ON HAMILTON CYCLES IN REGULAR GRAPHS

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KUALA LUMPUR

2016

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DISSERTATION SUBMITTED IN FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

INSTITUTE OF MATHEMATICAL SCIENCES
FACULTY OF SCIENCE
UNIVERSITY OF MALAYA
KUALA LUMPUR

2016

UNIVERSITY OF MALAYA

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Name of Degree: **MASTER OF SCIENCE**

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ABSTRACT

The purpose of this dissertation is to discuss the hamiltonicity of r-regular 3-connected planar graphs (rR3CPs) with faces of given types, in particular, $r \in \{3, 4\}$. In general, let $G_r(k_1, k_2, \ldots, k_t)$ denotes the class of all rR3CPs whose faces are of only t types, namely k_1-, k_2-, $\ldots$, k_t-gons where $k_i \geq 3$, $k_i \neq k_j \; \forall \; i \neq j$ and $i, j \in \{1, 2, \ldots, t\}$. The problem related to the hamiltonicity of 3R3CPs with only two types of faces are widely discussed and many results have been found. These results are reviewed in Chapter 2. Chapter 3 is devoted to the constructions of non-hamiltonian 3R3CPs with only three types of faces. Here, we show that $G_3(3, k, l)$ is empty if $11 \leq k < l$. We also show that for $h \neq k \neq l$, there exist non-hamiltonian members in (1) $G_3(3, k, l)$ for $4 \leq k \leq 10$ and $l \geq 7$; (2)(i) $G_3(4, k, l)$ for $k \in \{3, 5, 7, 9, 11\}$ and $l \geq 8$; and $(k, l) \in \{(3, 7), (6, 7), (6, 9), (6, 11)\}$; (2)(ii) $G_3(4, k, k+5)$ and $G_3(4, k+2, k+5)$ for $k \geq 3$; (3) $G_3(5, k, l)$ for $k = 3$ and $l \geq 7$; $k = 4$ and $l \geq 8$; and $6 \leq k < l$. Results (1), (2) and (3) are presented in Sections 3.3, 3.4 and 3.5, respectively. Chapter 4 deals with the hamiltonicity of 4R3CPs with faces of given types. We construct non-hamiltonian members of $G_4(3, 7)$ and $G_4(3, 8)$. Additionally, we show that for $k \neq l$ and $(k, l) \notin \{(6, 9), (9, 10), (9, 11)\}$, there exist non-hamiltonian members in $G_4(3, k, l)$ for $k \geq 4$ and $l \geq 7$.

ABSTRAK

Tujuan disertasi ini adalah untuk membincangkan kehamiltonan graf r-sekata 3-terkait satahan (rS3TS) dengan jenis muka tertentu, khususnya, $r \in \{3, 4\}$. Secara amnya, biar $G_r(k_1, k_2, \ldots, k_t)$ mewakili kelas semua graf rS3TS yang mempunyai t jenis muka sahaja, iaitu k_1-, k_2-, …, k_t-gon di mana $k_i \geq 3$, $k_i \neq k_j$ $\forall$ $i \neq j$ dan $i, j \in \{1, 2, \ldots, t\}$. Masalah yang melibatkan kehamiltonan graf 3S3TS dengan dua jenis muka sahaja telah dibincangkan secara meluas dan banyak keputusan telah dijumpai. Keputusan-keputusan ini diulas dalam Bab 2. Bab 3 ditumpukan pada pembinaan graf bukan hamiltonan 3S3TS dengan tiga jenis muka sahaja. Dalam bab ini, kami menunjukkan bahawa $G_3(3, k, l)$ tidak mempunyai ahli sekiranya $11 \leq k < l$. Kami juga menunjukkan bahawa untuk $h \neq k \neq l$, terdapat ahli bukan hamiltonan dalam (1) $G_3(3, k, l)$ bagi $4 \leq k \leq 10$ dan $l \geq 7$; (2)(i) $G_3(4, k, l)$ bagi $k \in \{3, 5, 7, 9, 11\}$ dan $l \geq 8$; dan $(k, l) \in \{(3, 7), (6, 7), (6, 9), (6, 11)\}$; (2)(ii) $G_3(4, k, k{+}5)$ dan $G_3(4, k{+}2, k{+}5)$ bagi $k \geq 3$; (3) $G_3(5, k, l)$ bagi $k = 3$ dan $l \geq 7$; $k = 4$ dan $l \geq 8$; dan $6 \leq k < l$. Keputusan (1), (2) dan (3) masing-masing dibentangkan dalam Seksyen 3.3, 3.4 dan 3.5. Bab 4 adalah berkaitan dengan kehamiltonan graf 4S3TS dengan jenis muka tertentu. Kami telah membina ahli yang bukan hamiltonan dalam $G_4(3, 7)$ dan $G_4(3, 8)$. Tambahan pula, kami menunjukkan bahawa untuk $k \neq l$ dan $(k, l) \notin \{(6, 9), (9, 10), (9, 11)\}$, terdapat ahli bukan hamiltonan dalam $G_4(3, k, l)$ bagi $k \geq 4$ dan $l \geq 7$.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to my supervisor, Dr. Ong Siew Hui, for her unparallelled guidance, continuous encouragement and invaluable advice which have helped me tremendously throughout my candidature.

Special thanks to my parents for their immeasurable love and care, without whom none of this would be possible. On a similar note, I would also like to extend my highest appreciation to my sister. I am deeply grateful for her assistance during the course of my study.

Last but not least, I also wish to thank the Head of Department, academic and general staffs of the Institute of Mathematical Sciences, University of Malaya, for their help and support.

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rR3CP	:	r-regular 3-connected planar graphs where $r \in \{3, 4, 5\}$.
$G_r(k_1, k_2, \ldots, k_t)$	:	A class of all rR3CPs whose faces are of only t types, namely k_1-, k_2-, $\ldots$, k_t-gons where $k_i \geq 3$, $k_i \neq k_j$ $\forall$ $i \neq j$ and $i, j \in \{1, 2, \ldots, t\}$.
$S^{n_1,n_2,n_3}_{(k_1,k_2,\ldots,k_t)}$	:	A 3-piece whose inner faces are of only t types, namely k_1-, k_2-, $\ldots$, k_t-gons where $k_i \geq 3$, $k_i \neq k_j$ $\forall$ $i \neq j$ and $i, j \in \{1, 2, \ldots, t\}$, that contributes n_1, n_2 and n_3 vertices to the three adjoining faces of any graph in which it occurs.
$S^{n_1,n_2,n_3,n_4}_{(k_1,k_2,\ldots,k_t)}$	:	A 4-piece whose inner faces are of only t types, namely k_1-, k_2-, $\ldots$, k_t-gons where $k_i \geq 3$, $k_i \neq k_j$ $\forall$ $i \neq j$ and $i, j \in \{1, 2, \ldots, t\}$, that contributes n_1, n_2, n_3 and n_4 vertices to the four adjoining faces of any graph in which it occurs.
$S^{n}_{(k_1,k_2,\ldots,k_t)}$	:	A simplified version of $S^{n_1,n_2,n_3}_{(k_1,k_2,\ldots,k_t)}$ and $S^{n_1,n_2,n_3,n_4}_{(k_1,k_2,\ldots,k_t)}$ if $n = n_1 = n_2 = n_3$ and $n = n_1 = n_2 = n_3 = n_4$, respectively.
$I^{m_1,m_2,m_3}_{(k_1,k_2,\ldots,k_t)}$	:	A I-piece whose inner faces are of only t types, namely k_1-, k_2-, $\ldots$, k_t-gons where $k_i \geq 3$, $k_i \neq k_j$ $\forall$ $i \neq j$ and $i, j \in \{1, 2, \ldots, t\}$, that contributes m_1, m_2 and m_3 edges to the three adjoining faces of any graph in which it occurs.
$II^{m_1,m_2,m_3}_{(k_1,k_2,\ldots,k_t)}$	:	A II-piece whose inner faces are of only t types, namely k_1-, k_2-, $\ldots$, k_t-gons where $k_i \geq 3$, $k_i \neq k_j$ $\forall$ $i \neq j$ and $i, j \in \{1, 2, \ldots, t\}$, that contributes m_1, m_2 and m_3 edges to the three adjoining faces of any graph in which it occurs.
$I^{m}_{(k_1,k_2,\ldots,k_t)}$	:	A simplified version of $I^{m_1,m_2,m_3}_{(k_1,k_2,\ldots,k_t)}$ if $m = m_1 = m_2 = m_3$.
$II^{m}_{(k_1,k_2,\ldots,k_t)}$	:	A simplified version of $II^{m_1,m_2,m_3}_{(k_1,k_2,\ldots,k_t)}$ if $m = m_1 = m_2 = m_3$.
P_{uv}	:	A path with end vertices u and v where $u, v \in V(G)$.

CHAPTER 1: INTRODUCTION

1.1 Introduction

A graph is hamiltonian if it has a cycle that contains all of its vertices; such a cycle is called a Hamilton cycle. Hamilton cycles are named after Sir William Rowan Hamilton, who in 1857 devised a mathematical game called the Icosian Game. The game consisted of a dodecahedron whose twenty vertices were labelled with the names of twenty cities. The objective of the game is to travel along the edges of the dodecahedron such that every city is visited exactly once and the end point is the same as the initial point. In graph theoretical terms, the aim is to find a Hamilton cycle in a dodecahedron.

The study of Hamilton cycles in cubic planar graphs was originally motivated by Tait (1880). In his attempt to prove the Four-Colour Theorem (see Theorem 1.1), Tait (1880) observed that the Four-Colour Theorem is equivalent to the assertion that every simple 2-edge-connected cubic planar graph is 3-edge-colourable.

Theorem 1.1. *(Four-Colour Theorem)* *Every planar graph is 4-colourable.*

By assuming that every 3-regular 3-connected planar graph (hereinafter abbreviated to 3R3CP) is hamiltonian, which became known as Tait's conjecture (see Conjecture 1.1), he gave a proof of the Four-Colour Theorem. Unfortunately, Tait's proof is invalid.

Conjecture 1.1. *(Tait's conjecture)* *Every $3R3CP$ is hamiltonian.*

Tutte (1946) refuted Tait's conjecture by constructing the counterexample shown in Figure 1.1, which is a non-hamiltonian 3R3CP with 46 vertices. Once this graph was discovered, the search for the smallest counterexample intensified. Holton and McKay (1988) showed that the smallest non-hamiltonian 3R3CP has 38 vertices. They also confirmed that the six non-isomorphic graphs discovered independently by Lederberg (1965), Bosák (1966) and Barnette (1969) are the only such graphs with 38 vertices.

Additionally, the hamiltonicity of various classes of 3R3CPs has been investigated. Tutte (1972) posed Question 1.1, which concerns the existence of Hamilton cycles in 3R3CPs with faces of given types. Grünbaum and Zaks (1974) also asked a related question (see Question 1.2). Answers to these questions and an account of the hamiltonicity of 3R3CPs whose faces are of only two types will be presented in Chapter 2.

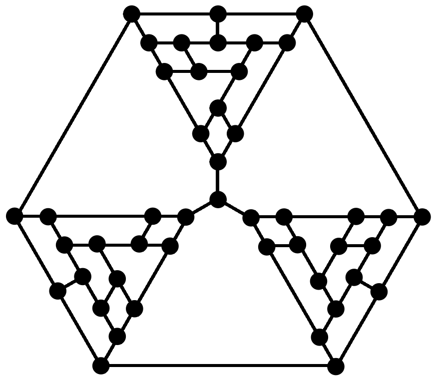


Figure 1.1: The first non-hamiltonian 3R3CP (Tutte, 1946).

Question 1.1. *(Tutte, 1972) Is every* $3R3CP$ *with all faces whose number of sides are congruent to* 2 *modulo* 3 *hamiltonian?*

Question 1.2. *(Grünbaum & Zaks, 1974) Do Hamilton cycles exist in all* $3R3CPs$ *whose faces are of only two types except those graphs that are forbidden by the Grinberg's Theorem (see Theorem 1.5)?*

One can also ask a similar question: Does every 3R3CP whose faces are of only three types hamiltonian? Chapter 3 details the constructions of non-hamiltonian 3R3CPs with only three types of faces. Here, we will construct a large number of such graphs. All the techniques and subgraphs that we use to construct these graphs will be thoroughly described.

The last chapter covers the discussion on the hamiltonicity of all 4-regular 3-connected planar graphs (hereinafter abbreviated to 4R3CPs) with faces of given types. We will show some known results and briefly describe the method of construction. In addition to that, we will also construct a number of non-hamiltonian 4R3CPs whose faces are of only two and three types.

1.2 Definitions

In this section, we shall introduce some basic graph theoretical terms, notations along with some fundamental results that will be used in the dissertation. Other graph theoretical terms that are not included in this section will be defined later as they are needed. Further information can be found in any introductory book on graph theory (see for example, Beineke & Wilson, 1978; Bondy & Murty, 1976).

1.2.1 Graphs

A *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set of elements called *vertices* and $E(G)$ is a finite set of unordered pairs of elements of $V(G)$ called *edges*. We call $V(G)$ the *vertex set* of G and $E(G)$ the *edge set* of G. A *subgraph* of G is a graph $S = (V(S), E(S))$ where $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$.

Two vertices $u, v \in V(G)$ are said to be *adjacent* if there is an edge $e \in E(G)$ that joins u and v. In this case, we also say that e is *incident* to u and v. A *loop* is an edge incident to only one vertex. *Multiple edges* are edges that join a pair of vertices more than once. A graph is called *simple* if it has neither loops nor multiple edges. In this dissertation, we are mainly concerned with simple graphs.

The *degree of a vertex* v in a graph G, denoted by $d(v)$, is the number of edges incident to v. If the degree of every vertex in G is r, then G is called an *r-regular* graph. A 3-regular graph is also called a *cubic graph*.

An *edge sequence* in a graph G is a sequence of edges of the form $e_1e_2 \ldots e_t$ where $e_i = v_iv_{i+1}$ is an edge of G that is incident to the vertices $v_i, v_{i+1} \in V(G)$ for $1 \leq i \leq t$. If these edges and vertices are all distinct, then the edge sequence $e_1e_2 \ldots e_t$ is called a *path* and v_1 and v_{t+1} are called the *end vertices* of the path. We denote a path with end vertices v_1 and v_{t+1} by $P_{v_1v_{t+1}}$. A path is said to be *closed* if $v_1 = v_{t+1}$ and is *open* otherwise. A closed path is called a *cycle*.

A *spanning path through subgraph* S is a path that contains every vertex of S. A *Hamilton path* of a graph G is a path that contains every vertex of G. Similarly, a *Hamilton cycle* of G is a cycle that contains every vertex of G. A graph that contains a Hamilton path is called a *traceable graph* and a graph that contains a Hamilton cycle is called a *hamiltonian graph*.

An edge in a hamiltonian graph G is called a *compulsory edge* if it belongs to every Hamilton cycle of G, whereas it is called an *impossible edge* if it does not belong to any Hamilton cycle of G. The concept of such edges was introduced by Bosák (1966).

1.2.2 Connectivity

A graph G is said to be *connected* if there is a path between every pair of vertices of G and it is *disconnected* otherwise. Any disconnected graph G can be expressed as the union of connected graphs, each of which is called a component of G.

A set V' of k vertices in a connected graph G is called a *k-vertex cut* if $G - V'$ is disconnected. Likewise, a set E' of k edges in a connected graph G is called a *k-edge cut* if $G - E'$ is disconnected. A component of G that is disconnected from G by the removal of E' is called a *k-piece*. An edge in E' is called an *edge of attachment* of a k-piece. We call a vertex in a k-piece that is incident to an edge of attachment an a-*vertex*. For an a-vertex u of a 3-piece S, if u is an end vertex of every open spanning path through S, then the edge of attachment incident to u is called a *compulsory edge of attachment* of S. If no open spanning path through S ends at u, then the edge of attachment incident to u is called an *impossible edge of attachment* of S.

The *connectivity* $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal from G disconnects G. When $\kappa(G) \geq k$, G is said to be *k-connected*. Analogously, the *edge connectivity* $\lambda(G)$ of a connected graph G is the minimum number of edges whose removal from G disconnects G. When $\lambda(G) \geq k$, G is said to be *k-edge-connected*. Let $\delta(G)$ be a minimum vertex degree of a connected graph G, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

A graph is called *cyclically k-connected* (*cyclically k-edge-connected*) if it cannot be disconnected into at least two components, each of which contains a cycle, by the removal of fewer than k vertices (k edges).

1.2.3 Planar Graphs

A *planar graph* is a graph that can be embedded on the plane in such a way that no two edges intersect geometrically except at a vertex to which they are both incident. A graph embedded on the plane in this way is called a *plane graph*. A plane graph partitions the plane into a set of regions called *faces*. A *k-gon* is a face of a plane graph bounded by k edges. Every face of a plane graph G must be bounded by at least three edges, that is $k \geq 3$, if G is simple.

In the 18th century, Leonhard Euler discovered a relationship between the numbers of vertices, edges and faces in any connected plane graph. This relationship is known today as Euler's formula.

Theorem 1.2. *(Euler's formula)* *Let G be a connected plane graph with n vertices, m edges and f faces. Then*

$$n + f = m + 2. \tag{1.1}$$

It can be verified from Euler's formula that every planar graph contains a vertex of degree at most 5. Thus, $3 \leq r \leq 5$ if G is an r-regular 3-connected planar graph and we denote such a graph by rR3CP. In addition, let $G_r(k_1, k_2, \ldots, k_t)$ denotes the class of all rR3CPs whose faces are of only t types, namely k_1-, k_2-, ..., k_t-gons where $k_i \geq 3$, $k_i \neq k_j \ \forall\, i \neq j$ and $i, j \in \{1, 2, \ldots, t\}$.

The following lemma states a well-known necessary condition for the existence of rR3CPs with f_k k-gons where $r \in \{3, 4, 5\}$ and $k \geq 3$.

Lemma 1.3. *(Grünbaum & Zaks, 1974)* *For $r \in \{3, 4, 5\}$ and $k \geq 3$, let G be an rR3CP and let f_k denote the number of k-gons in G. Then*

$$\sum_{k \geq 3} (2r + 2k - rk) f_k = 4r. \tag{1.2}$$

Observe that Equation 1.2 places no restriction on the number of f_6 when $r = 3$ and f_4 when $r = 4$. Nevertheless, they may not be taken arbitrarily since it can be shown that there is no 3R3CP with $f_3 = 4$, $f_6 = 1$, $f_k = 0$ for $k \notin \{3, 6\}$ and no 4R3CP with $f_3 = 8$, $f_4 = 1$, $f_k = 0$ for $k > 4$ (Malkevitch, 1988). There is also no 5R3CP with $f_3 = 22$, $f_4 = 1$, $f_k = 0$ for $k > 4$ (Malkevitch, 1988). Thus, this necessary condition is not sufficient.

1.2.4 Hamiltonian Planar Graphs

The question of the existence of Hamilton cycles is partially answered for planar graphs by one of the most celebrated theorems of Tutte (1956) stated below.

Theorem 1.4. *(Tutte, 1956)* *Every 4-connected planar graph is hamiltonian.*

Grinberg (1968) discovered the following necessary condition for a planar graph to be hamiltonian.

Theorem 1.5. *(Grinberg's theorem)* *Let C be a Hamilton cycle of a planar graph G. Let f'_k denote the number of k-gons in the interior of C and let f''_k denote the number of k-gons in the exterior of C. Then*

$$\sum_{k\geq 3}(k-2)(f'_k - f''_k) = 0. \tag{1.3}$$

Grinberg's theorem has led to the construction of some non-hamiltonian 3R3CPs (see Grinberg, 1968; Tutte, 1972). One of the applications of Equation 1.3 for constructing non-hamiltonian planar graphs is stated in the following example:

Example 1.1. *Let G be a planar graph. Suppose $f_i = 0$ whenever $i \not\equiv 2(mod\ 3)$ with exactly one exception that $f_j = 1$ for one particular $j \not\equiv 2(mod\ 3)$. Then G is non-hamiltonian.*

1.2.5 Bipartite Graphs

A *bipartite* graph is a graph G where the vertex set $V(G)$ can be partitioned into two non-empty sets V_1 and V_2 such that no two vertices within the same vertex set are adjacent. A *complete bipartite* graph is a bipartite graph in which every vertex in V_1 is adjacent to every vertex in V_2 and is denoted by $K_{m,n}$ if $|V_1| = m$ and $|V_2| = n$.

Since any cycle in a bipartite graph alternates between the vertices of the two sets, V_1 and V_2, all cycles are of even length. This implies that all faces in a planar bipartite graph are $2k$-gons where $k \geq 2$. If a bipartite graph is hamiltonian, then $|V_1| = |V_2| \geq 2$.

1.3 Some Graphs and k-pieces

In this section, we introduce some graphs and k-pieces that will be used in the constructions of non-hamiltonian graphs in subsequent chapters.

A k-piece will be denoted by a capital letter and represented in a labelled circle. The numbers around the circumference of a labelled circle are the numbers of vertices, which the k-piece contributes to the adjoining faces of any graph in which it occurs.

1.3.1 Herschel Graph H

We denote the Herschel graph shown in Figure 1.2 by H. H is a bipartite graph that has 11 vertices, 18 edges and nine 4-gons.

H:

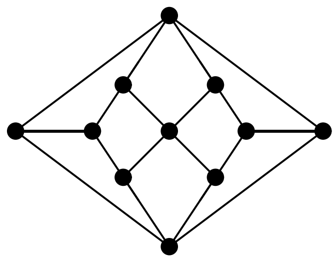


Figure 1.2: Herschel graph H.

H is non-hamiltonian since it is a bipartite graph with an odd number of vertices. Furthermore, it is the smallest non-hamiltonian 3-connected planar graph. H can also be shown to be non-hamiltonian by establishing that it does not satisfy Equation 1.3. By Grinberg's theorem, let C be a Hamilton cycle of H, then $2(f_4' - f_4'') = 0$. This implies that $f_4' = f_4''$. However, this is impossible since $f_4' + f_4'' = f_4 = 9$ is odd. It follows that no such C exists. Hence, H is non-hamiltonian.

1.3.2 Pentagonal Prism $\mathcal{P}$

We denote the pentagonal prism shown in Figure 1.3 by $\mathcal{P}$. $\mathcal{P}$ has five 4-gons and two 5-gons. Let e_i, $i = 0, 1, 2, 3, 4$ be the five edges of $\mathcal{P}$ that join a vertex of the inner pentagon to a vertex of the outer pentagon.

Lemma 1.6. *No Hamilton cycle in $\mathcal{P}$ contains both edges e_i and e_{i+2}, $i = 0, 1, 2, 3, 4$ with the subscripts reduced modulo 5.*

$\mathcal{P}$:

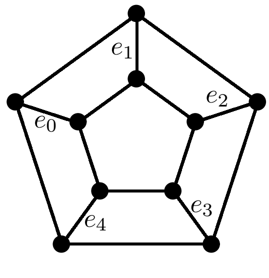


Figure 1.3: Pentagonal prism $\mathcal{P}$.

PROOF: By Grinberg's theorem, let C be a Hamilton cycle of $\mathcal{P}$, then $2(f_4' - f_4'') + 3(f_5' - f_5'') = 0$. Suppose C contains both edges e_i and e_{i+2}. This implies that $\{f_4', f_4''\} = \{2, 3\}$. Thus, we have $3(f_5' - f_5'') = \pm 2$, which is impossible since the right-hand side is not congruent to $0(mod\ 3)$. It follows that no such C exists. Hence, the lemma follows.

■

1.3.3 Graph Q

We denote the graph shown in Figure 1.4 by Q. Q has four 4-gons, four 5-gons and one 6-gon. Let e_1 and e_2 be the edges of Q as indicated.

Lemma 1.7. *Every Hamilton cycle of Q must contain exactly one of the edges e_1 and e_2, as shown in Figure 1.4.*

Q:

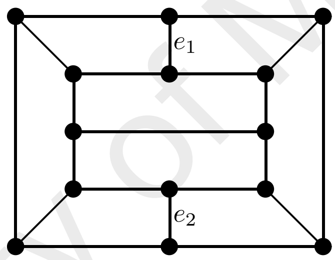


Figure 1.4: Graph Q.

PROOF: By Grinberg's theorem, let C be Hamilton cycle of Q, then $2(f_4' - f_4'') + 3(f_5' - f_5'') + 4(f_6' - f_6'') = 0$. Since there is only one 6-gon in Q, $\{f_6', f_6''\} = \{0, 1\}$. Therefore, the equation reduces to $2(f_4' - f_4'') + 3(f_5' - f_5'') \pm 4 = 0$.

If C contains both edges e_1 and e_2, then the edges separate the two pairs of 4-gons and so $f_4' = f_4'' = 2$. Thus, we have $3(f_5' - f_5'') = \pm 4$, which is impossible since the right-hand side is not congruent to $0(mod\ 3)$.

If C does not contain both edges e_1 and e_2, then all 4-gons lie on the same side of C and so $\{f_4', f_4''\} = \{0, 4\}$. Thus, we have either $3(f_5' - f_5'') = \pm 4$ or $3(f_5' - f_5'') = \pm 12$. The former has been dealt with. For $3(f_5' - f_5'') = \pm 12$, we must have $\{f_5', f_5''\} = \{0, 4\}$, which means all 5-gons must lie on the same side of C. This is impossible too. Hence, the lemma follows. ■

1.3.4 Tutte's Triangle T and T_i

Let T be the Tutte's triangle shown in Figure 1.5. Tutte (1946) constructed T and used it to construct the first non-hamiltonian 3R3CP (see Figure 1.1). The property of T is stated in Lemma 1.8.

Lemma 1.8. *(Tutte, 1946) The edge of attachment labelled e_c, as shown in Figure 1.5, is a compulsory edge of attachment of T.*

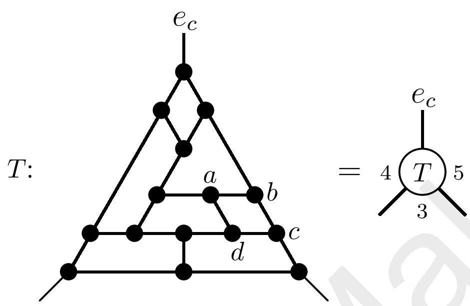


Figure 1.5: Tutte's triangle T (Tutte, 1946).

Lemma 1.9 extends Lemma 1.8.

Lemma 1.9. *For $i \geq 1$ and i is odd, let T_i be the 3-piece shown in Figure 1.6. Then the edge of attachment labelled e_c is a compulsory edge of attachment of T_i.*

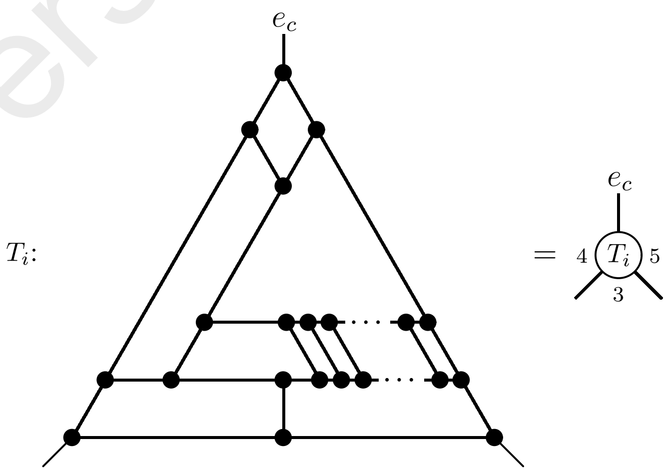


Figure 1.6: 3-piece T_i for $i \geq 1$ and i is odd.

PROOF: For $i \geq 1$ and i is odd, T_i is obtained by replacing the 4-gon $abcd$ of the Tutte's triangle T (see Figure 1.5) with a copy of L_i shown in Figure 1.7. L_i has i 4-gon(s). Clearly, T_1 is T.

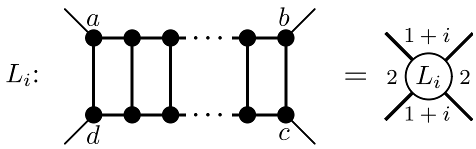


Figure 1.7: 4-piece L_i for $i \geq 1$.

Let P^o be an open spanning path through T_i and let P be $L_i \cap P^o$. Then for all odd $i \geq 1$, P are of the following four forms only: P_{ab} (or by symmetry, P_{cd}), P_{ad} (or by symmetry, P_{bc}), $P_{ab} \cup P_{cd}$ and $P_{ad} \cup P_{bc}$. Thus, L_i retains the property of T. Hence, the lemma follows. ■

1.3.5 3-pieces S_a^* and S_b^*

Lemma 1.10. *For $i \in \{a, b\}$, let S_i^* be a 3-piece shown in Figure 1.8 where S_i is a 3-piece with a compulsory edge of attachment e_i. Then the edge of attachment labelled e_i^* is a compulsory edge of attachment of S_i^*.*

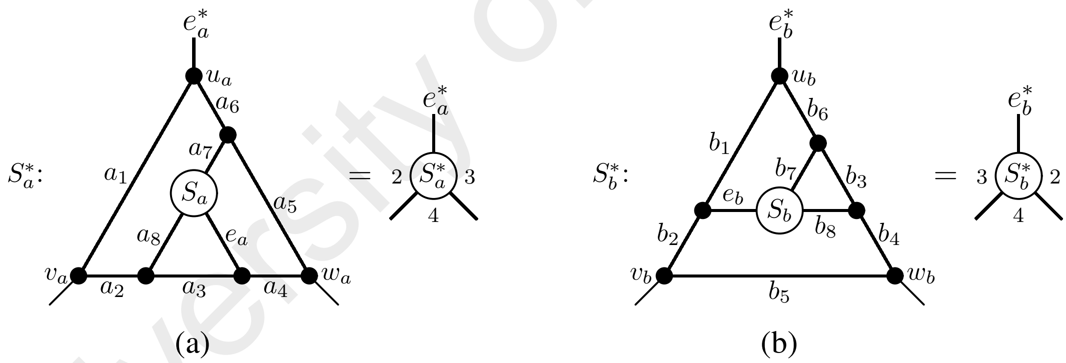


Figure 1.8: 3-pieces S_a^* and S_b^*.

PROOF: For $i \in \{a, b\}$, let u_i, v_i and w_i be a-vertices of S_i^* and let i_t for $1 \leq t \leq 8$ be some edges of S_i^*, as shown in Figure 1.8.

First, we show that there exists $P_{S_i^*}$, a spanning path through S_i^*, with an end vertex u_i that is incident to the edge of attachment labelled e_i^* for $i \in \{a, b\}$. Since e_i is a compulsory edge of attachment of S_i, there exists P_{S_i}, a spanning path through S_i, with an end vertex incident to either i_7 or i_8. It is easy to see that $a_1a_2a_3e_aP_{S_a}a_7a_5$ and $a_6a_5a_4e_aP_{S_a}a_8a_2$ are spanning paths through S_a^* with an end vertex u_a and $b_1e_bP_{S_b}b_7b_3b_4b_5$ and $b_6b_3b_8P_{S_b}e_bb_2b_5$ are spanning paths through S_b^* with an end vertex u_b.

Suppose the edge of attachment labelled e_i^* is not a compulsory edge of attachment of S_i^*. Then there exists $P_{S_i^*}$, a spanning path through S_i^*, with end vertices v_i, w_i and $i_1, i_6 \in P_{S_i^*}$.

Let P_{S_i} be $S_i \cap P_{S_i^*}$.

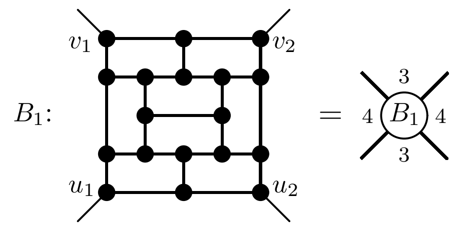
For $i = a$, this implies that $a_2 \notin P_{S_a^*}$. Thus, $a_3, a_8 \in P_{S_a^*}$. However, $a_3 a_8 P_{S_a} e_a$ forms a cycle in $P_{S_a^*}$, which is impossible.

For $i = b$, this implies that $b_7 \notin P_{S_b^*}$, otherwise $b_1 e_b P_{S_b} b_7 b_6$ forms a cycle in $P_{S_b^*}$. Thus, $b_3, b_8 \in P_{S_b^*}$. However, this forces $b_1 e_b P_{S_b} b_8 b_3 b_6$ to also form a cycle in $P_{S_b^*}$, which is impossible. Hence, the lemma follows. ■

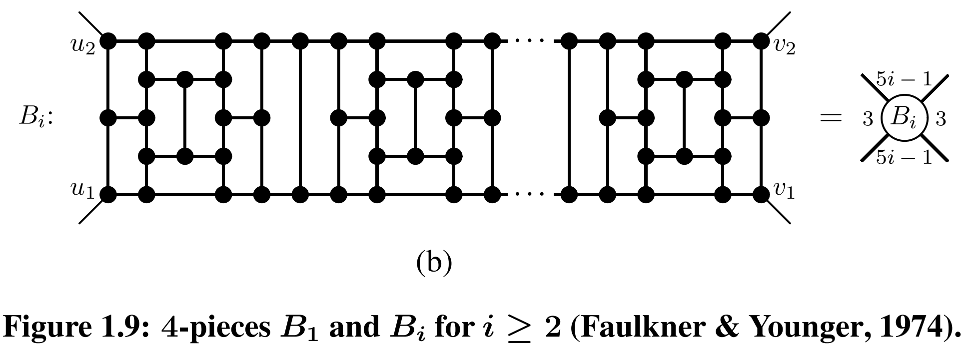
1.3.6 4-pieces B_1, B_i, B_i' and B_i''

Let B_1 be the 4-piece shown in Figure 1.9(a). Faulkner and Younger (1974) obtained B_1 by removing any two adjacent vertices from a dodecahedron. The properties of B_1 are stated in Lemma 1.11(1).

Faulkner and Younger (1974) also described the 4-piece B_2, which can be easily extended to B_i for $i \geq 2$, as shown in Figure 1.9(b). B_i is obtained by stacking i copies of B_1 and adding an edge between every two consecutive copies of B_1. The property of B_i for $i \geq 2$ is stated in Lemma 1.11(2), which extends Lemma 2.3 of (Faulkner & Younger, 1974).



(a)



(b)

Figure 1.9: 4-pieces B_1 and B_i for $i \geq 2$ (Faulkner & Younger, 1974).

Lemma 1.11. *(Faulkner & Younger, 1974)* *For $i \geq 1$, let C be a Hamilton cycle of a 3R3CP that contains the 4-piece B_i and let P be $B_i \cap C$. Then*

1. *for $i = 1$, P are of the following three forms only*

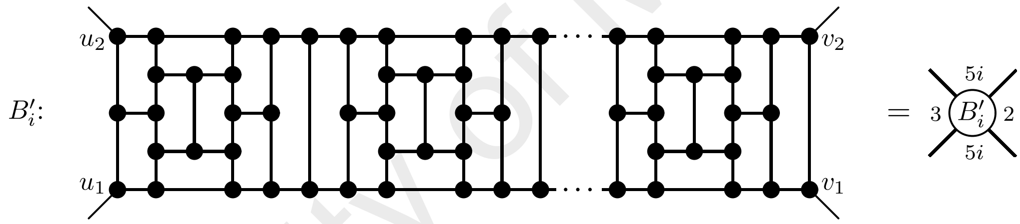
   (a) *$P_{u_1v_2}$ (or by symmetry, $P_{u_2v_1}$),*

   (b) *$P_{u_1u_2}$ (or by symmetry, $P_{v_1v_2}$) and*

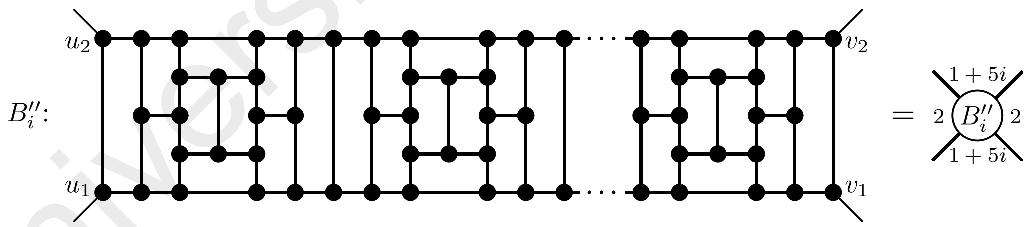
   (c) *$P_{u_1u_2} \cup P_{v_1v_2}$.*

2. *for $i \geq 2$, P is of the form $P_{u_1v_2}$ (or by symmetry, $P_{u_2v_1}$) only.*

Let B_i' and B_i'' for $i \geq 2$ be the 4-pieces shown in Figure 1.10. Zaks (1980) obtained B_i' and B_i'' by adding one and two edges, respectively, to the sides of B_i. The properties of B_i' and B_i'' for $i \geq 2$ are stated in Lemma 1.12.



