2t-1+(k-1)}) \\ =& f^\beta \binom{n-1}{k-1} \left(\sum_{r=0}^{t} \binom{k-1}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) d_{n-k} + O_t(n^{2t-2+k}). \end{aligned} \tag{4.5}$$

Suppose $t = 1$. Then $\sum_{r=0}^{t-1} \binom{k-1}{r} \frac{(-1)^{t-1-r}}{(t-1-r)!} = 1$, $q = 2$ and $\mu_2 = (n-1)$. By Theorem 4.2.2, $\eta_{\mu_2}(k-1) = \binom{n-1}{k-1} d_{n-k}$. Since $f^{\mu_2} = 1 = f^\beta$ (Theorem 1.2.33), we have

$$\begin{aligned} \sum_{j=2}^{q} f^{\mu_j}\eta_{\mu_j}(k-1) &= f^{\mu_2}\eta_{\mu_2}(k-1) \\ &= \binom{n-1}{k-1} d_{n-k} \\ &= f^\beta \binom{n-1}{k-1} d_{n-k} + O_t(n^{2t-4+k}) \\ &= f^\beta \binom{n-1}{k-1} \left(\sum_{r=0}^{t-1} \binom{k-1}{r} \frac{(-1)^{t-1-r}}{(t-1-r)!} \right) d_{n-k} + O_t(n^{2t-4+k}). \end{aligned} \tag{4.6}$$

Suppose $t > 1$. Then $t-1 > 0$. Note also that $(n-1) > (k-1) + 2(t-1)$. Therefore, by the induction hypothesis, for $2 \le j \le q$,

$$f^{\mu_j}\eta_{\mu_j}(k-1) = f^{\beta^{(t_{j-1}-1)-}} \binom{n-1}{k-1} \left(\sum_{r=0}^{t-1} \binom{k-1}{r} \frac{(-1)^{t-1-r}}{(t-1-r)!} \right) d_{n-k} + O_t(n^{2t-4+k}).$$

By equation (1.2),

$$f^{\beta} = \sum_{j=2}^{q} f^{\beta^{(t_{j-1}-1)-}}.$$

Since $q \leq t$,

$$\sum_{j=2}^{q} f^{\mu_j}\eta_{\mu_j}(k-1) = f^{\beta}\binom{n-1}{k-1}\left(\sum_{r=0}^{t-1}\binom{k-1}{r}\frac{(-1)^{t-1-r}}{(t-1-r)!}\right)d_{n-k} + O_t(n^{2t-4+k}).$$

Hence, equation (4.6) always hold for $t > 0$.

Next, note that (see equation (4.4))

$$\begin{aligned}
&\sum_{r=0}^{t-1}\binom{k-1}{r}\frac{(-1)^{t-1-r}}{(t-1-r)!} + \sum_{r=0}^{t}\binom{k-1}{r}\frac{(-1)^{t-r}}{(t-r)!} \\
&= \sum_{r=1}^{t}\binom{k-1}{r-1}\frac{(-1)^{t-r}}{(t-r)!} + \binom{k}{0}\frac{(-1)^{t}}{t!} + \sum_{r=1}^{t}\binom{k-1}{r}\frac{(-1)^{t-r}}{(t-r)!} \\
&= \binom{k}{0}\frac{(-1)^{t}}{t!} + \sum_{r=1}^{t}\left(\binom{k-1}{r-1} + \binom{k-1}{r}\right)\frac{(-1)^{t-r}}{(t-r)!} \\
&= \binom{k}{0}\frac{(-1)^{t}}{t!} + \sum_{r=1}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!} \\
&= \sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f^{\lambda}\eta_{\lambda}(k) &= \frac{n}{k}\left(f^{\beta}\binom{n-1}{k-1}\left(\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)d_{n-k} + O_t(n^{2t-2+k})\right) \\
&= f^{\beta}\binom{n}{k}\left(\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)d_{n-k} + O_t(n^{2t-1+k}).
\end{aligned}$$

Hence, the theorem follows. $\square$

Lemma 4.2.7. *Let $A = A(t,k)$ be a positive constant depending only on the variables t,k and p be a positive integer. Then there exists a positive integer $n_1 = n_1(t,k,p)$ such that for all $n \geq n_1$,*

$$d_{n-k} > An^{p}.$$

Proof. By Lemma 3.2.1,

$$\begin{aligned} d_{n-k} > \frac{(n-k)!}{3} &= \left(\frac{1}{3}\right)(n-k)(n-k-1)(n-k-2)\dots(n-k-p)(n-k-p-1)! \\ &\geq \left(\frac{n^{p+1}}{3}\right)\left(1-\frac{k}{n}\right)\left(1-\frac{k+1}{n}\right)\left(1-\frac{k+2}{n}\right)\dots\left(1-\frac{k+p}{n}\right) \\ &> \frac{n^{p+1}}{3(2)^{p+1}}, \end{aligned}$$

provided that $n \geq 2(k+p)$. Hence, the lemma follows by taking $n_1 = \max\{3(2)^{p+1}A, 2(k+p)\}$. □

Given any real number x, we define $\operatorname{sign}(x) = 1$ if $x > 0$ and $\operatorname{sign}(x) = -1$ if $x < 0$.

Corollary 4.2.8. *Let n,k,t be integers with $k \geq 0$, $t > 0$ and $n > k+2t$, and $\lambda = (n-t, \lambda_2, \dots, \lambda_r) \vdash n$ with $\sum_{i=2}^{r} \lambda_i = t$. If*

$$\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \neq 0,$$

then there exists a positive integer $n_1 = n_1(t,k)$ such that for all $n \geq n_1$,

$$\operatorname{sign}(\eta_\lambda(k)) = \operatorname{sign}\left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!}\right).$$

Proof. By Theorem 4.2.6,

$$f^\lambda \eta_\lambda(k) = f^\beta \binom{n}{k} \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!}\right) d_{n-k} + O_t(n^{2t-1+k}),$$

where $\beta = (\lambda_2, \dots, \lambda_r) \vdash t$. If $\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} > 0$, then

$$f^\beta \binom{n}{k} \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!}\right) \geq f^\beta \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!}\right) = A_1 > 0.$$

Note that

$$f^\lambda \eta_\lambda(k) \geq A_1 d_{n-k} - B_1 n^{2t-1+k},$$

for some positive constant B_1. By Lemma 4.2.7, for sufficiently large n, $d_{n-k} > \frac{B_1 n^{2t-1+k}}{A_1}$. Hence, $f^\lambda \eta_\lambda(k) > 0$, i.e., $\eta_\lambda(k) > 0$.

If $\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}<0$, then

$$f^{\beta}\binom{n}{k}\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)\leq f^{\beta}\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)=A_1<0.$$

Note that

$$f^{\lambda}\eta_{\lambda}(k)\leq A_1 d_{n-k}+B_1 n^{2t-1+k}.$$

By Lemma 4.2.7, for sufficiently large n, we have $d_{n-k}<\frac{-B_1 n^{2t-1+k}}{A_1}$ since $A_1<0$. Therefore $f^{\lambda}\eta_{\lambda}(k)\leq A_1 d_{n-k}+B_1 n^{2t-1+k}<0$, i.e., $\eta_{\lambda}(k)<0$.

This completes the proof of the corollary. □

Let n,k,t be integers with $0\leq k<n$ and $0\leq 2t<n$. We define

$$V(n,t)=\left\{\lambda\vdash n\ :\ \lambda=(n-t,\lambda_2,\ldots,\lambda_r)\ \text{with}\ \sum_{i=2}^{r}\lambda_i=t\right\}.$$

For instance,

$$\begin{aligned}
V(n,0)&=\{(n)\},\\
V(n,1)&=\{(n-1,1)\},\\
V(n,2)&=\{(n-2,2),(n-2,1,1)\},\\
V(n,3)&=\{(n-3,3),(n-3,2,1),(n-3,1,1,1)\}.
\end{aligned}$$

Let $t>0$. For sufficiently large n, $\operatorname{sign}(\eta_{\lambda}(k))=\operatorname{sign}\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)$ for all $\lambda\in V(n,t)$ (Corollary 4.2.8). So, $\eta_{\lambda}(k)$ has the same sign for all $\lambda\in V(n,t)$. Note that $\mu=(n-t,t)\in V(n,t)$. In the next theorem, we will show that $|\eta_{\mu}(k)|$ is the largest among all $|\eta_{\lambda}(k)|$, $\lambda\in V(n,t)$.

Theorem 4.2.9. *Let $t>0$ and $\mu=(n-t,t)\vdash n$. If*

$$\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\neq 0,$$

then there exists a positive integer $n_0=n_0(t,k)$ such that for all $n\geq n_0$,

$$|\eta_{\lambda}(k)|<|\eta_{\mu}(k)|,$$

for all $\lambda \in V(n,t) \setminus \{\mu\}$.

