Chapter Three

Gamma Processes

3.1 Introduction

The need for non-Gaussian models has long been felt from the fact that most stream flows are non-Gaussian. It is generally accepted that most stream flows can be fitted to Gamma distribution (Markoviv, 1965; Matalas Wallis 1973). Therefore this study concentrates mainly on Gamma processes.

Recently, several non-Gaussian models (e.g., Gaver and Lewis, 1980; Tavares, 1980b and etc.) have been developed for describing correlated time series which have Gamma marginal distribution. They are found superior to Gaussian models, especially in describing time-irreversible Gamma processes.

Firstly in this chapter, we shall devote ourselves to the study of some well established correlated autoregressive, moving average and mixed autoregressive-moving average Gamma processes. We shall also discuss some further properties of the $GAR(1)$ process of Sim (1990) in Section 3.2.2.

Secondly, we shall introduce a new time-irreversible Gamma-like $ARMA(1, 1)$ process. The autocorrelation structure, conditional moment and the bivariate distribution of the proposed model are discussed and given in Section 3.4.2. The applicability of this model for skewed river flows is investigated and the discussion is presented in Chapter Four.

For simplicity, in the following discussions, we shall denote a Gamma random variable (*r.v.*) with shape parameter $k > 0$ and scale parameter $\beta > 0$ by $G(k, \beta)$, we shall also denote an Exponential *r.v.* with parameter $\alpha > 0$ by $Exp(\alpha)$ and a Beta *r.v.* with parameters $m > 0$, $n > 0$ by $B(m, n)$.

The probability density function (*p.d.f.*) of a $G(k, \beta)$ is

$$f_G(x) = \frac{\beta^k x^{k-1} e^{-\beta x}}{\Gamma(k)}, \quad x \geq 0,\ \beta > 0,\ k > 0,$$

with corresponding Laplace-Stieltjes transform (*L.T.*)

$$\phi_G(s) = \left(\frac{\beta}{\beta+s}\right)^k ,$$

and the *p.d.f.* of a $B(m, n)$ is

$$f_B(x) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} x^{m-1} (1-x)^{n-1} .$$

The Gamma and Beta variates are intimately related and their relations are established as follows :

<u>*THEOREM 3.1.*</u>

Let G be a $G(a, \theta)$ and B be a $B(b, c)$ which is independent of G, then

$$E\{\exp[-(s_1+s_2B)G]\} = \left(\frac{\theta}{\theta+s_1}\right)^a {}_2F_1(a,\ b;\ b+c;\ \frac{-s_2}{\theta+s_1}),$$

where ${}_2F_1(\alpha,\ \beta;\ \gamma;\ z)$ is the hypergeometric function defined by

$${}_2F_1(\alpha,\ \beta;\ \gamma;\ z) = \sum_{\ell=0}^{\infty} \frac{(\alpha)_\ell(\beta)_\ell}{(\gamma)_\ell\ \ell!} z^\ell$$

with $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$.

The proof of Theorem 3.1 is given in Appendix 3A.

By taking $s_1 = 0$, $a = b+c$ in Theorem 3.1 and using the result that

$${}_2F_1(\alpha,\ \beta;\ \alpha;\ z) = (1-z)^{-\beta},$$

we then have $E[\exp(-s_2 BG)] = \left(1 + \frac{s_2}{\theta}\right)^{-b} = \left(\frac{\theta}{\theta+s_2}\right)^{b}$ which leads to the following Lemma of Lewis et. al. (1986) :

LEMMA 3.1 : BETA-GAMMA TRANSFORMATION.

The product of two independent $r.v.$'s $G(b+c,\ \theta)$ and $B(b,\ c)$ is a $G(b,\ \theta)$, namely,

$$G(b,\ \theta) = B(b,\ c)G(b+c,\ \theta).$$

3.2 Autoregressive Processes

Some well-known Gamma first-order autoregressive models are the models of Yevjevich (1966), Thomas-Fiering (1967), Klemes-Boruvka (1974), Gaver and Lewis (1980) etc. In this section, we shall describe the Beta-Gamma first-order autoregressive process $BGAR(1)$ of Lewis, et. al. (1986) and the Gamma first-order autoregressive process $GAR(1)$ of Sim (1990). The $GAR(1)$ process of Sim (1990) is used to construct a new Gamma-like mixed autoregressive-moving average process in Section 3.4.2.