(a)



(b)

Figure 1.10: 4-pieces B_i' and B_i'' for $i \geq 2$ (Zaks, 1980).

Lemma 1.12. *(Zaks, 1980)*

1. *Let C be a Hamilton cycle of a 3R3CP that contains the 4-piece B_i' for $i \geq 2$ and let P be $B_i' \cap C$. Then P is of the form $P_{u_1v_1}$ (or by symmetry, $P_{u_2v_2}$) only.*

2. *Let C be a Hamilton cycle of a 3R3CP that contains the 4-piece B_i'' for $i \geq 2$ and let P be $B_i'' \cap C$. Then P is of the form $P_{u_1v_2}$ (or by symmetry, $P_{u_2v_1}$) only.*

CHAPTER 2: 3-REGULAR 3-CONNECTED PLANAR GRAPHS WITH ONLY TWO TYPES OF FACES

2.1 Introduction

This chapter deals with the hamiltonicity of $G_3(h, k)$, $h \neq k$, which is the class of all 3-regular 3-connected planar graphs (3R3CPs) whose faces are of only two types, namely h-gons and k-gons. From Lemma 1.3, it is clear that every 3R3CP must have some 3-, 4- or 5-gons.

Let us first consider the class of all rR3CPs whose faces are of only one type, $G_r(k)$. $G_r(k)$ exists only if $k \in \{3, 4, 5\}$ for $r = 3$ and $k = 3$ for $r \in \{4, 5\}$. Each of the classes $G_3(3)$, $G_3(4)$, $G_3(5)$, $G_4(3)$ and $G_5(3)$ contains only one member, namely the tetrahedron, cube, dodecahedron, octahedron and icosahedron, respectively. All of these members, as shown in Figure 2.1, are hamiltonian.

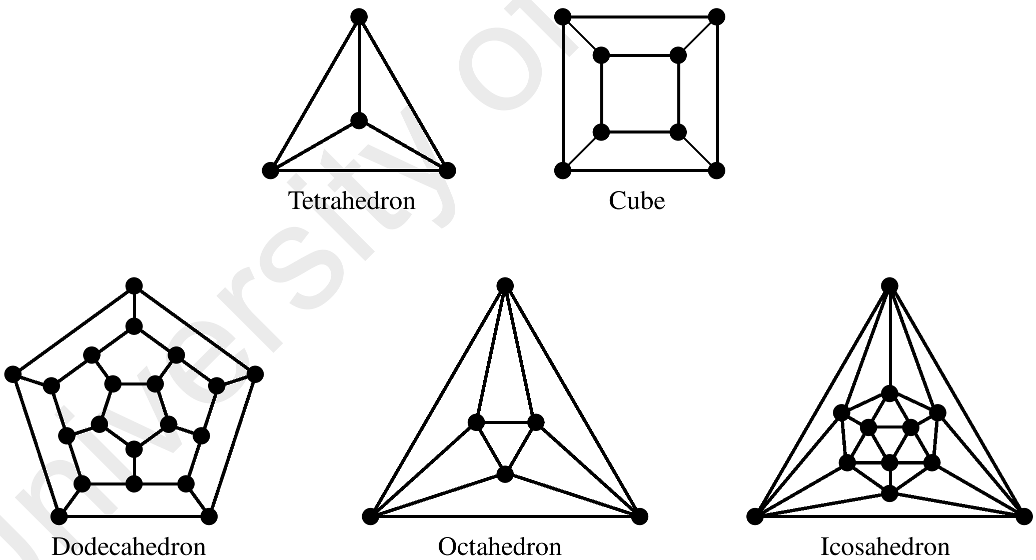


Figure 2.1: The members of $G_3(3)$, $G_3(4)$, $G_3(5)$, $G_4(3)$ and $G_5(3)$.

In the following sections, we will present a survey on the hamiltonicity of $G_3(h, k)$ where $h \in \{3, 4, 5\}$ and $h < k$. The survey will include some answers to Questions 1.1 and 1.2 stated in Chapter 1 and will be divided into three sections, namely Sections 2.2, 2.3 and 2.4, each of which focuses on the classes $G_3(3, k)$, $G_3(4, k)$ and $G_3(5, k)$, respectively. In the last section, we will present a brief account of the hamiltonicity of $G_3(k_1, k_2, \ldots, k_t)$ where $3 \leq k_1 < k_2 < \ldots < k_t \leq 6$.

2.2 $G_3(3, k)$

In this section, we consider the class $G_3(3, k)$ where $k \geq 4$. We begin with the result on the existence of $G_3(3, k)$ by Malkevitch (1970).

Theorem 2.1. *(Malkevitch, 1970)* $G_3(3, k)$ *is non-empty only if* $k \leq 10$.

Theorem 2.2. $G_3(3, 4)$ *contains only hamiltonian member.*

PROOF: Let G be a member of $G_3(3, 4)$. From Equation 1.2, we have $3f_3 + 2f_4 = 12$, which can be satisfied only if $f_3 = 2$ and $f_4 = 3$. Since G is 3-connected, no two 3-gons in G have a common edge. Thus, a 3-gon in G is adjacent to three 4-gons. Furthermore, a 4-gon is adjacent to at most two 3-gons.

It is then checked that the only member of $G_3(3, 4)$ is the graph shown in Figure 2.2, which has two 3-gons and three 4-gons. The graph is hamiltonian. ■

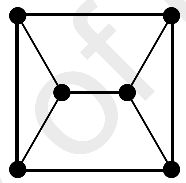


Figure 2.2: The member of $G_3(3, 4)$.

Theorem 2.3. *(Grünbaum, 1967, p. 272) Let* G *be a member of* $G_3(5, k)$ *and let* f_k *denote the number of* k*-gons in* G.

1. *Then* $f_k \neq 1$.
2. *When* $f_k = 2$*, no two* k*-gons in* G *have a common edge.*

Theorem 2.4. $G_3(3, 5)$ *contains only hamiltonian member.*

PROOF: Let G be a member of $G_3(3, 5)$. From Equation 1.2, we have $3f_3 + f_5 = 12$, which can be satisfied if $(f_3, f_5) \in \{(1, 9), (2, 6), (3, 3)\}$. By Theorem 2.3, there is no G with $f_3 = 1$ and $f_5 = 9$. Since G is 3-connected, no two 3-gons in G have a common edge. Thus, a 3-gon in G is adjacent to three 5-gons. Furthermore, a 5-gon is adjacent to at most two 3-gons. If $f_3 = f_5 = 3$, then all three 3-gons are adjacent to at least five 5-gons, which is impossible.

It is then checked that the only member of $G_3(3,5)$ is the graph shown in Figure 2.3, which has two 3-gons and six 5-gons. The graph is hamiltonian. ■

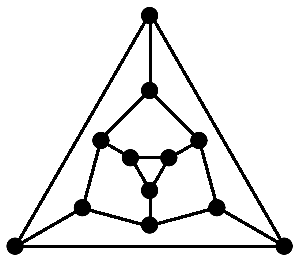


Figure 2.3: The member of $G_3(3,5)$.

Theorem 2.5. *(Goodey, 1977)* $G_3(3,6)$ *contains only hamiltonian members.*

Goodey (1977) proved that every member of $G_3(3,6)$ is hamiltonian. This result gives an affirmative answer to Question 1.2 posed by Grünbaum and Zaks (1974).

Theorem 2.6. *(Tkáč, 1994)* *There exists a non-hamiltonian member of* $G_3(3,7)$.

Tkáč (1994) constructed the 4-piece M shown in Figure 2.4. Here and in later diagrams, a small unlabelled white circle that has number two around the circumference of the circle represents a 3-gon. M contains two copies of B_1 (see Figure 1.9(a)) in which some of its vertices are replaced by 3-gons.

Let C be a Hamilton cycle of a 3R3CP that contains the 4-piece M and let P be $M \cap C$. Then P are of the following three forms only: $P_{u_1u_2}$ (or by symmetry, $P_{u_3u_4}$), $P_{u_1u_4}$ and $P_{u_1u_2} \cup P_{u_3u_4}$.

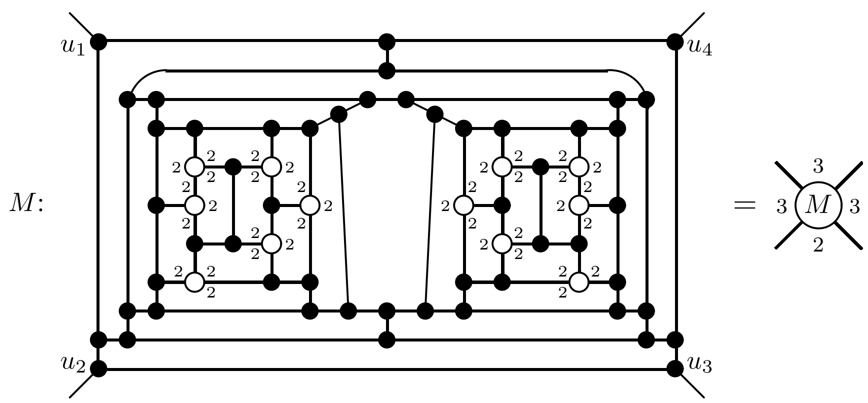


Figure 2.4: 4-piece M (Tkáč, 1994).

Tkáč (1994) obtained the graph G shown in Figure 2.5 by replacing three vertices of the pentagonal prism $\mathcal{P}$ (see Figure 1.3) with a copy of X, Y and Z. Each of the 3-pieces X and Y contains two copies of M and as indicated, each has a compulsory edge of attachment e_c. By inspection, $G \in G_3(3,7)$. Suppose C is a Hamilton cycle of G. Then C contains both e_c. By shrinking X, Y and Z to single vertices, G is converted into $\mathcal{P}$ and C into a Hamilton cycle of $\mathcal{P}$ that contains the edges e_c. However, this contradicts Lemma 1.6. It follows that no such C exists. Hence, G is non-hamiltonian.

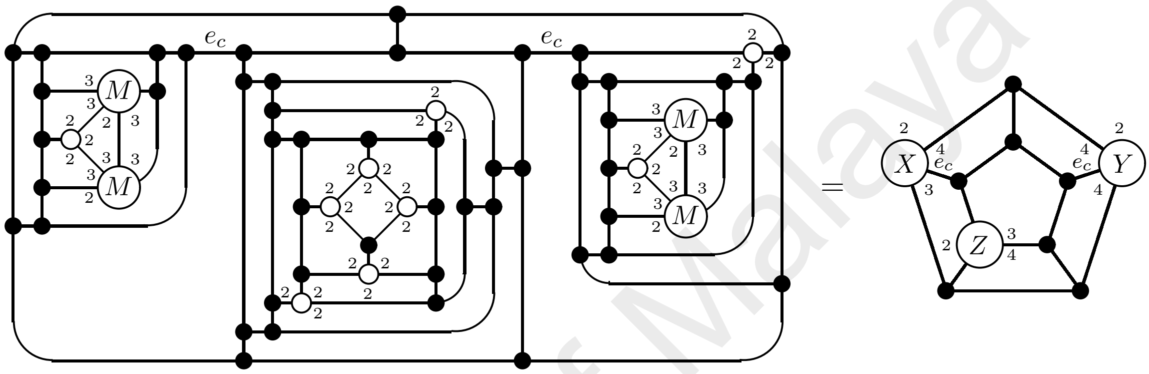


Figure 2.5: A non-hamiltonian member of $G_3(3,7)$ (Tkáč, 1994).

Theorem 2.7. *(Owens, 1984a) There exist non-hamiltonian members of $G_3(3,k)$ for $k \in \{8,9,10\}$.*

Owens (1984a) constructed non-hamiltonian members of $G_3(3,8)$ and $G_3(3,9)$ shown in Figures 2.6 and 2.7, respectively.

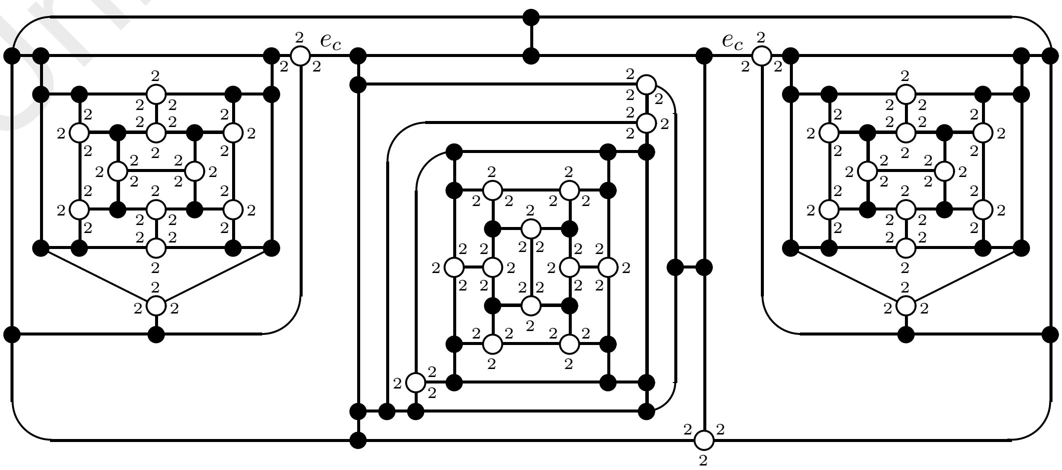


Figure 2.6: A non-hamiltonian member of $G_3(3,8)$ (Owens, 1984a).

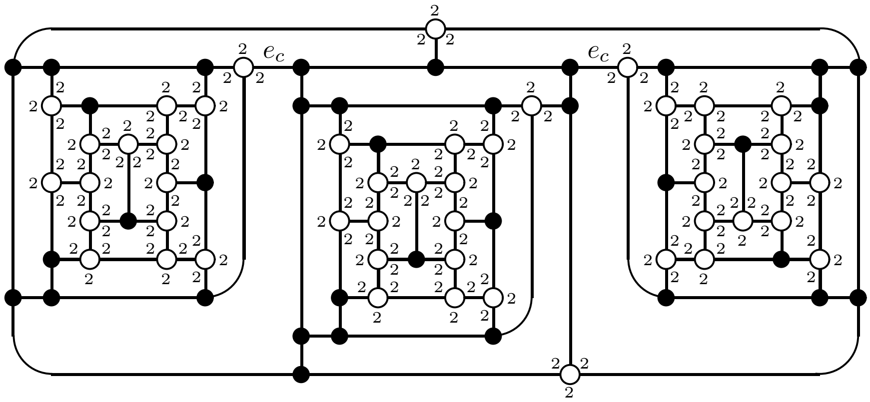
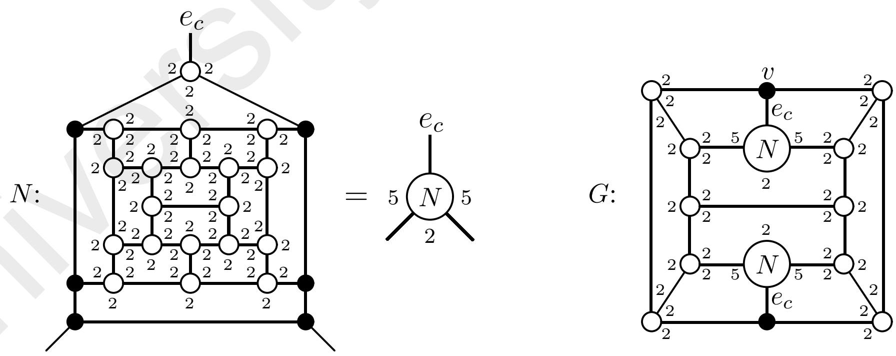


Figure 2.7: A non-hamiltonian member of $G_3(3, 9)$ (Owens, 1984a).

Owens (1984a) also constructed a non-hamiltonian member of $G_3(3, 10)$. The 3-piece N, as shown in Figure 2.8(a), contains a copy of B_1 (see Figure 1.9(a)) in which all of its vertices are replaced by 3-gons. As indicated, N has a compulsory edge of attachment e_c. He obtained the graph G shown in Figure 2.8(b) by replacing twelve vertices of the graph Q (see Figure 1.4) with two copies of N and ten copies of 3-gons. By inspection, $G \in G_3(3, 10)$. Suppose C is a Hamilton cycle of G. Then C contains both e_c. By shrinking N and 3-gons to single vertices, G is converted into Q and C into a Hamilton cycle of Q that contains the edges e_c. However, this contradicts Lemma 1.7. It follows that no such C exists. Hence, G is non-hamiltonian.



(a) (b)

Figure 2.8: 3-piece N and a non-hamiltonian member of $G_3(3, 10)$ (Owens, 1984a).

A summary of results for the class $G_3(3, k)$ is given in Theorem 2.8.

Theorem 2.8.

1. *$G_3(3, 4)$, $G_3(3, 5)$ and $G_3(3, 6)$ (Goodey, 1977) contain only hamiltonian members.*

2. *There exist non-hamiltonian members of* $G_3(3,k)$ *for* $k = 7$ *(Tkáč, 1994) and* $k \in \{8, 9, 10\}$ *(Owens, 1984a).*

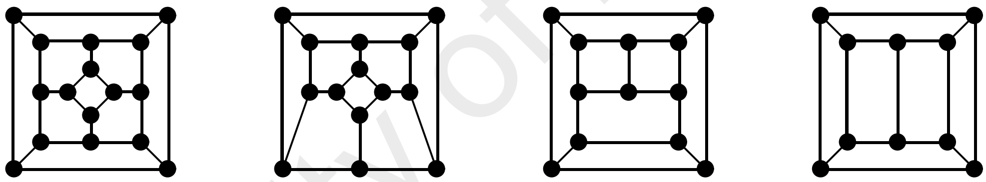
3. $G_3(3,k)$ *is empty for* $k \geq 11$ *(Malkevitch, 1970).*

2.3 $G_3(4,k)$

Now, we consider the class $G_3(4,k)$ where $k \geq 5$.

Theorem 2.9. $G_3(4,5)$ *contains only hamiltonian members.*

From Equation 1.2, we have $2f_4 + f_5 = 12$, which can be satisfied if $(f_4, f_5) \in \{(1,10), (2,8), (3,6), (4,4), (5,2)\}$. By Theorem 2.3, there is no 3R3CP with $f_4 = 1$ and $f_5 = 10$. Jendroľ and Jucovič (1989) showed that to every remaining four cases, there exists exactly one 3R3CP having those f_k k-gons for $k \in \{4, 5\}$, as shown in Figure 2.9. Each of these graphs contains a Hamilton cycle. Hence, these graphs are hamiltonian.



(a) (b) (c) (d)

Figure 2.9: Members of $G_3(4,5)$ (Jendroľ & Jucovič, 1989).

A conjecture regarding the existence of Hamilton cycles in bipartite 3R3CPs, which is known as Barnette's conjecture (Barnette, 1969), remains open.

Conjecture 2.1. *(Barnette's conjecture)* *Every bipartite* 3*R*3*CP is hamiltonian.*

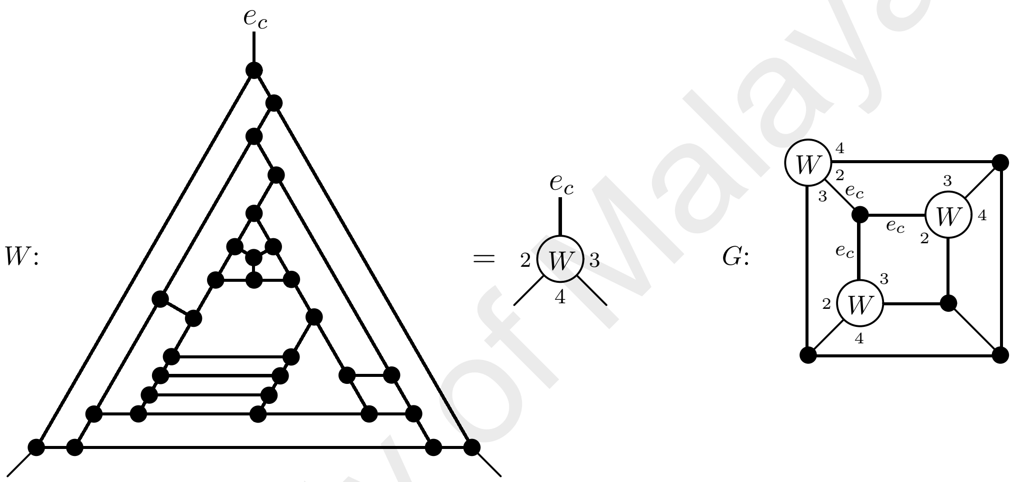
As already stated in Section 1.2.5, each face in a planar bipartite graph is a $2k$-gon where $k \geq 2$. If Barnette's conjecture is true, then all members of $G_3(4,k)$ are hamiltonian for every even value of k.

Theorem 2.10. *(Goodey, 1975)* $G_3(4,6)$ *contains only hamiltonian members.*

Goodey (1975) verified Conjecture 2.1 for the class $G_3(4,6)$ in which he showed that all members are hamiltonian. At present, there appear to be no known results concerning the hamiltonicity of $G_3(4,2k)$ for $k \geq 4$.

Theorem 2.11. *(Owens, 1984b) There exists a non-hamiltonian member of* $G_3(4, 7)$.

Owens (1984b) constructed the 3-piece W shown in Figure 2.10(a). W contains a modified form of the 3-piece T_3 (defined in Lemma 1.9). As indicated, W has a compulsory edge of attachment e_c. He obtained the graph G shown in Figure 2.10(b) by replacing three vertices of a cube (see Figure 2.1) with three copies of W, in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces W are incident to the same vertex. By inspection, $G \in G_3(4, 7)$. Since the three e_c are incident to the same vertex, no cycle in G can contain them all. Hence, G is non-hamiltonian.



(a) (b)

Figure 2.10: 3-piece W and a non-hamiltonian of member of $G_3(4, 7)$ (Owens, 1984b).

Theorem 2.12. *(Owens, 1984b; Walther, 1981) There exist non-hamiltonian members of* $G_3(4, 9)$.

Walther (1981) constructed the 3-pieces S, H, I and J shown in Figure 2.11(a) – (d). Some of these 3-pieces contain L_2, L_3 or L_4 (see Figure 1.7). He obtained the non-hamiltonian graph G shown in Figure 2.11(e) by replacing ten vertices of a non-hamiltonian graph constructed by Grinberg (1968) with six copies of H, three copies of I and one copy of J. By inspection, $G \in G_3(4, 9)$.

Owens (1984b) also constructed a non-hamiltonian member of $G_3(4, 9)$, which has far fewer vertices than the one constructed by Walther (1981). The graph that was due to Walther (1981), as shown in Figure 2.11(e), has about 700 vertices while the graph constructed by Owens (1984b) has 158 vertices only.

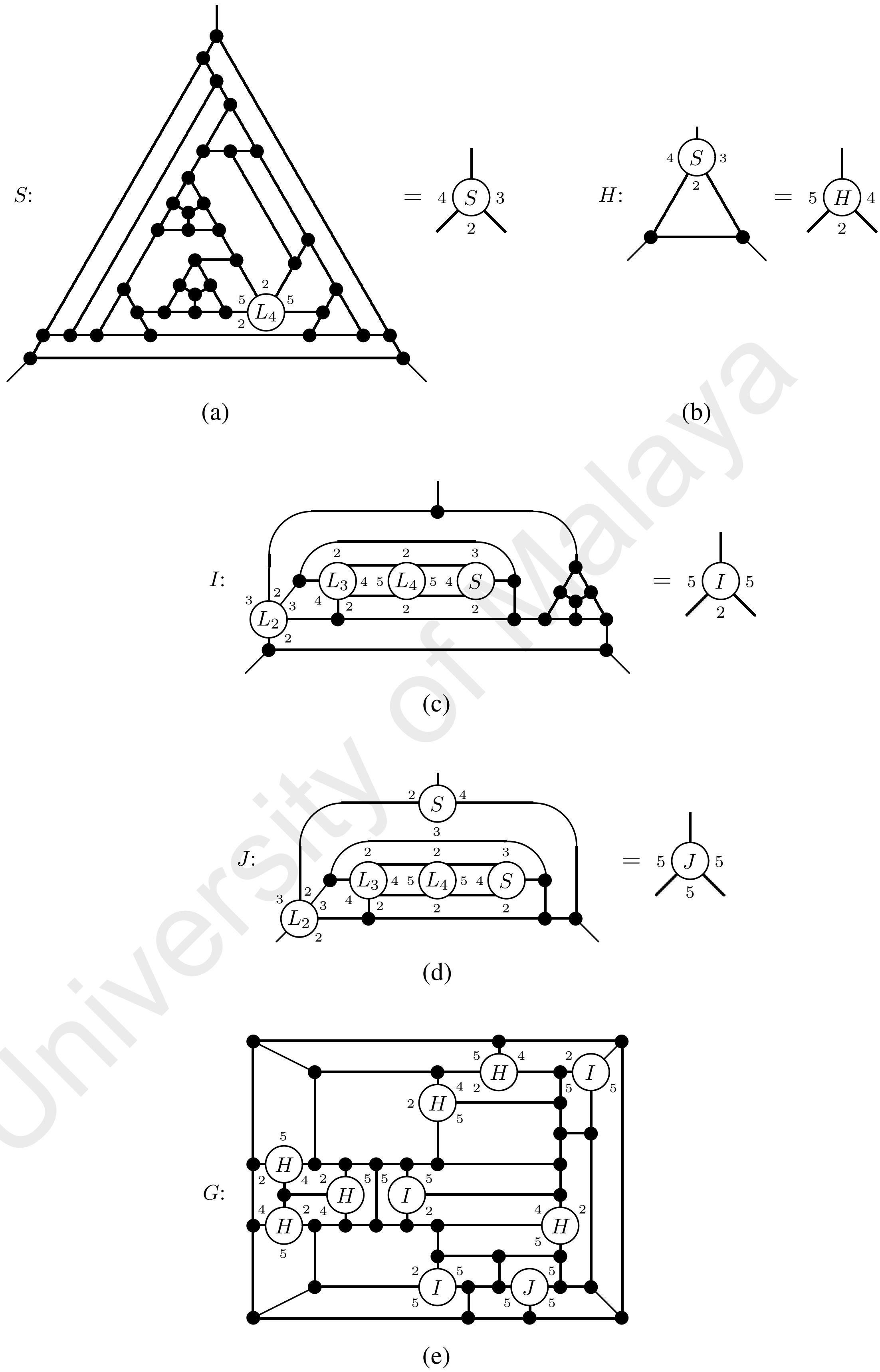


Figure 2.11: 3-pieces S, H, I, J and a non-hamiltonian member of $G_3(4, 9)$ (Walther, 1981).

Theorem 2.13. *(Walther, 1981)* *There exist non-hamiltonian members of* $G_3(4,k)$ *for all odd* $k \geq 11$.

According to Walther (1981), non-hamiltonian members of $G_3(4,k)$ for all odd $k \geq 11$ can be constructed in a similar way to that of a non-hamiltonian member of $G_3(4,9)$ (see Figure 2.11).

A summary of results for the class $G_3(4,k)$ is given in Theorem 2.14.

Theorem 2.14.

1. *$G_3(4,5)$ and $G_3(4,6)$ (Goodey, 1975) contain only hamiltonian members.*

2. *There exist non-hamiltonian members of $G_3(4,k)$ for $k = 7$ (Owens, 1984b), $k = 9$ (Owens, 1984a; Walther, 1981) and for all odd $k \geq 11$ (Walther, 1981).*

2.4 $G_3(5,k)$

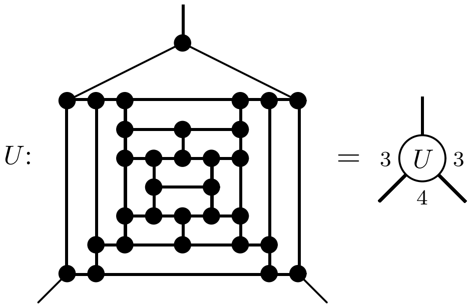
It has been conjectured that every member of $G_3(5,6)$ is hamiltonian (see Owens, 1999, Conjecture 5; Walther, 1997, Remarks).

Ewald (1973) proved that every member G of $G_3(5,6)$ contains a cycle that meets every face of G. This implies that there is a cycle through at least $\frac{n}{3}$ vertices of G on n vertices. Jendroľ and Owens (1995) gave a better bound of $\frac{4n}{5}$. Král', Pangrác, Sereni, and Škrekovski (2009) improved the bound to $\frac{5n}{6} - \frac{2}{3}$. Erman, Kardoš, and Miškuf (2009) further improved it to $\frac{6n}{7} + \frac{2}{7}$.

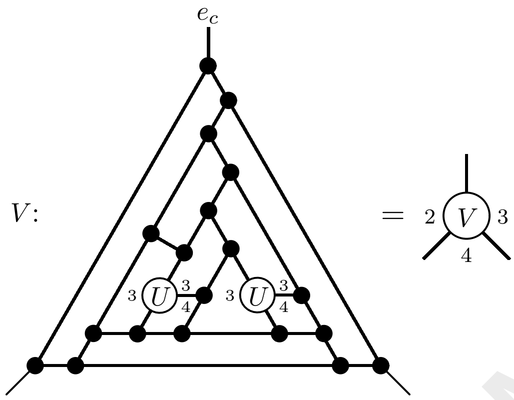
Brinkmann and Dress (1998) checked the hamiltonicity of all members of $G_3(5,6)$ on at most 150 vertices and all were found to have a Hamilton cycle. Brinkmann, Goedgebeur, and McKay (2012) checked up to 336 vertices.

Theorem 2.15. *(Owens, 1981)* *There exists a non-hamiltonian member of* $G_3(5,7)$.

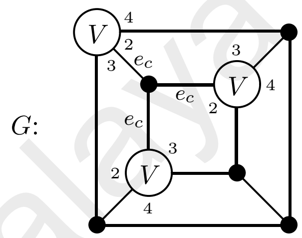
Owens (1981) constructed the 3-pieces U and V shown in Figure 2.12(a) and (b), respectively. V contains a copy of the Tutte's triangle T (see Figure 1.5) in which two of its vertices are replaced by two copies of U. As indicated, V has a compulsory edge of attachment e_c. He obtained the non-hamiltonian graph G shown in Figure 2.12(c) in a similar way to that of a non-hamiltonian member of $G_3(4,7)$ (see Figure 2.10(b)) described earlier. By inspection, $G \in G_3(5,7)$.



(a)



(b)



(c)

Figure 2.12: 3-pieces U, V and a non-hamiltonian member of $G_3(5,7)$ (Owens, 1981).

Theorem 2.16. *(Zaks, 1977) There exists a non-hamiltonian member of* $G_3(5,8)$.

Zaks (1977) showed that a non-hamiltonian member of $G_3(5,8)$, as shown in Figure 2.14, can be obtained from a non-hamiltonian member of $G_3(5,6,8)$ (Grinberg, 1968; Tutte, 1972) (see Figure 2.13) by removing the vertex labelled u from the graph shown in Figure 2.13 and joined two such copies to six new vertices, v_i where $1 \leq i \leq 6$ to obtain the graph shown in Figure 2.14.

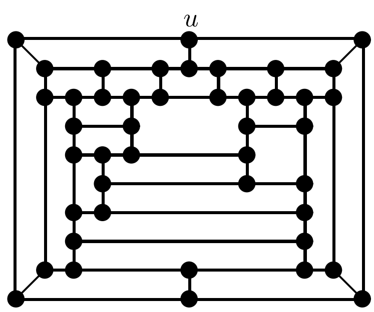


Figure 2.13: A non-hamiltonian member of $G_3(5,6,8)$ (Grinberg, 1968; Tutte, 1972).

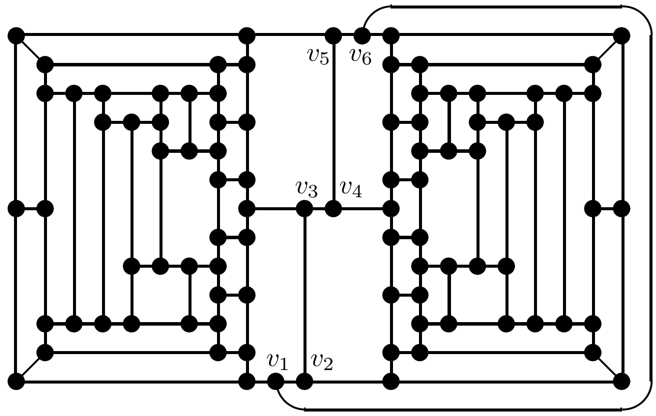


Figure 2.14: A non-hamiltonian member of $G_3(5, 8)$ (Zaks, 1977).

Additionally, Zaks (1977) coined the term *non-grinbergian*, which refers to a planar graph that satisfies the condition $f_k = 0$ for all $k \not\equiv 2(mod\, 3)$. The non-hamiltonian graph shown in Figure 2.14 is non-grinbergian and it provides negative answer to Question 1.1.

Theorem 2.17. *(Zaks, 1982a) There exists a non-hamiltonian member of* $G_3(5, 9)$.

Zaks (1982a) constructed a non-hamiltonian member of $G_3(5, 9)$ shown in Figure 2.15. The graph contains three copies of B_1 (see Figure 1.9(a)). He showed that the graph can be shown non-hamiltonian by applying Grinberg's Theorem (see Theorem 1.5). He pointed out that the graph can also be shown non-hamiltonian by considering all possible spanning paths through B_1 (see Lemma 1.11(1)) from which one easily gets a contradiction.

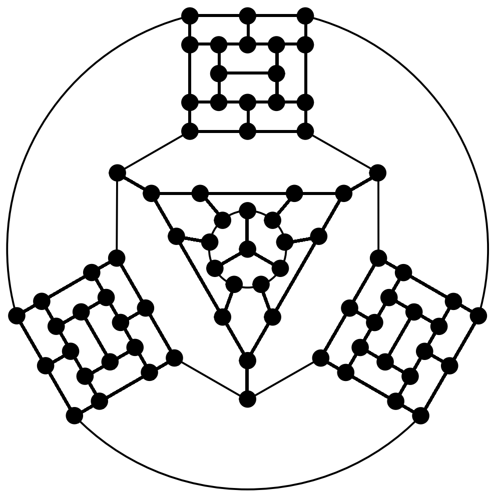


Figure 2.15: A non-hamiltonian member of $G_3(5, 9)$ (Zaks, 1982a).

Theorem 2.18. *(Owens, 1982b) There exists a non-hamiltonian member of* $G_3(5,10)$*.*

Owens (1982b) constructed the 3-piece E shown in Figure 2.16(a). E contains two copies of B_1 . As indicated, E has a compulsory edge of attachment e_c. The graph G shown in Figure 2.16(b) contains three copies of E that are placed in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces E are incident to the same vertex. By inspection, $G \in G_3(5,10)$. Since the three e_c are incident to the same vertex, no cycle in G can contain them all. Hence, G is non-hamiltonian.

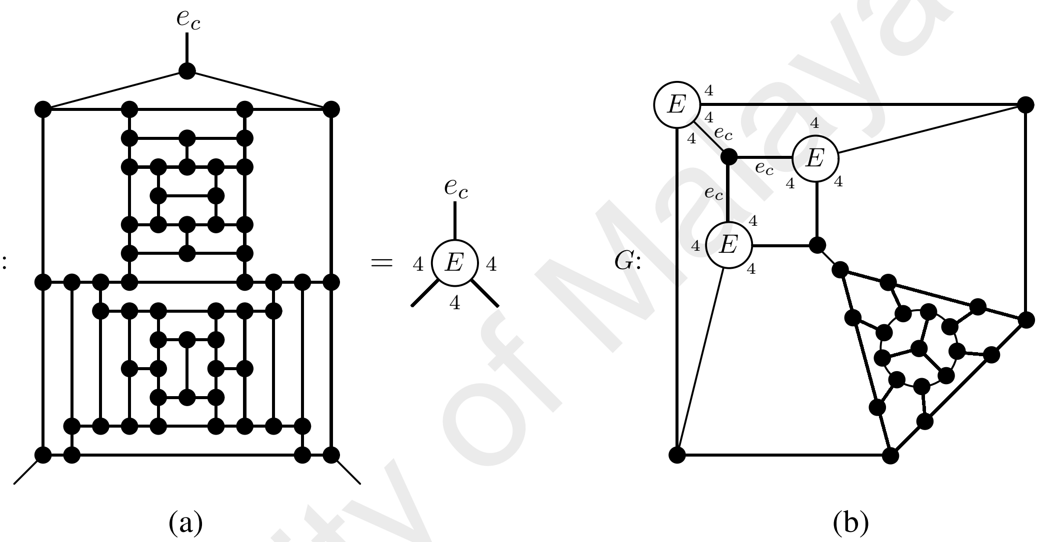


Figure 2.16: 3-piece E and a non-hamiltonian member of $G_3(5,10)$ (Owens, 1982b).