Proof. Let $\lambda = (n-t, \lambda_2, \ldots, \lambda_r) \vdash n$ with $\sum_{i=2}^{r} \lambda_i = t$. Since $\lambda \neq \mu$, $r \geq 3$. By Theorem 4.2.6,

$$f^{\lambda} \eta_{\lambda}(k) = f^{\beta} \binom{n}{k} \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) d_{n-k} + O_t(n^{2t-1+k}),$$

where $\beta = (\lambda_2, \ldots, \lambda_r) \vdash t$. So,

$$f^{\mu} \eta_{\lambda}(k) = \frac{f^{\mu} f^{\beta}}{f^{\lambda}} \binom{n}{k} \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) d_{n-k} + \left(\frac{f^{\mu}}{f^{\lambda}} \right) O_t(n^{2t-1+k}).$$

By part (a) of Lemma 4.1.3, $\frac{f^{\mu}}{f^{\lambda}} \leq \frac{1}{f^{\beta}}$. Therefore,

$$f^{\mu} \eta_{\lambda}(k) = \frac{f^{\mu} f^{\beta}}{f^{\lambda}} \binom{n}{k} \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) d_{n-k} + O_t(n^{2t-1+k}),$$

and

$$f^{\mu} |\eta_{\lambda}(k)| \leq \frac{f^{\mu} f^{\beta}}{f^{\lambda}} \binom{n}{k} \left| \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) \right| d_{n-k} + B_1(n^{2t-1+k}),$$

for some positive constant B_1. By Theorem 4.2.6,

$$f^{\mu} \eta_{\mu}(k) = \binom{n}{k} \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) d_{n-k} + O_t(n^{2t-1+k}).$$

From $|x+y| \geq |x| - |y|$, we deduce that

$$f^{\mu} |\eta_{\mu}(k)| \geq \binom{n}{k} \left| \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) \right| d_{n-k} - B_2(n^{2t-1+k}),$$

for some constant B_2. So, it is sufficient to show that

$$\left(1 - \frac{f^{\mu} f^{\beta}}{f^{\lambda}} \right) \binom{n}{k} \left| \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) \right| d_{n-k} > (B_1 + B_2) n^{2t-1+k},$$

or equivalently (by Lemma 4.1.3),

$$\begin{aligned} &\left(\sum_{i=0}^{l} c_i n^i \right) \binom{n}{k} \left| \left(\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \right) \right| d_{n-k} \\ &\qquad > (n-t+1) \left(\prod_{i=1}^{u} (n-t-i+1) \right) (B_1 + B_2) n^{2t-1+k}, \end{aligned}$$

where u is the number of columns in the Ferrers diagram of β, and

$$\sum_{i=0}^{l} c_i n^i = (n-t+1)\prod_{i=1}^{u}(n-t-i+1) - (n-2t+1)\prod_{i=1}^{u}(n-t+a_i-i+1),$$

where a_i is the number of boxes in the ith column in the Ferrers diagram of β, $l \le u-1 < t$, $c_l > 0$ and $|c_i| \le 2(4t)^{t+1}$ for $0 \le i \le l$.

Since $u < t$ (for $r \ge 3$),

$$(n-t+1)\left(\prod_{i=1}^{u}(n-t-i+1)\right)(B_1+B_2)n^{2t-1+k} \le (B_1+B_2)n^{2t+u+k} < (B_1+B_2)n^{3t+k}.$$

By part (c) of Lemma 4.1.3, $\binom{n}{k}\sum_{i=0}^{l} c_i n^i \ge \binom{n}{k} \ge 1$. So, it is sufficient to show that

$$d_{n-k} > \frac{(B_1+B_2)n^{3t+k}}{\left|\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)\right|},$$

which is true for sufficiently large n (Lemma 4.2.7).

This completes the proof of the theorem. $\square$

Lemma 4.2.10. *Let $t > 0$, $\mu = (n-t,t) \vdash n$ and $\lambda \vdash n$. If*

$$\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!} \ne 0,$$

and $\eta_\lambda(k) = O_t(n^p)$ for some positive integer p, then there exists a positive integer $n_0 = n_0(t,k,p)$ such that for all $n \ge n_0$,

$$|\eta_\lambda(k)| < |\eta_\mu(k)|.$$

Proof. Note that $|\eta_\lambda(k)| \le B_1 n^p$ for some positive constant B_1. By Theorem 4.2.6,

$$f^\mu |\eta_\mu(k)| \ge \binom{n}{k}\left|\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)\right| d_{n-k} - B_2 n^{2t-1+k},$$

for some positive constant B_2. By Lemma 4.1.4,

$$f^\mu < \left(\frac{2^t n^t}{t!}\right).$$

Therefore,

$$|\eta_\mu(k)| > \frac{t!}{2^t n^t}\binom{n}{k}\left|\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)\right| d_{n-k} - \frac{B_2 t!}{2^t} n^{t-1+k}.$$

It is sufficient to show that

$$\frac{t!}{2^t n^t}\binom{n}{k}\left|\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)\right| d_{n-k} > \frac{B_2 t!}{2^t} n^{t-1+k} + B_1 n^p,$$

which is equivalent to

$$\binom{n}{k}\left|\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)\right| d_{n-k} > B_2 n^{2t-1+k} + \frac{B_1 2^t}{t!} n^{t+p}.$$

Note that for sufficiently large n,

$$B_2 n^{2t-1+k} + \frac{B_1 2^t}{t!} n^{t+p} \leq 2n^q,$$

where $q = \max\{2t+k, t+p+1\}$. By Lemma 4.2.7,

$$\binom{n}{k}\left|\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)\right| d_{n-k} > 2n^q,$$

Hence, $|\eta_\lambda(k)| < |\eta_\mu(k)|$. $\square$

Let $\lambda \vdash n$. Recall that λ^T is the conjugate partition of λ.

Lemma 4.2.11. *Let $t \geq 0$ and $\lambda \in V(n,t)$. Then*

$$\eta_{\lambda^T}(0) = O_t(n).$$

Proof. If $t = 0$, then $\lambda = (n)$. By part (b) of Theorem 4.2.2, $\eta_{\lambda^T}(0) = (-1)^{n-1}(n-1) = O_t(n)$.

Suppose $t > 0$. Let $\lambda = (n-t, \lambda_2, \ldots, \lambda_r) \vdash n$ with $\sum_{i=2}^{r} \lambda_i = t$ and $\beta = (\lambda_2, \ldots, \lambda_r) \vdash t$. Note that $\lambda^T - \widehat{h}_{\lambda^T} = (\beta - \widehat{c}_\beta)^T = (\lambda_2 - 1, \ldots, \lambda_r - 1)^T$ and $\lambda^T - \widehat{c}_{\lambda^T} = \beta^T$. Therefore,

$\eta_{\lambda^T - \widehat{h}_{\lambda^T}}(0) = O_t(1) = \eta_{\lambda^T - \widehat{c}_{\lambda^T}}(0)$. By Theorem 1.4.2,

$$|\eta_{\lambda^T}(0)| \leq |\eta_{\lambda^T - \widehat{h}_{\lambda^T}}(0)| + n|\eta_{\lambda^T - \widehat{c}_{\lambda^T}}(0)| = O_t(n).$$

□

Lemma 4.2.12. *Let $t \geq 0$ and $\lambda \in V(n,t)$. Then*

$$\eta_{\lambda^T}(k) = O_t(n^{k+1}).$$

Proof. We shall prove by induction on k. The case $k = 0$ follows from Lemma 4.2.11. Suppose $k > 0$. Assume that the lemma true for $k-1$.

Let $\mu_1, \ldots, \mu_q$ be the only possible partitions of $(n-1)$ that correspond to the Ferrers diagrams obtained by removing 1 node from the right hand side from any row of the Ferrers diagram of λ. Then $\mu_1^T, \ldots, \mu_q^T$ are the only possible partitions of $(n-1)$ that correspond to the Ferrers diagrams obtained by removing 1 node from the right hand side from any row of the Ferrers diagram of λ^T. By Theorem 2.1.6,

$$\eta_{\lambda^T}(k) = \frac{n}{kf^{\lambda^T}} \sum_{j=1}^{q} f^{\mu_j^T} \eta_{\mu_j^T}(k-1).$$

By equation (1.2),

$$f^{\lambda^T} = \sum_{j=1}^{q} f^{\mu_j^T}.$$

So, $\frac{f^{\mu_j^T}}{f^{\lambda^T}} \leq 1$. By the induction hypothesis, $\eta_{\mu_j^T}(k-1) = O_t(n^k)$. Note that $q \leq t$. Hence, $\eta_{\lambda^T}(k) = O_t(n^{k+1})$. □

Theorem 4.2.13. *Let $t > 0$ and $\mu = (n-t,t) \vdash n$. If*

$$\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} \neq 0,$$

then there exists a positive integer $n_0 = n_0(t,k)$ such that for all $n \geq n_0$,

$$|\eta_{\lambda^T}(k)| < |\eta_{\mu}(k)|,$$

for all $\lambda \in \bigcup_{j=0}^{t} V(n,j)$.

Proof. If $\lambda \in \bigcup_{j=0}^{t} V(n,j)$, then by Lemma 4.2.12, $\eta_{\lambda^T}(k) = O_t(n^{k+1})$. By Lemma 4.2.10, $|\eta_{\lambda^T}(k)| < |\eta_{\mu}(k)|$. □

4.3 Eigenvalues with large dimension

Let t_0 be a fixed positive integer, and

$$U(n,t_0) = \left\{ \lambda \vdash n \;:\; \lambda \in \bigcup_{j=0}^{t_0} V(n,j) \text{ or } \lambda^T \in \bigcup_{j=0}^{t_0} V(n,j) \right\}.$$

Let $1 \le j \le t_0$ and $\beta \vdash j$. By Theorem 1.2.33,

$$f^{\beta} = \frac{j!}{\prod h_{\beta}(a,b)} \le j! \le t_0!.$$

By Lemma 4.1.4, for $n \ge 4t_0$,

$$f^{\lambda} < t_0! \left(\frac{2^j n^j}{j!} \right) \le t_0! 2^{t_0} n^{t_0},$$

for all $\lambda \in V(n,j)$. Furthermore, $f^{\lambda} = 1$ for $\lambda \in V(n,0) = \{(n)\}$. So, $f^{\lambda} < t_0!2^{t_0}n^{t_0}$ for all $\lambda \in \bigcup_{j=0}^{t_0} V(n,j)$, provided $n \ge 4t_0$. Since $f^{\lambda} = f^{\lambda^T}$ (Lemma 4.2.1),

$$f^{\lambda} < t_0! 2^{t_0} n^{t_0}, \tag{4.7}$$

for all $\lambda \in U(n,t_0)$, provided $n \ge 4t_0$.

Lemma 4.3.1. *Let* $n \ge \max\{(4t_0+4)^2, (2t_0+2)!t_0!2^{5t_0+4}\}$ *and* $\lambda \vdash n$*. Then*

$$f^{\lambda} \le \frac{n^{t_0+1}}{2^{4t_0+4}(2t_0+2)!},$$

if and only if $\lambda \in U(n,t_0)$.