3.2.1 The BGAR(1) Process

The Beta-Gamma first-order autoregressive process $BGAR(1)$ of Lewis, et. al. (1986) is defined as

$$Y_n = B_n Y_{n-1} + \varepsilon_n \tag{3.1}$$

for $n = 0,\ \pm1,\ \ldots,$ where

(a) $\{B_n\}$ is a sequence of $i.i.d.$ $r.v.$'s $B(kp,\ k\bar{p})$,

(b) $\{\varepsilon_n\}$ is a sequence of $i.i.d.$ $r.v.$'s $G(k\bar{p},\ \beta)$,

(c) $k > 0,\ \beta > 0,\ 0 \le p < 1,\ \bar{p} = 1-p$ and

(d) the two sequences $\{B_n\}$ and $\{\varepsilon_n\}$ are mutually independent.

If Y_{n-1} has Gamma(k, β) marginal distribution, then from Lemma 3.1, the resulting Y_n is a $G(k, \beta)$ also.

By taking $a = b+c$ with $b = kp$ and $c = k\bar{p}$ in Theorem 3.1, we have

$$E\{\exp[-(s_1+s_2B_n)Y_{n-1}]\} = \left(\frac{\beta}{\beta+s_1}\right)^{k\bar{p}}\left(\frac{\beta}{\beta+s_1+s_2}\right)^{kp},$$

and the joint $L.T.$ of Y_n and Y_{n-1} is then given by

$$\begin{aligned}
\phi_2(s_1, s_2) &= E[\exp(-s_2Y_n-s_1Y_{n-1})] \\
&= E\{\exp[-s_2(B_nY_{n-1}+\varepsilon_n) - s_1Y_{n-1}]\} \\
&= E[\exp(-s_2\varepsilon_n)]\ E\{\exp[-(s_1+s_2B_n)Y_{n-1}]\} \\
&= \left(\frac{\beta}{\beta+s_1}\right)^{k\bar{p}}\left(\frac{\beta}{\beta+s_2}\right)^{k\bar{p}}\left(\frac{\beta}{\beta+s_1+s_2}\right)^{kp}. \qquad (3.2)
\end{aligned}$$

Since B_n is independent of Y_{n-j} for $j = 1, 2, \ldots,$ the autocovariance function of Y_n and Y_{n-j} can be written as

$$\mathrm{Cov}(Y_n, Y_{n-j}) = E(B_n)\ \mathrm{Cov}(Y_{n-1}, Y_{n-j}),$$

thereby, the autocorrelation functions $(a.c.f.)$ of the process (3.1) is obtained as

$$\mathrm{Corr}(Y_n, Y_{n-j}) = p^{|j|},\quad j = 0, \pm1, \ldots, \qquad (3.3)$$

From the definition of Y_n in (3.1), the conditional expectation of Y_n given $Y_{n-1} = y$ is established as

$$\begin{aligned}
E(Y_n|Y_{n-1} = y) &= E[(B_nY_{n-1} + \varepsilon_n)|Y_{n-1} = y] \\
&= yE(B_n) + E(\varepsilon_n) = (1-p)k/\beta + py.
\end{aligned}$$

The conditional $p.d.f.$ of Y_n given $Y_{n-1} = y$ is given as

$$f_{Y_n|Y_{n-1}}(x|y)$$

$$= \frac{\Gamma(k)}{\Gamma(kp)[\Gamma(k\bar{p})]^2} k^{k\bar{p}} \exp(-kx)$$

$$\cdot \int_0^{\min(1,\ x/y)} w^{kp-1} (1-w)^{k\bar{p}-1} (x-yw) \exp(kyw)\ dw \qquad (3.4)$$

where $x,\ y \geq 0$.

For an observed series $\{y_i\}$ of length n, the moment estimates of the parameters k, β and p are as follows :

$$\hat{k} = (\bar{y})^2/s^2,$$

$$\hat{\beta} = \hat{k}/\bar{y}$$

and

$$\hat{p} = \frac{1}{(n-1)s^2} \sum_{i=1}^{n-1} (y_i - \bar{y})(y_{i+1} - \bar{y})$$

where $\bar{y}$ and s^2 are respectively the sample mean and variance.

The Gamma first-order autoregressive process of Sim (1986) is a special case of the $BGAR(1)$ process of Lewis et. al. (1986). It is constructed according to the following stochastic equation :

$$Y_n = V_n Y_{n-1} + \varepsilon_n \qquad n = 1,\ 2,\ \ldots,$$

where

(a) $\{\varepsilon_n\}$ is a sequence of $i.i.d.$ $r.v.$'s $Exp(\lambda)$,

(b) $\{V_n\}$ is a sequence of $i.i.d.$ random coefficients with standard Power-function distribution, $F(V) = v^{\alpha}$, $\alpha > 0$ defined on the interval $[0,\ 1)$,

(c) the two sequences $\{\varepsilon_n\}$ and $(V_n\}$ are mutually independent.