Theorem 2.19. *(Zaks, 1980) There exist non-hamiltonian members of* $G_3(5,k)$ *for all* $k \geq 11$*.*

Zaks (1980) constructed non-hamiltonian members of $G_3(5,k)$ for all $k \geq 11$ shown in Figures 2.17 – 2.23. Each graph in Figures 2.17, 2.18 and 2.19 contains three copies of B_2, B_2' and B_i, $i \geq 2$, respectively. The graphs in Figures 2.20 and 2.23 contain three copies of B_i'', $i \geq 2$ each, whereas the ones in Figures 2.21 and 2.22 contain five copies of B_i'', $i \geq 2$ each.

A summary of results for the class $G_3(5,k)$ is given in Theorem 2.20.

Theorem 2.20. *There exist non-hamiltonian members of* $G_3(5,k)$ *for* $k = 7$ *(Owens, 1981),* $k = 8$ *(Zaks, 1977),* $k = 9$ *(Zaks, 1982a),* $k = 10$ *(Owens, 1982b) and* $k \geq 11$ *(Zaks, 1980).*

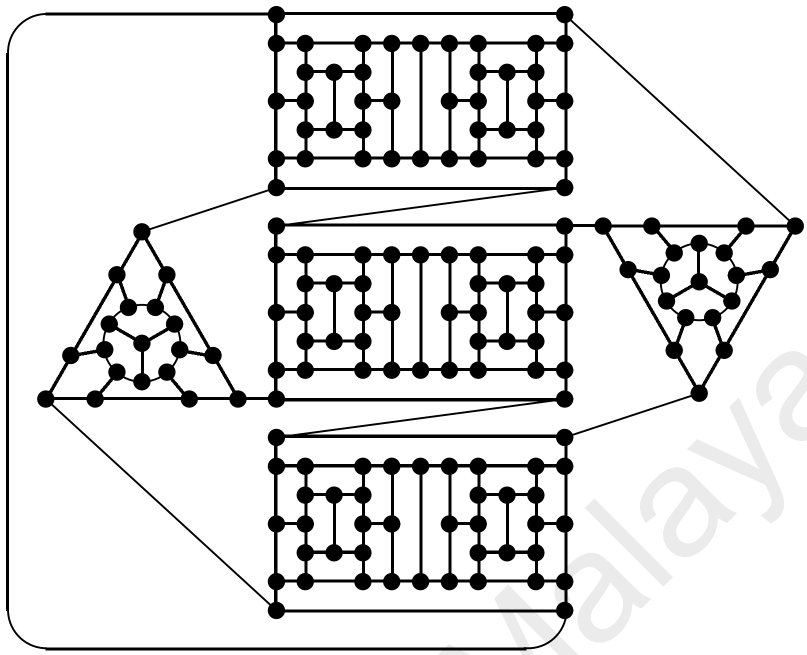


Figure 2.17: A non-hamiltonian member of $G_3(5, 11)$ (Zaks, 1980).

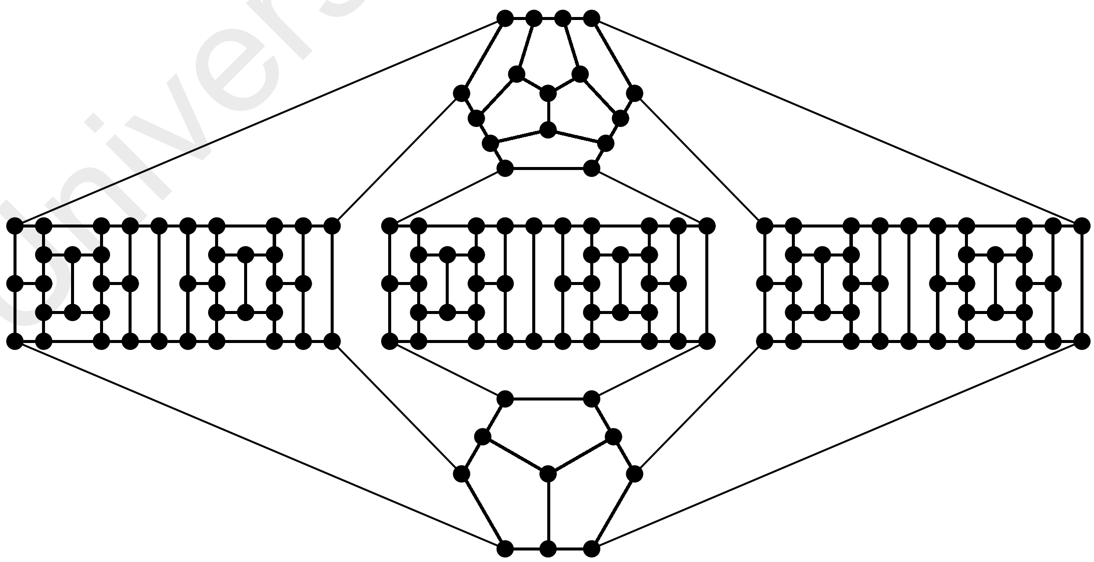


Figure 2.18: A non-hamiltonian member of $G_3(5, 12)$ (Zaks, 1980).

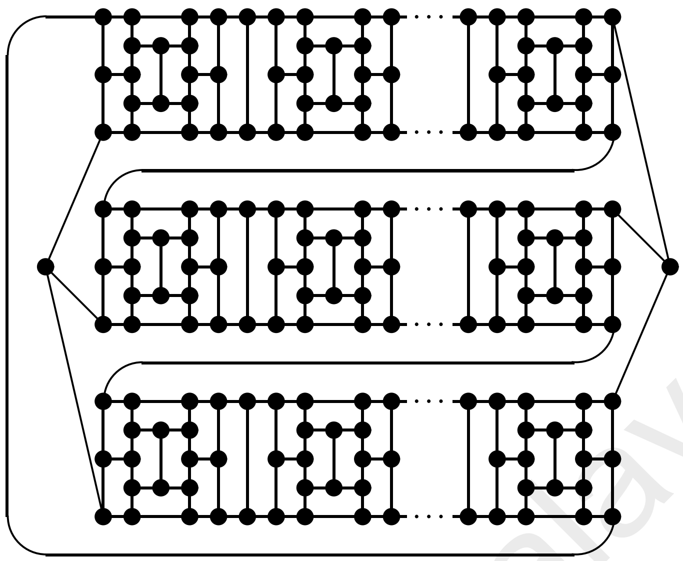


Figure 2.19: A non-hamiltonian member of $G_3(5, 3 + 5i)$ for $i \geq 2$ (Zaks, 1980).

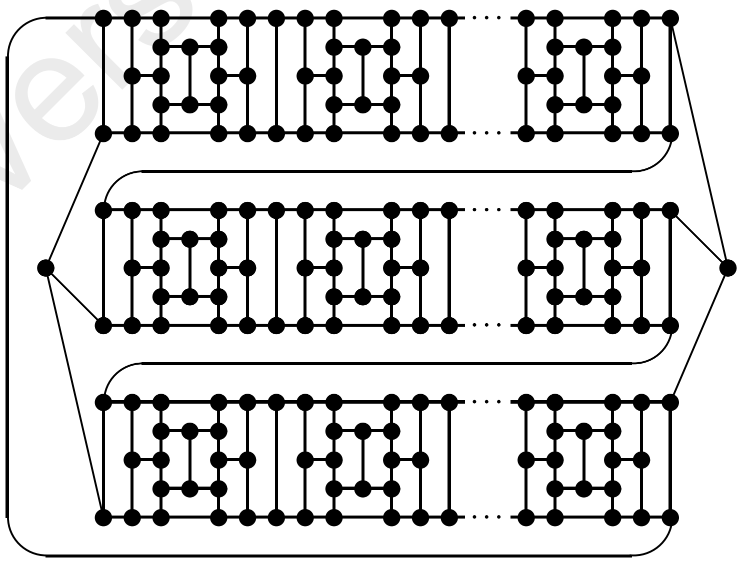


Figure 2.20: A non-hamiltonian member of $G_3(5, 4 + 5i)$ for $i \geq 2$ (Zaks, 1980).

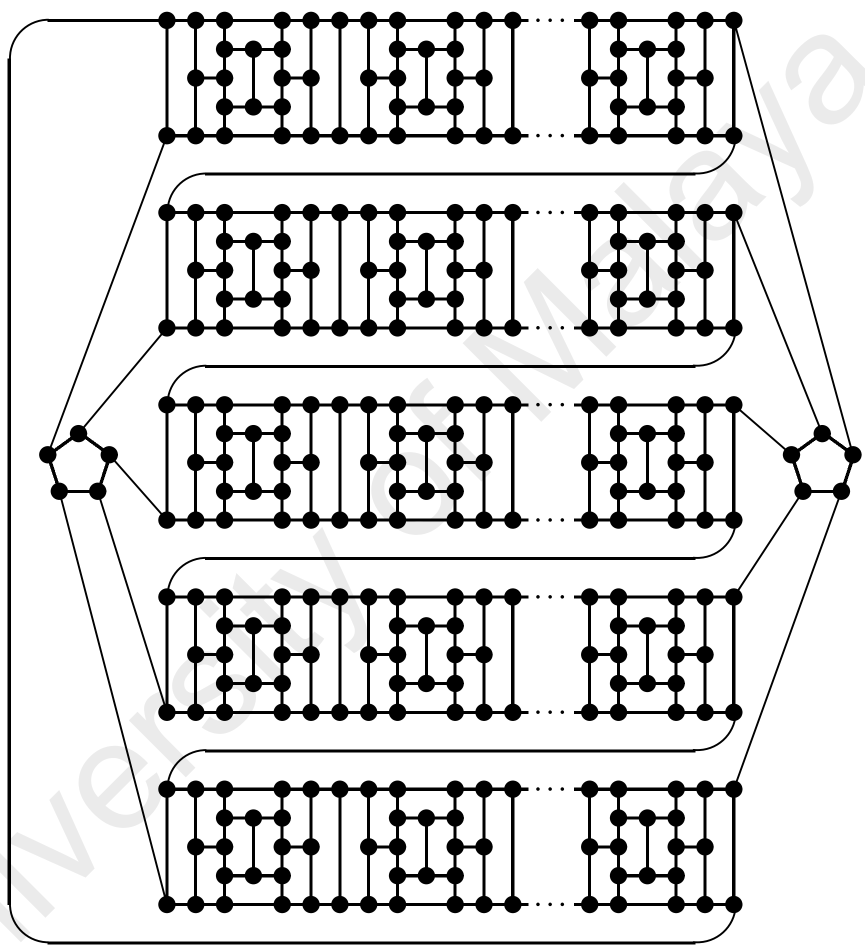


Figure 2.21: A non-hamiltonian member of $G_3(5, 5+5i)$ for $i \geq 2$ (Zaks, 1980).

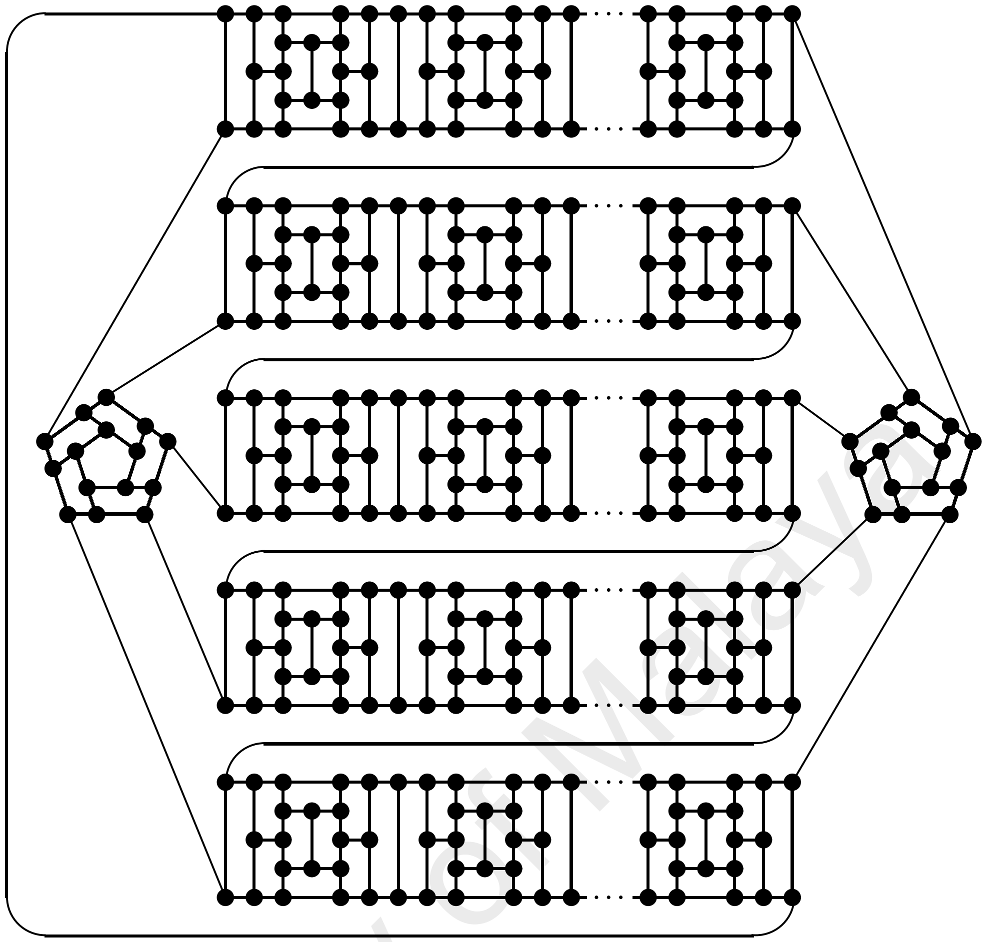


Figure 2.22: A non-hamiltonian member of $G_3(5, 6+5i)$ for $i \geq 2$ (Zaks, 1980).

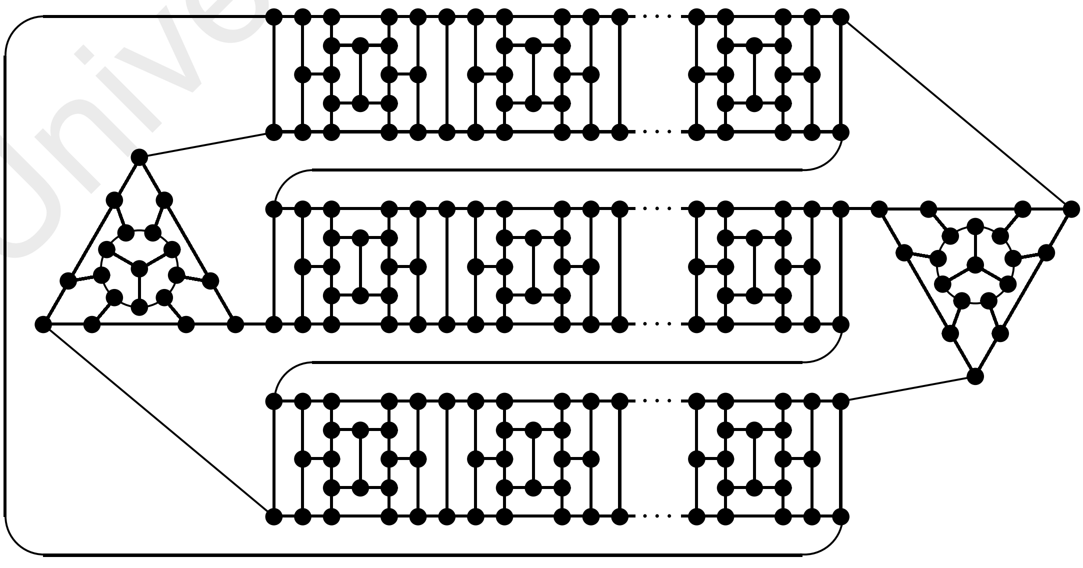


Figure 2.23: A non-hamiltonian member of $G_3(5, 7+5i)$ for $i \geq 2$ (Zaks, 1980).

As discussed in Sections 2.2 to 2.4, the hamiltonicity of $G_3(h, k)$ are known for all h and k, except those stated in the following two conjectures:

Conjecture 2.2. *$G_3(5, 6)$ contains only hamiltonian members.*

Conjecture 2.3. *$G_3(4, 2k)$ for $k \geq 4$ contains only hamiltonian members.*

The search for non-hamiltonian members of $G_3(h, k)$ is further refined by imposing additional restriction on the connectivity of the members. Here, we give a brief summary of the existence of non-hamiltonian cyclically 5-connected members of $G_3(5, k)$.

Theorem 2.21. *There exist non-hamiltonian cyclically 5-connected members of $G_3(5, k)$ for*

1. *$k = 8$ (Walther, 1997);*
2. *$k \geq 8$ and $k \notin \{13, 16, 18, 21\}$ (Fabrici, Owens, & Walther, 2000); and*
3. *$k = 20 + 8s$ for $s \geq 0$ (Owens, 1982a).*

Fabrici et al. (2000) only showed the constructions for $k = 10$ and $k \geq 17$ and noted that each of the remaining cases needs a special treatment.

2.5 Additional Result

In addition to the classes discussed earlier, here, we consider $G_3(k_1, k_2, \ldots, k_t)$ for $3 \leq k_1 < k_2 < \ldots < k_t \leq 6$.

Barnette (see Malkevitch, 1988) posed the following open question in which he asked if a 3R3CP whose biggest faces are 6-gons is hamiltonian.

Question 2.1. *Is every member of $G_3(k_1, k_2, \ldots, k_t)$ for $3 \leq k_1 < k_2 < \ldots < k_t \leq 6$ hamiltonian?*

Note that Question 2.1 covers the class $G_3(h, k)$ for $3 \leq h < k \leq 6$ where all members are known to be hamiltonian (see Theorems 2.8 and 2.14), except $h = 5$ and $k = 6$.

Aldred, Bau, Holton, and McKay (2000) generated all such graphs on at most 176 vertices and found them all to be hamiltonian. Brinkmann, McKay, and von Nathusius (2003) extended the computation to 250 vertices with the same outcome. Brinkmann et al. (2012) confirmed that all such graphs on at most 316 vertices are hamiltonian.

Grünbaum and Walther (1973) (see also Goodey, 1977, Remarks) conjectured that every 3R3CP whose biggest faces are 6-gons is hamiltonian.

Conjecture 2.4. *(Grünbaum & Walther, 1973)* $G_3(k_1, k_2, \ldots, k_t)$ *where* $3 \leq k_1 < k_2 < \ldots < k_t \leq 6$ *contains only hamiltonian members.*

CHAPTER 3: 3-REGULAR 3-CONNECTED PLANAR GRAPHS WITH ONLY THREE TYPES OF FACES

3.1 Introduction

This chapter is devoted to the constructions of non-hamiltonian members of $G_3(h, k, l)$, $h \neq k \neq l$, which is the class of all 3-regular 3-connected planar graphs (3R3CPs) whose faces are of only three types, namely h-gons, k-gons and l-gons. By Lemma 1.3, every 3R3CP must have some 3-, 4- or 5-gons. We construct a large number of non-hamiltonian members of $G_3(h, k, l)$ where $h \in \{3, 4, 5\}$. We will divide our results into three sections, namely Sections 3.3, 3.4 and 3.5, each of which focuses on the classes $G_3(3, k, l)$, $G_3(4, k, l)$ and $G_3(5, k, l)$, respectively. The techniques and subgraphs used in the constructions will be discussed in Section 3.2.

In general, let $S^{n_1,n_2,n_3}_{(k_1,k_2,\ldots,k_t)}$ ($S^{n_1,n_2,n_3,n_4}_{(k_1,k_2,\ldots,k_t)}$) denotes any 3-piece (4-piece) whose inner faces are of only t types, namely k_1-, k_2-, $\ldots$, k_t-gons where $k_i \geq 3$, $k_i \neq k_j$ $\forall$ $i \neq j$ and $i, j \in \{1, 2, \ldots, t\}$, that contributes n_1, n_2 and n_3 (n_1, n_2, n_3 and n_4) vertices to the three (four) adjoining faces of any graph in which it occurs. If $n = n_1 = n_2 = n_3$ ($n = n_1 = n_2 = n_3 = n_4$), then the notation is simplified to $S^{n}_{(k_1,k_2,\ldots,k_t)}$.

We represent an $S^{n_1,n_2,n_3}_{(k_1,k_2,\ldots,k_t)}$ ($S^{n_1,n_2,n_3,n_4}_{(k_1,k_2,\ldots,k_t)}$) by a labelled circle whose circumference is surrounded by n_1, n_2 and n_3 (n_1, n_2, n_3 and n_4). $S^2_{(3)}$, whose only interior face is a 3-gon, is represented by a small unlabelled white circle that has number two around the circumference of the circle, as shown in Figure 3.1(a). $S^3_{(4)}$ and $S^4_{(5)}$, as shown in Figures 3.1(b) and (c), are obtained by removing a vertex from a cube and a dodecahedron (see Figure 2.1), respectively.

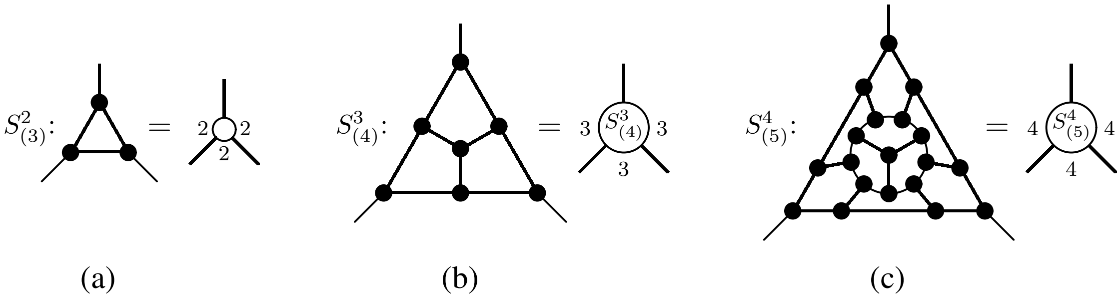
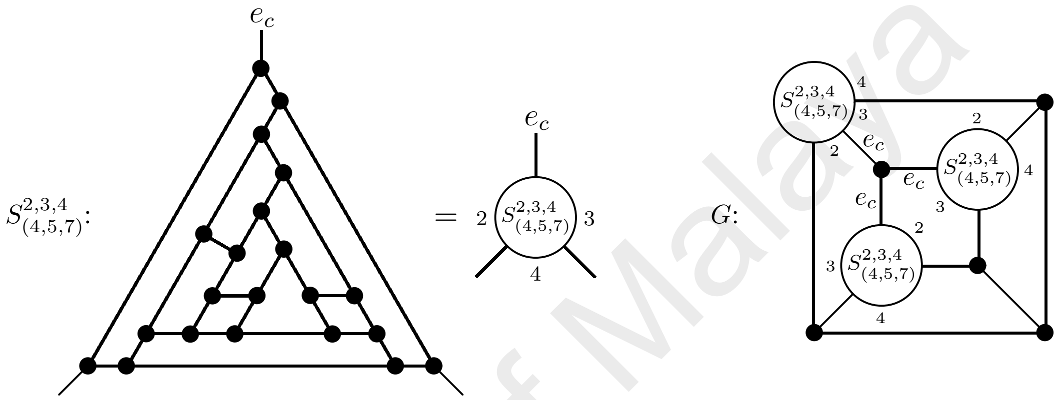


Figure 3.1: 3-pieces $S^2_{(3)}$, $S^3_{(4)}$ and $S^4_{(5)}$.

It is well known that there are 3R3CPs with no Hamilton cycle and among the non-hamiltonian 3R3CPs that have been constructed, there are graphs that have exactly three types of faces. The following paragraphs contain a brief discussion of such graphs.

Owens (1981) constructed a non-hamiltonian member of $G_3(4,5,7)$ shown in Figure 3.2(b). The non-hamiltonian graph G contains three copies of $S^{2,3,4}_{(4,5,7)}$ (see Figure 3.2(a)), whose compulsory edges of attachment e_c are incident to the same vertex. G is then used to obtain a non-hamiltonian member of $G_3(5,7)$ shown in Figure 2.12(c).



(a) (b)

Figure 3.2: 3-piece $S^{2,3,4}_{(4,5,7)}$ and a non-hamiltonian member of $G_3(4,5,7)$ (Owens, 1981).

Hunter (1962) constructed a non-hamiltonian cyclically 4-connected 3R3CP with 58 vertices. The graph is a member of $G_3(5,6,9)$. Another such graphs with only 42 vertices that were due to Grinberg (1968) and Faulkner and Younger (1974) are members of $G_3(4,5,8)$ and $G_3(4,5,11)$, respectively.

Grinberg (1968) constructed two non-hamiltonian cyclically 5-connected 3R3CPs. One is a member of $G_3(5,8,9)$ and another, which was also constructed independently by Tutte (1972), is a member of $G_3(5,6,8)$ (see Figure 2.13). Zaks (1982b) presented a non-hamiltonian non-grinbergian cyclically 5-connected 3R3CP, which is a member of $G_3(5,14,20)$.

The girth of a graph G is the length of a shortest cycle in G. In search of a non-hamiltonian 3R3CP of girth 5 with minimum number of vertices, Aldred et al. (2000) discovered three non-hamiltonian cyclically 4-connected 3R3CPs whose faces are of only three types, each of which is a member of $G_3(5,6,9)$, $G_3(5,6,11)$ and $G_3(5,8,12)$.

3.2 Lemmas

In this section, we discuss the techniques that will be used in the proof of theorems in subsequent sections. This also includes discussions on subgraphs with properties that are useful in the constructions of non-hamiltonian members of $G_3(h, k, l)$.

Lemma 3 of (Owens, 1984a) states that if a graph H is obtained from a graph G by shrinking a triangular face of G to a single vertex, then G and H are either both hamiltonian or both non-hamiltonian. Lemma 3.1 extends Lemma 3 of (Owens, 1984a).

Lemma 3.1. *Let S be a 3-piece in a graph G. S has a spanning path through S that contains any two of its edges of attachment. Suppose G^* is a graph obtained from G by shrinking S to a single vertex. Then G and G^* are either both hamiltonian or both non-hamiltonian.*

Lemmas 3.2 and 3.3 are easily obtained.

Lemma 3.2. *For each $i = 1, 2, 3$, let S_i be a 3-piece obtained by removing a set of three edges from a 3R3CP. Suppose S_3 can be obtained from S_1 by replacing a vertex with a copy of S_2. Then an edge of attachment of S_3 is a compulsory edge of attachment if it is a compulsory edge of attachment of S_1.*

Lemma 3.3. *Let C be a Hamilton cycle of a 3R3CP that contains a k-piece S for $k \in \{3, 4\}$. Suppose S^* is obtained from S by replacing a vertex with a copy $S^2_{(3)}$ (see Figure 3.1(a)). Then the resulting 3R3CP contains a Hamilton cycle C^*. Furthermore, if there is a spanning path through S that ends at its* a*-vertices u and v, then there is a spanning path through S^* that also ends at u and v.*

PROOF: For a path that passes through any two edges incident to a vertex in S, there is a corresponding path through the two edges in S^*, as shown in Figure 3.3. Hence the lemma follows. ■

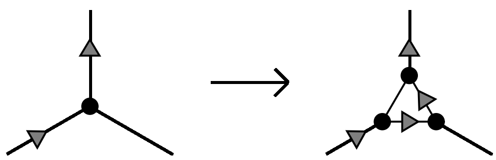


Figure 3.3: Paths through a vertex and a 3-piece $S^2_{(3)}$.

Lemma 3.4 can be found in (Owens, 1984b).

Lemma 3.4. *(Owens, 1984b)* *For each $i = 1, 2, 3$, let S_i be a 3-piece with a compulsory edge of attachment e_i. If a graph G contains S_1, S_2 and S_3 such that e_1, e_2 and e_3 are incident to the same vertex, then G is non-hamiltonian.*

Lemmas 3.5–3.10 exploit the presence of B_1, B_{2+i}, B'_{2+i} and B''_{2+i} for $i \geq 0$ (see Figures 1.9 and 1.10) in some k-pieces.

Lemma 3.5. *Let $S^{3,5,5}_{(4,5)}$, $S^{3,6,6}_{(3,5)}$, $S^{4,7,7}_{(3,6)}$ and $S^{4,5,5}_{(4,5,8)}$ be the 3-pieces shown in Figure 3.4. Then the edge of attachment labelled e_c is a compulsory edge of attachment of each 3-piece.*

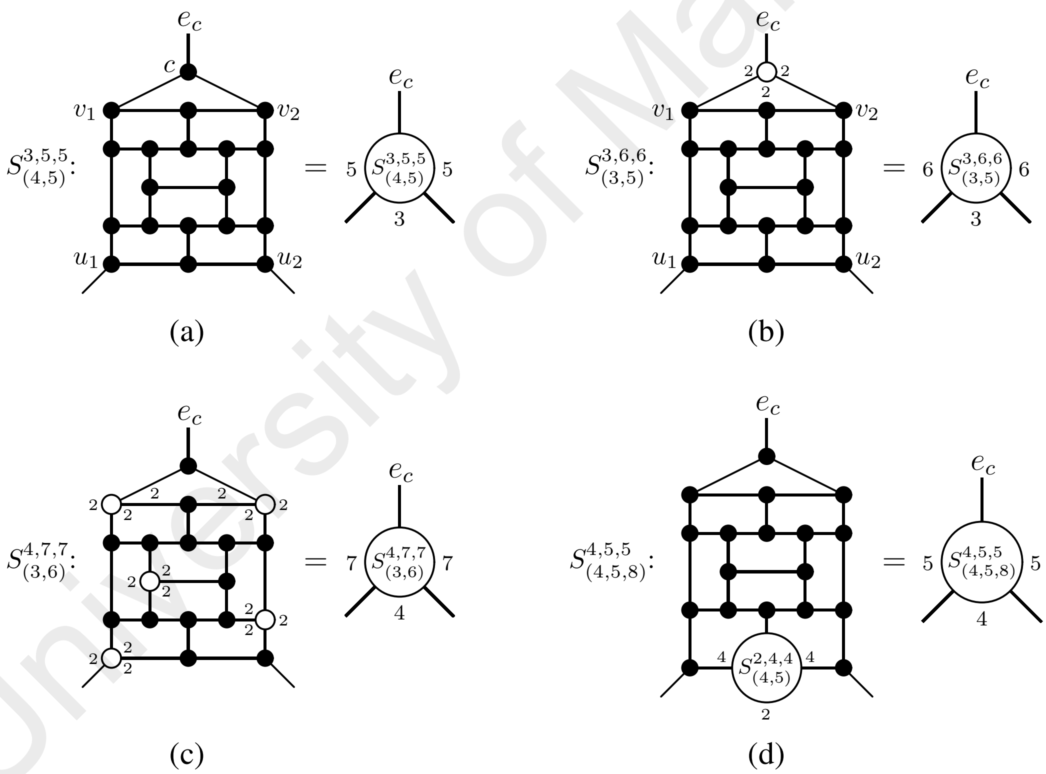


Figure 3.4: 3-pieces $S^{3,5,5}_{(4,5)}$, $S^{3,6,6}_{(3,5)}$, $S^{4,7,7}_{(3,6)}$ and $S^{4,5,5}_{(4,5,8)}$.

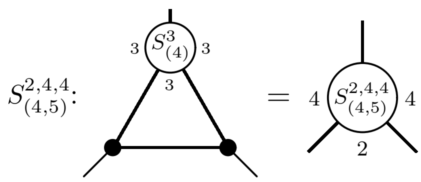


Figure 3.5: 3-piece $S^{2,4,4}_{(4,5)}$.

PROOF: Let c, u_1 and u_2 be a-vertices of $S^{3,5,5}_{(4,5)}$, as shown in Figure 3.4(a). Note that $S^{3,5,5}_{(4,5)} - c$ is the 4-piece B_1 (see Figure 1.9(a)) described by Faulkner and Younger (1974). Let C be a Hamilton cycle of a 3R3CP that contains B_1 and $S^{3,5,5}_{(4,5)}$. Let P and P^* be $B_1 \cap C$ and $S^{3,5,5}_{(4,5)} \cap C$, respectively. By Lemma 1.11(1), P are of the following three forms only: $P_{u_1 v_2}$ (or by symmetry, $P_{u_2 v_1}$), $P_{u_1 u_2}$ (or by symmetry, $P_{v_1 v_2}$) and $P_{u_1 u_2} \cup P_{v_1 v_2}$. Now, consider the following three cases:

Case (1): If P is of the form $P_{u_1 v_2}$ (or by symmetry, $P_{u_2 v_1}$), then P^* takes the form $cP_{v_2 u_1}$ (or by symmetry, $cP_{v_1 u_2}$).

Case (2): If P is of the form $P_{u_1 u_2}$, then P^* misses the a-vertex c, which is impossible. If P is of the form $P_{v_1 v_2}$, then $cP_{v_1 v_2}c$ forms a cycle in P^*, which is impossible too.

Case (3): If P is of the form $P_{u_1 u_2} \cup P_{v_1 v_2}$, then $cP_{v_1 v_2}c$ forms a cycle in P^*, which is also impossible.

Hence, by *Case (1)*, e_c is a compulsory edge of attachment of $S^{3,5,5}_{(4,5)}$.

$S^{3,6,6}_{(3,5)}$ and $S^{4,5,5}_{(4,5,8)}$, as shown in Figure 3.4(b) and (d), are obtained by replacing a vertex of $S^{3,5,5}_{(4,5)}$ with a copy of $S^{2}_{(3)}$ and $S^{2,4,4}_{(4,5)}$ (see Figure 3.5), respectively. $S^{4,7,7}_{(3,6)}$, as shown in Figure 3.4(c), is obtained by replacing five vertices of $S^{3,5,5}_{(4,5)}$ with five copies of $S^{2}_{(3)}$. By Lemma 3.2, the edge of attachment labelled e_c is a compulsory edge of attachment of each of the 3-pieces $S^{3,6,6}_{(3,5)}$, $S^{4,7,7}_{(3,6)}$ and $S^{4,5,5}_{(4,5,8)}$. ■

Lemma 3.6. *Let $S^{2,6,6}_{(4,5)}$ and $S^{2,7,7}_{(3,5)}$ be the 3-pieces shown in Figure 3.6. Then the edge of attachment labelled e_c is a compulsory edge of attachment of each 3-piece.*

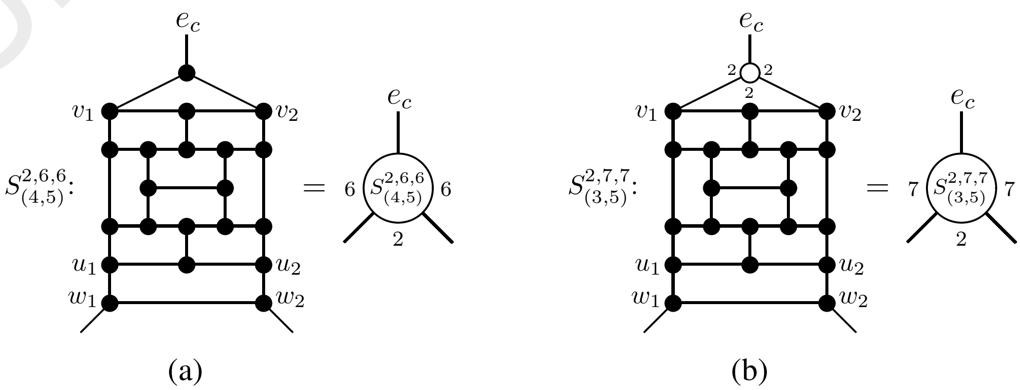


Figure 3.6: 3-pieces $S^{2,6,6}_{(4,5)}$ and $S^{2,7,7}_{(3,5)}$.