Proof. If $\lambda \in U(n,t_0)$, then by equation (4.7),

$$f^{\lambda} < t_0! 2^{t_0} n^{t_0} \le \frac{n^{t_0+1}}{2^{4t_0+4}(2t_0+2)!},$$

provided $n \geq 2^{5t_0+4} t_0!(2t_0+2)!$. Thus, $f^\lambda \leq \frac{n^{t_0+1}}{2^{4t_0+4}(2t_0+2)!}$ for all $\lambda \in U(n,t_0)$.
If $\lambda \in V(n,t_0+1)$ or $\lambda^T \in V(n,t_0+1)$, then by Lemmas 4.1.4 and 4.2.1,

$$f^\lambda > \frac{n^{t_0+1}}{2^{2t_0+1}(t_0+1)!} > \frac{n^{t_0+1}}{2^{4t_0+4}(2t_0+2)!},$$

since $n \geq (4t_0+4)^2 \geq 4(t_0+1)$. If $\lambda \in \bigcup_{j=t_0+2}^{2t_0+2} V(n,j)$ or $\lambda^T \in \bigcup_{j=t_0+2}^{2t_0+2} V(n,j)$, then by Lemmas 4.1.4 and 4.2.1,

$$f^\lambda > \frac{n^j}{2^{2j-1} j!} \geq \frac{n^{t_0+2}}{2^{4t_0+3}(2t_0+2)!} > \frac{n^{t_0+1}}{2^{4t_0+4}(2t_0+2)!},$$

since $n \geq (4t_0+4)^2 \geq 4(2t_0+2)$.
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ and $\lambda, \lambda^T \notin \bigcup_{j=0}^{2t_0+2} V(n,j)$. Since $f^\lambda = f^{\lambda^T}$ (Lemma 4.2.1), we may assume that $\lambda_1 \geq r$. Note that $r\lambda_1 \geq n$. This implies that $\lambda_1 \geq \sqrt{n}$. Note that $\sum_{i=2}^{r} \lambda_i > (2t_0+2)$. Let $\mu_1 = (\lambda_1, \lambda_2, \ldots, \lambda_{r-1}, \lambda_r - 1)$. Then by equation (1.2),

$$f^\lambda \geq f^{\mu_1}.$$

If we remove a node from the last row of μ_1 and denote the resulting partition by μ_2, then $f^\lambda \geq f^{\mu_1} \geq f^{\mu_2}$. We shall continue removing a node from the last row until we obtain $\alpha = (\lambda_1, \lambda_2, \ldots, \lambda_{r'}, q)$ where $q + \sum_{i=2}^{r'} \lambda_i = 2t_0+2$. Note that $f^\lambda \geq f^\alpha$. By Lemma 4.1.5,

$$f^\lambda \geq f^\alpha > \frac{n^{t_0+1}}{2^{4t_0+4}(2t_0+2)!},$$

provided that $n \geq (4t_0+4)^2$. Hence, $f^\lambda \leq \frac{n^{t_0+1}}{2^{4t_0+4}(2t_0+2)!}$ if and only if $\lambda \in U(n,t_0)$. □

Lemma 4.3.2. $|\mathscr{S}(n,k)| = \binom{n}{k} d_{n-k}$.

Proof. By Theorem 1.2.17 and part (a) of Theorem 4.2.2, $|\mathscr{S}(n,k)| = \eta_{(n)}(k) = \binom{n}{k} d_{n-k}$. □

Lemma 4.3.3. *Let* $n \geq \max\{(4t_0+4)^2, (2t_0+2)!t_0!2^{5t_0+4}\}$ *and* $\lambda \vdash n$. *If* $\lambda \notin U(n,t_0)$, *then*

$$|\eta_\lambda(k)| < \frac{2^{4t_0+4}(2t_0+2)!\sqrt{n!\binom{n}{k} d_{n-k}}}{n^{t_0+1}}.$$

Proof. By Theorem 1.2.17, Lemmas 3.1.4 and 4.3.2,

$$\sum_{\lambda \vdash n} \left(f^{\lambda} \eta_{\lambda}(k) \right)^2 = 2e(\mathscr{F}(n,k)) = n!|\mathscr{S}(n,k)| = n!\binom{n}{k} d_{n-k}.$$

Therefore,

$$|\eta_{\lambda}(k)| \leq \frac{\sqrt{n!\binom{n}{k} d_{n-k}}}{f^{\lambda}} < \frac{2^{4t_0+4}(2t_0+2)!\sqrt{n!\binom{n}{k} d_{n-k}}}{n^{t_0+1}},$$

where the last inequality follows from Lemma 4.3.1. $\square$

Theorem 4.3.4. *Let* $\mu = (n-t_0, t_0) \vdash n$. *If*

$$\sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} \neq 0,$$

then there exists a positive integer $n_0 = n_0(t_0, k)$ *such that for all* $n \geq n_0$,

$$|\eta_{\lambda}(k)| < |\eta_{\mu}(k)|,$$

for all $\lambda \vdash n$ *and* $\lambda \notin U(n, t_0)$.

Proof. Throughout, we shall make n sufficiently large whenever necessary. By Lemma 4.3.3,

$$|\eta_{\lambda}(k)| < \frac{2^{4t_0+4}(2t_0+2)!\sqrt{n!\binom{n}{k} d_{n-k}}}{n^{t_0+1}}.$$

By Theorem 4.2.6,

$$f^{\mu}|\eta_{\mu}(k)| \geq \binom{n}{k} \left| \sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} \right| d_{n-k} - Bn^{2t_0-1+k},$$

for some positive constant B. By Lemma 4.2.7, we have

$$\frac{1}{2}\binom{n}{k} \left| \sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} \right| d_{n-k} > Bn^{2t_0-1+k}.$$

Thus,

$$f^{\mu}|\eta_{\mu}(k)| \geq \frac{1}{2}\binom{n}{k}\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right| d_{n-k}.$$

By Lemma 4.1.4,

$$f^{\mu} < \left(\frac{2^{t_0}n^{t_0}}{t_0!}\right).$$

Therefore,

$$|\eta_{\mu}(k)| > \frac{t_0!}{2^{t_0+1}n^{t_0}}\binom{n}{k}\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right| d_{n-k}.$$

So, in view of Lemma 4.3.3, it is sufficient to show that

$$\frac{t_0!}{2^{t_0+1}n^{t_0}}\binom{n}{k}\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right| d_{n-k} > \frac{2^{4t_0+4}(2t_0+2)!\sqrt{n!\binom{n}{k}d_{n-k}}}{n^{t_0+1}},$$

or equivalently,

$$d_{n-k} > \frac{2^{10t_0+10}(2t_0+2)!^2(n-k)!k!}{\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right|^2 t_0!^2n^2}. \tag{4.8}$$

For sufficiently large n,

$$\frac{2^{10t_0+10}(2t_0+2)!^2(n-k)!k!}{\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right|^2 t_0!^2n^2} < \frac{(n-k)!}{3}.$$

Since $d_{n-k} > \frac{(n-k)!}{3}$ (Lemma 3.2.1), equation (4.8) holds.

This completes the proof of the theorem. $\square$

4.4 Smallest eigenvalues of $\mathscr{F}(n,k)$

Let $\mathbb{N}$ be the set of positive integers, $k \geq 0$ be an integer and

$$P = \left\{ t \in \mathbb{N} \;:\; \sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!} < 0 \right\}.$$

We shall show that P is non-empty.

Lemma 4.4.1. *If $t \geq 2k+1$, then*

$$\sum_{r=0}^{k} \binom{t}{r} \frac{(-1)^{k-r}}{(k-r)!} > 0.$$

Proof. When $k=0$, $\sum_{r=0}^{k} \binom{t}{r} \frac{(-1)^{k-r}}{(k-r)!} = 1 > 0$. When $k=1$, $\sum_{r=0}^{k} \binom{t}{r} \frac{(-1)^{k-r}}{(k-r)!} = -1+t > 0$.

Assume that $k \geq 2$.

Suppose $k = 2l$ for some positive integer l. Then

$$\begin{aligned}\sum_{r=0}^{k} \binom{t}{r} \frac{(-1)^{k-r}}{(k-r)!} &= \frac{1}{(2l)!} + \sum_{r=1}^{l} \binom{t}{2r} \frac{1}{(2l-2r)!} - \sum_{r=1}^{l} \binom{t}{2r-1} \frac{1}{(2l-2r+1)!} \\ &= \frac{1}{(2l)!} + \sum_{r=1}^{l} \left(\binom{t}{2r} \frac{1}{(2l-2r)!} - \binom{t}{2r-1} \frac{1}{(2l-2r+1)!} \right).\end{aligned}$$

Since $t \geq 2k+1 = 4l+1$, $\binom{t}{2r} > \binom{t}{2r-1}$. Clearly, $\frac{1}{(2l-2r)!} > \frac{1}{(2l-2r+1)!}$. Thus,

$$\sum_{r=0}^{k} \binom{t}{r} \frac{(-1)^{k-r}}{(k-r)!} > 0.$$

Suppose $k = 2l+1$ for some positive integer l. Then

$$\begin{aligned}\sum_{r=0}^{k} \binom{t}{r} \frac{(-1)^{k-r}}{(k-r)!} &= \sum_{r=0}^{l} \binom{t}{2r+1} \frac{1}{(2l-2r)!} - \sum_{r=0}^{l} \binom{t}{2r} \frac{1}{(2l+1-2r)!} \\ &= \sum_{r=0}^{l} \left(\binom{t}{2r+1} \frac{1}{(2l-2r)!} - \binom{t}{2r} \frac{1}{(2l+1-2r)!} \right).\end{aligned}$$

Since $t \geq 2k+1 = 4l+3$, $\binom{t}{2r+1} > \binom{t}{2r}$. Clearly, $\frac{1}{(2l-2r)!} > \frac{1}{(2l-2r+1)!}$. Thus,

$$\sum_{r=0}^{k} \binom{t}{r} \frac{(-1)^{k-r}}{(k-r)!} > 0.$$

□

Lemma 4.4.2. *P is non-empty.*

Proof. Note that $\binom{k}{r} = 0$ if $r > k$. So,

$$\begin{aligned}
\sum_{r=0}^{t} \binom{k}{r} \frac{(-1)^{t-r}}{(t-r)!} &= \sum_{r=0}^{k} \left(\frac{k!}{(k-r)!r!} \right) \left(\frac{(-1)^{t-r}}{(t-r)!} \right) \\
&= (-1)^{t+k} \left(\frac{k!}{t!} \right) \sum_{r=0}^{k} \left(\frac{t!}{(t-r)!r!} \right) \left(\frac{(-1)^{k-r}}{(k-r)!} \right) \\
&= (-1)^{t+k} \left(\frac{k!}{t!} \right) \sum_{r=0}^{k} \binom{t}{r} \frac{(-1)^{k-r}}{(k-r)!}.
\end{aligned}$$

By Lemma 4.4.1, $\sum_{r=0}^{k} \binom{t}{r} \frac{(-1)^{k-r}}{(k-r)!} > 0$ provided that $t \geq 2k+1$. So, if $t' \geq 2k+1$ and $t'+k$ is odd, then $t' \in P$. Hence, P is non-empty. $\square$

Now, we choose the smallest positive integer in P, say $t_0 = t_0(k)$. We are ready to show that the smallest eigenvalue of $\mathscr{F}(n,k)$ occurs at $(n-t_0, t_0)$.