The resulting Y_n is a $G(\alpha+1,\ \lambda)$.

The lag-1 joint *p.d.f.* of $\{Y_n\}$ is

$$f_{Y_{n+1},Y_n}(x,\ y) = \begin{cases} \lambda^{\alpha+2} y^{\alpha} \exp[-\lambda(x+y)]\,{}_1F_1(\alpha;\alpha+1;\lambda y)/\Gamma(\alpha+1), & x>y \\ \lambda^{\alpha+2} x^{\alpha} \exp[-\lambda(x+y)]\,{}_1F_1(\alpha;\alpha+1;\lambda x)/\Gamma(\alpha+1), & x\le y \end{cases}$$

where ${}_1F_1(\alpha;\ \beta;\ z) = \sum_{\ell=0}^{\infty} \frac{(\alpha)_\ell}{(\beta)_\ell\ \ell!} z^\ell$ is the confluent Hypergeometric function. Figure 3.1 exhibits the three dimensional representation of this joint *p.d.f.*

3.2.2 The GAR(1) Process

The Gamma first-order autoregressive process $GAR(1)$ of Sim (1990) is constructed according to the autoregressive representation

$$Y_n = p * Y_{n-1} + \varepsilon_n \tag{3.5}$$

where the operator '$*$ ' is defined as

$$p * Y = \sum_{i=1}^{N(Y)} W_i$$

and

(a) the ε_n's are *i.i.d. r.v.*'s $G(\nu,\ \alpha)$ with $\alpha,\ \nu > 0$,

(b) the W_i's are *i.i.d. r.v.*'s $Exp(\alpha)$ and

(c) for each fixed value of y, $N(y)$ is a Poisson *r.v.* with $\lambda = p\alpha,\ 0 \le p < 1$.

From the definition of the operator '$*$ ', we have the following theorem :

<u>*Theorem 3.2*</u>

(a) $0 * Y = 0$,

(b) $E(p * Y) = pE(Y)$,

(c) $E(p_1 * p_2 * Y) = E(p_1p_2 * Y) = p_1p_2\ E(Y)$,

(d) $E[p * (X+Y)] = E[(p * X) + (p * Y)]$,

(e) $E\left(\exp[-s(p^j * Y)]\right) = \left[\frac{\alpha(1-p) + (1-p^j)s}{\alpha(1-p) + s}\right]^{\nu}$

where $p^j * Y = \underbrace{p * p * \ldots * p}_{\leftarrow\ j \text{ times}\ \rightarrow} * Y$.

The proof of Theorem 3.2 is given in Appendix 3B.

Since $p * Y_{n-1}$ is independent of ε_n and by using the result that the $L.T.$ of $p * Y$ given $Y = y$ is

$$E\left\{\exp[-s(p * Y)] \,|\, Y = y\right\} = E\left\{\exp\left[-s \sum_{i=1}^{N(Y)} W_i\right] |Y = y\right\}$$

$$= \sum_{n=0}^{\infty} E\left\{\exp\left[-s \sum_{j=1}^{n} W_i\right]\right\} \Pr\left[N(y) = n\right]$$

$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{\alpha+s}\right)^n \frac{(\lambda y)^n}{n!} \exp(-\lambda y)$$

$$= \exp\left(\frac{-p\alpha s y}{\alpha+s}\right), \qquad (3.6)$$

the $L.T.$ of the $p.d.f.$ of Y_n can then be expressed recursively as

$$\phi_{Y_n}(s) = E\{E[\exp(-sY_n)|Y_{n-1}]\}$$

$$= E\left\{E[\exp(-s\varepsilon_n)]\ E\left[\exp[-s(p * Y_{n-1})]\,|Y_{n-1}\right]\right\}$$

$$= E\left[\left(\frac{\alpha}{\alpha+s}\right)^{\nu} \exp\left(\frac{-\alpha p s}{\alpha+s} Y_{n-1}\right)\right]$$

$$= \left(\frac{\alpha}{\alpha+s}\right)^{\nu} \phi_{Y_{n-1}}\left(\frac{\alpha p s}{\alpha+s}\right) \qquad (3.7)$$

which on solving recursively, yields

$$\phi_{Y_n}(s) = [1 + \theta(1-p^n)s]^{-\nu}\ \phi_{Y_0}\left[\frac{p^n s}{1 + \theta(1-p^n)s}\right]$$

where $\theta = 1/\alpha(1-p)$. Note that since $0 \le p < 1$ and $\phi_{Y_n}(s) \to (1+\theta s)^{-\nu}$ as $n \to \infty$, the Y_n's are asymptotically distributed $G(\nu, \theta^{-1})$. If Y_0 has the equilibrium Gamma distribution, then $\phi_{Y_n}(s) = (1+\theta s)^{-\nu}$ for all n, implying that the Y_n's are identically Gamma distributed $r.v.$'s.