PROOF: $S^{2,6,6}_{(4,5)}$ and $S^{2,7,7}_{(3,5)}$, as shown in Figure 3.6, are obtained from $S^{3,5,5}_{(4,5)}$ and $S^{3,6,6}_{(3,5)}$ (see Figure 3.4(a) and (b)), respectively, by adding the vertices w_1 and w_2 and edges u_1w_1, u_2w_2 and w_1w_2. This is an easy consequence of Lemma 3.5. Hence, the lemma follows. ■

Lemma 3.7. *Let $S^{3,3,5}_{(5,6)}$, $S^{3,4,6}_{(3,7)}$, $S^{4,4,6}_{(3,8)}$, $S^{4,4,7}_{(3,9)}$ and $S^{4,4,8}_{(3,10)}$ be the 3-pieces shown in Figure 3.7. Then the edge of attachment labelled e_c is a compulsory edge of attachment of each 3-piece.*

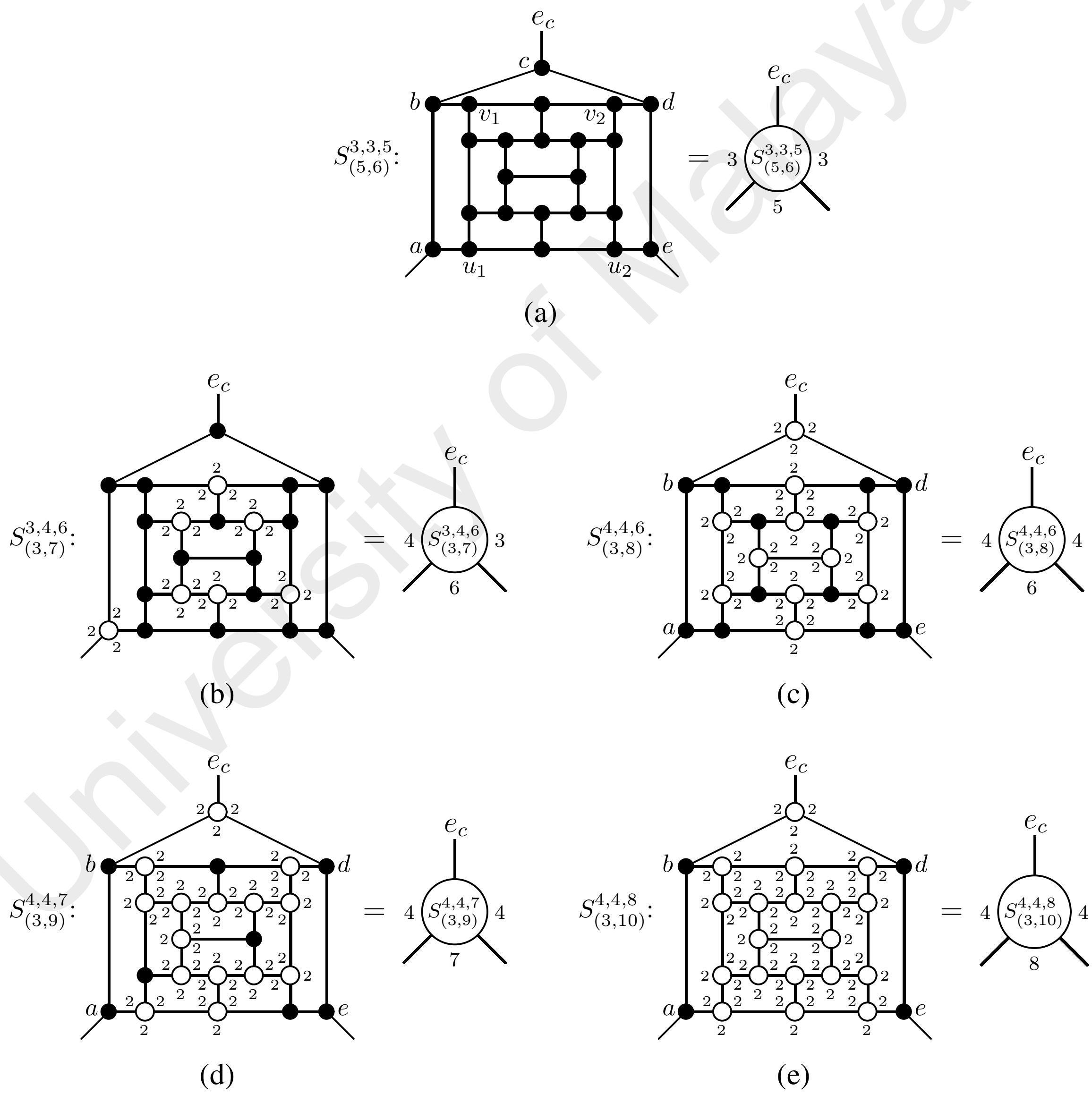


Figure 3.7: 3-pieces $S^{3,3,5}_{(5,6)}$, $S^{3,4,6}_{(3,7)}$, $S^{4,4,6}_{(3,8)}$, $S^{4,4,7}_{(3,9)}$ and $S^{4,4,8}_{(3,10)}$.

PROOF: Let a, c and e be a-vertices of $S^{3,5,5}_{(5,6)}$, as shown in Figure 3.7(a). Note that $S^{3,3,5}_{(5,6)}$ contains a copy of B_1 (see Figure 1.9(a)). Let C be a Hamilton cycle of a 3R3CP that contains B_1 and $S^{3,5,5}_{(5,6)}$. Let P and P^* be $B_1 \cap C$ and $S^{3,5,5}_{(5,6)} \cap C$, respectively. By Lemma 1.11(1), P are of the following three forms only: $P_{u_1v_2}$ (or by symmetry, $P_{u_2v_1}$), $P_{u_1u_2}$ (or by symmetry, $P_{v_1v_2}$) and $P_{u_1u_2} \cup P_{v_1v_2}$. Now, consider the following three cases:

Case (1): If P is of the form $P_{u_1v_2}$ (or by symmetry, $P_{u_2v_1}$), then $bv_1, eu_2 \notin P^*$ (or by symmetry, $au_1, dv_2 \notin P^*$). This implies that $cba, de \in P^*$ (or by symmetry, $cde, ba \in P^*$). Thus, P^* takes the form $cbaP_{u_1v_2}de$ (or by symmetry, $cdeP_{u_2v_1}ba$).

Case (2): If P is of the form $P_{u_1u_2}$, then $bv_1, dv_2 \notin P^*$. This implies that $abcde \in P^*$. However, $abcdeP_{u_2u_1}a$ forms a cycle in P^*, which is impossible. If P is of the form $P_{v_1v_2}$, then $au_1, eu_2 \notin P^*$. This implies that exactly one of ab and ed is in P^*, otherwise $cbP_{v_1v_2}dc$ forms a cycle in P^* or P^* misses the a-vertex c. However, if $abP_{v_1v_2}dc \in P^*$ (or by symmetry, $edP_{v_2v_1}bc \in P^*$), then P^* misses the a-vertex e (or by symmetry, P^* misses the a-vertex a), which is impossible too.

Case (3): If P is of the form $P_{u_1u_2} \cup P_{v_1v_2}$, then $au_1, bv_1, dv_2, eu_2 \in P^*$. This implies that exactly one of ab and ed is in P^*, otherwise $cbP_{v_1v_2}dc$ or $abP_{v_1v_2}deP_{u_2u_1}a$ forms a cycle in P^*. Thus, P^* takes the form $cbP_{v_1v_2}deP_{u_2u_1}a$ (or by symmetry, $cdP_{v_2v_1}baP_{u_1u_2}e$).

Hence, by *Case (1)* and *Case (3)*, e_c is a compulsory edge of attachment of $S^{3,5,5}_{(5,6)}$.

$S^{3,4,6}_{(3,7)}$, $S^{4,4,6}_{(3,8)}$, $S^{4,4,7}_{(3,9)}$ and $S^{4,4,8}_{(3,10)}$, as shown in Figure 3.7(b), (c), (d) and (e), respectively, are obtained by replacing some vertices of $S^{3,5,5}_{(5,6)}$ with some copies of $S^{2}_{(3)}$. By Lemma 3.2, the edge of attachment labelled e_c is a compulsory edge of attachment of each of the 3-pieces $S^{3,4,6}_{(3,7)}$, $S^{4,4,6}_{(3,8)}$, $S^{4,4,7}_{(3,9)}$ and $S^{4,4,8}_{(3,10)}$. ■

Lemma 3.8. *Let $S^{2,4,4}_{(5,6,7)}$, $S^{2,5,5}_{(3,8)}$, $S^{2,5,5}_{(3,9)}$ and $S^{2,5,5}_{(3,10)}$ be the 3-pieces shown in Figure 3.8. Then the edge of attachment labelled e_c is a compulsory edge of attachment of each 3-piece.*

PROOF: $S^{2,4,4}_{(5,6,7)}$, $S^{2,5,5}_{(3,8)}$, $S^{2,5,5}_{(3,9)}$ and $S^{2,5,5}_{(3,10)}$, as shown in Figure 3.8, are obtained from $S^{3,3,5}_{(5,6)}$, $S^{4,4,6}_{(3,8)}$, $S^{4,4,7}_{(3,9)}$ and $S^{4,4,8}_{(3,10)}$ (see Figure 3.7(a), (c), (d) and (e)), respectively, by adding the vertices f and g and edges af, eg and fg. This is an easy consequence of Lemma 3.7. Hence, the lemma follows. ■

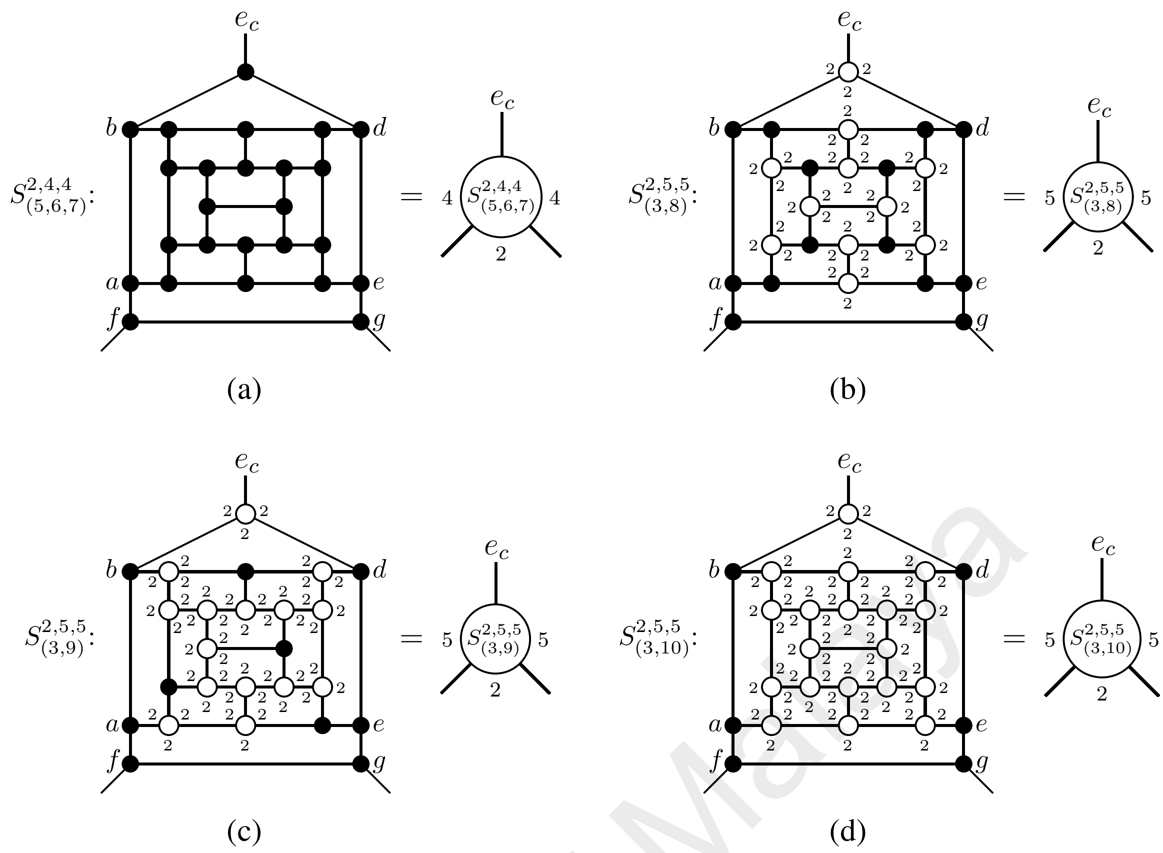


Figure 3.8: 3-pieces $S^{2,4,4}_{(5,6,7)}$, $S^{2,5,5}_{(3,8)}$, $S^{2,5,5}_{(3,9)}$ and $S^{2,5,5}_{(3,10)}$.

Lemma 3.9. *Let $S^{3,3,9+5i,9+5i}_{(5)}$, $S^{3,3,10+6i,11+6i}_{(3,6)}$, $S^{4,4,11+6i,11+6i}_{(3,6)}$, $S^{3,3,12+7i,12+7i}_{(3,7)}$, $S^{4,4,14+8i,14+8i}_{(3,8)}$, $S^{4,4,20+9i,20+9i}_{(3,9)}$, $S^{4,5,15+9i,15+9i}_{(3,9)}$, $S^{5,5,16+9i,16+9i}_{(3,9)}$ and $S^{6,6,18+10i,18+10i}_{(3,10)}$ for $i \geq 0$ be the 4-pieces shown in Figure 3.9. Let C be a Hamilton cycle of a 3R3CP that contains each of these 4-pieces and let P be the intersection of each 4-piece and C. Then P is of the form $P_{u_1v_2}$ (or by symmetry, $P_{u_2v_1}$) only.*

PROOF: For the convenience of writing, we shall denote the 4-piece B_{2+i} for $i \geq 0$ (see Figure 1.9(b)) by $S^{3,3,9+5i,9+5i}_{(5)}$, as shown in Figure 3.9(a). By Lemma 1.11(2), $S^{3,3,9+5i,9+5i}_{(5)} \cap C$ is of the form $P_{u_1v_2}$ (or by symmetry, $P_{u_2v_1}$) only.

All 4-pieces, as shown in Figure 3.9(b)–(e) and (g)–(i), are obtained by replacing some vertices of $S^{3,3,9+5i,9+5i}_{(5)}$ with some copies of $S^{2}_{(3)}$. By Lemma 3.3, we can apply Lemma 1.11(2) to these 4-pieces. Hence, the lemma follows.

$S^{4,4,20+9i,20+9i}_{(3,9)}$, as shown in Figure 3.9(f), is obtained by replacing some vertices of B''_{2+i} for $i \geq 0$ (see Figure 1.10(b)) with some copies of $S^{2}_{(3)}$. By Lemma 3.3, we can apply Lemma 1.12(2) to $S^{4,4,20+9i,20+9i}_{(3,9)}$. Hence, the lemma follows. ∎

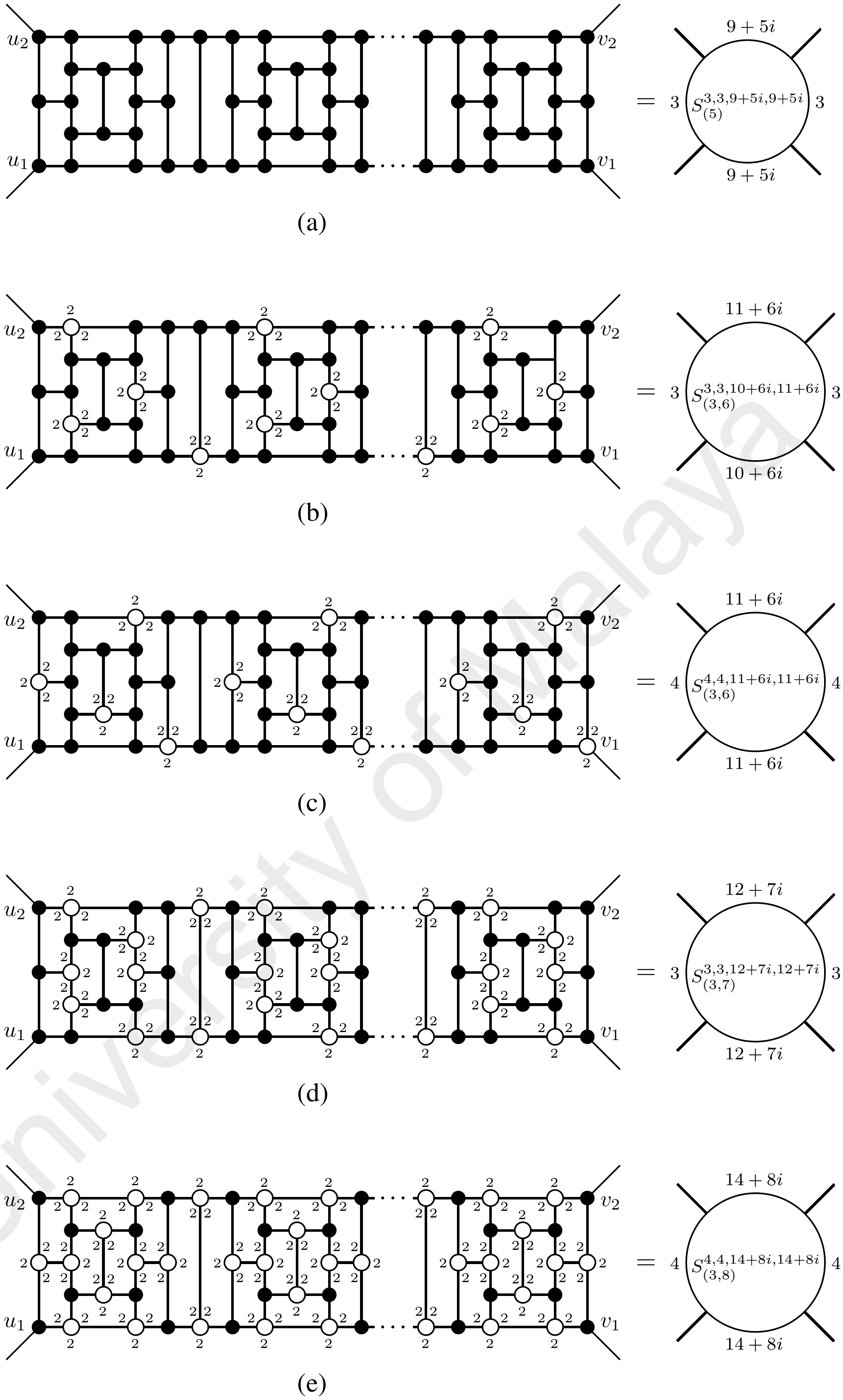
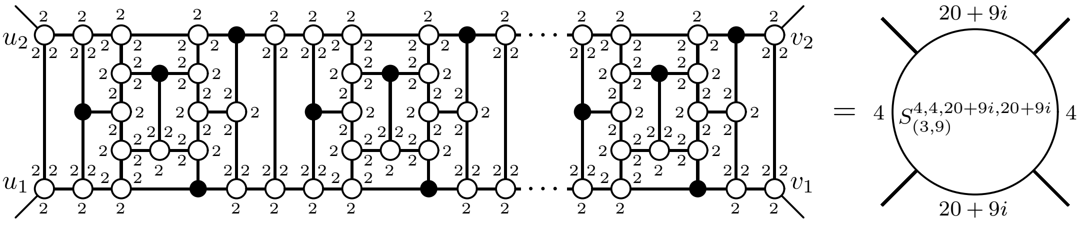
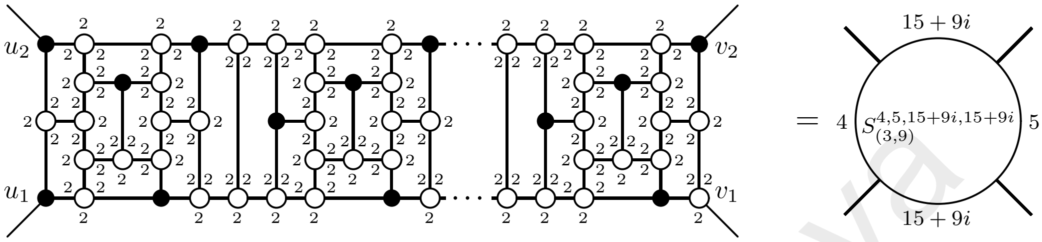


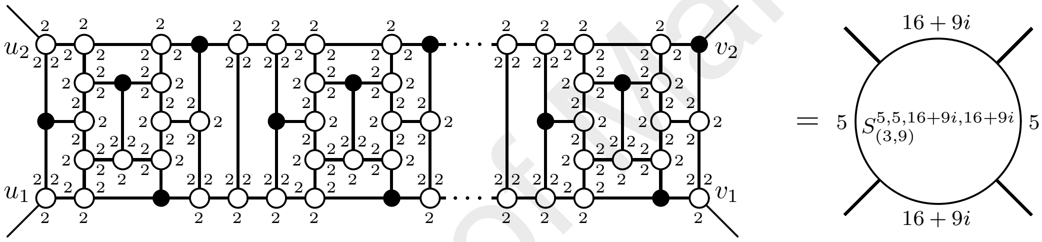
Figure 3.9: 4-pieces $S^{3,3,9+5i,9+5i}_{(5)}$, $S^{3,3,10+6i,11+6i}_{(3,6)}$, $S^{4,4,11+6i,11+6i}_{(3,6)}$, $S^{3,3,12+7i,12+7i}_{(3,7)}$ and $S^{4,4,14+8i,14+8i}_{(3,8)}$ for $i \geq 0$.



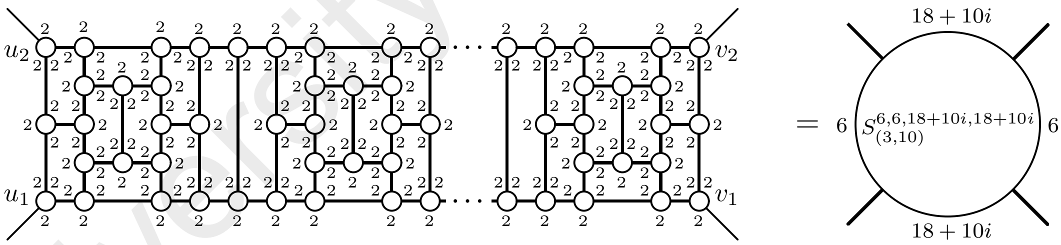
(f)



(g)



(h)



(i)

Figure 3.9 (Continued): $S_{(3,9)}^{4,4,20+9i,20+9i}$, 4-pieces $S_{(3,9)}^{4,5,15+9i,15+9i}$, $S_{(3,9)}^{5,5,16+9i,16+9i}$ and $S_{(3,10)}^{6,6,18+10i,18+10i}$ for $i \geq 0$.

Lemma 3.10. *Let $S_{(3,9)}^{4,5,18+9i,18+9i}$ for $i \geq 0$ be the 4-piece shown in Figure 3.10. Let C be a Hamilton cycle of a 3R3CP that contains $S_{(3,9)}^{4,5,18+9i,18+9i}$ and let P be $S_{(3,9)}^{4,5,18+9i,18+9i} \cap C$. Then P is of the form $P_{u_1v_1}$ (or by symmetry, $P_{u_2v_2}$) only.*

PROOF: $S_{(3,9)}^{4,5,18+9i,18+9i}$ is obtained by replacing some vertices of B'_{2+i} for $i \geq 0$ (see Figure 1.10(a)) with some copies of $S_{(3)}^{2}$. By Lemma 3.3, we can apply Lemma 1.12(1) to $S_{(3,9)}^{4,5,18+9i,18+9i}$. Hence, the lemma follows. ■

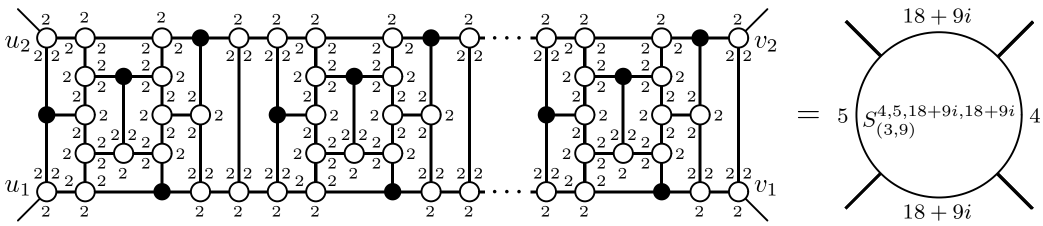


Figure 3.10: 4-piece $S^{4,5,18+9i,18+9i}_{(3,9)}$ for $i \geq 0$.

The following lemma is a simplified version of Lemma 4 of (Zaks, 1980).

Lemma 3.11. *Let D be the graph shown in Figure 3.11. D contains three copies of 4-piece S. If S is one of the 4-pieces shown in Figures 3.9 and 3.10, then D is non-hamiltonian.*

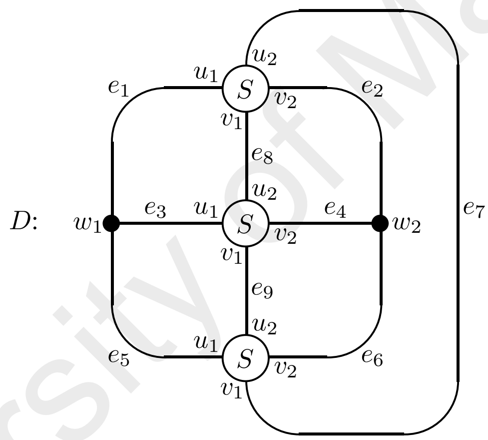


Figure 3.11: Graph D.

PROOF: Let u_1, u_2, v_1 and v_2 be a-vertices of S, w_1 and w_2 be some vertices of D and e_t for $1 \leq t \leq 9$ be some edges of D, as shown in Figure 3.11. We shall denote the top, middle and bottom copies of S by S_t, S_m and S_b, respectively. Let C be a Hamilton cycle of D and let P_i be $S_i \cap C$ where $i \in \{t, m, b\}$. Now, consider the following two cases:

Case (1): S_i is one of the 4-pieces shown in Figure 3.9.

By Lemma 3.9, P_i is of the form $P_{u_1v_2}$ (or by symmetry, $P_{u_2v_1}$) only. If P_t is of the form $P_{u_1v_2}$, then $e_1, e_2 \in E(C)$ and $e_7, e_8 \notin E(C)$. This implies that both P_m and P_b must also take the form $P_{u_1v_2}$. Thus, $e_3, e_4, e_5, e_6 \in E(C)$. Then all three edges incident to each of the vertices w_1 and w_2 are in $E(C)$, which is impossible. If P_t is of the form $P_{u_2v_1}$, then $e_7, e_8 \in E(C)$. This implies that both P_m and P_b must also take the form

$P_{u_2v_1}$. Thus, $e_9 \in E(C)$. However, this says that C contains a smaller cycle, which is impossible too.

Case (2): S_i is the 4-piece shown in Figure 3.10.

By Lemma 3.10, P_i is of the form $P_{u_1v_1}$ (or by symmetry, $P_{u_2v_2}$) only. If P_t is of the form $P_{u_1v_1}$, then $e_1, e_8 \in E(C)$ and $e_2, e_7 \notin E(C)$. This implies that P_b must take the form $P_{u_2v_2}$. Thus, $e_6, e_9 \in E(C)$ and $e_5 \notin E(C)$. Then $e_3, e_4 \in E(C)$. In this case, P_m is a union of two disjoint paths. This contradicts Lemma 3.10. A similar conclusion can be drawn if P_t is of the form $P_{u_2v_2}$.

It follows that no such C exists. Hence, the lemma follows. ■

3.3 $G_3(3, k, l)$

This section concerns the class of all 3R3CPs whose faces are 3-, k- and l-gons where $k \neq l \neq 3$. In addition to showing that $G_3(3, k, l)$ is empty if $11 \leq k < l$, we also construct non-hamiltonian members of $G_3(3, k, l)$ for k and l as stated in Theorem 3.18.

Theorem 3.12 extends Lemma 1 of (Owens, 1984a).

Theorem 3.12. *For $k < l$, $G_3(3, k, l)$ is non-empty only if $k \leq 10$.*

PROOF: Let G be a member of $G_3(3, k, l)$. Since G is 3-connected, no two 3-gons in G have a common edge. Furthermore, a k-gon is adjacent to at most $\left\lfloor \frac{k}{2} \right\rfloor$ 3-gons. Suppose G^* is a graph obtained from G by shrinking all 3-gons to single vertices. No face of G^* has less than $k - \left\lfloor \frac{k}{2} \right\rfloor$ edges. By Lemma 1.3, every 3R3CP must have some faces with less than 6 edges. Thus, $k - \left\lfloor \frac{k}{2} \right\rfloor < 6$, which implies that $k \leq 10$. ■

Theorem 3.13. *There exist non-hamiltonian members of $G_3(3, k, k+1)$ for $6 \leq k \leq 10$.*

PROOF: For $7 \leq k \leq 10$, let G_k be a non-hamiltonian member of $G_3(3, k)$ shown in Figures 2.5–2.8. Note that each G_k contains more than one 3-gon and more than three k-gons. Furthermore, there is at least one 3-gon in G_k that is adjacent to three k-gons.

Suppose G'_k is a graph obtained from G_k by shrinking a 3-gon to a single vertex. Then the three k-gons that are adjacent to the 3-gon in G_k are now reduced to $(k-1)$-gons in G'_k. Thus, $G'_k \in G_3(3, k-1, k)$ for $7 \leq k \leq 10$. Hence, by Lemma 3.1, G'_k is non-hamiltonian.

Let v be a vertex in G_{10} that is shared by three adjacent 10-gons, as shown in Figure 2.8. Suppose G''_{10} is a graph obtained from G_{10} by replacing the vertex v with a 3-gon. Then the three adjacent 10-gons in G_{10} have now become 11-gons in G''_{10}. Thus, $G''_{10} \in G_3(3, 10, 11)$. If G''_{10} is hamiltonian, then G_{10} is hamiltonian by Lemma 3.1. This a contradiction since G_{10} is non-hamiltonian. Hence, G''_{10} is non-hamiltonian. ■

Theorem 3.14. *There exist non-hamiltonian members of* $G_3(3, 4, l)$ *for* $l \geq 7$.

PROOF: The construction of a non-hamiltonian member of $G_3(3, 4, l)$ for $l \geq 7$ is shown in Figure 3.12. Let V and W be the 3-pieces shown in Figure 3.12(a) and (b), respectively. V is obtained by replacing three vertices of T_3 (defined in Lemma 1.9) with two copies of suitable 3-piece X and one copy of suitable 3-piece Y. Here, X and Y are to be defined later. W is obtained from S_a^* (see Figure 1.8(a)) by replacing S_a and a vertex with a copy of V and X, respectively. Note that by Lemma 3.2, we can apply Lemma 1.9 to V and Lemma 1.10 to W to conclude that the edge of attachment labelled e_c is a compulsory edge of attachment of each of the 3-pieces V and W.

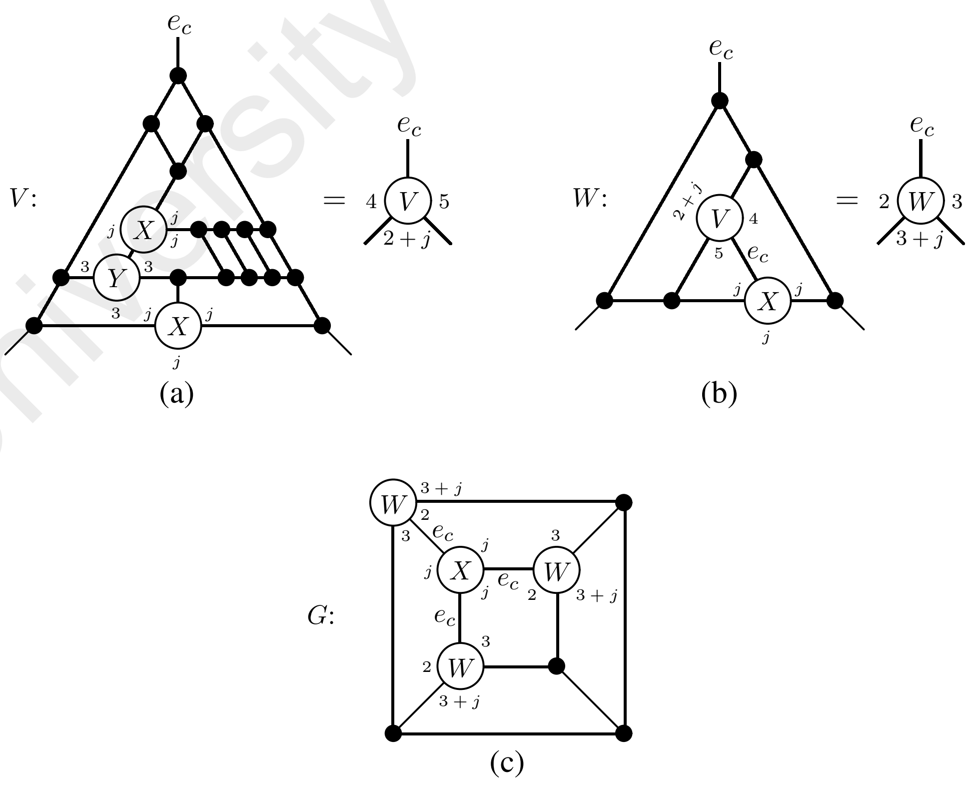


Figure 3.12: 3-pieces V, W and a non-hamiltonian member of $G_3(3, 4, 6 + j)$ for $j \geq 1$.

Let G be the graph shown in Figure 3.12(c). G is obtained by replacing four vertices of a cube with three copies of W and one copy of X, in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces W are incident to X. If X is a 3-piece listed in Table 3.1, it can be checked that X has the property that there is a spanning path through X that contains any two of its edges of attachment. We shrink X to a single vertex and denote the resulting graph by G^*. By Lemma 3.4, G^* is non-hamiltonian. Hence, G is non-hamiltonian by Lemma 3.1. By substituting the 3-pieces X and Y listed in Table 3.1 for those 3-pieces shown in Figure 3.12, we obtain G, a non-hamiltonian member of $G_3(3,4,l)$ for $l \geq 7$. ■

Table 3.1: 3-pieces X and Y for the construction of non-hamiltonian members of $G_3(3,4,l)$ for $l \geq 7$.

l	X	Y
7	A vertex	$S^3_{(3,4,7)}$ (Figure 3.13(a))
8	$S^2_{(3)}$ (Figure 3.1(a))	$S^3_{(3,4,8)}$ (Figure 3.13(b))
9	$S^3_{(4)}$ (Figure 3.1(b))	$S^3_{(3,4,9)}$ (Figure 3.13(c))
$10+i$, $i \geq 0$	$S^{4+i}_{(3,4,10+i)}$ (Figure 3.14(c))	$S^3_{(4)}$ (Figure 3.1(b))

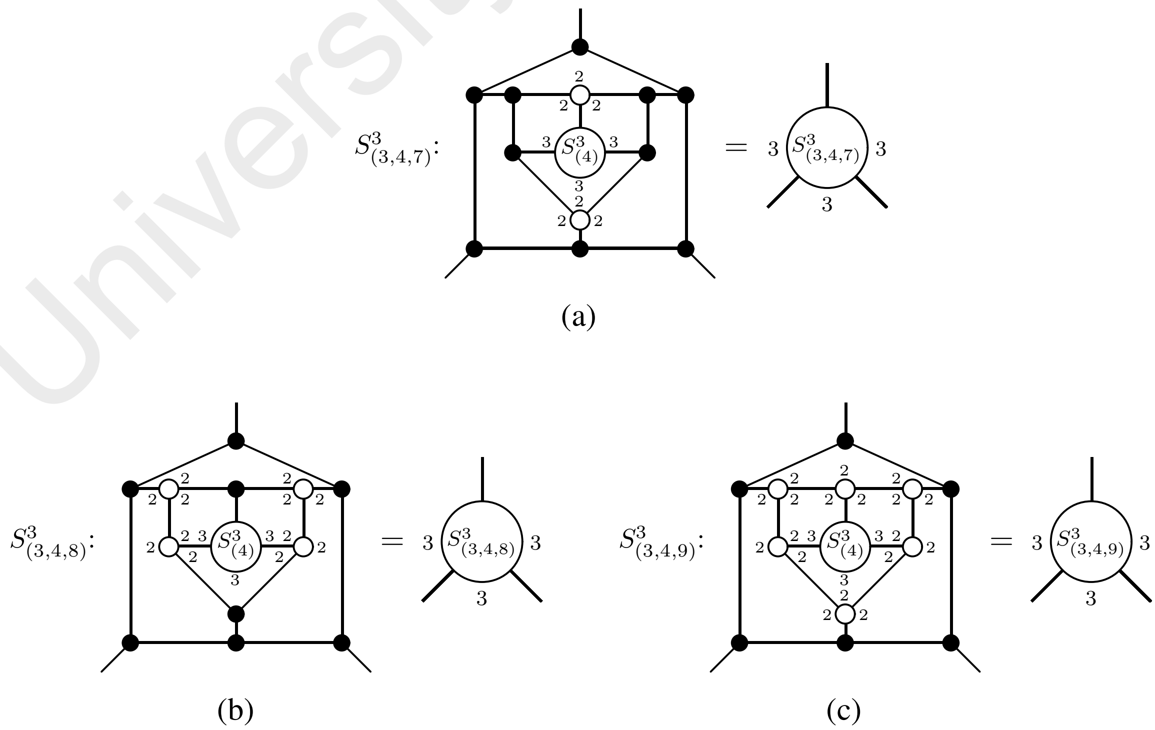


Figure 3.13: 3-pieces $S^3_{(3,4,7)}$, $S^3_{(3,4,8)}$ and $S^3_{(3,4,9)}$.