Theorem 4.4.3. *Let n,k be integers with $0 \leq k < n$. Let $t_0 = t_0(k)$ be the smallest positive integer such that*

$$\sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} < 0.$$

Then there exists a positive integer $n_0 = n_0(t_0,k)$ such that for all $n \geq n_0$, $(n-t_0,t_0)$ is the only partition associated to the smallest eigenvalue of $\mathscr{F}(n,k)$.

Proof. Let $\mu = (n-t_0,t_0)$. By Corollary 4.2.8,

$$\operatorname{sign}\left(\eta_\mu(k)\right) = \operatorname{sign}\left(\sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} \right) = -1.$$

By part (a) of Theorem 4.2.2, $\eta_{(n)}(k) = \binom{n}{k} d_{n-k} > 0$. So,

$$\eta_\mu(k) < 0 < \eta_{(n)}(k). \tag{4.9}$$

Let

$$J_0 = \left\{ j \in \mathbb{N} \,:\, 1 \le j \le t_0 - 1, \sum_{r=0}^{j} \binom{k}{r} \frac{(-1)^{j-r}}{(j-r)!} = 0 \right\};$$

$$J_1 = \left\{ j \in \mathbb{N} \,:\, 1 \le j \le t_0 - 1, \sum_{r=0}^{j} \binom{k}{r} \frac{(-1)^{j-r}}{(j-r)!} > 0 \right\}.$$

By the choice of t_0, $J_0 \cup J_1 = \{ j \in \mathbb{N} \,:\, 1 \le j \le t_0 - 1\}$. If $j \in J_0$, then by Theorem 4.2.6,

$$f^\lambda \eta_\lambda(k) = O_t(n^{2t-1+k}),$$

for all $\lambda \in V(n,j)$. By Lemma 4.1.4,

$$\frac{n^t}{2^{2t-1}t!} < f^\lambda.$$

Therefore,

$$\eta_\lambda(k) = O_t(n^{t-1+k}).$$

By Lemma 4.2.10,

$$|\eta_\lambda(k)| < |\eta_\mu(k)|.$$

Since $\eta_\mu(k)$ is negative, this implies that

$$\eta_\mu(k) < \eta_\lambda(k), \tag{4.10}$$

for all $\lambda \in V(n,j)$, $j \in J_0$.

If $j \in J_1$, then by Corollary 4.2.8,

$$\operatorname{sign}(\eta_\lambda(k)) = \operatorname{sign}\left(\sum_{r=0}^{j} \binom{k}{r} \frac{(-1)^{j-r}}{(j-r)!} \right) = 1,$$

for all $\lambda \in V(n,j)$. Thus,

$$\eta_\mu(k) < 0 < \eta_\lambda(k), \tag{4.11}$$

for all $\lambda \in V(n, j)$, $j \in J_1$.

By Theorem 4.2.9,

$$|\eta_\lambda(k)| < |\eta_\mu(k)|,$$

for all $\lambda \in V(n,t_0) \setminus \{\mu\}$. Since $\operatorname{sign}(\eta_\lambda(k)) = \operatorname{sign}\big(\eta_\mu(k)\big)$ (Corollary 4.2.8),

$$\eta_\mu(k) < \eta_\lambda(k) < 0, \tag{4.12}$$

for all $\lambda \in V(n,t_0) \setminus \{\mu\}$.

By Theorem 4.2.13, $|\eta_{\lambda^T}(k)| < |\eta_\mu(k)|$ for all $\lambda \in \bigcup_{j=0}^{t_0} V(n, j)$. Thus, by equations (4.9), (4.10), (4.11) and (4.12), $\eta_\mu(k) < \eta_\lambda(k)$ for all $\lambda \in U(n,t_0) \setminus \{\mu\}$. By Theorem 4.3.4, $|\eta_\lambda(k)| < |\eta_\mu(k)|$ for all $\lambda \notin U(n,t_0)$. So, $\eta_\mu(k) < \eta_\lambda(k)$ for all $\lambda \vdash n$ and $\lambda \neq \mu$. Hence, $\eta_\mu(k)$ is the smallest eigenvalue of $\mathscr{F}(n,k)$ and μ is the only partition associated to the smallest eigenvalue of $\mathscr{F}(n,k)$.

This completes the proof of the theorem. $\square$

Now we can deduce a bound for the largest independent number in $\mathscr{F}(n,k)$.

Theorem 4.4.4. *Let n, k be integers with $0 \le k \le n$. Let $t_0 = t_0(k)$ be the smallest positive integer such that*

$$\sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} < 0.$$

Then there exists a positive integer $n_0 = n_0(t_0,k)$ such that for all $n \ge n_0$, the size of a largest independent set in $\mathscr{F}(n,k)$ is less than

$$2^{2t_0} t_0! \left| \sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} \right| (n-t_0)!.$$

Proof. Let $\mu = (n-t_0,t_0)$. By Theorem 4.2.6 and Lemma 4.1.4,

$$\frac{n^{t_0}}{2^{2t_0-1} t_0!} |\eta_\mu(k)| < f^\mu |\eta_\mu(k)| \le \binom{n}{k} \left| \sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} \right| d_{n-k} + Bn^{2t-1+k},$$

for some positive constant B. By Lemma 4.2.7,

$$\binom{n}{k} \left| \sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} \right| d_{n-k} > Bn^{2t-1+k}.$$

Thus,

$$|\eta_\mu(k)| < \frac{2^{2t_0}t_0!}{n^{t_0}}\binom{n}{k}\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right| d_{n-k}.$$

By Lemma 4.3.2, $|\mathscr{S}(n,k)| = \binom{n}{k}d_{n-k}$. Since $\eta_\mu(k) < 0$ (Corollary 4.2.8),

$$\begin{aligned}
1 - \frac{|\mathscr{S}(n,k)|}{\eta_\mu(k)} &= 1 + \frac{|\mathscr{S}(n,k)|}{|\eta_\mu(k)|} \\
&> 1 + \frac{1}{\frac{2^{2t_0}t_0!}{n^{t_0}}\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right|} \\
&> \frac{n^{t_0}}{2^{2t_0}t_0!\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right|}.
\end{aligned}$$

By Theorem 1.4.1 and Theorem 4.4.3 , if I is an independent set in $\mathscr{F}(n,k)$, then

$$\begin{aligned}
|I| &\leq \frac{n!}{\frac{n^{t_0}}{2^{2t_0}t_0!\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right|}} \\
&= 2^{2t_0}t_0!\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right|\left(\prod_{i=1}^{t_0}\left(1-\frac{i-1}{n}\right)\right)(n-t_0)! \\
&< 2^{2t_0}t_0!\left|\sum_{r=0}^{t_0}\binom{k}{r}\frac{(-1)^{t_0-r}}{(t_0-r)!}\right|(n-t_0)!.
\end{aligned}$$

□

We now list out the values of $t_0 = t_0(k)$ for small values of k:

Table 4.1: $t_0 = t_0(k)$ for $0 \leq k \leq 23$

k	$t_0(k)$	k	$t_0(k)$	k	$t_0(k)$
0	1	8	6	16	13
1	2	9	7	17	13
2	2	10	8	18	14
3	3	11	9	19	15
4	3	12	9	20	16
5	4	13	10	21	17
6	5	14	11	22	17
7	6	15	12	23	18

Notice that in the case $k = 1$, we have $t_0 = 2$. Hence, Theorem 4.4.4 implies that the size of an independent set in $\mathscr{F}(n,1)$ is $O_1((n-2)!)$. In fact, as proved by Ellis (2014, [18]), the optimal upper bound is $(n-2)!$. For general k, we have the following bounds on t_0.

Theorem 4.4.5.
$$\frac{k-1}{2} < t_0 \le k+1.$$

Proof. By Lemma 4.4.1, if $k \ge 2t_0+1$, then

$$\sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} > 0.$$

So, $t_0 > \frac{k-1}{2}$.

Let

$$I = \{\sigma \in \mathscr{S}_n \ : \ \sigma(i) = i, \ \ \forall 1 \le i \le k+1\}.$$

Note that I is an independent set in $\mathscr{F}(n,k)$. By Lemma 4.4.4,

$$(n-k-1)! = |I| < 2^{2t_0} t_0! \left| \sum_{r=0}^{t_0} \binom{k}{r} \frac{(-1)^{t_0-r}}{(t_0-r)!} \right| (n-t_0)!.$$

Hence, $t_0 \le k+1$. $\square$

It was conjectured by Ellis (2014, [18]) that if $\mathscr{A}$ is an independent set in $\mathscr{F}(n,k)$, then $|\mathscr{A}| \le (n-k-1)!$ for sufficiently large n. Our result implies that $|\mathscr{A}| \le O_k\left(\left(n-\frac{k-1}{2}\right)!\right)$ for sufficiently large n.