The $L.T.$ of the joint $p.d.f.$ of $Y_1, Y_2, \ldots, Y_n$ was obtained by Sim (1990) as

$$\phi_n(s_1, s_2, \ldots, s_n) = |I_n + \theta S_n V_n|^{-\nu} \tag{3.8}$$

where $S_n = \text{diag}(s_1, s_2, \ldots, s_n)$, I_n is a nxn identity matrix, V_n is a nxn positive definite matrix with elements $v_{ij} = p^{|i-j|/2}$, $i, j = 1, 2, \ldots, n$ and $|A|$ denotes the determinant of A.

Consequently, the $L.T.$ of Y_n and Y_{n+j} is derived as

$$\phi_{j+1}(s_{j+1}, 0, \ldots, 0, s_1)$$
$$= [1 + \theta(s_1+s_{j+1}) + \theta^2(1-p^j)s_1 s_{j+1}]^{-\nu}$$

with corresponding bivariate joint $p.d.f.$,

$$f_{Y_{n+j}, Y_n}(x, y) = \left[\theta^{\nu+1}(1-p^j)\Gamma(\nu)\right]^{-1} \left[xy/p^j\right]^{(\nu-1)/2}$$
$$\cdot\exp\left[-(x+y)/\theta(1-p^j)\right] I_{\nu-1}\left[2\sqrt{p^j xy}/\theta(1-p^j)\right]$$

where $I_r(z)$ is the modified Bessel function of the first kind and of order r. Figure 3.2 is a graphical representation of this function.

By using Theorem 3.2, we have

$$\begin{aligned}\text{Cov}(Y_{n+j}, Y_n) &= \text{Cov}(p * Y_{n+j-1} + G_{n+1}, Y_n)\\ &= \text{Cov}(p * Y_{n+j-1}, Y_n)\\ &= E[Y_n(p * Y_{n+j-1})] - E(Y_n)E(p * Y_{n+j-1})\\ &= pE(Y_n Y_{n+j-1}) - pE(Y_n)E(Y_{n+j-1})\\ &= p\text{Cov}(Y_{n+j-1}, Y_n).\end{aligned} \tag{3.9}$$

Hence, the $a.c.f.$ of the Gamma $GAR(1)$ process is established as

$$\mathrm{Corr}(Y_{n+j}, Y_n) = p^j, \qquad j \geq 0.$$

The ℓth order conditional moment and the conditional variance of Y_{n+j} given $Y_n = y$ are

$$E(Y_{n+j}^{\ell} | Y_n = y) = \ell! \left[\theta(1-p^j)\right]^{\ell} L_{\ell}^{\nu-1}\left[-p^j y/\theta(1-p^j)\right]$$

and

$$\mathrm{Var}(Y_{n+j} | Y_n = y) = \theta^2(1-p^j)^2 \; [\nu + 2p^j y/\theta(1-p^j)]$$

respectively, where $L_{\ell}^{\nu-1}(z)$ is the generalized Laguerre polynomial of degree ℓ. The first-order conditional variance and moment are also illustrated respectively in Figures 3.3 and 3.4.