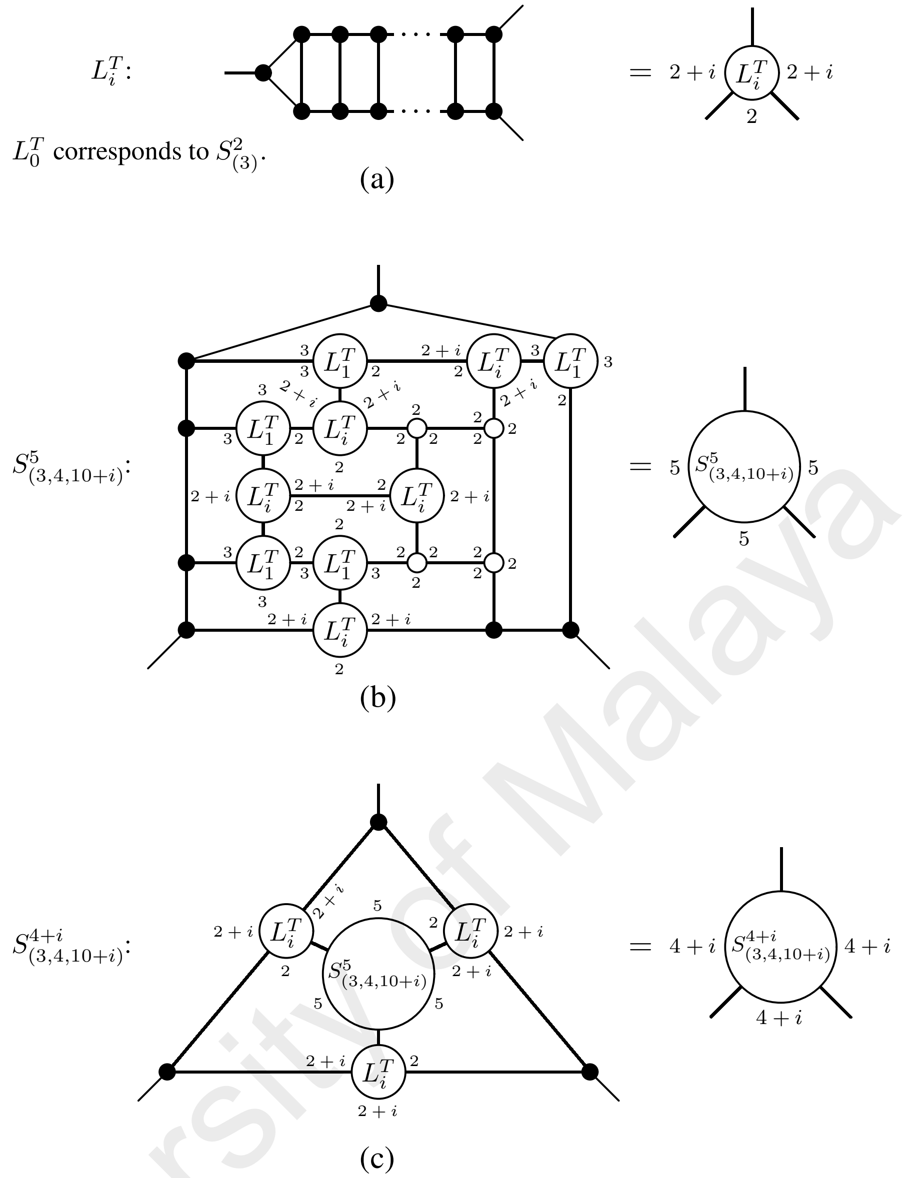


Figure 3.14: 3-pieces L_i^T, $S^5_{(3,4,10+i)}$ and $S^{4+i}_{(3,4,10+i)}$ for $i \geq 0$.

Theorem 3.15. *There exist non-hamiltonian members of* $G_3(3,5,l)$ *for* $l \geq 7$.

PROOF: A non-hamiltonian member of $G_3(3,5,7)$ is obtained from a non-hamiltonian member of $G_3(5,7)$ (Owens, 1981) (see Figure 2.12(b)) by replacing two copies of U with two copies of $S^{3,3,4}_{(3,5,7)}$ shown in Figure 3.15.

For $l \in \{8,9,10\}$, non-hamiltonian members of $G_3(3,5,l)$ are shown in Figure 3.16. Let G_a, G_b and G_c be the graphs shown in Figure 3.16(a), (b) and (c), respectively. G_a is obtained by replacing two vertices of the graph Q (see Figure 1.4) with two copies of $S^{2,5,5}_{(3,8)}$ (see Figure 3.8(b)); G_b is obtained by replacing four vertices of Q with a copy of $S^{3,6,6}_{(3,5)}$ (see Figure 3.4(b)), $S^{2,5,5}_{(3,9)}$ (see Figure 3.8(c)), $S^{4,5,5}_{(3,5)}$ (see Figure 3.17) and $S^2_{(3)}$; and

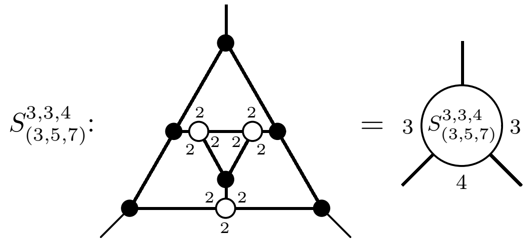
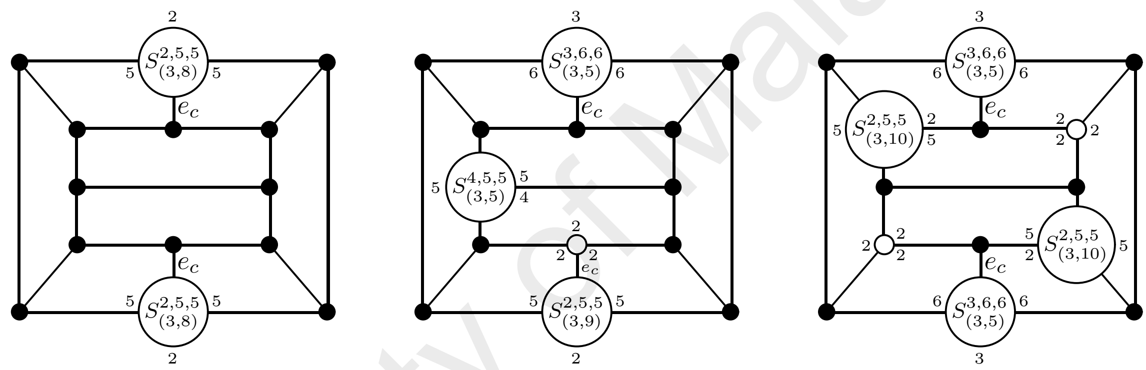


Figure 3.15: 3-piece $S^{3,3,4}_{(3,5,7)}$.

G_c is obtained by replacing six vertices of Q with two copies of $S^{3,6,6}_{(3,5)}$, $S^{2,5,5}_{(3,10)}$ (see Figure 3.8(d)) and $S^{2}_{(3)}$. The edge of attachment labelled e_c is a compulsory edge of attachment of each of the 3-pieces $S^{3,6,6}_{(3,5)}$ (by Lemma 3.5), $S^{2,5,5}_{(3,8)}$ and $S^{2,5,5}_{(3,9)}$ (by Lemma 3.8).



$G_a \in G_3(3,5,8)$ $G_b \in G_3(3,5,9)$ $G_c \in G_3(3,5,10)$

(a) (b) (c)

Figure 3.16: Non-hamiltonian members of $G_3(3,5,l)$ for $l \in \{8,9,10\}$.

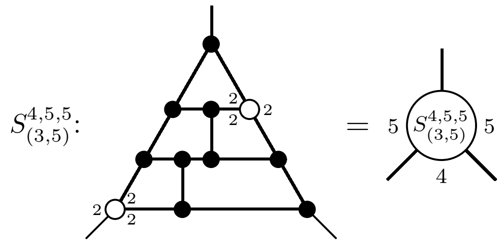


Figure 3.17: 3-piece $S^{4,5,5}_{(3,5)}$.

Suppose C_i is a Hamilton cycle of G_i for $i \in \{a,b,c\}$. Then C_i contains both e_c. By shrinking all 3-pieces to single vertices, G_i is converted into Q and C_i into a Hamilton cycle of Q that contains the edges e_c. However, this contradicts Lemma 1.7. Hence, G_i is non-hamiltonian.

For $l \in \{11, 12, 13\}$, a non-hamiltonian member of $G_3(3, 5, l)$ is shown in Figure 3.18(b). Let W be the 3-piece shown in Figure 3.18(a). W is obtained from S_b^* (see Figure 1.8(b)) by replacing S_b and two other vertices with a copy of $S_{(3,5)}^{2,7,7}$ (see Figure 3.6(b)), $S_{(3)}^2$ and suitable 3-piece X. Here, X is to be defined later. By Lemma 3.6, the edge of attachment labelled e_c' is a compulsory edge of attachment of $S_{(3,5)}^{2,7,7}$. Note that by Lemma 3.2, we can apply Lemma 1.10 to W to conclude that the edge of attachment labelled e_c is a compulsory edge of attachment of W.

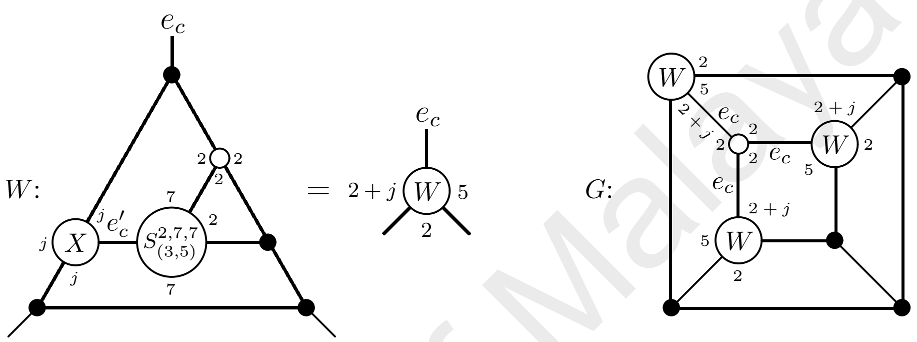


Figure 3.18: 3-piece W and a non-hamiltonian member of $G_3(3, 5, 10 + j)$ for $j \in \{1, 2, 3\}$.

Let G be the graph shown in Figure 3.18(b). G is obtained by replacing four vertices of a cube with three copies of W and one copy of $S_{(3)}^2$, in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces W are incident to $S_{(3)}^2$. It can be checked that $S_{(3)}^2$ has the property that there is a spanning path through $S_{(3)}^2$ that contains any two of its edges of attachment. We shrink $S_{(3)}^2$ to a single vertex and denote the resulting graph by G^*. By Lemma 3.4, G^* is non-hamiltonian. Hence, G is non-hamiltonian by Lemma 3.1. We obtain G, a non-hamiltonian member of $G_3(3, 5, 10+j)$ if X is a vertex for $j = 1$, $X = S_{(3)}^2$ for $j = 2$ and $X = S_{(3,5)}^3$ (see Figure 3.19(a)) for $j = 3$.

For $l \geq 14$, a non-hamiltonian member of $G_3(3, 5, l)$ is shown in Figure 3.20. Let us denote the graph by G. G is obtained from the graph D (see Figure 3.11) by replacing three copies of S and two vertices labelled w_1 and w_2 with three copies of $S_{(5)}^{3,3,9+5i,9+5i}$ (see Figure 3.9(a)) and two copies of suitable 3-piece X. For $i \geq 0$, we obtain G, a member of $G_3(3, 5, 14 + 5i)$ if $X = S_{(3)}^2$ and a member of $G_3(3, 5, 12 + j + 5i)$ if

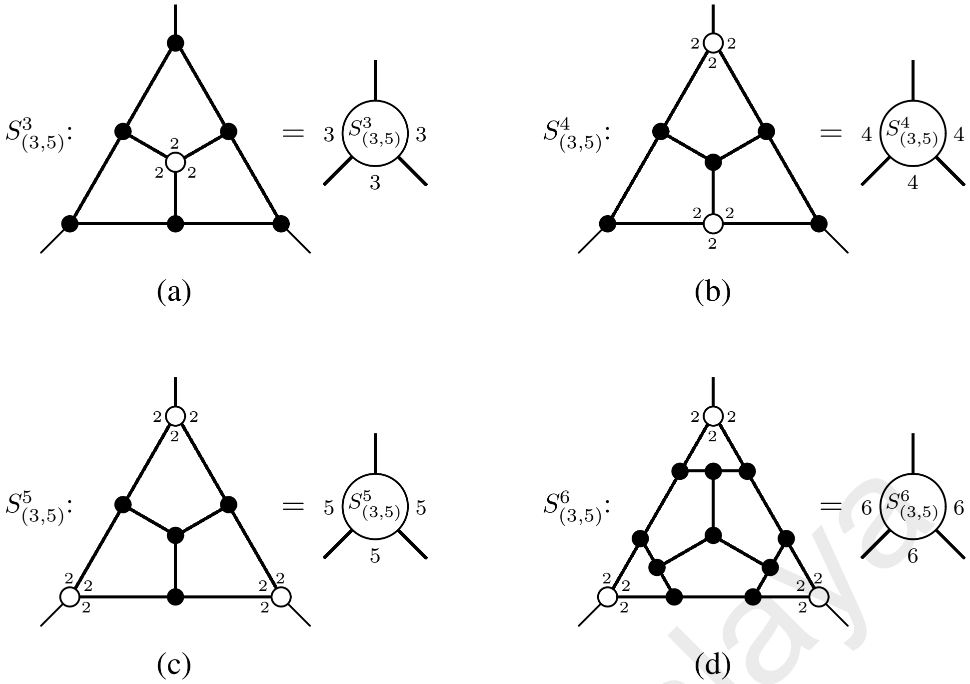


Figure 3.19: 3-pieces $S^j_{(3,5)}$ for $3 \leq j \leq 6$.

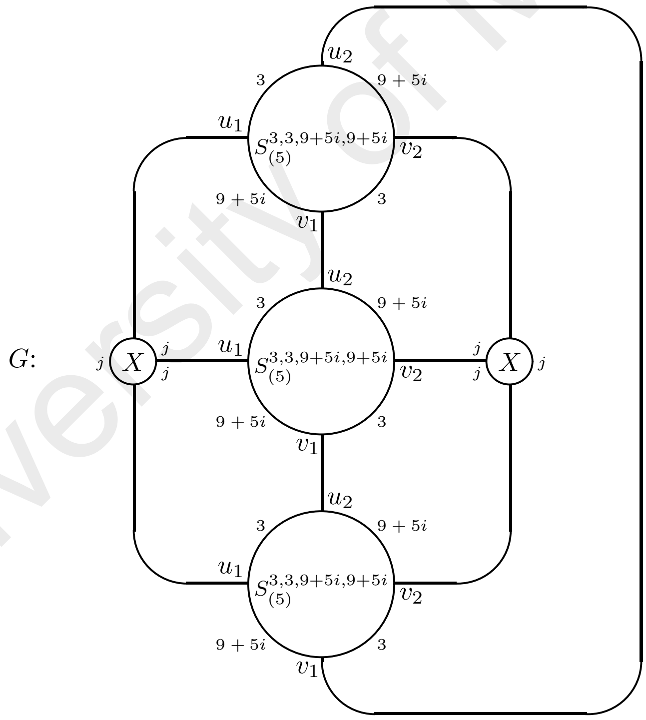


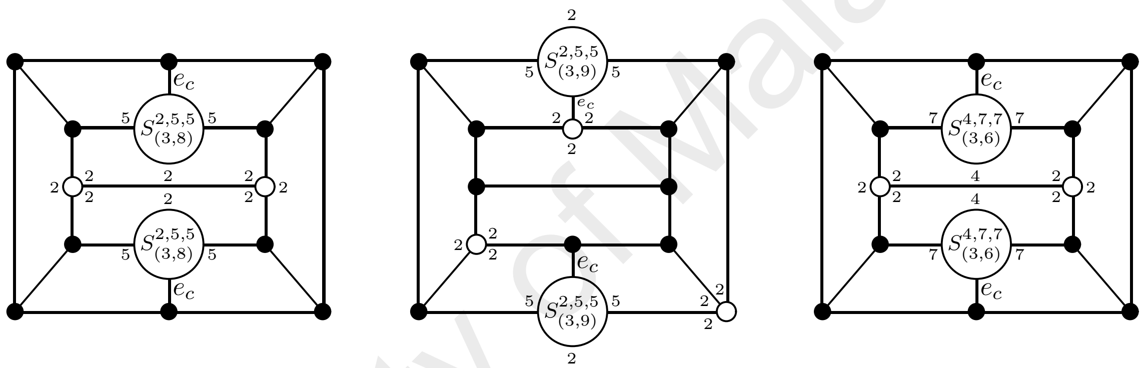
Figure 3.20: A non-hamiltonian member of $G_3(3, 5, 12 + j + 5i)$ for $2 \leq j \leq 6$ and $i \geq 0$.

$X = S^j_{(3,5)}$ (see Figure 3.19) for $3 \leq j \leq 6$. It can be checked that X has the property that there is a spanning path through X that contains any two of its edges of attachment. We shrink both X to single vertices and denote the resulting graph by G^*. By Lemma 3.11, G^* is non-hamiltonian. Hence, G is non-hamiltonian by Lemma 3.1. ■

Theorem 3.16. *There exist non-hamiltonian members of $G_3(3,6,l)$ for $l \geq 7$.*

PROOF: For $l = 7$, the result follows from Theorem 3.13.

For $l \in \{8, 9, 10\}$, non-hamiltonian members of $G_3(3,6,l)$ are shown in Figure 3.21. These graphs are constructed in a similar way to that of non-hamiltonian members of $G_3(3,5,l)$ for $l \in \{8,9,10\}$ (see Figure 3.16). The graph in Figure 3.21(a) is obtained by replacing four vertices of the graph Q (see Figure 1.4) with two copies of $S^{2,5,5}_{(3,8)}$ (see Figure 3.8(b)) and $S^2_{(3)}$; the graph in Figure 3.21(b) is obtained by replacing five vertices of Q with two copies of $S^{2,5,5}_{(3,9)}$ (see Figure 3.8(c)) and three copies of $S^2_{(3)}$; and the graph in Figure 3.21(c) is obtained by replacing four vertices of Q with two copies of $S^{4,7,7}_{(3,6)}$ (see Figure 3.4(c)) and $S^2_{(3)}$.



(a) (b) (c)

Figure 3.21: Non-hamiltonian members of $G_3(3,6,l)$ for $l \in \{8,9,10\}$.

For $l \in \{11, 12, 13, 14\}$, a non-hamiltonian member of $G_3(3,6,l)$ is shown in Figure 3.22(b). The graph is constructed in a similar way to that of non-hamiltonian members of $G_3(3,5,l)$ for $l \in \{11,12,13\}$ (see Figure 3.18). Let W be the 3-piece shown in Figure 3.22(a). W is obtained from S^*_b (see Figure 1.8(b)) by replacing S_b and two other vertices with a copy of $S^{4,7,7}_{(3,6)}$ (see Figure 3.4(c)), $S^2_{(3)}$ and suitable 3-piece X. Here, X is to be defined later.

Let G be the graph shown in Figure 3.22(b). G is obtained by replacing four vertices of a cube with three copies of W and one copy of $S^2_{(3)}$, in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces W are incident to the same vertex. We obtain G, a non-hamiltonian member of $G_3(3,6,10+j)$ if X is a vertex for $j=1$, $X = S^2_{(3)}$ for $j=2$, $X = S^3_{(3,6,13)}$ (see Figure 3.23(d)) for $j=3$ and $X = S^4_{(3,6)}$ (see Figure 3.23(a)) for $j=4$.

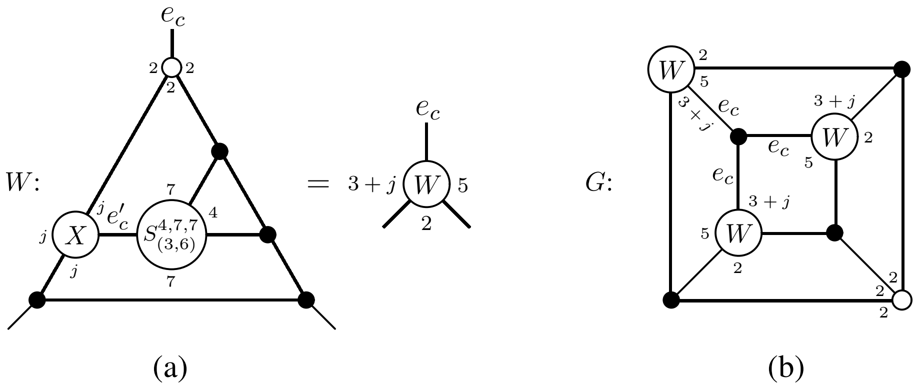


Figure 3.22: 3-piece W and a non-hamiltonian member of $G_3(3, 6, 10 + j)$ for $j \in \{1, 2, 3, 4\}$.

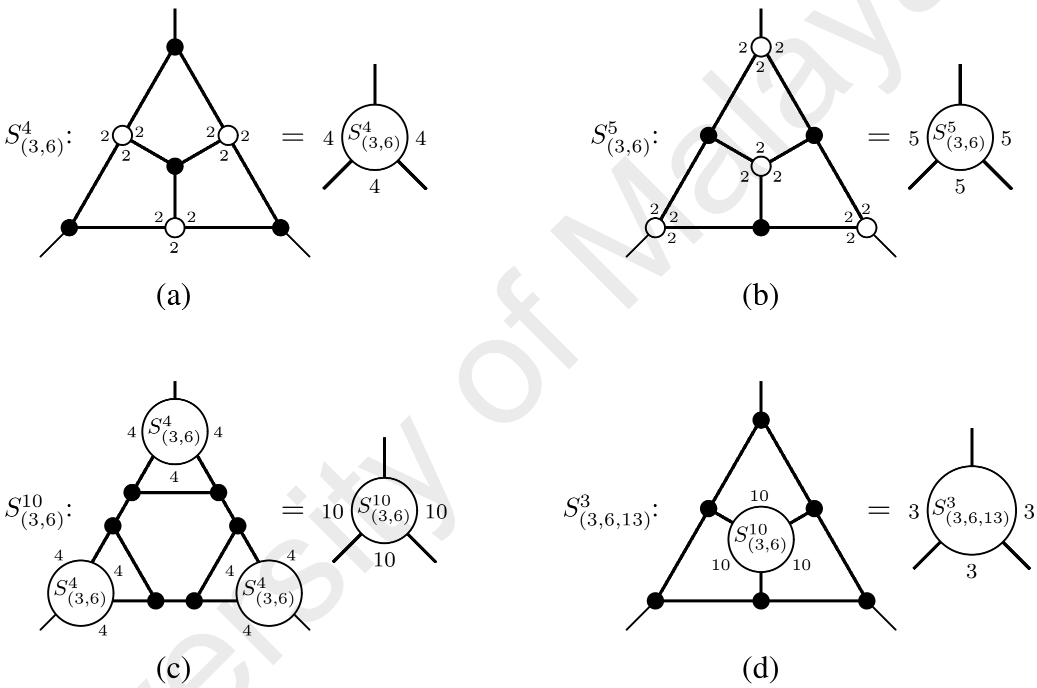


Figure 3.23: 3-pieces $S^4_{(3,6)}$, $S^5_{(3,6)}$, $S^{10}_{(3,6)}$ and $S^3_{(3,6,13)}$.

For $l \geq 15$, non-hamiltonian members of $G_3(3, 6, l)$ are shown in Figure 3.24. These graphs are constructed in a similar way to that of non-hamiltonian members of $G_3(3, 5, l)$ for $l \geq 14$ (see Figure 3.20).

The graph in Figure 3.24(a) is obtained from the graph D (see Figure 3.11) by replacing three copies of S and two vertices labelled w_1 and w_2 with three copies of $S^{3,3,10+6i,11+6i}_{(3,6)}$ (see Figure 3.9(b)), one copy of suitable 3-piece X and one copy of suitable 3-piece Y. For $i \geq 0$, we obtain a non-hamiltonian member of $G_3(3, 6, 15 + 6i)$ if X is a vertex and $Y = S^2_{(3)}$ and a non-hamiltonian member $G_3(3, 6, 18 + 6i)$ if $X = S^4_{(3,6)}$ (see Figure 3.23(a)) and $Y = S^5_{(3,6)}$ (see Figure 3.23(b)).

The graph in Figure 3.24(b) is obtained from D by replacing three copies of S and two vertices labelled w_1 and w_2 with three copies of $S^{4,4,11+6i,11+6i}_{(3,6)}$ (see Figure 3.9(c)) and two copies of suitable 3-piece X. For $i \geq 0$, we obtain a non-hamiltonian member of $G_3(3, 6, 16+6i)$ if X is a vertex, a non-hamiltonian member $G_3(3, 6, 17+6i)$ if $X = S^2_{(3)}$ and a non-hamiltonian member $G_3(3, 6, 15 + j + 6i)$ if $X = S^j_{(3,6)}$ for $j \in \{4, 5\}$. ■

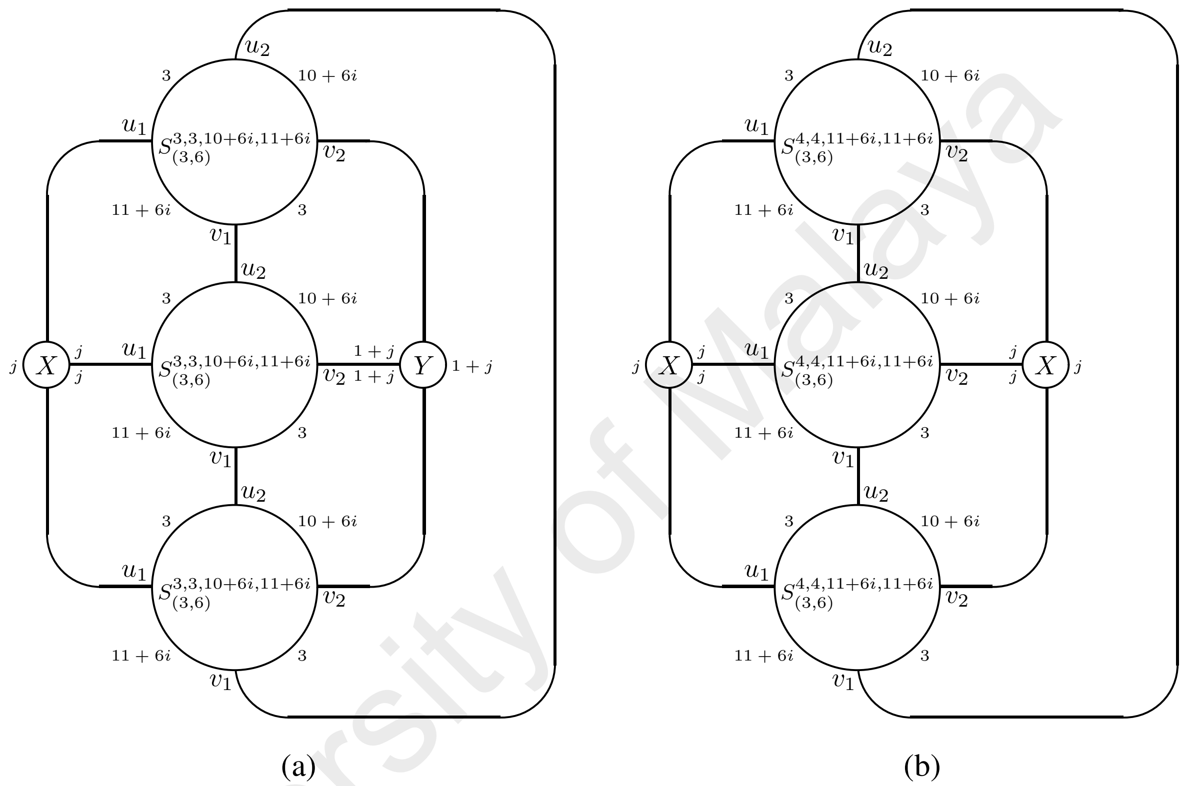


Figure 3.24: Non-hamiltonian members of (a) $G_3(3, 6, 14 + j + 6i)$ for $j \in \{1, 4\}$ and (b) $G_3(3, 6, 15 + j + 6i)$ for $j \in \{1, 2, 4, 5\}$ and $i \geq 0$.

Theorem 3.17. *There exist non-hamiltonian members of $G_3(3, k, l)$ for $k \in \{7, 8, 9, 10\}$ and $l \geq k + 1$.*

PROOF: For $k \in \{7, 8, 9, 10\}$ and $l = k + 1$, the results follow from Theorem 3.13.

For $k \in \{7, 8, 9, 10\}$ and $k + 2 \leq l \leq 3(k - 2)$, except $k = 9$ and $l \in \{12, 15, 21\}$, a non-hamiltonian member of $G_3(3, k, l)$ is shown in Figure 3.25. Let us denote the graph by G. G is obtained by replacing all vertices of a cube with three copies of W, one copy of suitable 3-piece X, one copy of suitable 3-piece Y and three copies of suitable 3-piece Z, in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces W are incident to X. Here, W, X, Y and Z are to be defined later.

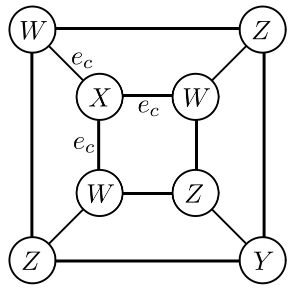


Figure 3.25: A non-hamiltonian member of $G_3(3, k, l)$ for $k \in \{7, 8, 9, 10\}$ and $k + 2 \leq l \leq 3(k - 2)$, except $k = 9$ and $l \in \{12, 15, 21\}$.

If X is a 3-piece listed in Table 3.2, it can be checked that X has the property that there is a spanning path through X that contains any two of its edges of attachment. We shrink X to a single vertex and denote the resulting graph by G^*. By Lemma 3.4, G^* is non-hamiltonian. Hence, G is non-hamiltonian by Lemma 3.1.

Table 3.2: 3-pieces W, X, Y and Z for the construction of non-hamiltonian members of $G_3(3, k, l)$ for $k \in \{7, 8, 9, 10\}$ and $k + 2 \leq l \leq 3(k - 2)$, except $k = 9$ and $l \in \{12, 15, 21\}$.

<table>
<tr><th>k</th><th>l</th><th>W</th><th>X</th><th>Y</th><th>Z</th></tr>
<tr><td rowspan="3">7</td><td>9</td><td rowspan="3">$S^{3,4,6}_{(3,7)}$ (Fig. 3.7(b))</td><td colspan="2">A vertex</td><td rowspan="3">A vertex</td></tr>
<tr><td>10</td><td colspan="2">$S^2_{(3)}$ (Fig. 3.1(a))</td></tr>
<tr><td>$8 + j, 3 \leq j \leq 7$</td><td colspan="2">$S^j_{(3,7)}$ (Fig. 3.26)</td></tr>
<tr><td rowspan="3">8</td><td>10</td><td rowspan="3">$S^{4,4,6}_{(3,8)}$ (Fig. 3.7(c))</td><td>A vertex</td><td>$S^2_{(3)}$ (Fig. 3.1(a))</td><td rowspan="3">A vertex</td></tr>
<tr><td>11</td><td>$S^2_{(3)}$ (Fig. 3.1(a))</td><td>$S^3_{(3,8)}$ (Fig. 3.27(a))</td></tr>
<tr><td>$9 + j, 3 \leq j \leq 9$</td><td>$S^j_{(3,8)}$ (Fig. 3.27)</td><td>$S^{1+j}_{(3,8)}$ (Fig. 3.27)</td></tr>
<tr><td rowspan="8">9</td><td>11</td><td rowspan="2">$S^{4,4,7}_{(3,9)}$ (Fig. 3.7(d))</td><td colspan="2">$S^2_{(3)}$ (Fig. 3.1(a))</td><td rowspan="2">A vertex</td></tr>
<tr><td>13</td><td colspan="2">$S^4_{(3,9,13)}$ (Fig. 3.28(c))</td></tr>
<tr><td>14</td><td>$S^{2,5,5}_{(3,9)}$ (Fig. 3.8(c))</td><td>$S^2_{(3)}$ (Fig. 3.1(a))</td><td>$S^8_{(3,9)}$ (Fig. 3.28(c))</td><td>$S^2_{(3)}$ (Fig. 3.1(a))</td></tr>
<tr><td>16</td><td rowspan="4">$S^{4,4,7}_{(3,9)}$ (Fig. 3.7(d))</td><td colspan="2">$S^7_{(3,9)}$ (Fig. 3.28(a))</td><td rowspan="2">A vertex</td></tr>
<tr><td>17</td><td colspan="2">$S^8_{(3,9)}$ (Fig. 3.28(b))</td></tr>
<tr><td>18</td><td>$S^8_{(3,9)}$ (Fig. 3.28(c))</td><td>$S^7_{(3,9)}$ (Fig. 3.28(b))</td><td>$S^2_{(3)}$ (Fig. 3.1(a))</td></tr>
<tr><td>19</td><td colspan="2">$S^{10}_{(3,9,19)}$ (Fig. 3.28(d))</td><td>A vertex</td></tr>
<tr><td>20</td><td>$S^{2,5,5}_{(3,9)}$ (Fig. 3.8(c))</td><td colspan="2">$S^2_{(3)}$ (Fig. 3.1(a))</td><td>$S^8_{(3,9)}$ (Fig. 3.28(c))</td></tr>
<tr><td rowspan="2">10</td><td>12</td><td rowspan="2">$S^{4,4,8}_{(3,10)}$ (Fig. 3.7(e))</td><td rowspan="2">A vertex</td><td>$S^2_{(3)}$ (Fig. 3.1(a))</td><td rowspan="2">A vertex</td></tr>
<tr><td>$10 + j, 3 \leq j \leq 14$</td><td>$S^j_{(3,10)}$ (Fig. 3.29)</td></tr>
</table>

By substituting the 3-pieces W, X, Y and Z listed in Table 3.2 for those 3-pieces shown in Figure 3.25, we obtain G, a non-hamiltonian member of $G_3(3,k,l)$ for $k \in \{7,8,9,10\}$ and $k+2 \le l \le 3(k-2)$, except $k=9$ and $l \in \{12,15,21\}$.