CHAPTER 5: CERTAIN REGULAR SUBGRAPHS OF $\mathscr{F}(n,0)$

Let $V = \{v_1, \dots, v_n\}$ be the vertex set of a graph Γ. Let $A \subseteq V$. The *characteristic function* of A, $1_A : V \to \{0,1\}$ is defined as

$$1_A(b) = \begin{cases} 1 & \text{if } b \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The *characteristic vector* of A is the $|V|$-tuple, $\mathbf{1}_A = (1_A(v_1), 1_A(v_2), \dots, 1_A(v_n))$.
Let $1 \leq i \leq n$. For each $\sigma \in S_n$, let $f_i(\sigma)$ denote the number of i-cycles appearing in the cyclic decomposition of σ. Let C_i be the subset of S_n, in which, each element in C_i contains at least an i-cycles in its cyclic decomposition, i.e.,

$$C_i = \{\sigma \in S_n \ : \ f_i(\sigma) > 0\}.$$

Note that C_i is a union of conjugacy classes of S_n. Let

$$C_{(i_1=1, i_2, \dots, i_s)} = \bigcup_{j=1}^{s} C_{i_j}.$$

Note that $S_n \setminus C_{(i_1, i_2, \dots, i_s)}$ is also a union of conjugacy classes of S_n, and $\Gamma(S_n, S_n \setminus C_{(i_1, i_2, \dots, i_s)})$ is a $|S_n \setminus C_{(i_1, i_2, \dots, i_s)}|$-regular Cayley graph.
In this chapter, we study the Cayley graph $\Gamma_n^{(k)} = \Gamma(S_n, S_n \setminus C_{(1,2,\dots,k)})$, where $1 \leq k < n$. Note that

$$S_n \setminus C_{(1,2,\dots,k)} \subseteq D_n = S_n \setminus C_{(1)}.$$

Therefore $\Gamma_n^{(k)}$ is a subgraph of the derangement graph $\Gamma_n^{(1)} = \Gamma_n$. In fact,

$$\Gamma_n^{(n-1)} \leq_{\text{sub}} \Gamma_n^{(n-2)} \leq_{\text{sub}} \cdots \leq_{\text{sub}} \Gamma_n^{(2)} \leq_{\text{sub}} \Gamma_n^{(1)},$$

where $H \leq_{\text{sub}} K$ means H is a subgraph of K.

We determine the smallest eigenvalue of $\Gamma_n^{(k)}$ and then we show that the set of all the largest independent sets in $\Gamma_n^{(k)}$ is equal to the set of all the largest independent sets in the derangement graph, $\mathscr{F}(n,0)$. The results of this chapter have been published in Ku, Lau and Wong (2016, [39]).

Since $S_n \setminus C_{(1,2,\ldots,k)}$ is closed under conjugation, the eigenvalue $\eta_{\chi_\lambda}^{(k)}$ of the Cayley graph $\Gamma_n^{(k)}$ can be denoted by $\eta_\lambda^{(k)}$. Throughout the chapter, we shall use this notation.

5.1 Some Eigenvalues of $\Gamma_n^{(k)}$

We shall use the following notations:

(a) $e_n^{(k)}$ is the number of even permutations in $S_n \setminus C_{(1,2,\ldots,k)}$;

(b) $o_n^{(k)}$ is the number of odd permutations in $S_n \setminus C_{(1,2,\ldots,k)}$;

(c) $s_n^{(k)} = e_n^{(k)} - o_n^{(k)}$;

(d) $d_n^{(k)} = |S_n \setminus C_{(1,2,\ldots,k)}|$;

(e) ε is the sign function for S_n, i.e., $\varepsilon(\sigma) = 1$ if σ is an even permutation and $\varepsilon(\sigma) = -1$ if σ is an odd permutation.

Lemma 5.1.1. *Let k,n be positive integers and $k < n$. Then*

(a) $\eta_{(n)}^{(k)} = d_n^{(k)}$;

(b) $\eta_{(1^n)}^{(k)} = s_n^{(k)}$;

(c) $\eta_{(n-1,1)}^{(k)} = -\frac{d_n^{(k)}}{n-1}$;

(d) $\eta_{(2,1^{n-1})}^{(k)} = -\frac{s_n^{(k)}}{n-1}$.

Proof. We shall use Theorem 1.2.17 to calculate these eigenvalues.

(a) $\eta_{(n)}^{(k)} = \frac{1}{f^{(n)}} \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} \chi_{(n)}(\sigma) = \frac{1}{1} \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} 1 = |S_n \setminus C_{(1,2,\ldots,k)}| = d_n^{(k)}$.

(b) $\eta_{(1^n)}^{(k)} = \frac{1}{f^{(1^n)}} \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} \chi_{(1^n)}(\sigma) = \frac{1}{1} \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} \varepsilon(\sigma) = e_n^{(k)} - o_n^{(k)} = s_n^{(k)}$.

(c) $\eta^{(k)}_{(n-1,1)} = \frac{1}{f^{(n-1,1)}} \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} \chi_{(n-1,1)}(\sigma) = \frac{1}{n-1} \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} (\#\{\text{fixed points of } \sigma\} - 1) = \frac{1}{n-1} \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} -1 = -\frac{d_n^{(k)}}{n-1}$.

(d) $\eta^{(k)}_{(2,1^{n-1})} = \frac{1}{f^{(2,1^{n-1})}} \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} \chi_{(2,1^{n-1})}(\sigma) = \frac{1}{n-1} \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} -\varepsilon(\sigma) = -\frac{s_n^{(k)}}{n-1}$. □

Lemma 5.1.2. *Let k,n be positive integers, $k < n$, $n \geq 4$, and $\lambda \vdash n$. If the dimension of the Specht module S^λ, $f^\lambda \geq \binom{n-1}{2} - 1 = \frac{n(n-3)}{2}$, then*

$$\left|\eta^{(k)}_\lambda\right| \leq 2\sqrt{\frac{d_n^{(k)}(n-1)(n-2)(n-4)!}{n(n-3)}}.$$

Proof. By Lemma 3.1.4,

$$\sum_{\lambda \vdash n} \left(f^\lambda \eta^{(k)}_\lambda\right)^2 = 2e\left(\Gamma_n^{(k)}\right) = n! d_n^{(k)}.$$

This implies that

$$\left|\eta^{(k)}_\lambda\right| \leq 2\sqrt{\frac{d_n^{(k)}(n-1)(n-2)(n-4)!}{n(n-3)}}.$$

□

5.2 $d_n^{(k)}$ and $s_n^{(k)}$

Note that $d_j^{(k)} = 0$ for $j = 1, \ldots, k$. For convenience, we set $d_0^{(k)} = 1$.

Lemma 5.2.1. *Let k,n be positive integers.*

(a) *If $n > k \geq \lfloor \frac{n}{2} \rfloor$, then $d_n^{(k)} = (n-1)!$.*

(b) *If $n > k \geq 2$, then*

$$d_n^{(k)} = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \frac{(-1)^i n!}{i! k^i (n-ki)!} d_{n-ki}^{(k-1)}; \tag{5.1}$$

$$d_n^{(k)} = d_n^{(k-1)} - \sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} \frac{n!}{i! k^i (n-ki)!} d_{n-ki}^{(k)}. \tag{5.2}$$

(c) *If $n \geq k+1$, then*

$$d_n^{(k)} = (n-1)d_{n-1}^{(k)} + \frac{(n-1)!}{(n-k-1)!}d_{n-k-1}^{(k)}. \tag{5.3}$$

Proof. (a) Note that $\sigma \in S_n \setminus C_{(1,2,\ldots,k)}$ if and only if σ is an n-cycles. Hence, $d_n^{(k)} = (n-1)!$.

(b) Let T be the set of all the k-cycles in S_n. Let

$$N(\alpha) = \{\sigma \in S_n \setminus C_{(1,2,\ldots,k-1)} \ : \ \sigma \text{ contains } \alpha \text{ in its cycle decomposition } \}.$$

Note that

$$S_n \setminus C_{(1,2,\ldots,k)} = \left(S_n \setminus C_{(1,2,\ldots,k-1)}\right) \setminus \left(\bigcup_{\alpha \in T} N(\alpha)\right).$$

Therefore

$$d_n^{(k)} = d_n^{(k-1)} - \left|\bigcup_{\alpha \in T} N(\alpha)\right|. \tag{5.4}$$

Now, if $\alpha_1, \ldots, \alpha_i \in T$ are disjoint cycles, then

$$|N(\alpha_1) \cap \cdots \cap N(\alpha_i)| = d_{n-ki}^{(k-1)}.$$

Furthermore, the number of subsets of T with exactly i disjoint cycles is equal to $\frac{n!}{i!k^i(n-ki)!}$. Hence, equation (5.1) follows from equation (5.4) and the Principal of Inclusion and Exclusion.

Recall that for each $\sigma \in S_n$, $f_k(\sigma)$ denotes the number of k-cycles appearing in the cyclic decomposition of σ. Let

$$M_i = \{\sigma \in S_n \setminus C_{(1,2,\ldots,k-1)} \ : \ f_k(\sigma) = i\}.$$

Note that

$$S_n \setminus C_{(1,2,\ldots,k)} = \left(S_n \setminus C_{(1,2,\ldots,k-1)}\right) \setminus \left(\bigcup_{i=1}^{\lfloor \frac{n}{k} \rfloor} M_i \right).$$

Furthermore, $M_j \cap M_{j'} = \varnothing$ for $j \neq j'$ and $|M_i| = \frac{n!}{i!k^i(n-ki)!} d_{n-ki}^{(k)}$. Hence, equation (5.2) follows from the following equation

$$d_n^{(k)} = d_n^{(k-1)} - \sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} |M_i|.$$

(c) Given a permutation $\sigma \in S_n \setminus C_{(1,2,\ldots,k)}$, the element n may appear in a t-cycles with $t \geq k+2$ or $t = k+1$. If $t \geq k+2$, then $\sigma = (n\ r)\rho$ for some 2-cycles $(n\ r)$ and $\rho \in S_{n-1}$ with $f_i(\rho) = 0$ for $i = 1, 2, \ldots, k$. If $t = k+1$, then $\sigma = \beta\rho$, where

$$\beta = (n\ j_1\ j_2\ \cdots\ j_k),$$

is a $(k+1)$-cycle and ρ is a permutation of $[n-1] \setminus \{j_1, j_2, \ldots, j_k\}$ with $f_i(\rho) = 0$ for $i = 1, 2, \ldots, k$. Note that the number of such $(n\ r)$ is $n-1$ and the number of such β is $\frac{(n-1)!}{(n-k-1)!}$. Hence, equation (5.3) follows. $\square$

Theorem 5.2.2. *Let k, n be positive integers and $k < n$.*

(a) *If $n \geq 2k$, then*

$$d_n^{(k)} < d_n^{(k-1)} < \cdots < d_n^{(1)}.$$

(b) *If $n \geq k+1$, then*

$$d_n^{(k)} \geq \frac{n!}{3k}.$$

Proof. (a) It follows from equation (5.2).