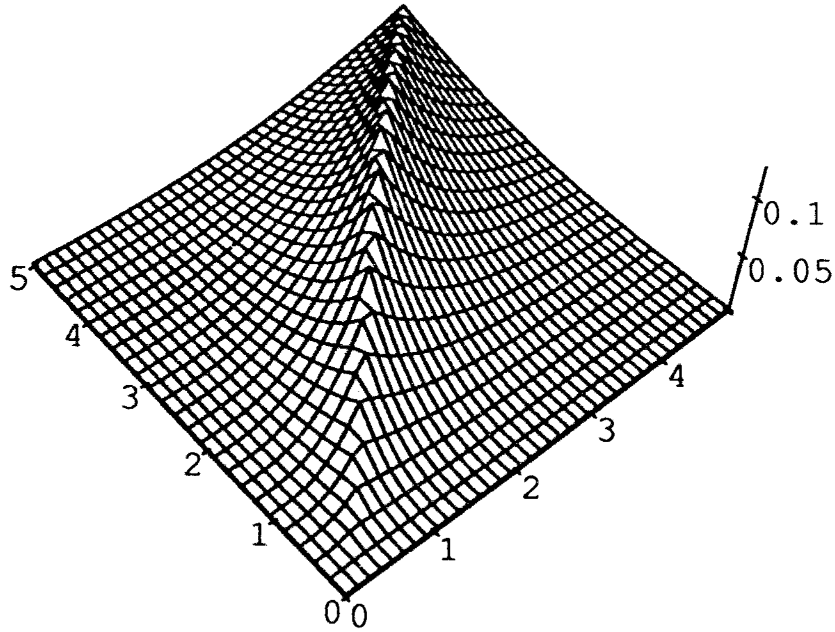

Figure 3.1 Plot Of The Joint p.d.f. Of $Y_n = V_n Y_{n-1} + \varepsilon_n$
$\lambda = 1.0$ and $\alpha = 3.0$

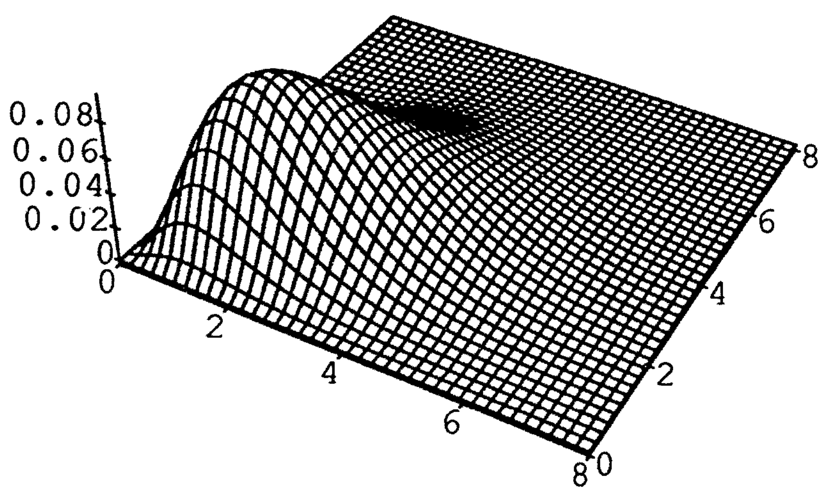

Figure 3.2 Plot Of The Joint p.d.f. Of $y_n = p * Y_{n-1} + \varepsilon_n$
with $p = 0.5$, $\nu = 3.0$, $\theta = 1.0$ and $j = 1.0$

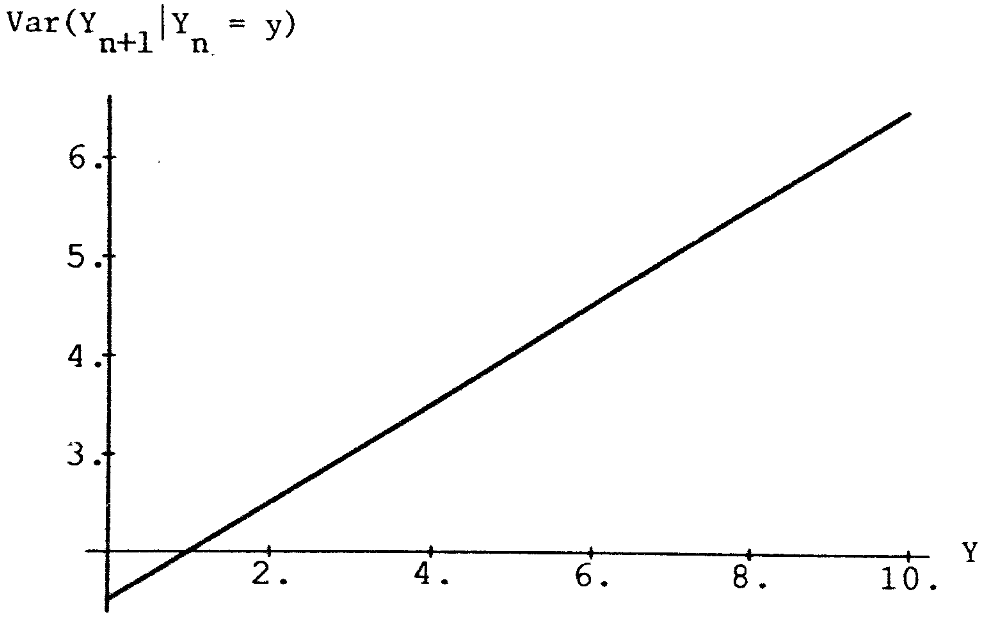

Figure 3.3 Plot Of The Conditional Variance Of $Y_n = P * Y_{n-1} + \varepsilon_n$