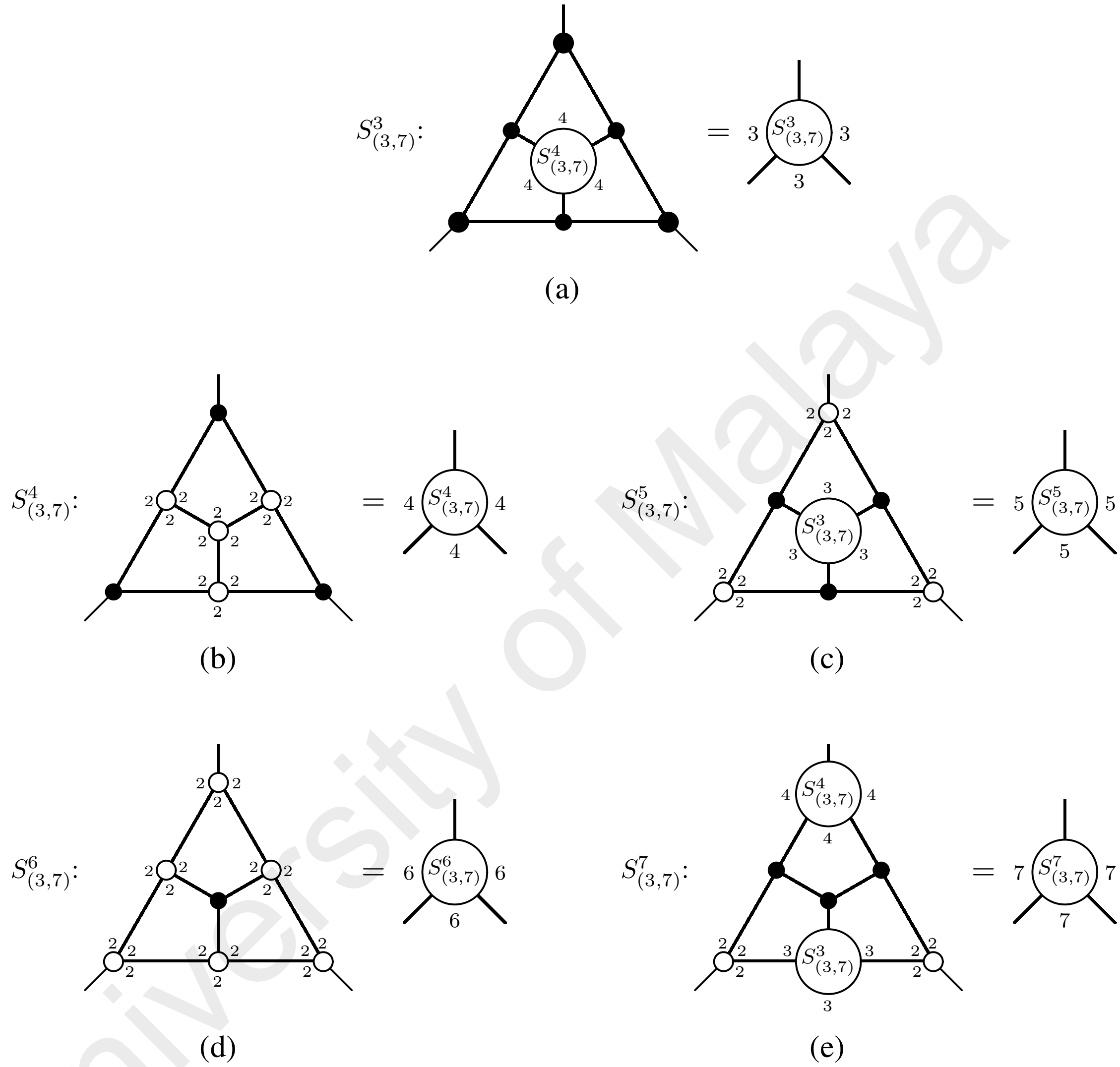


Figure 3.26: 3-pieces $S^j_{(3,7)}$ for $3 \le j \le 7$.

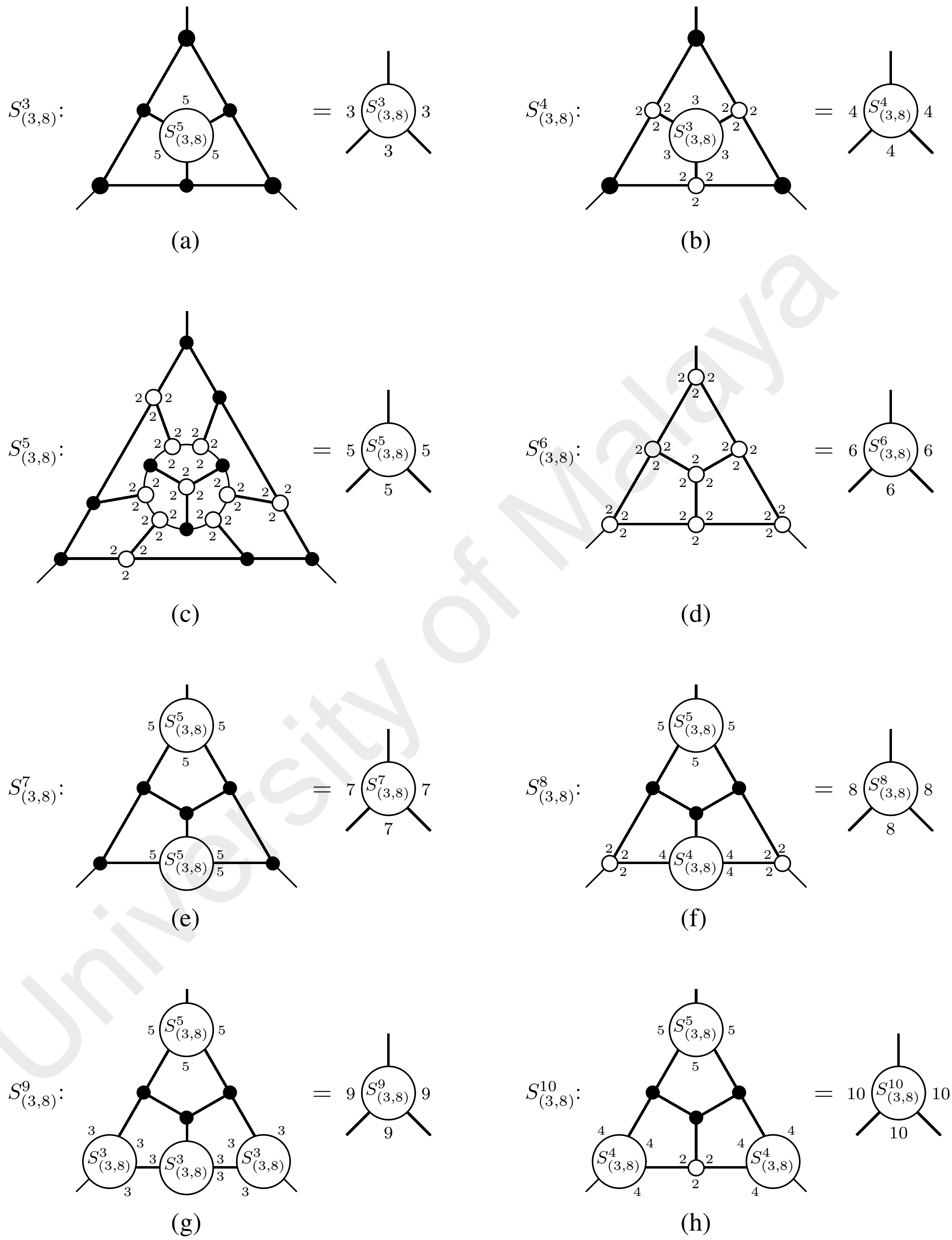


Figure 3.27: 3-pieces $S^j_{(3,8)}$ for $3 \leq j \leq 10$.

Each of the 3-pieces $S^{7}_{(3,9)}$ and $S^{8}_{(3,9)}$, as shown in Figure 3.28(a) and (b), respectively, contains three copies of $S^{2,5,5}_{(3,9)}$, which is shown in Figure 3.8(c).

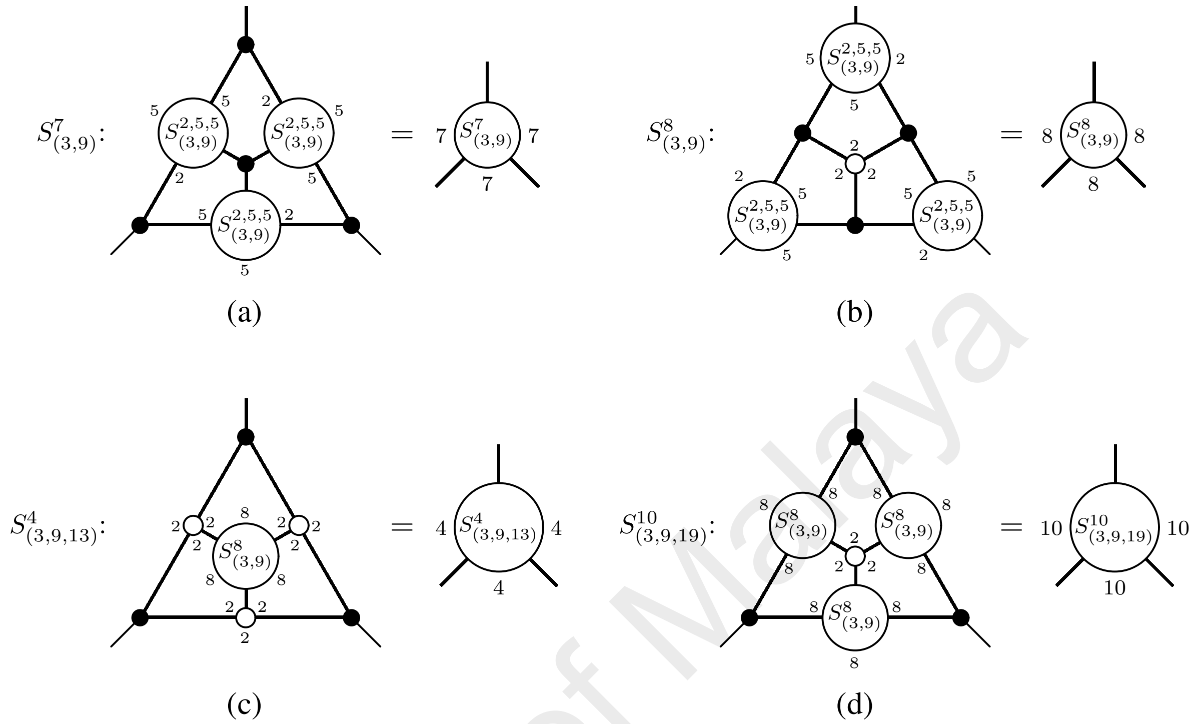


Figure 3.28: 3-pieces $S^{7}_{(3,9)}$, $S^{8}_{(3,9)}$, $S^{4}_{(3,9,13)}$ and $S^{10}_{(3,9,19)}$.

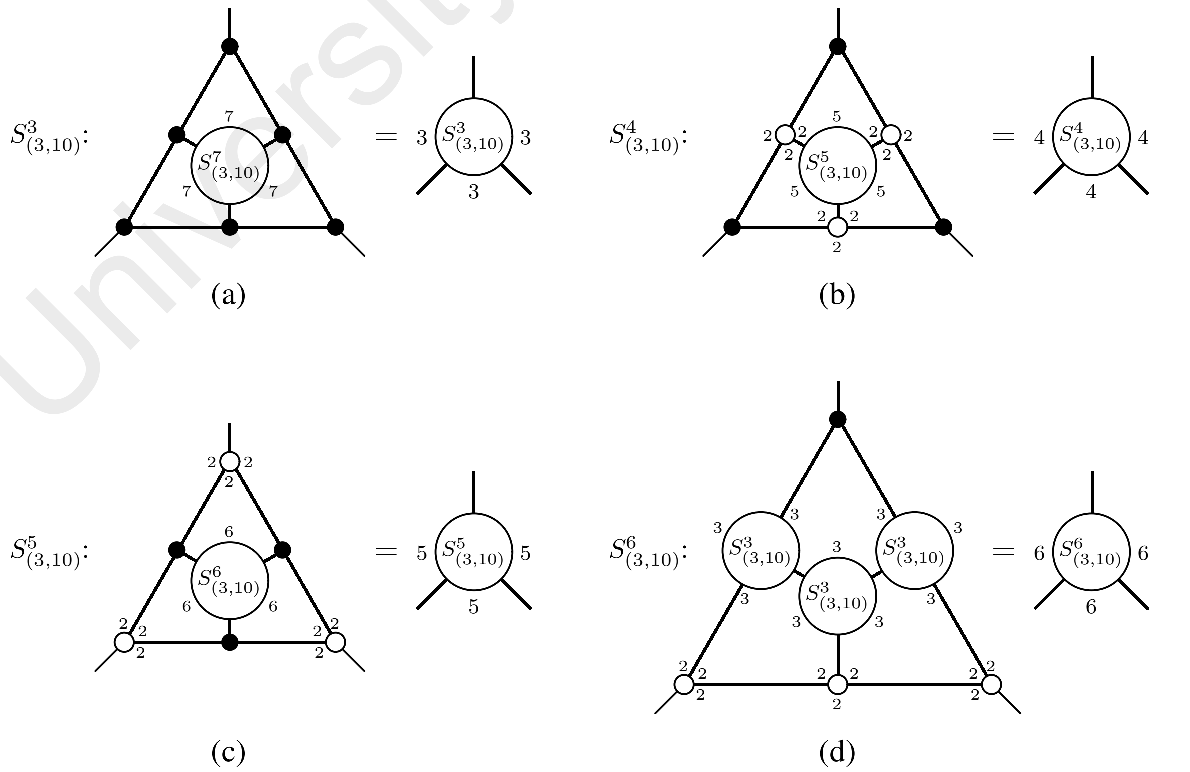


Figure 3.29: 3-pieces $S^{j}_{(3,10)}$ for $3 \leq j \leq 6$.

The 3-piece $S^{7}_{(3,10)}$, as shown in Figure 3.29(e), contains three copies of $S^{2,5,5}_{(3,10)}$, which is shown in Figure 3.8(d).

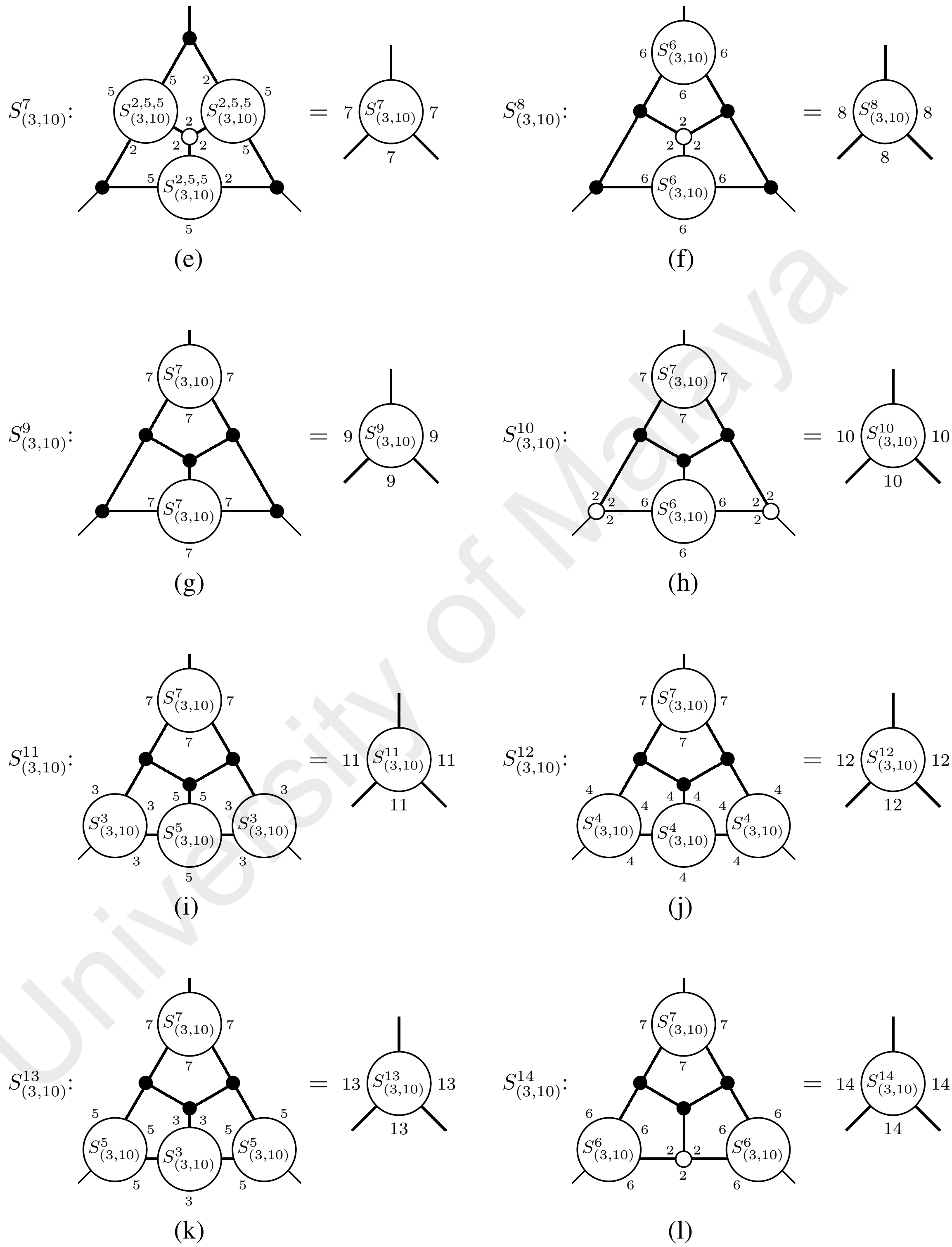


Figure 3.29 (Continued): 3-pieces $S^{j}_{(3,10)}$ for $7 \leq j \leq 14$.

Non-hamiltonian members of $G_3(3,9,12)$, $G_3(3,9,15)$ and $G_3(3,9,21)$ are shown in Figure 3.30. Let G_a, G_b and G_c be the graphs shown in Figure 3.30(a), (b) and (c), respectively. G_a contains three copies of $S^{4,4,7}_{(3,9)}$ (see Figure 3.7(d)) that are adjacent to the vertex labelled v and G_b contains three copies of $S^{2,5,5}_{(3,9)}$ (see Figure 3.8(c)) that are adjacent to $S^2_{(3)}$. By Lemmas 3.7 and 3.8, the edge of attachment labelled e_c is a compulsory edge of attachment of each of the 3-pieces $S^{4,7,7}_{(3,9)}$ and $S^{2,5,5}_{(3,9)}$, respectively. Suppose C_i is a Hamilton cycle of G_i for $i \in \{a,b\}$. Then C_i contains all three e_c and omit v and $S^2_{(3)}$ for $i = a$ and $i = b$, respectively. This is a contradiction. Hence, G_i is non-hamiltonian.

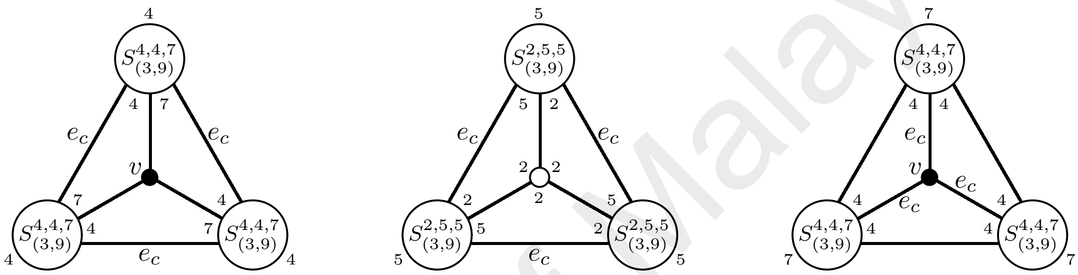


Figure 3.30: Non-hamiltonian members of $G_3(3,9,l)$ for $l \in \{12,15,21\}$.

G_c contains three copies of $S^{4,4,7}_{(3,9)}$ that are placed in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces $S^{4,4,7}_{(3,9)}$ are incident to the vertex labelled v. By Lemma 3.4, G_c is non-hamiltonian.

For $k \in \{7,8,9,10\}$ and $l > 3(k-2)$, a non-hamiltonian member of $G_3(3,k,l)$ is constructed in a similar way to that of non-hamiltonian members of $G_3(3,6,l)$ for $l \geq 15$ (see Figure 3.24). By substituting the 4-piece S, 3-pieces X and Y listed in Table 3.3 for those shown in Figure 3.24, we obtain a non-hamiltonian member of $G_3(3,k,l)$ for $k \in \{7,8,9,10\}$ and $l > 3(k-2)$. ■

Table 3.3: 4-piece S, 3-pieces X and Y for the construction of non-hamiltonian members of $G_3(3,k,l)$ for $k \in \{7,8,9,10\}$ and $l > 3(k-2)$.

<table>
<tr><th>k</th><th>l</th><th>S</th><th>X</th><th>Y</th></tr>
<tr><td rowspan="3">7</td><td>$16+7i, i \geq 0$</td><td rowspan="3">$S^{3,3,12+7i,12+7i}_{(3,7)}$ (Figure 3.9(d))</td><td colspan="2">A vertex</td></tr>
<tr><td>$17+7i, i \geq 0$</td><td colspan="2">$S^2_{(3)}$ (Figure 3.1(a))</td></tr>
<tr><td>$15+j+7i, 3 \leq j \leq 7, i \geq 0$</td><td colspan="2">$S^j_{(3,7)}$ (Figure 3.26)</td></tr>
<tr><td rowspan="3">8</td><td>$19+8i, i \geq 0$</td><td rowspan="3">$S^{4,4,14+8i,14+8i}_{(3,8)}$ (Figure 3.9(e))</td><td colspan="2">A vertex</td></tr>
<tr><td>$20+8i, i \geq 0$</td><td colspan="2">$S^2_{(3)}$ (Figure 3.1(a))</td></tr>
<tr><td>$18+j+8i, 3 \leq j \leq 8, i \geq 0$</td><td colspan="2">$S^j_{(3,8)}$ (Figure 3.27)</td></tr>
<tr><td rowspan="10">9</td><td>$22+9i, i \geq 0$</td><td rowspan="2">$S^{5,5,16+9i,16+9i}_{(3,9)}$ (Figure 3.9(h))</td><td colspan="2">A vertex</td></tr>
<tr><td>$23+9i, i \geq 0$</td><td colspan="2">$S^2_{(3)}$ (Figure 3.1(a))</td></tr>
<tr><td>$24+9i, i \geq 0$</td><td>$S^{4,5,18+9i,18+9i}_{(3,9)}$ (Figure 3.10)</td><td>A vertex</td><td>$S^2_{(3)}$ (Figure 3.1(a))</td></tr>
<tr><td>$25+9i, i \geq 0$</td><td rowspan="2">$S^{4,4,20+9i,20+9i}_{(3,9)}$ (Figure 3.9(f))</td><td colspan="2">A vertex</td></tr>
<tr><td>$26+9i, i \geq 0$</td><td colspan="2">$S^2_{(3)}$ (Figure 3.1(a))</td></tr>
<tr><td>$27+9i, i \geq 0$</td><td>$S^{4,5,15+9i,15+9i}_{(3,9)}$ (Figure 3.9(g))</td><td>$S^8_{(3,9)}$ (Figure 3.28(b))</td><td>$S^7_{(3,9)}$ (Figure 3.28(a))</td></tr>
<tr><td>$28+9i, i \geq 0$</td><td rowspan="2">$S^{5,5,16+9i,16+9i}_{(3,9)}$ (Figure 3.9(h))</td><td colspan="2">$S^7_{(3,9)}$ (Figure 3.28(a))</td></tr>
<tr><td>$29+9i, i \geq 0$</td><td colspan="2">$S^8_{(3,9)}$ (Figure 3.28(b))</td></tr>
<tr><td>$30+9i, i \geq 0$</td><td>$S^{4,5,18+9i,18+9i}_{(3,9)}$ (Figure 3.10)</td><td>$S^7_{(3,9)}$ (Figure 3.28(a))</td><td>$S^8_{(3,9)}$ (Figure 3.28(b))</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="3">10</td><td>$25+10i, i \geq 0$</td><td rowspan="3">$S^{6,6,18+10i,18+10i}_{(3,10)}$ (Figure 3.9(i))</td><td colspan="2">A vertex</td></tr>
<tr><td>$26+10i, i \geq 0$</td><td colspan="2">$S^2_{(3)}$ (Figure 3.1(a))</td></tr>
<tr><td>$24+j+10i, 3 \leq j \leq 10, i \geq 0$</td><td colspan="2">$S^j_{(3,10)}$ (Figure 3.29)</td></tr>
</table>

Combining Theorems 3.13–3.17, we have the following result.

Theorem 3.18. *For $k \neq l$, there exist non-hamiltonian members of $G_3(3,k,l)$ for $4 \leq k \leq 10$ and $l \geq 7$.*

3.4 $G_3(4, k, l)$

This section concerns the class of all 3R3CPs whose faces are 4-, k- and l-gons where $k \neq l \neq 4$. Here, we construct non-hamiltonian members of $G_3(4, k, l)$ for k and l as stated in Theorem 3.22.

As previously mentioned in Section 3.1, Owens (1981) constructed a non-hamiltonian member of $G_3(4, 5, 7)$ (see Figure 3.2). In the following theorem, we construct non-hamiltonian members of $G_3(4, 5, l)$ for $l \geq 8$. Our construction method of non-hamiltonian members of $G_3(4, 5, 8)$ and $G_3(4, 5, 11)$ is different from that of Grinberg's (1968) and Faulkner and Younger's (1974), respectively.

Theorem 3.19. *There exist non-hamiltonian members of* $G_3(4, 5, l)$ *for* $l \geq 8$.

PROOF: A non-hamiltonian member of $G_3(4, 5, 8)$ is shown in Figure 3.31. The graph is constructed in a similar way to that of non-hamiltonian members of $G_3(3, 5, l)$ (see Figure 3.16) and $G_3(3, 6, l)$ (see Figure 3.21) for $l \in \{8, 9, 10\}$. The graph is obtained by replacing two vertices of the graph Q (see Figure 1.4) with a copy of $S^{3,5,5}_{(4,5)}$ (see 3.4(a)) and $S^{4,5,5}_{(4,5,8)}$ (see 3.4(d)).

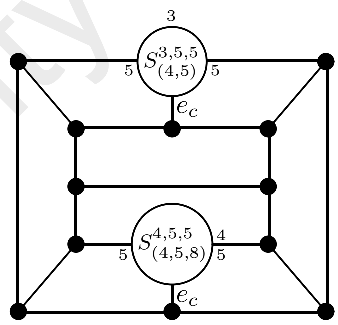


Figure 3.31: A non-hamiltonian member of $G_3(4, 5, 8)$.

For $l \geq 9$, a non-hamiltonian member of $G_3(4, 5, l)$ is shown in Figure 3.32(c). Let V and W be the 3-pieces shown in Figure 3.32(a) and (b), respectively. V is obtained by replacing a vertex of the Tutte's triangle T (see Figure 1.5) with a copy of suitable 3-piece X. W is obtained from S^*_a (see Figure 1.8(a)) by replacing S_a and a vertex with a copy of V and suitable 3-piece Y, respectively. Here, X and Y are to be defined later. Note that by Lemma 3.2, we can apply Lemma 1.8 to V and Lemma 1.10 to W to conclude that the edge of attachment labelled e_c is a compulsory edge of attachment of each of the 3-pieces V and W.

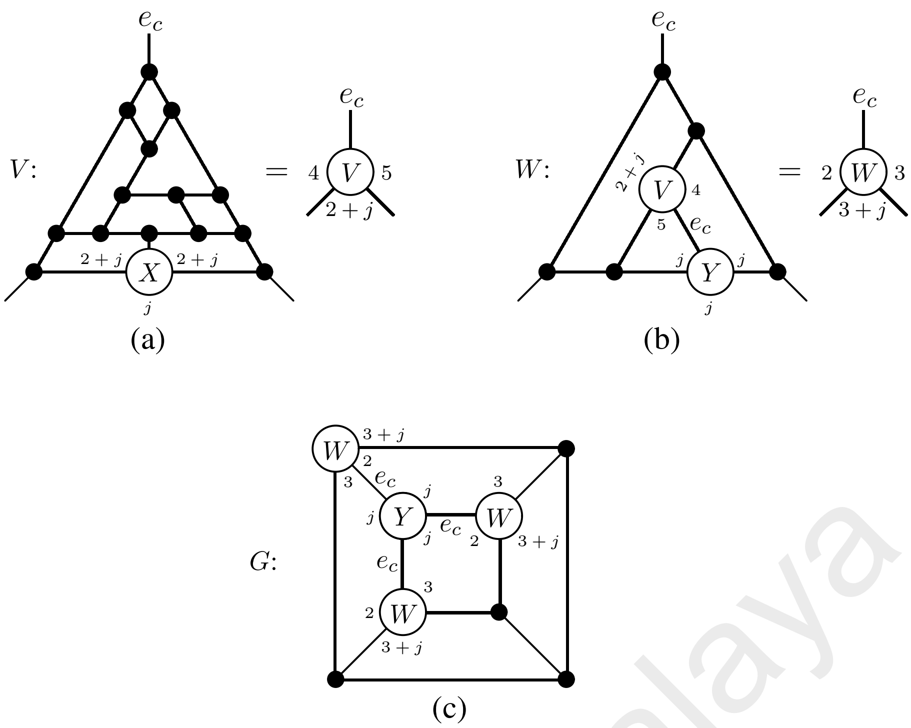


Figure 3.32: 3-pieces V, W and a non-hamiltonian member of $G_3(4, 5, 6 + j)$ for $j \geq 3$.

Let G be the graph shown in Figure 3.32(c). G is obtained by replacing four vertices of a cube with three copies of W and one copy of Y, in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces W are incident to Y. If Y is a 3-piece listed in Table 3.4, it can be checked that Y has the property that there is a spanning path through Y that contains any two of its edges of attachment. We shrink Y to a single vertex and denote the resulting graph by G^*. By Lemma 3.4, G^* is non-hamiltonian. Hence, G is non-hamiltonian by Lemma 3.1. By substituting the 3-pieces X and Y listed in Table 3.4 for those 3-pieces shown in Figure 3.32, we obtain G, a non-hamiltonian member of $G_3(4, 5, l)$ for $l \geq 9$. ■

Table 3.4: 3-pieces X and Y for the construction of non-hamiltonian members of $G_3(4, 5, l)$ for $l \geq 9$.

l	X	Y
9	$S^{3,5,5}_{(4,5)}$ (Figure 3.4(a))	$S^{3}_{(4)}$ (Figure 3.1(b))
10	$S^{4,6,6}_{(4,5,10)}$ (Figure 3.33(a))	$S^{4}_{(5)}$ (Figure 3.1(c))
$11+i, i \geq 0$	$S^{5+i,7+i,7+i}_{(4,5,11+i)}$ (Figure 3.33(d))	$S^{5+i}_{(4,5,11+i)}$ (Figure 3.33(c))

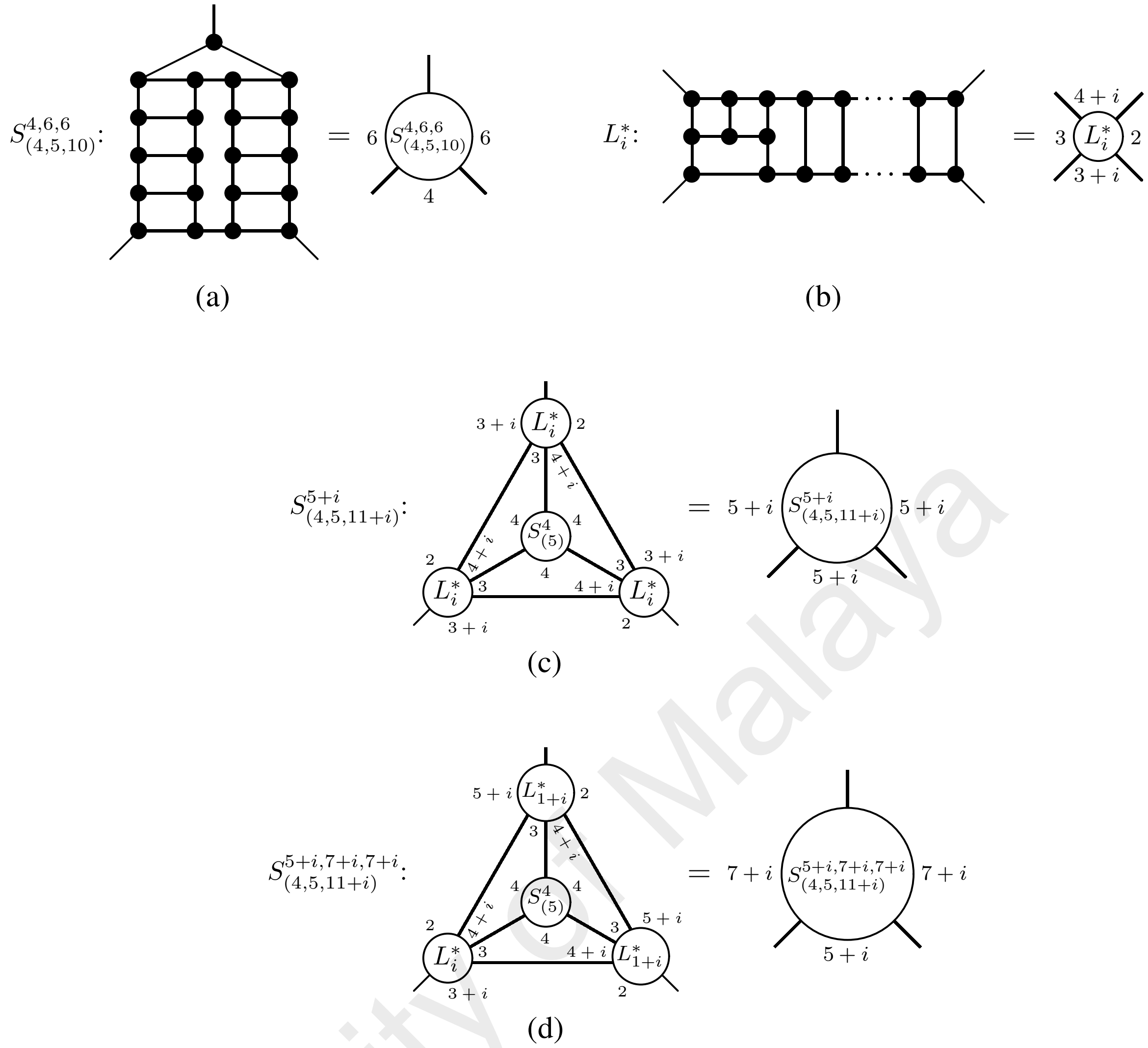


Figure 3.33: 3-piece $S^{4,6,6}_{(4,5,10)}$, 4-piece L_i^*, 3-pieces $S^{5+i}_{(4,5,11+i)}$ and $S^{5+i,7+i,7+i}_{(4,5,11+i)}$ for $i \geq 0$.

Theorem 3.20. *For $k \neq l \neq 4$, there exist non-hamiltonian members of*

1. *$G_3(4, k, l)$ for $k \in \{7, 9\}$ and $l \geq 3$; and $(k, l) \neq (7, 5)$; and*

2. *$G_3(4, k, k+5)$ and $G_3(4, k+2, k+5)$ for $k \geq 3$.*

PROOF: Non-hamiltonian members of $G_3(4, k, l)$ for $k \in \{7, 9\}$ and $(k, l) \neq (7, 5)$ can be obtained from Theorems 3.14 and 3.19 for $l = 3$ and 5, respectively. Similarly, non-hamiltonian members of $G_3(4, k, k+5)$ and $G_3(4, k+2, k+5)$ can be obtained from Theorems 3.14 and 3.19, respectively, for $k = 3$. For the remaining cases, the graphs are constructed in a similar way to that of non-hamiltonian members of $G_3(3, 4, l)$ for $l \geq 7$ (see Figure 3.12). By substituting the 3-pieces X and Y listed in Table 3.5 for those

3-pieces shown in Figure 3.12, we obtain a non-hamiltonian member of $G_3(4, k, l)$ for $k \in \{7, 9\}$ and $l \geq 6$; $G_3(4, k, k+5)$ and $G_3(4, k+2, k+5)$ for $k \geq 5$.