(b) We shall prove the inequality by induction on n. Suppose $k+1 \leq n \leq 2k+1$. Then by part (a) of Lemma 5.2.1, $d_n^{(k)} = (n-1)!$. On the other hand,

$$\frac{n!}{3k(n-1)!} = \frac{n}{3k} \leq \frac{(2k+1)}{3k} \leq 1.$$

Thus, $d_n^{(k)} \geq \frac{n!}{3k}$ for $k+1 \leq n \leq 2k+1$.

Suppose $n \geq 2k+2$. Assume that the inequality holds for all n_0 with $k+1 \leq n_0 < n$. By equation (5.3),

$$d_n^{(k)} = (n-1)d_{n-1}^{(k)} + \frac{(n-1)!}{(n-k-1)!}d_{n-k-1}^{(k)}.$$

By induction, $d_{n-1}^{(k)} \geq \frac{(n-1)!}{3k}$ and $d_{n-k-1}^{(k)} \geq \frac{(n-k-1)!}{3k}$. Therefore

$$d_n^{(k)} \geq \frac{(n-1)!}{3k}((n-1)+1) = \frac{n!}{3k}.$$

This completes the proof of part (b). □

Note that $e_j^{(k)} = 0 = o_j^{(k)}$, for $j = 1, \ldots, k$. For convenience, we set $e_0^{(k)} = 1$ and $o_0^{(k)} = 0$. Thus, $s_0^{(k)} = e_0^{(k)} - o_0^{(k)} = 1$.

Lemma 5.2.3. *Let k, n be positive integers and $k < n$.*

(a) $s_n^{(1)} = (-1)^{n-1}(n-1)$.

(b) *If $n > k \geq \left\lfloor \frac{n}{2} \right\rfloor$, then $s_n^{(k)} = (-1)^{n-1}(n-1)!$.*

(c) *If $n > k \geq 2$, then*

$$s_n^{(k)} = \begin{cases} \sum_{i=0}^{\left\lfloor \frac{n}{k} \right\rfloor} \frac{n!}{i!(n-ki)!k^i} s_{n-ki}^{(k-1)} & \text{if } k \text{ is even,} \\ \sum_{i=0}^{\left\lfloor \frac{n}{k} \right\rfloor} \frac{n!}{i!(n-ki)!k^i} (-1)^i s_{n-ki}^{(k-1)} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. (a) See equation 2.8 in Ellis (2012, [17]).

(b) Note that $\sigma \in S_n \setminus C_{(1,2,\ldots,k)}$ if and only if σ is an n-cycle. So, $s_n^{(k)} = -(n-1)!$ if n is even and $s_n^{(k)} = (n-1)!$ if n is odd.

(c) We shall use the Principal of Inclusion and Exclusion.

Suppose k is even. Note that a k-cycles has negative sign. When we remove a k-cycles from a permutation, the sign of the permutation changes. Therefore

$$\begin{aligned} e_n^{(k)} = e_n^{(k-1)} - \binom{n}{k}(k-1)!o_{n-k}^{(k-1)} + \binom{n}{k}\binom{n-k}{k}\frac{[(k-1)!]^2}{2!}e_{n-2k}^{(k-1)} \\ - \binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}\frac{[(k-1)!]^3}{3!}o_{n-3k}^{(k-1)} + \cdots. \end{aligned}$$

Similarly,

$$\begin{aligned} o_n^{(k)} = o_n^{(k-1)} - \binom{n}{k}(k-1)!e_{n-k}^{(k-1)} + \binom{n}{k}\binom{n-k}{k}\frac{[(k-1)!]^2}{2!}o_{n-2k}^{(k-1)} \\ - \binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}\frac{[(k-1)!]^3}{3!}e_{n-3k}^{(k-1)} + \cdots. \end{aligned}$$

Hence,

$$\begin{aligned} s_n^{(k)} &= e_n^{(k)} - o_n^{(k)} \\ &= \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \frac{n!}{i!(n-ki)!k^i} s_{n-ki}^{(k-1)}. \end{aligned}$$

Suppose k is odd. Note that a k-cycles has positive sign. When we remove a k-cycles from a permutation, the sign of the permutation does not change. Therefore

$$\begin{aligned} e_n^{(k)} = e_n^{(k-1)} - \binom{n}{k}(k-1)!e_{n-k}^{(k-1)} + \binom{n}{k}\binom{n-k}{k}\frac{[(k-1)!]^2}{2!}e_{n-2k}^{(k-1)} \\ - \binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}\frac{[(k-1)!]^3}{3!}e_{n-3k}^{(k-1)} + \cdots. \end{aligned}$$

Similarly,

$$\begin{aligned} o_n^{(k)} = o_n^{(k-1)} - \binom{n}{k}(k-1)!o_{n-k}^{(k-1)} + \binom{n}{k}\binom{n-k}{k}\frac{[(k-1)!]^2}{2!}o_{n-2k}^{(k-1)} \\ - \binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}\frac{[(k-1)!]^3}{3!}o_{n-3k}^{(k-1)} + \cdots. \end{aligned}$$

Hence,

$$\begin{aligned} s_n^{(k)} &= e_n^{(k)} - o_n^{(k)} \\ &= \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \frac{n!}{i!(n-ki)!k^i} (-1)^i s_{n-ki}^{(k-1)}. \end{aligned}$$

$\square$

Lemma 5.2.4. *Let k, n be positive integers and $k < n$.*

(a) $\operatorname{sign}\left(s_n^{(k)}\right) = (-1)^{n-1}$.

(b) *If $k \geq 2$, then*

$$\left|s_n^{(k)}\right| = \begin{cases} \sum_{i=0}^{l-1} \frac{n!}{i!(n-ki)!k^i} \left|s_{n-ki}^{(k-1)}\right| & \text{if } n-kl > 0, \\ \\ \sum_{i=0}^{l-2} \frac{n!}{i!(n-ki)!k^i} \left|s_{n-ki}^{(k-1)}\right| + \left(\frac{n!}{(l-1)!k^l} - \frac{n!}{l!k^l}\right) & \text{if } n-kl = 0, \end{cases}$$

where $l = \lfloor \frac{n}{k} \rfloor$.

Proof. (a) We shall prove by induction on k. The case $k = 1$ follows from part (a) of Lemma 5.2.3.

Suppose $k \geq 2$. Assume that it holds for $k-1$.

Case 1. Suppose k is even. Then by part (c) of Lemma 5.2.3,

$$s_n^{(k)} = \sum_{i=0}^{l-1} \frac{n!}{i!(n-ki)!k^i} s_{n-ki}^{(k-1)} + \frac{n!}{l!(n-kl)!k^l} s_{n-kl}^{(k-1)}.$$

Note that $0 \leq n-kl < k$. If $n-kl \neq 0$, then $s_{n-kl}^{(k-1)} = 0$ and

$$s_n^{(k)} = \sum_{i=0}^{l-1} \frac{n!}{i!(n-ki)!k^i} s_{n-ki}^{(k-1)}.$$

Now, for $0 \leq i \leq l-1$,

$$n-ki = \begin{cases} \text{odd} & \text{if } n \text{ is odd}, \\ \\ \text{even} & \text{if } n \text{ is even}. \end{cases}$$

This implies that $(-1)^{n-ki-1} = (-1)^{n-1}$. So, by induction, $\operatorname{sign}\left(s_{n-ki}^{(k-1)}\right) = (-1)^{n-1}$. Thus $\operatorname{sign}\left(s_n^{(k)}\right) = (-1)^{n-1}$.

Suppose $n-kl = 0$. Then $l \geq 2$, n is even, $n-ki$ is even for $0 \leq i \leq l-2$, and $s_{n-kl}^{(k-1)} = s_0^{(k-1)} = 1$. By part (b) of Lemma 5.2.3, $s_k^{(k-1)} = -(k-1)!$. Therefore

$$\begin{aligned} s_n^{(k)} &= \sum_{i=0}^{l-2} \frac{n!}{i!(n-ki)!k^i} s_{n-ki}^{(k-1)} + \frac{n!}{(l-1)!k!k^{l-1}} s_k^{(k-1)} + \frac{n!}{l!k^l} \\ &= \sum_{i=0}^{l-2} \frac{n!}{i!(n-ki)!k^i} s_{n-ki}^{(k-1)} - \left(\frac{n!}{(l-1)!k^l} - \frac{n!}{l!k^l} \right). \end{aligned}$$

Note that $\frac{n!}{(l-1)!k^l} - \frac{n!}{l!k^l} > 0$. By induction, $\operatorname{sign}\left(s_{n-ki}^{(k-1)}\right) = -1$ for $0 \leq i \leq l-2$. Thus, $\operatorname{sign}\left(s_n^{(k)}\right) = -1$.

This completes the proof of Case 1.

Case 2. Suppose k is odd. Then by part (c) of Lemma 5.2.3,

$$s_n^{(k)} = \sum_{i=0}^{l-1} \frac{n!}{i!(n-ki)!k^i} (-1)^i s_{n-ki}^{(k-1)} + \frac{n!}{l!(n-kl)!k^l} (-1)^l s_{n-kl}^{(k-1)}.$$

If $n-kl > 0$, then $s_{n-kl}^{(k-1)} = 0$ and

$$s_n^{(k)} = \sum_{i=0}^{l-1} \frac{n!}{i!(n-ki)!k^i} (-1)^i s_{n-ki}^{(k-1)}.$$

Now, for $0 \leq i \leq l-1$,

$$n-ki = \begin{cases} \text{even} & \text{if } i \text{ and } n \text{ are odd, or } i \text{ and } n \text{ are even,} \\ \text{odd} & \text{otherwise.} \end{cases}$$

This implies that $(-1)^{n-ki-1} = (-1)^{i+n-1}$. So, by induction, $\operatorname{sign}\left(s_{n-ki}^{(k-1)}\right) = (-1)^{i+n-1}$ for $0 \leq i \leq l-1$. Thus $\operatorname{sign}\left(s_n^{(k)}\right) = (-1)^{n-1}$.