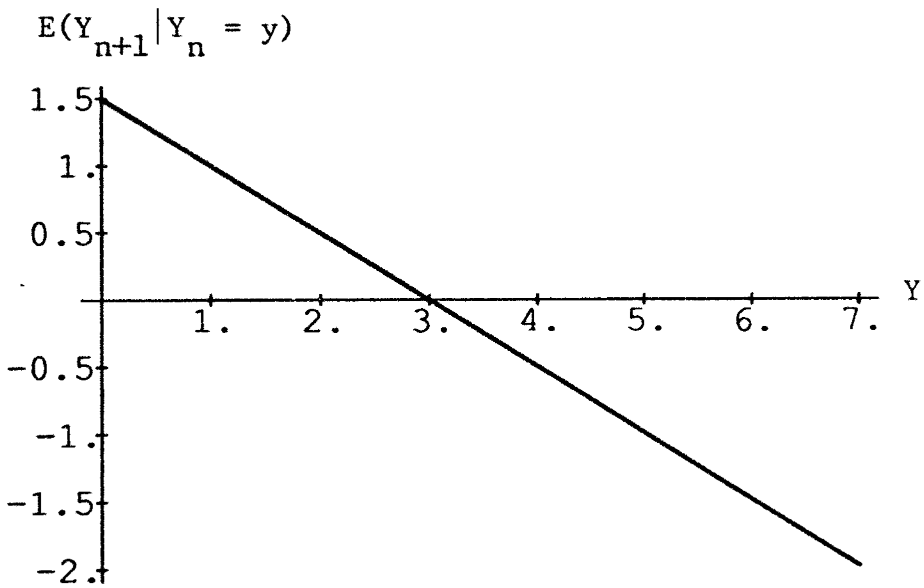

Figure 3.4 Plot Of The Conditional Moment Of $Y_n = P * Y_{n-1} + \varepsilon_n$

3.3 Moving Average Processes

In this section, we shall discuss two first-order moving average processes, they are the Beta-Gamma first-order moving average process $BGMA(1)$ of Lewis, et. al. (1986) and the Gamma first-order moving average process Gamma $MA(1)$ of Sim (1987b). The $BGMA(1)$ process of Lewis et. al. (1986) is used to construct a new Gamma-like mixed autoregressive-moving average process in Section 3.4.2.

3.3.1 The BGMA(1) Process

The $BGMA(1)$ process of Lewis, et. al. (1986) is defined as

$$X_n = G_n + B_n G_{n-1} \tag{3.13}$$

where

(a) $\{G_n\}$ is a sequence of $i.i.d.$ r.v.'s $G(k\bar{p}_1,\ \theta^{-1})$,

(b) $\{B_n\}$ is a sequence of $i.i.d.$ $r.v.$'s $B(kp_1,\ k(1-2p_1))$,

(c) the two processes $\{B_n\}$ and $\{G_n\}$ are mutually independent and

(d) $0 \le p_1 < 1/2$, $k > 0$, $\theta > 0$ and $\bar{p}_1 = 1-p_1$.

By using Lemma 3.1, the $L.T.$ of X_n is obtained as

$$\begin{aligned}\phi_{X_n}(s) &= E[\exp(-sX_n)]\\ &= \left(1+\theta s\right)^{-k\bar{p}_1}\left(1+\theta s\right)^{-kp_1}\\ &= (1+\theta s)^{-k}.\end{aligned}$$

Hence, all of the X_n's are identically distributed $G(k,\ \theta^{-1})$.

By taking $a = b+c$ with $b = kp_1$ and $c = -k\bar{p}_1+kp_1$ in Theorem 3.1, we have

$$E\{\exp[-(s_1+s_2B)G]\} = \left(1+\theta s_1\right)^{-k\bar{p}_1+kp_1}\left[1+\theta(s_1+s_2)\right]^{-kp_1}$$

and the joint $L.T.$ of X_n and X_{n+1} can then be obtained as

$$\begin{aligned}\phi_2(s_1, s_2) &= E[\exp(-s_2X_{n+1}-s_1X_n)] \\ &= E[\exp(-s_2G)]\ E\{\exp[-(s_1+s_2B)G]\}\ E[\exp(-s_1BG)] \\ &= \left(1+\theta s_1\right)^{-k\bar{p}_1}\left(1+\theta s_2\right)^{-k\bar{p}_1}\left[1+\theta(s_1+s_2)\right]^{-kp_1} . \end{aligned} \tag{3.14}$$

Note that the lag-1 joint $L.T.$ of (X_n, X_{n+1}) of this process obtained above has exactly the same form as the lag-1 joint $L.T.$ of (Y_n, Y_{n+1}) of the $BGAR(1)$ process obtained in Section 3.2.1.