Table 3.5: 3-pieces X and Y for the construction of non-hamiltonian members of $G_3(4, k, l)$ for $k \in \{7, 9\}$ and $l \geq 6$; $G_3(4, k, k+5)$ and $G_3(4, k+2, k+5)$ for $k \geq 5$.

k	l	X	Y
7	$6+i, i \geq 0$	A vertex	$S^3_{(4,6+i)}$ (Figure 3.34(b) and (d))
9		$S^3_{(4)}$ (Figure 3.1(b))	
$4+i, i \geq 1$	$9+i, i \geq 1$	$S^{3+i}_{(4,4+i)}$ (Figure 3.34(a))	$S^3_{(4)}$ (Figure 3.1(b))
$6+i, i \geq 1$		$S^{3+i}_{(4,6+i)}$ (Figure 3.34(c))	

Each of the 3-pieces $S^{3+i}_{(4,4+i)}$ and $S^{3+i}_{(4,6+i)}$ for $i \geq 1$, as shown in Figure 3.34(a) and (c), respectively, contains three copies of L_i, which is shown in Figure 1.7. ■

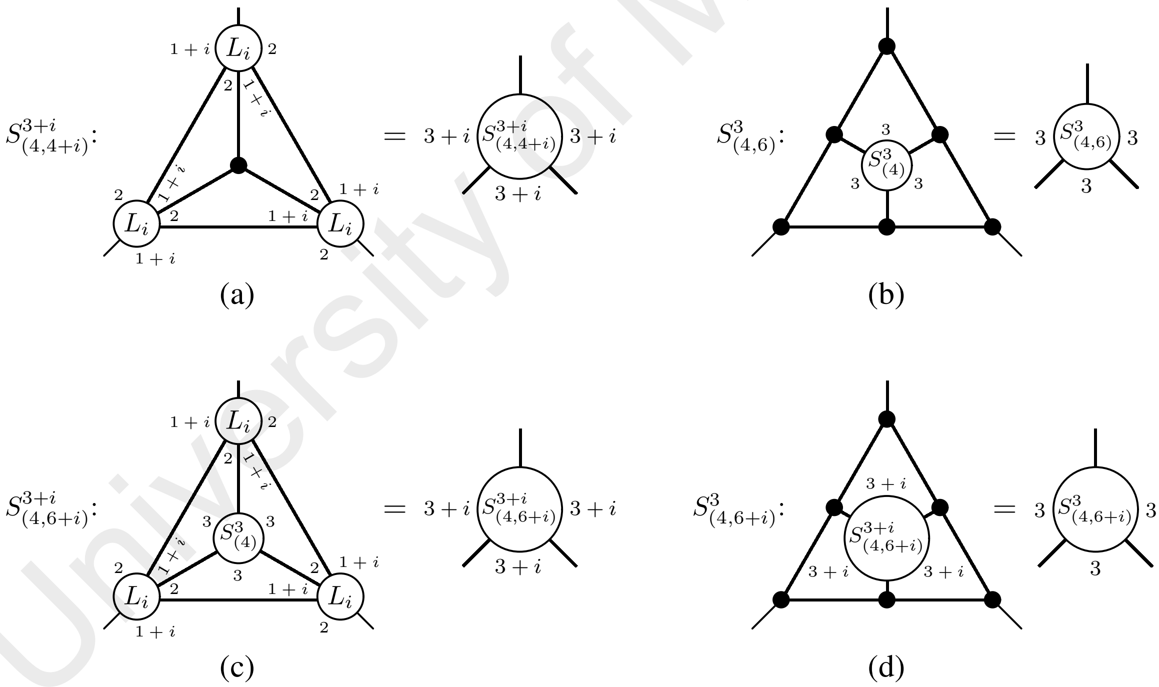
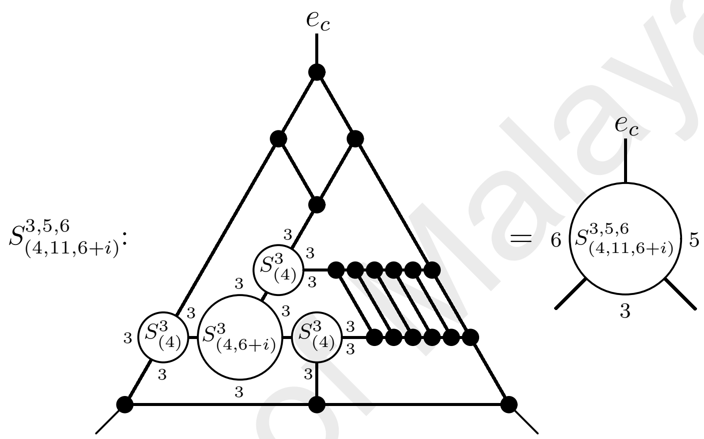


Figure 3.34: 3-pieces $S^{3+i}_{(4,4+i)}$, $S^3_{(4,6)}$, $S^{3+i}_{(4,6+i)}$ and $S^3_{(4,6+i)}$ for $i \geq 1$.

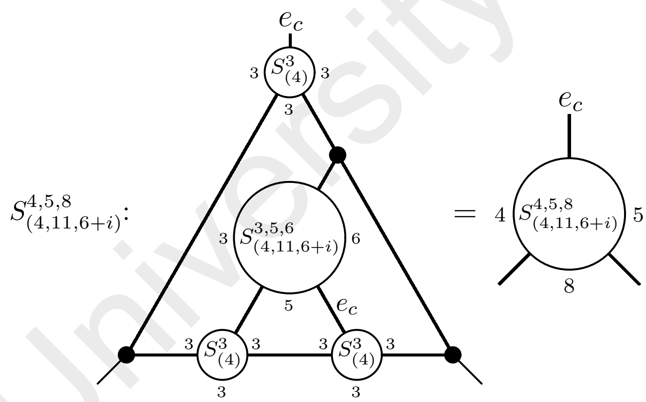
Theorem 3.21. *There exist non-hamiltonian members of $G_3(4, 11, l)$ for $l \geq 3$ and $l \neq \{4, 11\}$.*

PROOF: For $l = 3$, 5 and 6, non-hamiltonian members of $G_3(4, 11, l)$ can be obtained from Theorems 3.14, 3.19 and 3.20(2), respectively.

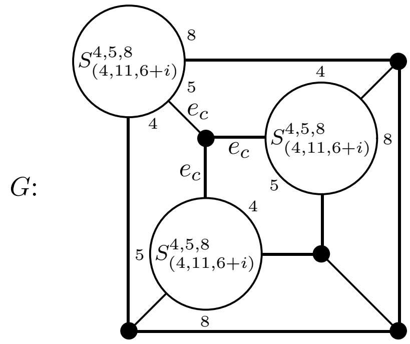
For $l \geq 7$, a non-hamiltonian member of $G_3(4, 11, l)$ is shown in Figure 3.35(c). The graph is constructed in a similar way to that of non-hamiltonian members of $G_3(3, 4, l)$ for $l \geq 7$ (see Figure 3.12). The 3-piece $S^{3,5,6}_{(4,11,6+i)}$, as shown in Figure 3.35(a), is obtained by replacing four vertices of T_5 (defined in Lemma 1.9) with one copy of $S^3_{(4,6+i)}$ (see Figure 3.34(d)) and three copies of $S^3_{(4)}$ (see Figure 3.1(b)). The 3-piece $S^{4,5,8}_{(4,11,6+i)}$, as shown in Figure 3.35(b), is obtained from S^*_a (see Figure 1.8(a)) by replacing S_a and three other vertices with one copy of $S^{3,5,6}_{(4,11,6+i)}$ and three copies of $S^3_{(4)}$, respectively.



(a)



(b)



(c)

Figure 3.35: 3-pieces $S^{3,5,6}_{(4,11,6+i)}$, $S^{4,5,8}_{(4,11,6+i)}$ and a non-hamiltonian member of $G_3(4, 11, 6+i)$ for $i \geq 1$.

The non-hamiltonian graph G, as shown in Figure 3.35(c), is obtained by replacing three vertices of a cube with three copies of $S^{4,5,8}_{(4,11,6+i)}$, in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces $S^{4,5,8}_{(4,11,6+i)}$ are incident to the same vertex. ■

Combining Theorems 3.14 and 3.19–3.21, we have the following result.

Theorem 3.22. *For $k \neq l \neq 4$, there exist non-hamiltonian members of*

1. $G_3(4,k,l)$ *for* $k \in \{3,5,7,9,11\}$ *and* $l \geq 8$*; and* $(k,l) \in \{(3,7),(6,7),(6,9),(6,11)\}$*; and*

2. $G_3(4,k,k+5)$ *and* $G_3(4,k+2,k+5)$ *for* $k \geq 3$.

3.5 $G_3(5,k,l)$

This section concerns the class of all 3R3CPs whose faces are 5-, k- and l-gons where $k \neq l \neq 5$. Here, we construct non-hamiltonian members of $G_3(5,k,l)$ for k and l as stated in Theorem 3.25.

Theorem 3.23. *There exists non-hamiltonian members of* $G_3(5,6,l)$ *for* $l \geq 7$.

PROOF: For $l \in \{7,9\}$, non-hamiltonian members of $G_3(5,6,l)$ are shown in Figure 3.36. These graphs are constructed in a similar way to that of non-hamiltonian members of $G_3(3,5,l)$ (see Figure 3.16) and $G_3(3,6,l)$ (see Figure 3.21) for $l \in \{8,9,10\}$ and $G_3(4,5,8)$ (see Figure 3.31). The graphs, as shown in Figure 3.36(a) and (b), are obtained by replacing two vertices of the graph Q (see Figure 1.4) with two copies of $S^{2,4,4}_{(5,6,7)}$ (Figure 3.8(a)) and $S^{3,3,5}_{(5,6)}$ (Figure 3.7(a)), respectively.

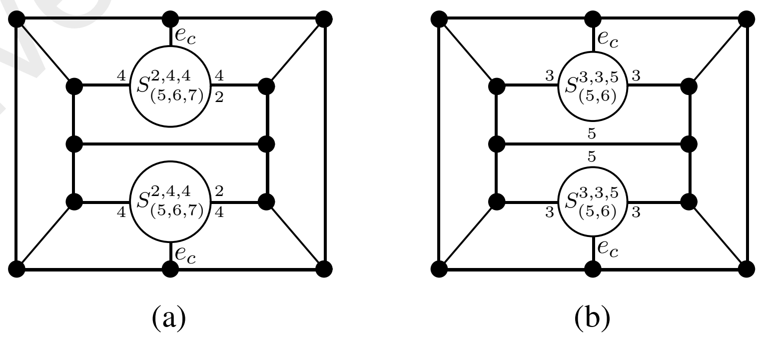


Figure 3.36: Non-hamiltonian members of $G_3(5,6,7)$ and $G_3(5,6,9)$.

For $l \geq 8$ and $l \neq 9$, a non-hamiltonian member of $G_3(5,6,l)$ is shown in Figure 3.37. The graph is constructed in a similar way to that of non-hamiltonian members of $G_3(3,k,l)$ for $k \in \{7,8,9,10\}$ and $k+2 \leq l \leq 3(k-2)$, except $k = 9$ and $l \in \{12,15,21\}$ (see Figure 3.25). The non-hamiltonian graph G, as shown in Figure 3.37, is obtained by replacing five vertices of a cube with three copies of $S^{3,3,5}_{(5,6)}$ (see Figure 3.7(a)) and two copies of suitable 3-piece X, in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces $S^{3,3,5}_{(5,6)}$ are incident to one of the 3-pieces X. By substituting the 3-pieces X listed in Table 3.6 for those 3-pieces shown in Figure 3.37, we obtain G, a non-hamiltonian member of $G_3(5,6,l)$ for $l \geq 8$ and $l \neq 9$.

G:

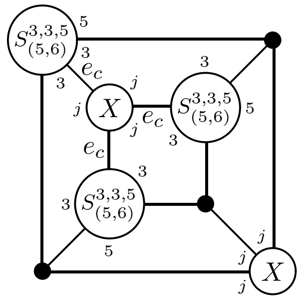


Figure 3.37: A non-hamiltonian member of $G_3(5,6,7+j)$ for $j \geq 1$ and $j \neq 2$.

Table 3.6: 3-piece X for the construction of non-hamiltonian members of $G_3(5,6,l)$ for $l \geq 8$ and $l \neq 9$.

l	X
8	A vertex
10	$S^{3}_{(5,6,10)}$ (Figure 3.40(a))
11	$S^{4}_{(5)}$ (Figure 3.1(c))
12	$S^{5}_{(5,6,12)}$ (Figure 3.40(b))
$13+i$, $i \geq 0$	$S^{6+i}_{(5,6,13+i)}$ (Figure 3.40(c))

$S^{7}_{(5,6)}$:

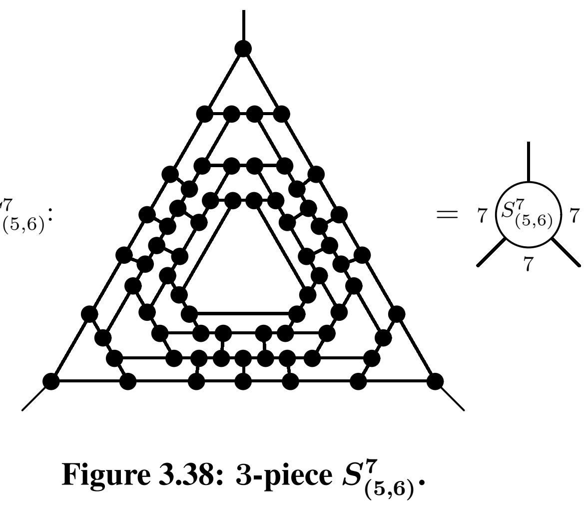


Figure 3.38: 3-piece $S^{7}_{(5,6)}$.

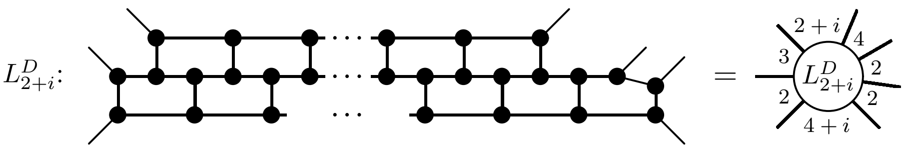


Figure 3.39: 7-piece L^D_{2+i} for $i \geq 0$.

Each of the 3-pieces shown in Figure 3.40 contains a copy of $S^7_{(5,6)}$, which is shown in Figure 3.38. The 3-piece $S^{6+i}_{(5,6,13+i)}$ for $i \geq 0$, as shown in Figure 3.40(c), contains three copies of L^D_{2+i}, which is shown in Figure 3.39. ■

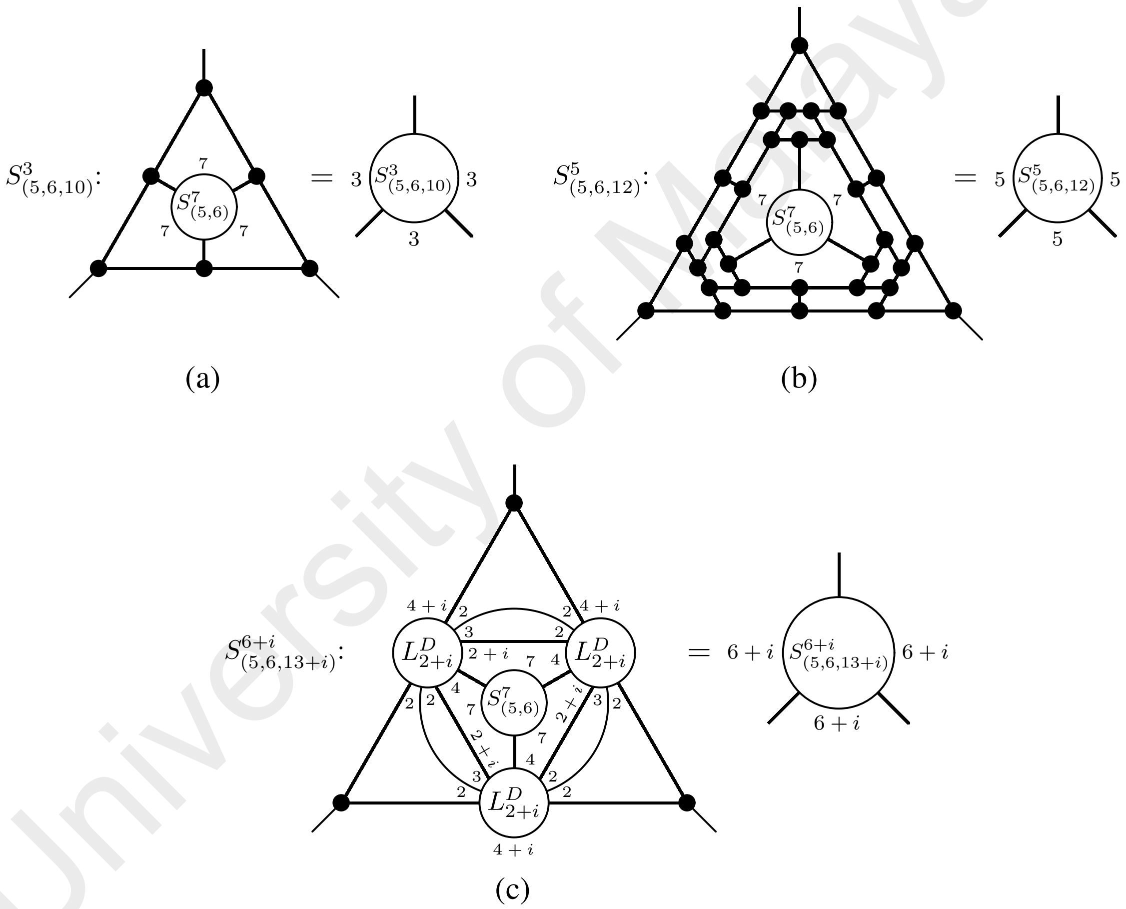


Figure 3.40: $S^3_{(5,6,10)}$, $S^5_{(5,6,12)}$ and $S^{6+i}_{(5,6,13+i)}$ for $i \geq 0$.

Theorem 3.24. *There exists non-hamiltonian members of $G_3(5,k,l)$ for $7 \leq k < l$.*

PROOF: For $k \geq 7$, let G_k be a non-hamiltonian member of $G_3(5,k)$ shown in Figures 2.12–2.23, except Figure 2.13. Note that each G_k contains more than three 5-gons. Furthermore, there is at least one vertex in G_k that is shared by three adjacent 5-gons. Let us denote one of these vertices by v.

Suppose G_k^* is a graph obtained from G_k by replacing the vertex v with a copy of $S^4_{(5)}$ (see Figure 3.1(c)). Then the three adjacent 5-gons in G_k has now become 8-gons in G_k^*. Thus, $G_k^* \in G_3(5, 8, k)$ for $k \geq 7$. If G_k^* is hamiltonian, then G_k is hamiltonian by Lemma 3.1. This is a contradiction since G_k is non-hamiltonian. Hence, G_k^* is non-hamiltonian.

Similarly, we obtain non-hamiltonian members of $G_3(5, 9, k)$ and $G_3(5, 10+i, k)$ for $i \geq 0$ and $k \geq 7$ by replacing the vertex v in each G_k with a copy of $S^5_{(5,9)}$ (see Figure 3.41(a)) and $S^{6+i}_{(5,10+i)}$ (see Figure 3.41(b)), respectively. ■

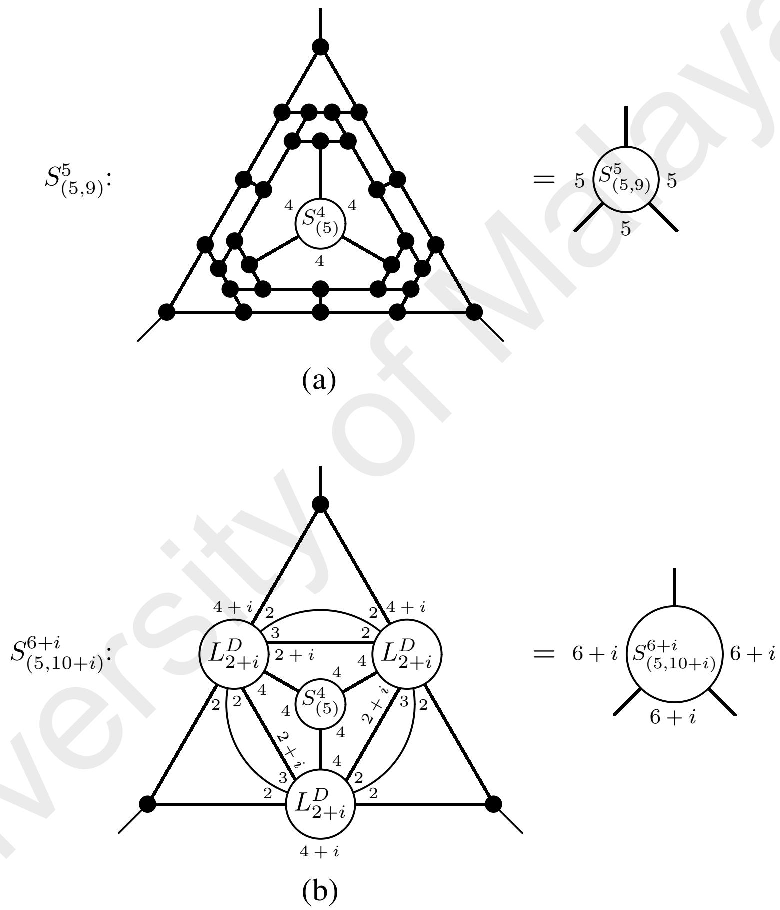


Figure 3.41: 3-pieces $S^5_{(5,9)}$ and $S^{6+i}_{(5,10+i)}$ for $i \geq 0$.

Combining Theorems 3.15, 3.19, 3.23 and 3.24, we have the following result.

Theorem 3.25. *There exist non-hamiltonian members of $G_3(5, k, l)$ for $k = 3$ and $l \geq 7$; $k = 4$ and $l \geq 8$; and $6 \leq k < l$.*

Combining Theorems 3.18, 3.22 and 3.25, we have the following result.

Theorem 3.26. *For $h \neq k \neq l$, there exist non-hamiltonian members of*

1. $G_3(3, k, l)$ *for* $4 \leq k \leq 10$ *and* $l \geq 7$;

2. (a) $G_3(4, k, l)$ *for* $k \in \{3, 5, 7, 9, 11\}$ *and* $l \geq 8$; *and* $(k, l) \in \{(3, 7), (6, 7), (6, 9), (6, 11)\}$;

   (b) $G_3(4, k, k + 5)$ *and* $G_3(4, k + 2, k + 5)$ *for* $k \geq 3$; *and*

3. $G_3(5, k, l)$ *for* $k = 3$ *and* $l \geq 7$; $k = 4$ *and* $l \geq 8$; *and* $6 \leq k < l$.

Remark. *The following are some open problems:*

1. *Is every member of* $G_3(3, 4, 5)$, $G_3(3, 4, 6)$, $G_3(3, 5, 6)$ *and* $G_3(4, 5, 6)$ *hamiltonian?*

2. *Is every member of* $G_3(4, 2k, 2l)$ *for* $3 \leq k < l$ *hamiltonain?*

CHAPTER 4: 4-REGULAR 3-CONNECTED PLANAR GRAPHS WITH FACES OF GIVEN TYPES

4.1 Introduction

This chapter deals with the hamiltonicity of $G_4(h, k)$ and $G_4(h, k, l)$, $h \neq k \neq l$, which are the classes of all 4-regular 3-connected planar graphs (4R3CPs) whose faces are of only two and three types, respectively. It follows easily from Lemma 1.3 that every 4R3CP must have some 3-gons.

We construct a number of non-hamiltonian members of $G_4(3, k)$ and $G_4(3, k, l)$. These graphs will be presented in Sections 4.2 and 4.3, each of which focuses on the classes $G_4(3, k)$ and $G_4(3, k, l)$, respectively. In Section 4.2, we will also present some known answers to Question 4.1 posed by Grünbaum and Zaks (1974), which concerns the existence of Hamilton cycles in 4R3CPs with only two types of faces.

Question 4.1. *(Grünbaum & Zaks, 1974) Do Hamilton cycles exist in all 4R3CPs whose faces are of only two types?*

In this chapter, an a-vertex is called an a′*-vertex* if it is of degree three and incident to exactly one edge of attachment, whereas it is called an a″*-vertex* if it is of degree two and incident to exactly two edges of attachment (see Figure 4.1).

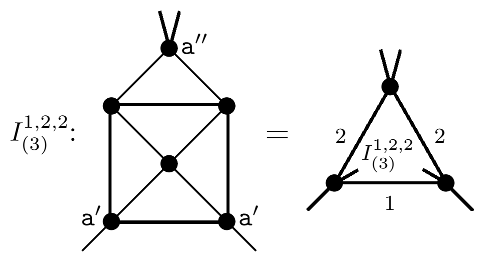


Figure 4.1: $I^{1,2,2}_{(3)}$.

Let I*-piece* and II*-piece* denote k-pieces that have exactly three a-vertices and other vertices of degree four (if they exist). For a I-piece, two of its a-vertices are a′-vertices and the other a-vertex is an a″-vertex, in contrast to that of a II-piece, whose a-vertices are all a″-vertices. I-piece and II-piece are sometimes abbreviated to I and II, respectively.

Additionally, let $I^{m_1,m_2,m_3}_{(k_1,k_2,\ldots,k_t)}$ ($II^{m_1,m_2,m_3}_{(k_1,k_2,\ldots,k_t)}$) denotes any I-piece (II-piece) whose inner faces are of only t types, namely k_1-, k_2-, …, k_t-gons where $k_i \geq 3$, $k_i \neq k_j$ $\forall$ $i \neq j$ and $i, j \in \{1, 2, \ldots, t\}$, that contributes m_1, m_2 and m_3 edges to the three adjoining faces of any graph in which it occurs. If $m = m_1 = m_2 = m_3$, then the notation is simplified to $I^{m}_{(k_1,k_2,\ldots,k_t)}$ ($II^{m}_{(k_1,k_2,\ldots,k_t)}$).

We represent $I^{m_1,m_2,m_3}_{(k_1,k_2,\ldots,k_t)}$ and $II^{m_1,m_2,m_3}_{(k_1,k_2,\ldots,k_t)}$ by labelled triangles whose perimeters are surrounded by m_1, m_2 and m_3. A dangling edge inside a labelled triangle of a I-piece that is incident to an a'-vertex (see Figure 4.1) is to indicate that the a'-vertex is a vertex of degree three.

$I^{1,2,2}_{(3)}$, as shown in Figure 4.1, is a I-piece, which is obtained by removing two edges of a face from an octahedron (see Figure 2.1). Figure 4.2(a) shows $II^{1}_{(3)}$, which is a II-piece whose only interior face is a 3-gon. Apart from the three a''-vertices, $II^{1}_{(3)}$ has no vertices of degree four. $II^{2}_{(3)}$, as shown in Figure 4.2(b), is II-piece, which is obtained by removing three edges of a face from an octahedron (see Figure 2.1).

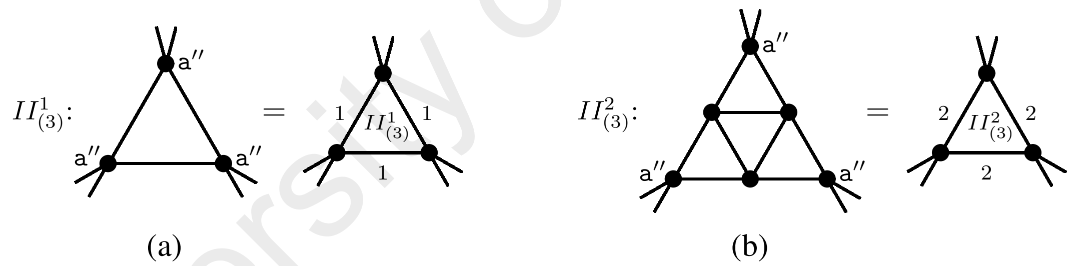


Figure 4.2: $II^{1}_{(3)}$ and $II^{2}_{(3)}$.

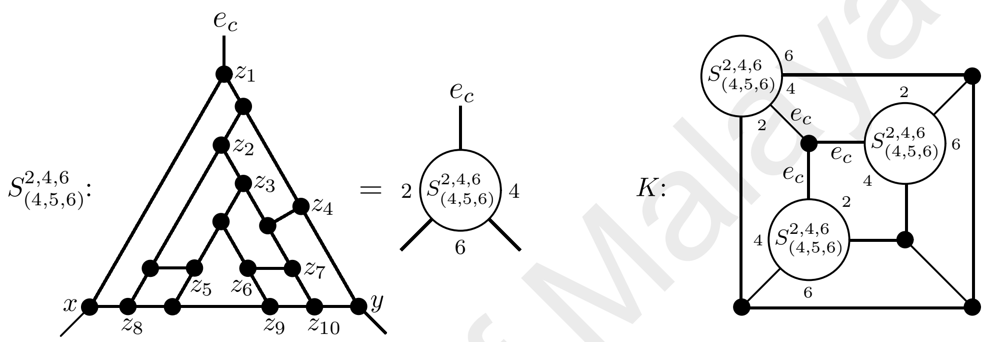
4.2 $G_4(3, k)$

This section concerns the class of all 4R3CPs whose faces are 3- and k-gons where $k > 3$. It was shown by Owens (1984b) that $G_4(3, k)$ contains non-hamiltonian members for $k \geq 12$. This result provides negative answer to Question 4.1.

Theorem 4.1. *(Owens, 1984b) There exist non-hamiltonian members of* $G_4(3, k)$ *for* $k \geq 12$.

Here, we construct non-hamiltonian members of $G_4(3, k)$ for certain k as stated in Theorem 4.5. First, we shall briefly describe the method used by Owens (1984b) to construct non-hamiltonian members of $G_4(3,k)$ for $k \geq 12$ and then show in detail the construction for $k = 12$.

Owens (1984b) constructed the 3-piece $S^{2,4,6}_{(4,5,6)}$ shown in Figure 4.3(a). $S^{2,4,6}_{(4,5,6)}$ contains a copy of the Tutte's triangle T (see Figure 1.5). Let C be a Hamilton cycle of a 3R3CP that contains the 3-piece $S^{2,4,6}_{(4,5,6)}$ and let P be $S^{2,4,6}_{(4,5,6)} \cap C$. By simple analysis, it can be shown that P are of the following two forms only: P_{xz_1} and P_{yz_1}. Thus, the edge of attachment labelled e_c is a compulsory edge of attachment of $S^{2,4,6}_{(4,5,6)}$.



(a) (b)

Figure 4.3: 3-piece $S^{2,4,6}_{(4,5,6)}$ and graph K (Owens, 1984b).

The graph K, as shown in Figure 4.3(b), is obtained from a cube by replacing three vertices with three copies of $S^{2,4,6}_{(4,5,6)}$, in such a way that the three compulsory edges of attachment e_c associated with the three 3-pieces $S^{2,4,6}_{(4,5,6)}$ are incident to the same vertex. By Lemma 3.4, K is non-hamiltonian. Furthermore, the thirty vertices labelled z_i in K for $i \in \{1, 2, \ldots, 10\}$, which Owens (1984b) called z-vertices, have the following property.

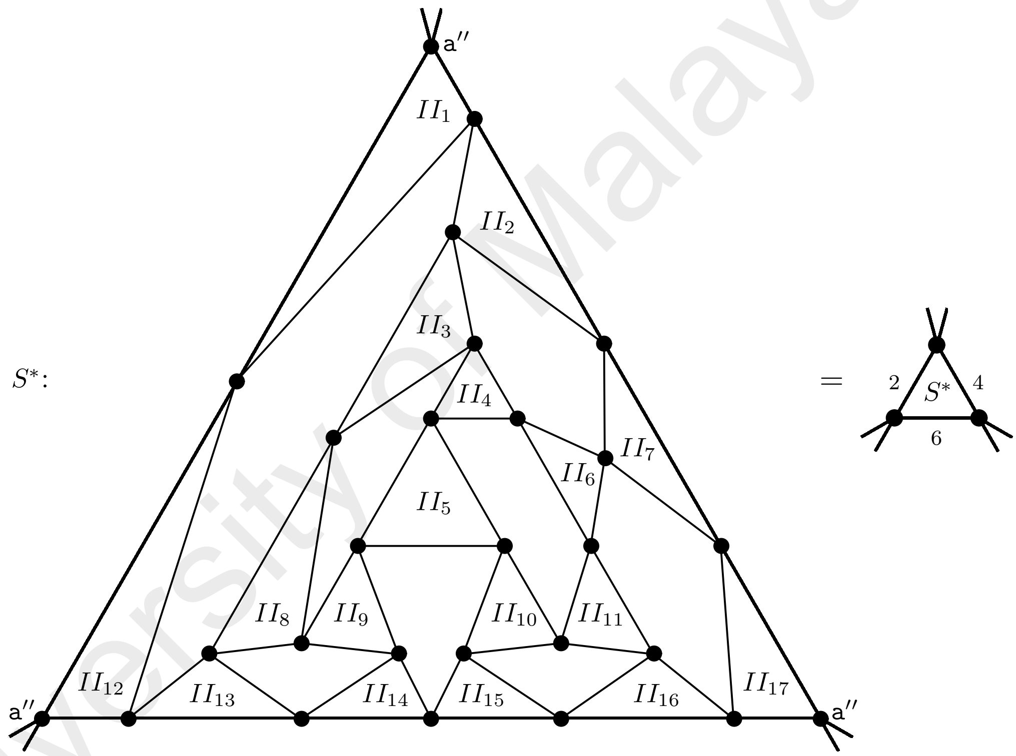
Lemma 4.2. *(Owens, 1984b) Every cycle in K omits at least one z-vertex.*

Since the three e_c are incident to the same vertex, no cycle C in K can contain them all. This implies that at least one e_c is omitted from C and for the corresponding copy of $S^{2,4,6}_{(4,5,6)}$, $S^{2,4,6}_{(4,5,6)} \cap C$ is of the form P_{xy}. However, it is impossible to have all ten z-vertices in P_{xy}. Thus, C does not contain every z-vertex of K. This property of K is essential in the construction of non-hamiltonian 4R3CPs.

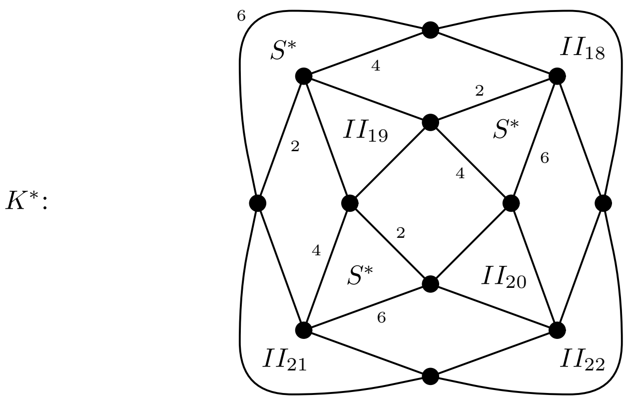
Walther (1969) introduced a method to transform non-hamiltonian 3R3CPs to non-hamiltonian 4R3CPs. In particular, he showed the transformation of the Tutte's graph (see Figure 1.1) to a non-hamiltonian 4R3CP. Owens (1984b) generalised this method and called it a *4-transform*. A 4-transform is a transformation of a 3-regular planar graph G to a 4-regular planar graph G^* where

1. every vertex v of G corresponds to a II-piece of G^*, denoted by II_v; and
2. every edge uv of G corresponds to a vertex of G^*, which is obtained by identifying an a$''$-vertex of II_u with an a$''$-vertex of II_v.

For instance, Figure 4.4 shows S^* and K^*, which are 4-transforms of $S^{2,4,6}_{(4,5,6)}$ and K (see Figure 4.3), respectively.



(a) A 4-transform of $S^{2,4,6}_{(4,5,6)}$ (see Figure 4.3(a))



(b) A 4-transform of K (see Figure 4.3(b))

Figure 4.4: II-piece S^* and graph K^* (Owens, 1984b).

In the construction of members of $G_4(3,k)$ for $k \geq 12$, every II-piece of K^* that corresponds to a z-vertex of K contains exactly one copy of $II^2_{(3)}$ (see Figure 4.2(b)). Thus, there are thirty copies of $II^2_{(3)}$ in K^* that corresponds to thirty z-vertices in K. Owens (1984b) called these copies of $II^2_{(3)}$, z-copies.