Suppose $n-kl = 0$. Then $l \geq 2$, $(-1)^l = (-1)^n$ and $s_{n-kl}^{(k-1)} = s_0^{(k-1)} = 1$. By part (b) of

Lemma 5.2.3, $s_k^{(k-1)} = (k-1)!$. Therefore

$$\begin{aligned} s_n^{(k)} &= \sum_{i=0}^{l-2} \frac{n!}{i!(n-ki)!k^i}(-1)^i s_{n-ki}^{(k-1)} + (-1)^{n-1}\left(\frac{n!}{(l-1)!k!k^{l-1}}s_k^{(k-1)} - \frac{n!}{l!k^l}\right) \\ &= \sum_{i=0}^{l-2} \frac{n!}{i!(n-ki)!k^i}(-1)^i s_{n-ki}^{(k-1)} + (-1)^{n-1}\left(\frac{n!}{(l-1)!k^l} - \frac{n!}{l!k^l}\right). \end{aligned}$$

Note that $\frac{n!}{(l-1)!k^l} - \frac{n!}{l!k^l} > 0$ and $(-1)^{n-ki-1} = (-1)^{i+n-1}$ for k is odd. By induction, $\operatorname{sign}\left(s_{n-ki}^{(k-1)}\right) = (-1)^{i+n-1}$ for $0 \le i \le l-2$. Thus $\operatorname{sign}\left(s_n^{(k)}\right) = (-1)^{n-1}$.
This completes the proof of Case 2.

(b) This follows from part (c) of Lemma 5.2.3 and part (a) of this lemma. □

Let $i, j \in [n]$ with $i \neq j$ and $\sigma \in S_n$. We define

$$\operatorname{dist}_{i,j}(\sigma) = m,$$

if $\sigma = \alpha\beta$ with

$$\alpha = (i\, x_1\, x_2\, \cdots x_{m-1}\, j\, x_{m+1}\, \cdots x_{r-1})$$

is a r-cycles and β is a permutation of $[n] \setminus \{i, j, x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_{r-1}\}$. If i, j are not in the same cycle in the cyclic decomposition of σ, then $\operatorname{dist}_{i,j}(\sigma) = \infty$. Note that $\operatorname{dist}_{i,j}(\sigma) \ge 1$. In general, $\operatorname{dist}_{i,j}(\sigma) \neq \operatorname{dist}_{j,i}(\sigma)$. For instance, $\operatorname{dist}_{1,2}((1,2,3)) = 1$, whereas $\operatorname{dist}_{2,1}((1,2,3)) = 2$.

Theorem 5.2.5. *Let k, n be positive integers and $k < n$.*

(a) *If $n \ge 2k$ and $k \ge 2$, then*

$$\left|s_n^{(1)}\right| < \left|s_n^{(2)}\right| < \cdots < \left|s_n^{(k)}\right|.$$

(b) $\left|s_n^{(j)}\right| \le (n-1)!$ *for* $j = 1, 2, \ldots, n-1$.

(c) *For sufficiently large n,*

$$\left|s_n^{(k)}\right| < 8k^2(n-3)!\ln n.$$

Proof. (a) Since $n \geq 2k$, $l \geq 2$. So, the inequality follows from part (b) of Lemma 5.2.4.

(b) By part (b) of Lemma 5.2.3, we may assume that $1 \leq j < \lfloor \frac{n}{2} \rfloor$. By part (a) of this lemma, $\left| s_n^{(j)} \right| < \left| s_n^{(\lfloor \frac{n}{2} \rfloor)} \right| = (n-1)!$.

(c) By part (a) of Lemma 5.2.3, we may assume that $k \geq 2$. Let

$$T_1 = \{\sigma \in S_n \setminus C_{(1,2,\ldots,k)} \ : \ (n\ 1)\sigma \in S_n \setminus C_{(1,2,\ldots,k)}\}.$$

Note that $\sigma \neq (n\ 1)\sigma$ and $\sigma \in T_1$ if and only if $(n\ 1)\sigma \in T_1$. So, the number of elements in T_1 is even. Furthermore, σ is even if and only if $(n\ 1)\sigma$ is odd. This implies that

$$s_n^{(k)} = \sum_{\sigma \in S_n \setminus C_{(1,2,\ldots,k)}} \varepsilon(\sigma) = \sum_{\sigma \in T_2} \varepsilon(\sigma),$$

where $T_2 = (S_n \setminus C_{(1,2,\ldots,k)}) \setminus T_1$.

Note that each element in T_2 is of the form $\alpha\beta$ where

(i) $\alpha = (n\ x_1\ x_2\ \cdots\ x_{r-1})$ is a r-cycles ($r \geq k+1$) with $1 \in \{x_1, x_2, \ldots, x_k\}$ or $1 \in \{x_{r-1}, x_{r-2}, \ldots, x_{r-k}\}$;

(ii) β is a permutation of $[n] \setminus \{n, x_1, x_2, \ldots, x_{r-1}\}$ and $f_i(\beta) = 0$ for $1 \leq i \leq k$.

Let

$$T_3 = \{\sigma \in T_2 \ : \ \mathrm{dist}_{n,1}(\sigma) \leq k \text{ and } \mathrm{dist}_{n,2}(\sigma) = \infty\};$$

$$T_4 = \{\sigma \in T_2 \ : \ \mathrm{dist}_{n,1}(\sigma) \leq k \text{ and } \mathrm{dist}_{n,2}(\sigma) \neq \infty\};$$

$$T_5 = \{\sigma \in T_2 \ : \ \mathrm{dist}_{n,1}(\sigma) > k, \ \mathrm{dist}_{1,n}(\sigma) \leq k \text{ and } \mathrm{dist}_{1,2}(\sigma) = \infty\};$$

$$T_6 = \{\sigma \in T_2 \ : \ \mathrm{dist}_{n,1}(\sigma) > k, \ \mathrm{dist}_{1,n}(\sigma) \leq k \text{ and } \mathrm{dist}_{1,2}(\sigma) \neq \infty\}.$$

Note that $T_2 = \bigcup_{i=3}^{6} T_i$ and $T_i \cap T_j = \varnothing$ for $3 \leq i < j \leq 6$. Therefore,

$$\sum_{\sigma \in T_2} \varepsilon(\sigma) = \sum_{i=3}^{6} \sum_{\sigma \in T_i} \varepsilon(\sigma).$$

Now, for each $\sigma \in T_3$, we have $(n\ 2)\sigma \in T_4$. This implies that

$$\sum_{\sigma\in T_3} \varepsilon(\sigma) + \sum_{\sigma\in T_4} \varepsilon(\sigma) = \sum_{\sigma\in T_7} \varepsilon(\sigma),$$

where

$$T_7 = \{\sigma \in T_2 \ : \ \mathrm{dist}_{n,1}(\sigma) \leq k \text{ and } (\mathrm{dist}_{n,2}(\sigma) \leq k \text{ or } \mathrm{dist}_{2,n}(\sigma) \leq k)\}.$$

Note that each element in T_7 is of the form $\alpha\beta$ where

(i) $\alpha = (n\, x_1\ x_2\ \cdots\ x_{r-1})$ is a r-cycles ($r \geq k+1$) with $1 \in \{x_1, x_2, \ldots, x_k\}$ and either $2 \in \{x_1, x_2, \ldots, x_k\}$ or $2 \in \{x_{r-1}, x_{r-2}, \ldots, x_{r-k}\}$;

(ii) β is a permutation of $[n] \setminus \{n, x_1, x_2, \ldots, x_{r-1}\}$ and $f_i(\beta) = 0$ for $1 \leq i \leq k$.

For such a fixed α_0, when β runs through all the possible permutations,

$$\sum_{\alpha_0\beta} \varepsilon(\alpha_0\beta) = \varepsilon(\alpha_0)\sum_{\beta} \varepsilon(\beta) = \varepsilon(\alpha_0)s_{n-r}^{(k)} = (-1)^{r-1}s_{n-r}^{(k)}.$$

Let M_r be the number of such r-cycles. Then

$$\sum_{\sigma\in T_7} \varepsilon(\sigma) = \sum_{r=k+1}^{n} (-1)^{r-1} M_r s_{n-r}^{(k)}.$$

Note that

$$|M_r| < 2k^2 \binom{n-3}{r-3}(r-3)! = 2k^2\frac{(n-3)!}{(n-r)!}.$$

If $k \geq n-r$ and $n-r \neq 0$, then $s_{n-r}^{(k)} = 0$. If $k < n-r$, then by part (b) of this lemma, $\left|s_{n-r}^{(k)}\right| \leq (n-r-1)!$. In either case, $\left|s_{n-r}^{(k)}\right| \leq (n-r-1)!$ if $n-r \neq 0$. If $n-r = 0$, then

$s_{n-r}^{(k)} = 1$. Note that $1 + \sum_{r=k+1}^{n-1} \frac{1}{(n-r)} < 2\ln n$ for sufficiently large n. Therefore,

$$\begin{aligned}
\left| \sum_{\sigma \in T_7} \varepsilon(\sigma) \right| &\leq \sum_{r=k+1}^{n} |M_r| \left| s_{n-r}^{(k)} \right| \\
&= \left(\sum_{r=k+1}^{n-1} |M_r| \left| s_{n-r}^{(k)} \right| \right) + |M_n| \\
&< \left(\sum_{r=k+1}^{n-1} \left(2k^2 \frac{(n-3)!}{(n-r)!} \right) ((n-r-1)!) \right) + 2k^2(n-3)! \\
&= 2k^2(n-3)! \left(1 + \sum_{r=k+1}^{n-1} \frac{1}{(n-r)} \right) \\
&< 4k^2(n-3)! \ln n.
\end{aligned}$$

Next, for each $\sigma \in T_5$, we have $(1\ 2)\sigma \in T_6$. This implies that

$$\sum_{\sigma \in T_5} \varepsilon(\sigma) + \sum_{\sigma \in T_6} \varepsilon(\sigma) = \sum_{\sigma \in T_8} \varepsilon(\sigma),$$

where

$$\begin{aligned}
T_8 = \{\sigma \in T_2 \ : \ &\mathrm{dist}_{n,1}(\sigma) > k, \ \mathrm{dist}_{1,n}(\sigma) \leq k \text{ and} \\
&(\mathrm{dist}_{1,2}(\sigma) \leq k \text{ or } \mathrm{dist}_{2,1}(\sigma) \leq k)\}.
\end{aligned}$$

Note that each element in T_8 is of the form $\alpha\beta$ where

(i) $\alpha = (1\ x_1\ x_2\ \cdots\ x_{r-1})$ is a r-cycles ($r \geq k+1$) with $n \in \{x_1, x_2, \ldots, x_k\}$, $n \notin \{x_{r-1}, x_{r-2}, \ldots, x_{r-k}\}$ and either $2 \in \{x_1, x_2, \ldots, x_k\}$ or $2 \in \{x_{r-1}, x_{r-2}, \ldots, x_{r-k}\}$;

(ii) β is a permutation of $[n] \setminus \{n, x_1, x_2, \ldots, x_{r-1}\}$ and $f_i(\beta) = 0$ for $1 \leq i \leq k$.