Similarly, the joint $L.T.$ of $X_1, X_2, \ldots, X_n$ can be obtained as

$$\begin{aligned}&\phi_n(s_1, s_2, \ldots, s_n) \\ &= E[\exp(-s_nX_n-s_{n-1}X_{n-1}-\ldots-s_1X_1)] \\ &= E[\exp(-s_nG_n)]\ E[s_1(BG)]\ \prod_{i=1}^{n-1} E\{\exp[-(s_i+s_{i+1}B_{i+1})G_i]\} \\ &= \left(1+\theta s_1\right)^{-kp_1}\left(1+\theta s_n\right)^{-k\bar{p}_1}\prod_{i=1}^{n-1}\left(1+\theta s_i\right)^{-k\bar{p}_1+kp_1}\left[1+\theta(s_i+s_{i+1})\right]^{-kp_1}\end{aligned}$$

The $a.c.f.$ of the $BGMA(1)$ process can also be determined directly from (3.13) as

$$\mathrm{Corr}(X_n, X_{n+j}) = \begin{cases} p_1 & j = 1, \\ 0 & j > 1. \end{cases}$$

Note that the transformation (3.14) is symmetric in s_1 and s_2, which implies that the process is time-reversible. However, a time-reversible process is transformable to a Gaussian process, then theoretically, time reversible models would have little significance in the modelling of non-Gaussian time-irreversible processes. In view of this, in the following sections, we shall discuss three time-irreversible Gamma processes.

3.3.2 The Gamma MA(1) Process

The Gamma *MA*(1) process of Sim (1987b) is constructed from an *i.i.d.* sequence of $G(\alpha+1,\ \gamma)$. The Gamma *MA*(1) process is defined by the following moving average representation

$$X_n = \lambda\varepsilon_n + V_n\varepsilon_{n+1} \tag{3.15}$$

where

(a) $\lambda > 0$,

(b) the V_n's are *i.i.d.* random coefficients defined on the interval $[0,\ 1)$ with standard Power-Function distribution $F_V(v) = v^{\alpha},\ \alpha > 0$,

(c) the ε_n's are *i.i.d.* *r.v.*'s $G(\alpha+1,\ \gamma)$ and

(d) the sequences $\{V_n\}$ and $\{\varepsilon_n\}$ are mutually independent.

By setting $b = \alpha$, $c = 1$ and $\theta = \gamma$ in Lemma 3.1, the *r.v.* $Y = V\varepsilon$ is obtained as a $G(\alpha,\ \gamma)$, and the *L.T.* of X_n is

$$\begin{aligned}
\phi_{X_n}(s) &= E[\exp(-sX_n)] \\
&= E\{\exp[-s(\lambda\varepsilon_n + V_n\varepsilon_{n+1})]\} \\
&= E[\exp(-s\lambda\varepsilon_n)]\ E[-s(V_n\varepsilon_{n+1})] \\
&= \left(\frac{\gamma}{\gamma+\lambda s}\right)^{\alpha+1}\left(\frac{\gamma}{\gamma+s}\right)^{\alpha}.
\end{aligned} \tag{3.16}$$

It is clearly shown in (3.16) that the X_n's are the sum of two independent *r.v.*'s $G(\alpha+1,\ \gamma/\lambda)$ and $G(\alpha,\ \gamma)$.

By inverting the *L.T.* (3.16), the *p.d.f.* of X_n is obtained as

$$f_{X_n}(x) = \frac{\gamma^{2\alpha+1}x^{2\alpha}}{\Gamma(2\alpha+1)\lambda^{\alpha+1}}\exp(-\gamma x)\ {}_1F_1[\alpha+1;\ 2\alpha+1;\ \gamma(\lambda-1)x/\lambda].$$

Note that the process is a generalization of the following three processes, i.e.,

(a) when $\lambda = 1$, $\{X_n\}$ is a first-order moving average Gamma$(2\alpha+1, \gamma)$,

(b) when $\lambda = 0$, $\{X_n\}$ is an *i.i.d.* sequence of $G(\alpha, \gamma)$,

(c) when $\lambda = 0$ and $\alpha = 1$, $\{X_n\}$ is an *i.i.d.* sequence of $Exp(\gamma)$.