Lemma 4.3. *(Owens, 1984b) No cycle in K^* enters every z-copy.*

Let C^* be a cycle in K^*. Let II_u and II_v be II-pieces of K^* that correspond to vertices u and v, respectively, of K. There is a corresponding cycle C in K that contains an edge $e = uv$ of K if and only if C^* contains two adjacent edges e_{II_u} and e_{II_v} where $e_{II_u} \in E(II_u)$ and $e_{II_v} \in E(II_v)$. Therefore, $v \in V(C)$ if and only if C^* enters II_v. By Lemma 4.2, C omits at least one z-vertex. Thus, C^* does not enter at least one z-copy. Hence, K^* is non-hamiltonian. Owens (1984b) also proved that K^* is 3-connected.

Here, we show the construction of a non-hamiltonian member of $G_4(3,k)$ for $k = 12$ only. For $k \geq 13$, the reader is referred to (Owens, 1984b).

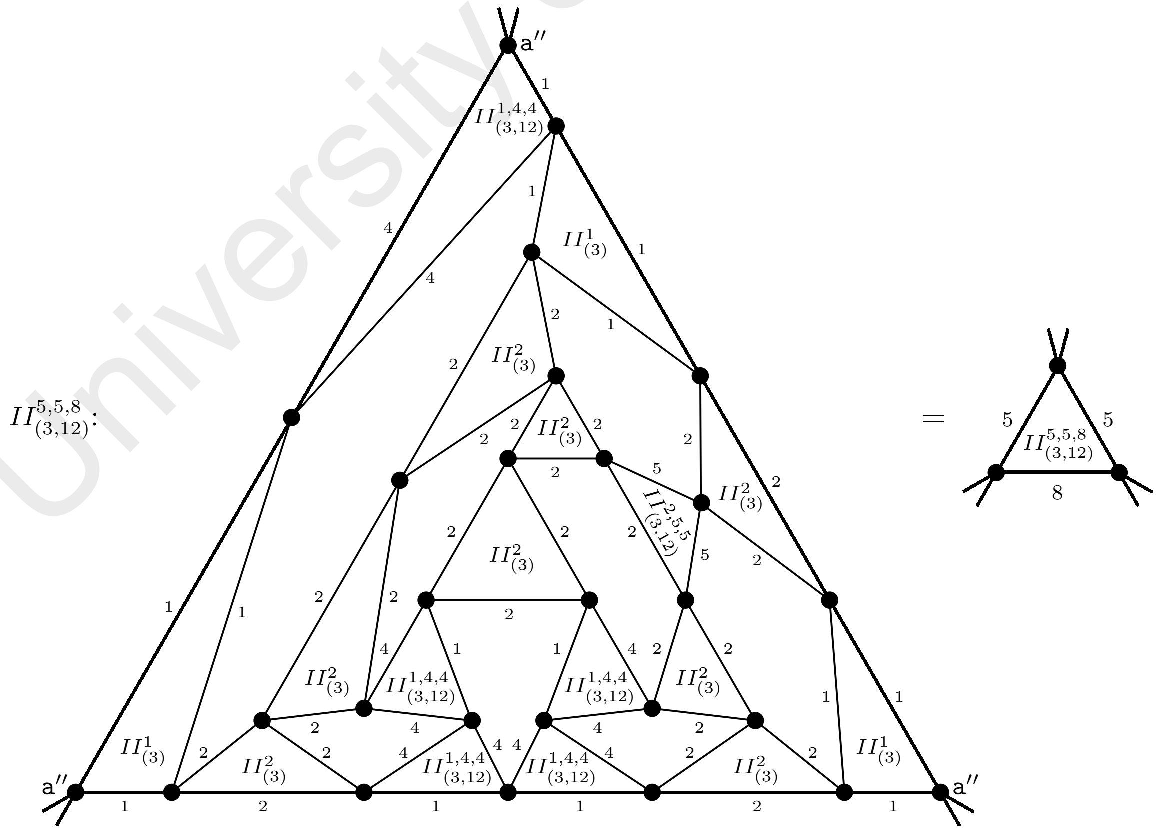


Figure 4.5: $II^{5,5,8}_{(3,12)}$ (Owens, 1984b).

$II^{5,5,8}_{(3,12)}$, as shown in Figure 4.5, is obtained from S^* (see Figure 4.4(a)) by replacing all II-pieces with three copies of $II^1_{(3)}$, eight copies of $II^2_{(3)}$ (see Figure 4.2), five copies of $II^{1,4,4}_{(3,12)}$ and one copy of $II^{2,5,5}_{(3,12)}$ (see Figure 4.6).

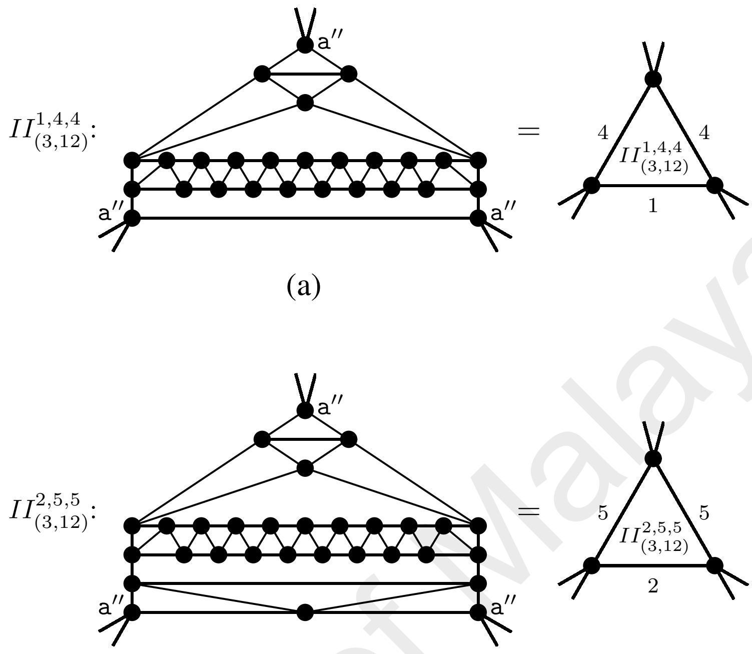


Figure 4.6: $II^{1,4,4}_{(3,12)}$ and $II^{2,5,5}_{(3,12)}$ (Owens, 1984b).

The graph G shown in Figure 4.7 is obtained from K^* (see Figure 4.4(b)) by replacing three copies of S^* and five II-pieces with three copies of $II^{5,5,8}_{(3,12)}$, four copies of $II^1_{(3)}$ and one copy of $II^2_{(3)}$. By inspection, $G \in G_4(3,12)$. By Lemma 4.3, G is non-hamiltonian.

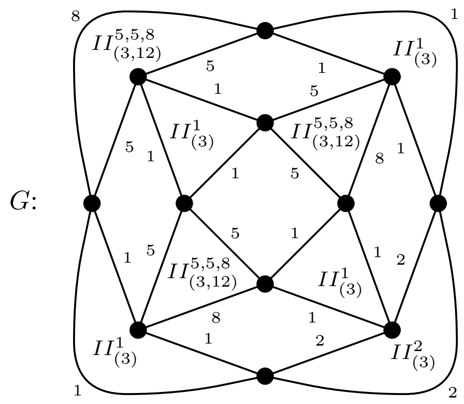
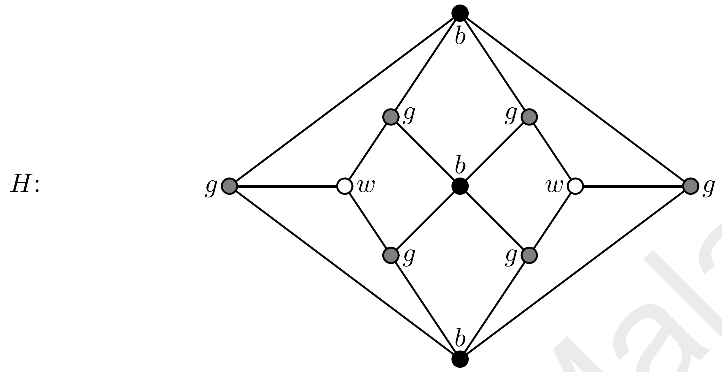


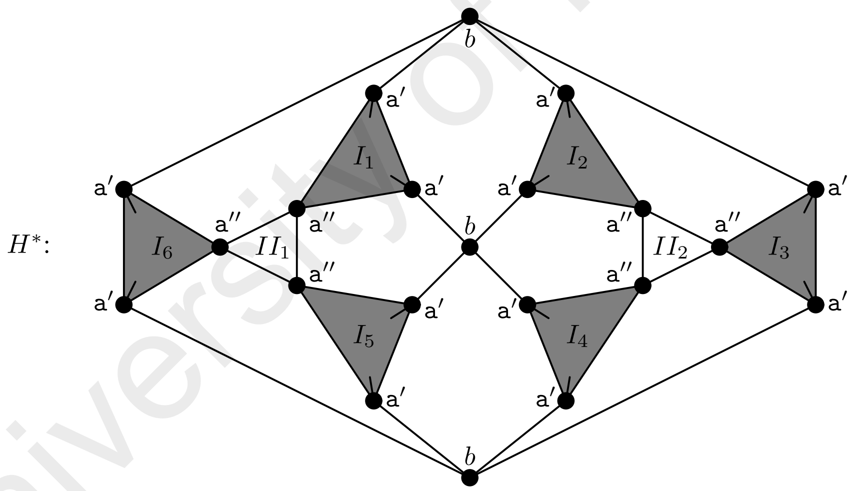
Figure 4.7: A non-hamiltonian member of $G_4(3,12)$ (Owens, 1984b).

Now, we shall describe the method of transformation that we used in the construction of non-hamiltonian members of $G_4(3,7)$ and $G_4(3,8)$. This method will be used again in Section 4.3.

Lemma 4.4. *The graph H^* shown in Figure 4.8(b) is a non-hamiltonian 4-regular planar graph.*



(a) Herschel graph H



(b) A 4-transform of H

Figure 4.8: The transformation of the Herschel graph H to a non-hamiltonian 4-regular planar graph H^*.

PROOF: Figure 4.8(a) shows the Herschel graph H. All vertices of degree four in H are labelled b and coloured black. Six vertices of degree three in H that are adjacent to exactly one vertex of degree three are coloured grey and labelled g. The remaining two vertices of degree three in H, which are adjacent to vertices of degree three only, are coloured white and labelled w.

The Herschel graph H is transformed to a 4-regular planar graph H^* shown in Figure 4.8(b) in such a way that

1. a vertex g of H corresponds to a I-piece, I_i where $i \in \{1,2,3,4,5,6\}$, of H^*; a vertex w of H corresponds to a II-piece, II_j where $j \in \{1,2\}$, of H^*; the vertices b remains unaltered; and

2. an edge gb of H corresponds to an a'-vertex of H^* and an edge gw of H corresponds to an a''-vertex of H^*, which is obtained by identifying an a''-vertex of a I-piece with an a''-vertex of a II-piece.

Every inner vertex of I and II is of degree four. Every a'-vertex of I is adjacent to a vertex b and every a''-vertex of I is identified with an a''-vertex of II. Thus, all vertices of H^* are of degree four. Hence, H^* is 4-regular.

Let C be a cycle in H^*. Let S be the set of two II-pieces and three vertices b in H^*. Since H is bipartite, successive open paths through I-pieces in C are separated either by a vertex b or by at least two vertices of a II-piece. However, there are six I-pieces and $|S| = 5$. Thus, C contains at most five out of six I-pieces. Hence, H^* is non-hamiltonian.

■

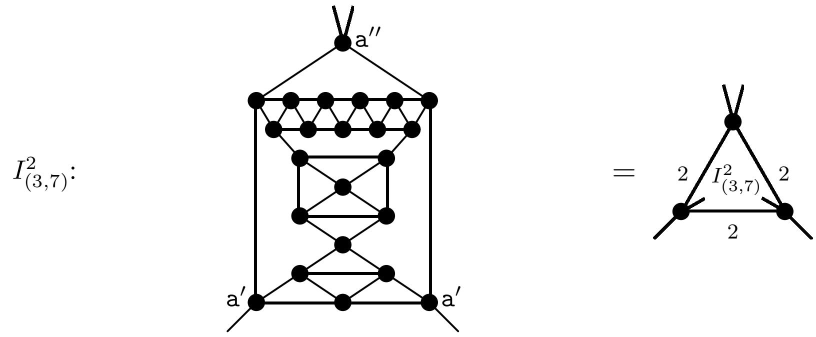
Theorem 4.5. *There exist non-hamiltonian members of $G_4(3,k)$ for $k \in \{7,8\}$.*

PROOF: We obtain a member of $G_4(3,k)$ for $k \in \{7,8\}$ from H^* (see Figure 4.8(b)) by replacing I_i where $i \in \{1,2,3,4,5,6\}$ and II_j where $j \in \{1,2\}$ with those I- and II-pieces listed in Table 4.1. By Lemma 4.4, the graphs obtained are non-hamiltonian.

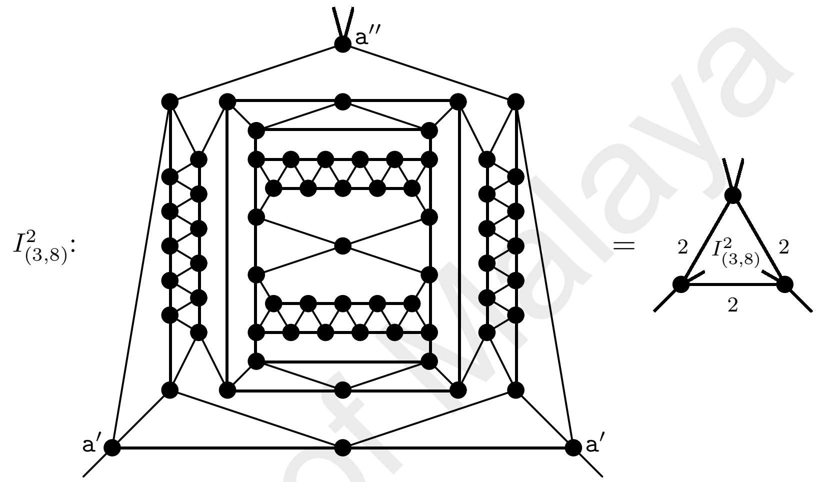
■

Table 4.1: I- and II-pieces for the construction of non-hamiltonian members of $G_4(3,k)$ for $k \in \{7,8\}$.

k	I_i for $i \in \{1,3,4\}$	I_i for $i \in \{2,5,6\}$	II_j for $j \in \{1,2\}$
7	$I^2_{(3,7)}$ (Figure 4.9(a))	$I^{1,2,2}_{(3)}$ (Figure 4.1)	$II^1_{(3)}$ (Figure 4.2(a))
8	$I^2_{(3,8)}$ (Figure 4.9(b))		$II^2_{(3)}$ (Figure 4.2(b))



(a)



(b)

Figure 4.9: $I^2_{(3,7)}$ and $I^2_{(3,8)}$.

4.3 $G_4(3, k, l)$

This section concerns the class of all 4R3CPs whose faces are 3-, k- and l-gons where $k \neq l \neq 3$. Owens (1982b) constructed non-hamiltonian members of $G_4(3, 4, 8)$, $G_4(3, 6, 7)$, $G_4(3, 6, 8)$ and $G_4(3, 6, 10)$. The graphs are shown in Figures 4.10 – 4.13.

Theorem 4.6. *(Owens, 1982b) There exist non-hamiltonian members of* $G_4(3, 4, 8)$ *and* $G_4(3, 6, l)$ *for* $l \in \{7, 8, 10\}$.

Here, we construct non-hamiltonian members of $G_4(3, k, l)$ for k and l as stated in Theorem 4.8 by using some non-hamiltonian members of $G_4(3, k)$. Note that every non-hamiltonian member of $G_4(3, k)$ for $k \in \{7, 8\}$ and $k \geq 12$ constructed in Theorems 4.5 and 4.1, respectively, contains at least one copy of Δ_i where $i = k - 3$ (see Figure 4.14). Δ_{k-3} contains only two types of faces, namely 3-gons and a k-gon.

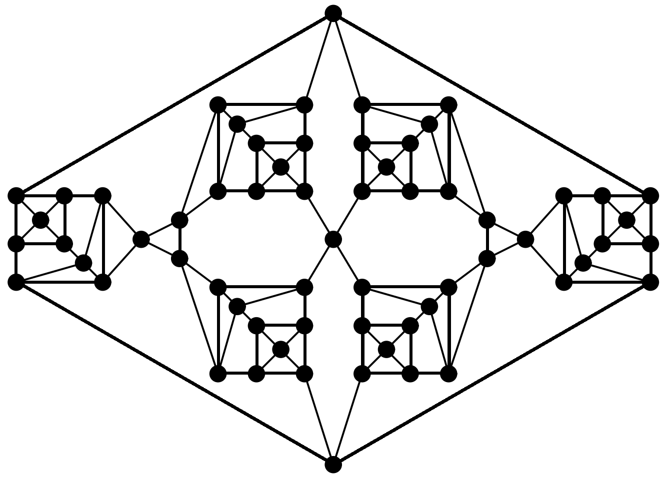


Figure 4.10: A non-hamiltonian member of $G_4(3, 4, 8)$ (Owens, 1982b).

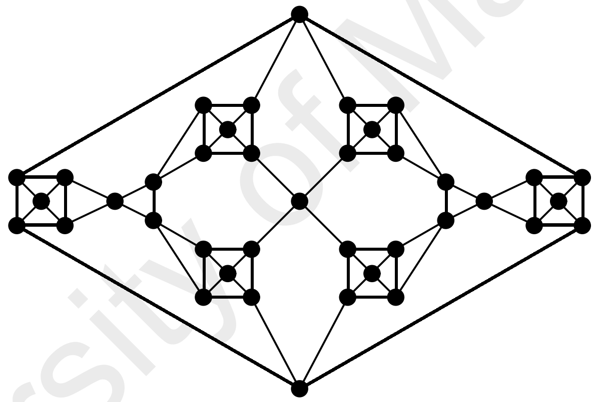


Figure 4.11: A non-hamiltonian member of $G_4(3, 6, 7)$ (Owens, 1982b).

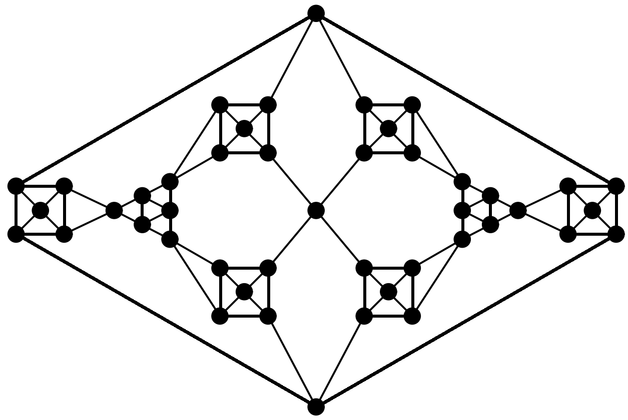


Figure 4.12: A non-hamiltonian member of $G_4(3, 6, 8)$ (Owens, 1982b).

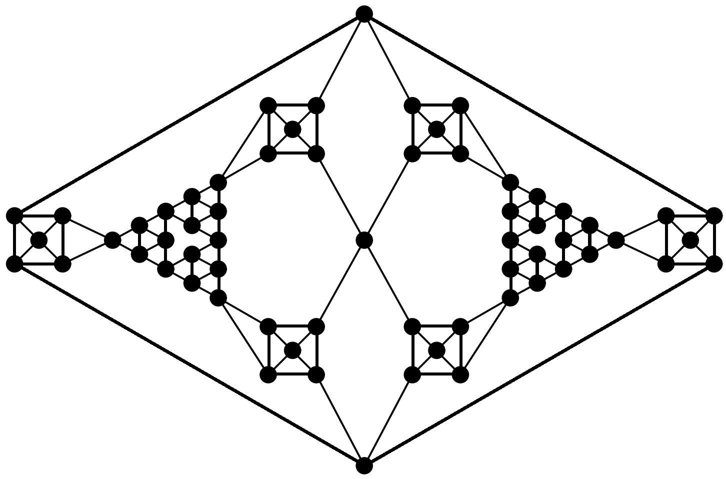


Figure 4.13: A non-hamiltonian member of $G_4(3, 6, 10)$ (Owens, 1982b).

Lemma 4.7. *Let G be a member of $G_4(3, k)$ for $k \geq 4$ that contains at least three k-gons and at least one copy of Δ_i where $i = k - 3$ (see Figure 4.14). Suppose G^* is a graph obtained from G by replacing exactly one copy of Δ_{k-3} with exactly one copy of Δ_i for $i \neq k - 3$. Then G^* is a member of $G_4(3, k, 3 + i)$ for $k \neq 3 + i$. Furthermore, G and G^* are either both hamiltonian or both non-hamiltonian.*

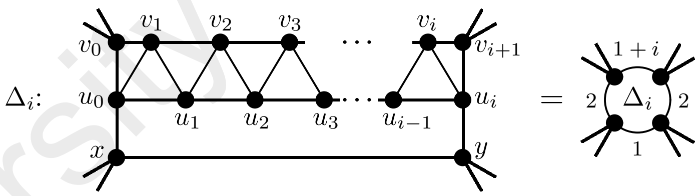


Figure 4.14: $\boldsymbol{\Delta_i}$ for $\boldsymbol{i \geq 1}$.

PROOF: Let C and C^* be Hamilton cycles of G and G^*, respectively. Let P and P^* be $\Delta_{k-3} \cap C$ and $\Delta_i \cap C^*$, respectively. By inspection, P and P^* take all the forms of single paths, union of two disjoint paths or union of three disjoint paths except $P_x \cup P_{v_0 v_{i+1}}$ (or by symmetry $P_y \cup P_{v_0 v_{i+1}}$) and $P_{x v_{i+1}} \cup P_{y v_0}$. Thus, Δ_i retains the hamiltonicity of G. Hence, the lemma follows. ■

Theorem 4.8. *For $k \neq l$, there exist non-hamiltonian members of $G_4(3, k, l)$ for $k \geq 4$ and $l \geq 7$; and $(k, l) \notin \{(6, 9), (9, 10), (9, 11)\}$.*

PROOF: The construction of a non-hamiltonian member of the above-mentioned classes is as follows. For $l \in \{7, 8\}$ and $l \geq 12$, let G_l be a non-hamiltonian member of $G_4(3, l)$

(see Theorems 4.5 and 4.1). Note that each G_l contains at least three l-gons and at least one copy of Δ_{l-3}. Suppose G_l^* is a graph obtained from G_l by replacing exactly one copy of Δ_{l-3} with exactly one copy of Δ_i for $i \geq 1$ and $i \neq l-3$ (see Figure 4.14). Then by Lemma 4.7, $G_l^* \in G_4(3, 3+i, l)$ for $i \geq 1$, $i \neq l-3$, $l \in \{7, 8\}$ and $l \geq 12$. Furthermore, G_l^* is non-hamiltonian since G_l is non-hamiltonian.

For $k \neq l$, the remaining cases are $k \in \{4, 5, 6, 9, 10, 11\}$ and $l \in \{9, 10, 11\}$; and $(k, l) \notin \{(6, 9), (9, 10), (9, 11)\}$. We obtain a member of $G_4(3, k, l)$ for k and l stated above from H^* (see Figure 4.8(b)) by replacing I_i where $i \in \{1, 2, 3, 4, 5, 6\}$ and II_j where $j \in \{1, 2\}$ with those I- and II-pieces listed in Table 4.2. By Lemma 4.4, the graphs obtained are non-hamiltonian. ■

Table 4.2: I- and II-pieces for the construction of non-hamiltonian members of $G_4(3, k, l)$ for $k \neq l$, $k \in \{4, 5, 6, 9, 10, 11\}$ and $l \in \{9, 10, 11\}$; and $(k, l) \notin \{(6, 9), (9, 10), (9, 11)\}$.

<table>
<tr><th>k</th><th>l</th><th>I_i for $i \in \{1, 3, 4\}$</th><th>I_i for $i \in \{2, 5, 6\}$</th><th>II_j for $j \in \{1, 2\}$</th></tr>
<tr><td>4</td><td rowspan="2">9</td><td rowspan="2">$I^{2,3,3}_{(3,k)}$
(Figures 4.15(a) and 4.16(a))</td><td rowspan="2">$I^{3}_{(3,k)}$
(Figures 4.15(b) and 4.16(b))</td><td rowspan="2">$II^{1}_{(3)}$ (Figure 4.2(a))</td></tr>
<tr><td>5</td></tr>
<tr><td>4</td><td rowspan="3">10</td><td colspan="2" rowspan="3">$I^{3}_{(3,k)}$
(Figures 4.15(b), 4.16(b) and 4.17(a))</td><td rowspan="3">$II^{2}_{(3)}$ (Figure 4.2(b))</td></tr>
<tr><td>5</td></tr>
<tr><td>6</td></tr>
<tr><td>4</td><td rowspan="4">11</td><td rowspan="2">$I^{3,4,4}_{(3,k)}$
(Figures 4.15(c) and 4.16(c))</td><td rowspan="2">$I^{4}_{(3,k)}$
(Figures 4.15(d) and 4.16(d))</td><td rowspan="2">$II^{1}_{(3)}$ (Figure 4.2(a))</td></tr>
<tr><td>5</td></tr>
<tr><td>6</td><td colspan="2">$I^{1,2,2}_{(3)}$ (Figure 4.1)</td><td>$II^{5}_{(3,6)}$ (Figure 4.17(b))</td></tr>
<tr><td>10</td><td colspan="2">$I^{3}_{(3,10,11)}$ (Figure 4.18)</td><td>$II^{2}_{(3)}$ (Figure 4.2(b))</td></tr>
</table>

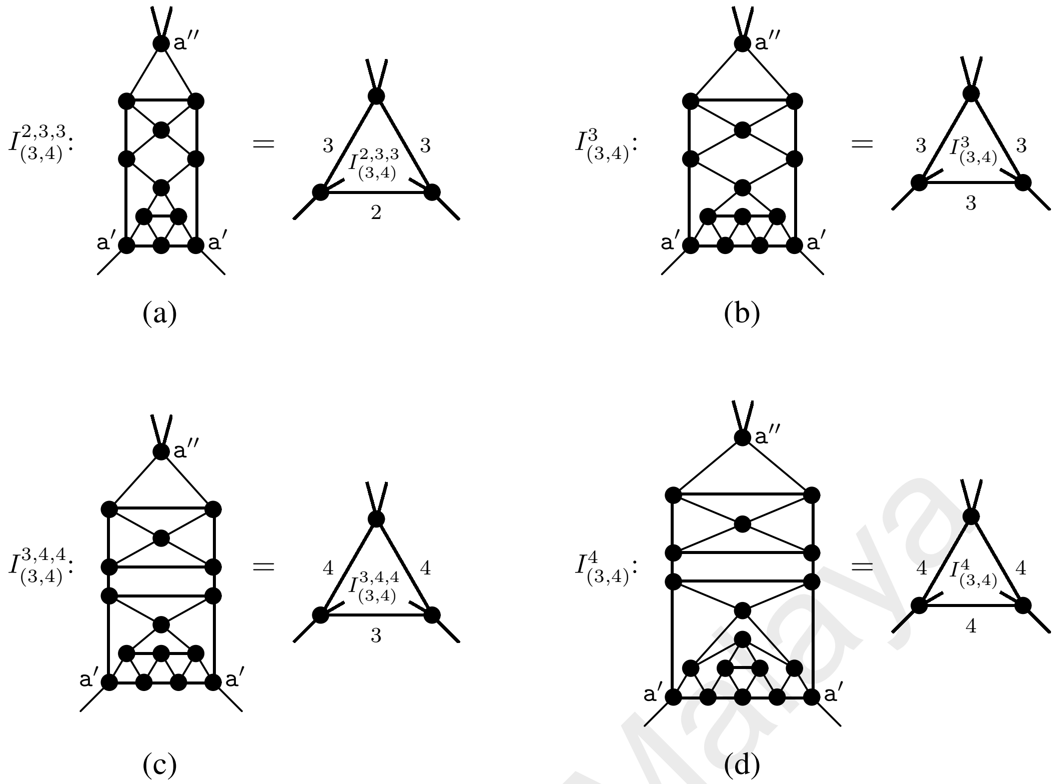


Figure 4.15: $I^{2,3,3}_{(3,4)}$, $I^{3}_{(3,4)}$, $I^{3,4,4}_{(3,4)}$ **and** $I^{4}_{(3,4)}$.

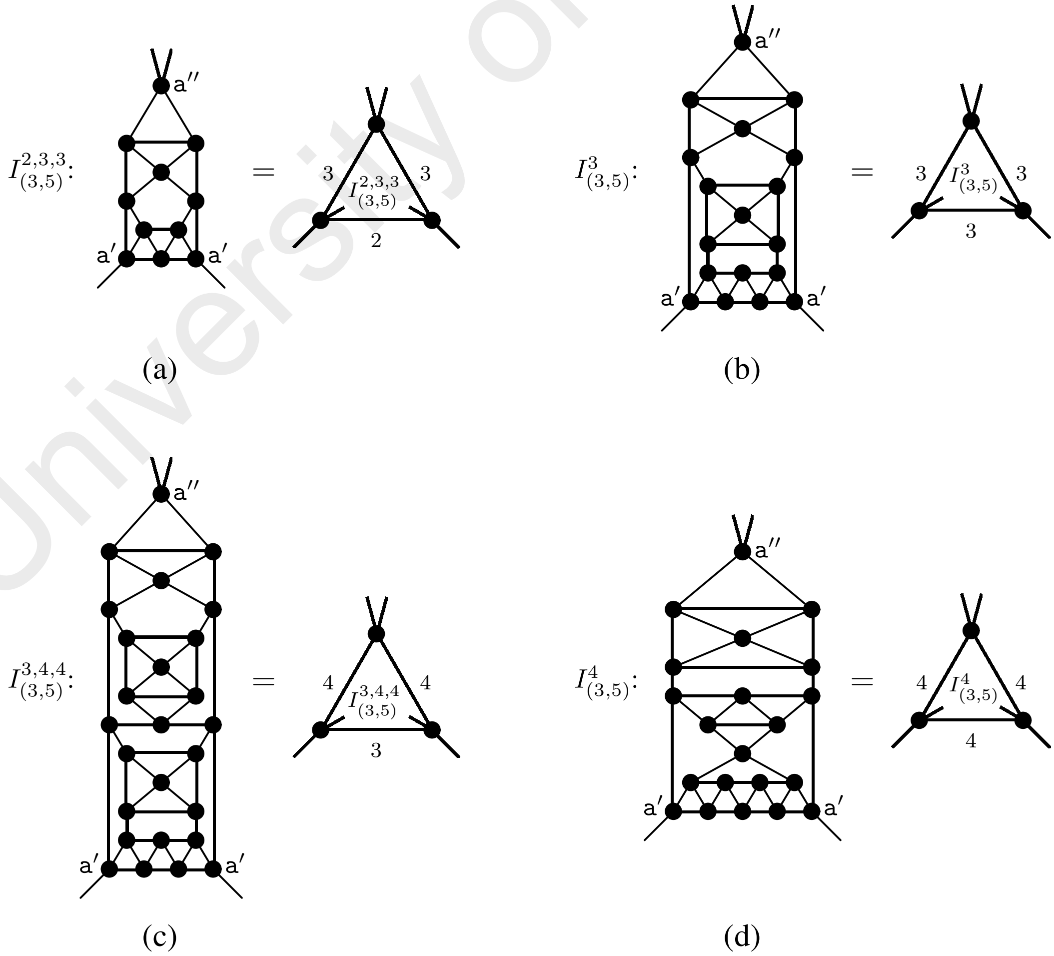


Figure 4.16: $I^{2,3,3}_{(3,5)}$, $I^{3}_{(3,5)}$, $I^{3,4,4}_{(3,5)}$ **and** $I^{4}_{(3,5)}$.

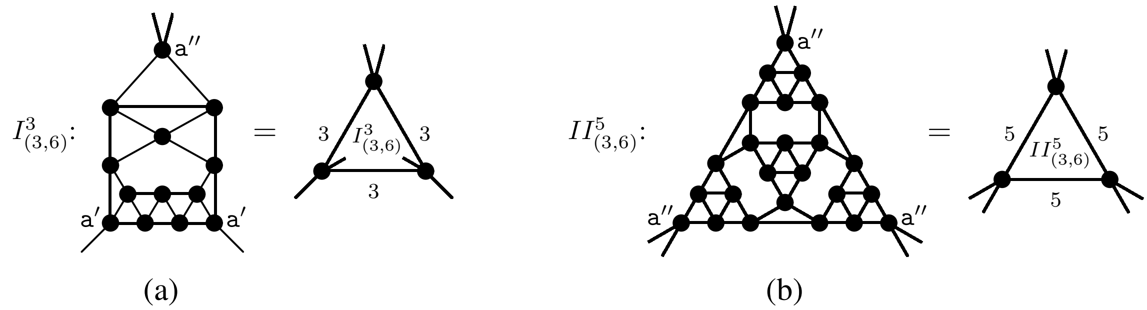


Figure 4.17: $\boldsymbol{I^3_{(3,6)}}$ **and** $\boldsymbol{II^5_{(3,6)}}$**.**

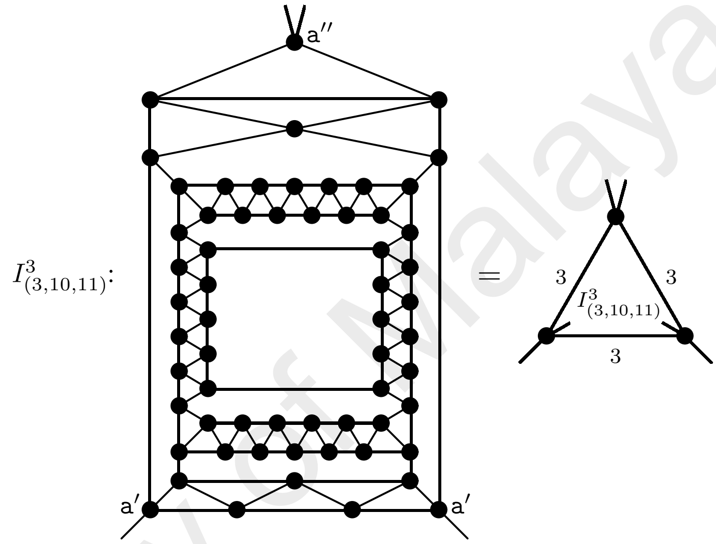


Figure 4.18: $\boldsymbol{I^3_{(3,10,11)}}$**.**

Combining Theorems 4.5 and 4.8, we have the following result.

Theorem 4.9. *For $k \neq l$, there exist non-hamiltonian members of*

1. $G_4(3,k)$ *for* $k \in \{7,8\}$*; and*

2. $G_4(3,k,l)$ *for* $k \geq 4$ *and* $l \geq 7$*; and* $(k,l) \notin \{(6,9),(9,10),(9,11)\}$.

Remark. *The following are some open problems:*

1. *Is every member of* $G_4(3,k)$ *for* $k \in \{4,5,6,9,10,11\}$ *hamiltonian?*

2. *Is every member of* $G_4(3,4,5)$, $G_4(3,4,6)$, $G_4(3,5,6)$, $G_4(3,6,9)$, $G_4(3,9,10)$ *and* $G_4(3,9,11)$ *hamiltonian?*

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LIST OF PUBLICATIONS AND PAPERS PRESENTED

Publication

Ong, S. H., & Nornadia Zainal Abidin. (in press). Non-hamiltonian 3-connected cubic planar graphs with only two types of faces besides 4-gons. *AIP Conference Proceedings Series.*

Papers Presented

Ong, S. H., & Nornadia Zainal Abidin. (2012, Dec). *Non-hamiltonian 4-regular 3-connected planar graphs with two types of faces.* Paper presented at the 8th Mathematics and Physical Sciences Graduate Congress, Chulalongkorn University, Bangkok, Thailand.

Ong, S. H., & Nornadia Zainal Abidin. (2014, Nov). *Non-hamiltonian 3-connected cubic planar graphs with only two types of faces besides 4-gons.* Paper presented at the 22nd National Symposium on Mathematical Sciences, Grand BlueWave Hotel, Shah Alam, Malaysia.