By a similar argument as above, we may conclude that

$$\left| \sum_{\sigma \in T_8} \varepsilon(\sigma) \right| < 4k^2(n-3)! \ln n.$$

Hence,

$$\left| s_n^{(k)} \right| = \left| \sum_{i=7}^{8} \sum_{\sigma \in T_i} \varepsilon(\sigma) \right| \leq \sum_{i=7}^{8} \left| \sum_{\sigma \in T_i} \varepsilon(\sigma) \right| < 8k^2(n-3)! \ln n.$$

□

5.3 Smallest Eigenvalue of $\Gamma_n^{(k)}$

Lemma 5.3.1. *Let k, n be positive integers, $k < n$, and $\lambda \vdash n$. If the dimension of the Specht module S^λ, $f^\lambda \geq \binom{n-1}{2} - 1 = \frac{n(n-3)}{2}$, then for sufficiently large n,*

$$(n-1)\left|\eta_\lambda^{(k)}\right| < d_n^{(k)}.$$

Proof. By Lemma 5.1.2,

$$\left|\eta_\lambda^{(k)}\right| \leq 2\sqrt{\frac{d_n^{(k)}(n-1)(n-2)(n-4)!}{n(n-3)}}.$$

So, it is sufficient to show that

$$\frac{4(n-1)^3(n-2)(n-4)!}{n(n-3)} < d_n^{(k)} \tag{5.5}$$

By part (b) of Theorem 5.2.2, $d_n^{(k)} \geq \frac{n!}{3k}$. Therefore equation (5.5) holds if

$$\frac{4(n-1)^3(n-2)(n-4)!}{n(n-3)} < \frac{n!}{3k},$$

which is equivalent to

$$\frac{12k(n-1)^2}{n^2(n-3)^2} < 1.$$

Note that

$$\frac{12k(n-1)^2}{n^2(n-3)^2} = \left(\frac{12}{n}\right)\left(\frac{k}{n}\right)\left(\frac{n-1}{n-3}\right)^2 < \left(\frac{12}{n}\right)\left(\frac{n-1}{n-3}\right)^2 < 1,$$

for sufficiently large n. Hence, the lemma holds. □

Lemma 5.3.2. *Let k, n be positive integers and $k \leq n^\delta$ with $0 < \delta < \frac{2}{3}$. Then for sufficiently large n,*

$$(n-1)\left|s_n^{(k)}\right| < d_n^{(k)}.$$

Proof. By part (c) of Theorem 5.2.5,

$$\left|s_n^{(k)}\right| < 8k^2(n-3)!\ln n.$$

By part (b) of Theorem 5.2.2, $d_n^{(k)} \geq \frac{n!}{3k}$. Since $k \leq n^{\delta}$, $d_n^{(k)} \geq \frac{n!}{3n^{\delta}}$ and $\left|s_n^{(k)}\right| < 8n^{2\delta}(n-3)!\ln n$. Therefore, it is sufficient to show that

$$8n^{2\delta}(n-1)(n-3)!\ln n < \frac{n!}{3n^{\delta}},$$

which is equivalent to

$$\frac{24n^{3\delta}\ln n}{n(n-2)} < 1.$$

Now, $2-3\delta > 0$. So, for sufficiently large n,

$$\frac{24n^{3\delta}\ln n}{n(n-2)} = 24\left(\frac{\ln n}{n^{2-3\delta}}\right)\left(\frac{n}{n-2}\right) < 1.$$

Hence, the lemma holds. □

Theorem 5.3.3. *Let k,n be positive integers and $k \leq n^{\delta}$ with $0 < \delta < \frac{2}{3}$. Then for sufficiently large n, the smallest eigenvalue of $\Gamma_n^{(k)}$ is equal to*

$$\eta_{(n-1,1)}^{(k)} = -\frac{d_n^{(k)}}{n-1}$$

where $d_n^{(k)} = |S_n \setminus C_{(1,2,\ldots,k)}|$. Furthermore, $\eta_{\lambda}^{(k)} = -\frac{d_n^{(k)}}{n-1}$ if and only if $\lambda = (n-1,1)$.

Proof. It follows from Lemmas 5.1.1, 5.1.2, 5.3.1 and 5.3.2 that for sufficiently large n, $-\frac{d_n^{(k)}}{n-1}$ is the smallest eigenvalue of $\Gamma_n^{(k)}$. Furthermore, $\eta_{\lambda}^{(k)} = -\frac{d_n^{(k)}}{n-1}$ if and only if $\lambda = (n-1,1)$.

This completes the proof of Theorem 5.3.3. □

5.4 Largest Independent set of $\Gamma_n^{(k)}$

Note that for each $\lambda \vdash n$, the Specht module S^λ is an irreducible $\mathbb{C}S_n$-module. Let $\mathscr{U}_\lambda$ be the sum of all copies of S^λ in $\mathbb{C}S_n$. Note that $\mathscr{U}_\lambda$ is the $\eta_\lambda^{(k)}$-eigenspace of the adjacency matrix of $\Gamma_n^{(k)}$, i.e.,

$$\mathscr{U}_\lambda = \left\{ \mathbf{x} \in \mathbb{C}^{n!} \ : \ B\mathbf{x} = \eta_\lambda^{(k)} \mathbf{x} \right\},$$

where B is the adjacency matrix of $\Gamma_n^{(k)}$ (see Diaconis and Shahshahani,1981 [15] and Ellis, 2012 [17, Theorem 2.3]). In the proof of Theorem 1.4.9, Ellis (2012, [17]) used the following two lemmas which will be needed in this chapter.

Lemma 5.4.1. (Ellis, 2012 [17, Lemma 2.7])

$$\mathscr{U}_{(n)} \oplus \mathscr{U}_{(n-1,1)} = \mathrm{Span}\{\mathbf{1}_{S_{i,j}} \ : \ i, j \in [n]\}.$$

Lemma 5.4.2. (Ellis, Friedgut and Pilpel, 2011 [19, Theorem 8]) *Let $A \subseteq S_n$. If $\mathbf{1}_A \in$ Span$\{\mathbf{1}_{S_{i,j}} \ : \ i, j \in [n]\}$, then A is a disjoint union of some of the $S_{i,j}$'s.*

It follows from Lemmas 5.1.1, 5.1.2, 5.3.1 and 5.3.2 that for sufficiently large n, $\mathscr{U}_{(n)}$ and $\mathscr{U}_{(n-1,1)}$ are the $d_n^{(k)}$-eigenspace and $-\frac{d_n^{(k)}}{n-1}$-eigenspace, respectively, of the adjacency matrix of $\Gamma_n^{(k)}$.

Claim. Every element in $\mathfrak{S}_n^{(k)}$ is of size $(n-1)!$.

Proof. Let $B \in \mathfrak{S}_n^{(k)}$. By Theorems 1.4.1 and 5.3.3,

$$|B| \leq \frac{n!}{1 - \frac{d_n^{(k)}}{-\frac{d_n^{(k)}}{n-1}}} = (n-1)!.$$

Note that $S_{1,2} = \{\pi \in S_n : \pi(1) = 2\}$ is an independent set in $\Gamma_n^{(k)}$ and $\left|S_{1,2}\right| = (n-1)!$. This establishes the claim. □

Recall that the set

$$S_{i,j} = \{\pi \in S_n : \pi(i) = j\},$$

for $i, j \in [n]$ is the coset of stabiliser of a point. Define $\mathfrak{S}_n^{(k)}$ to be the set consisting of all the largest independent sets in $\Gamma_n^{(k)}$. Cameron and Ku (2003, [11]), Godsil and Meagher (2009, [27]), and Larose and Malvenuto (2004, [53]) showed that $\mathfrak{S}_n^{(1)} = \{S_{i,j} : i, j \in [n]\}$. The following theorem is a generalization of this result.

Theorem 5.4.3. *Let k, n be positive integers and $k \le n^{\delta}$ with $0 < \delta < \frac{2}{3}$. Then for sufficiently large n,*

$$\mathfrak{S}_n^{(k)} = \{S_{i,j} : i, j \in [n]\}.$$

In particular, the largest independent set in $\Gamma_n^{(k)}$ is of size $(n-1)!$.

Proof. In fact, $\{S_{i,j} : i, j \in [n]\} \subseteq \mathfrak{S}_n^{(k)}$. Let $S \in \mathfrak{S}_n^{(k)}$. Then $|S| = (n-1)!$. By Theorem 1.4.1,

$$\mathbf{1}_S \in \mathscr{U}_{(n)} \oplus \mathscr{U}_{(n-1,1)}.$$

It then follows from Lemmas 5.4.1 and 5.4.2 that $S \in \{S_{i,j} : i, j \in [n]\}$. Hence,

$$\mathfrak{S}_n^{(k)} = \{S_{i,j} : i, j \in [n]\}.$$

This completes the proof of Theorem 5.4.3. $\square$

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List of Publications and Papers Presented

Papers Presented

T. Lau, Choices of Power Sum Symmetric Functions, *The 22nd National Symposium on Mathematical Sciences (SKSM22)*, Shah Alam, 24-26 November 2014.

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