Since $\{\varepsilon_n\}$ is an *i.i.d.* sequence of $G(\alpha+1, \gamma)$, then by conditioning on V_n, the double *L.T.* of the joint *p.d.f.* of X_n and X_{n+1} can be obtained from (3.15) as

$$\begin{aligned}\phi_2(s_1, s_2) &= E\{\exp(-s_1 X_{n+1} - s_2 X_n)\}\\ &= E[\exp(-s_1 V_{n+1}\varepsilon_{n+2})]\; E[\exp(-\lambda s_2 \varepsilon_n)]\\ &\quad \cdot E\{\exp[-(\lambda s_1 + s_2 V_n)\varepsilon_{n+1}]\}\\ &= \left(\frac{\gamma}{\gamma+\lambda s_1}\right)\left(\frac{\gamma}{\gamma+\lambda s_2}\right)^{\alpha+1}\left[\left(\frac{\gamma}{\gamma+s_1}\right)\left(\frac{\gamma}{\gamma+\lambda s_1+s_2}\right)\right]^{\alpha}.\end{aligned}$$

The joint *L.T.* obtained above is non-symmetrical in s_1 and s_2, which indicates that the process $\{X_n\}$ of (3.15) is time-irreversible.

By using the fact that ε_n's are mutually independent *r.v.*'s, the covariance structure of X_n and X_{n+j} for this moving average process is established as

$$\operatorname{Cov}(X_{n+j}, X_n) = \operatorname{Cov}(\lambda\varepsilon_{n+j} + V_{n+j}\varepsilon_{n+j+1},\ \lambda\varepsilon_n + V_n\varepsilon_{n+1})$$

$$= \begin{cases} \lambda^2 \operatorname{Var}(\varepsilon_n) + \operatorname{Var}(V_n\varepsilon_{n+1}) & j = 0 \\ \operatorname{Cov}(\lambda\varepsilon_{n+j},\ V_n\varepsilon_{n+1}) & j \ge 1 \end{cases}$$

$$= \begin{cases} [\alpha+(\alpha+1)\lambda^2]/\gamma^2 & j = 0 \\ \alpha\lambda/\gamma^2 & j = 1 \\ 0 & j \ge 2 \end{cases}$$

By dividing the variance of X_n, i.e., when $j = 0$ into its autocovariance function, the *a.c.f.* of the process (3.15) is obtained as

$$\rho_j = \begin{cases} \alpha\lambda/[\alpha+(\alpha+1)\lambda^2] & j = 1 \\ 0 & j \geq 2 \end{cases}$$

The conditional expectation of X_{n+1} given $X_n = x$ is obtained by evaluating $\frac{\partial}{\partial s_1}\phi_{X_{n+1}, X_n}(s_1, s_2)|_{s_1=0}$ with respect to s_2 and then dividing it by the marginal density function of X_n. The conditional expectation takes the rather complicated form

$$E(X_{n+1}|X_n = y) = \frac{\alpha+\lambda}{\gamma} + \frac{\alpha\lambda x}{2\alpha+1} \frac{{}_1F_1[\alpha+1;\ 2\alpha+2;\ \gamma(1-\lambda)x/\lambda]}{{}_1F_1[\alpha;\ 2\alpha+1;\ \gamma(1-\lambda)x/\lambda]} .$$

3.4 Mixed Autoregressive-Moving Average Processes

In this section, we shall first of all, study the first-order mixed Gamma autoregressive-moving average process *MGARMA* (1, 1) of Sim (1987a) in Section 3.4.1. This model is constructed by combining the Gamma first-order autoregressive process of Sim (1986) and the Gamma *MA*(1) process of Section 3.3.2.

Secondly, we shall introduce a new Gamma mixed autoregressive-moving average process in Section 3.4.2. This new process can be used as an alternative to the former process and is constructed by combining the *GAR*(1) process of Sim (1990) in Section 3.2.2 and the *BGMA*(1) process of Lewis, et. al. in Section 3.3.1.

3.4.1 The MGARMA(1, 1) Process

The mixed Gamma *MGARMA*(1, 1) process of Sim (1987a) is defined by the following stochastic difference equations

$$\begin{aligned} X_n &= V_n Y_{n-1} + K_n Z_n \\ Y_n &= U_n Y_{n-1} + Z_n \end{aligned} \tag{3.17}$$

for $n = 1, 2, \ldots,$ where

(a) the Z_n's are *i.i.d. r.v.*'s $Exp(\lambda)$ with $\lambda > 0$,

(b) the K_n are *i.i.d.* Bernoulli *r.v.*'s with

$$K_n = \begin{cases} 0 & w.p.\ \theta, \\ \beta & w.p.\ 1-\theta. \end{cases}$$

(c) the U_n and V_n are *i.i.d.* random coefficients defined on the interval [0, 1) with distribution function $F_V(v) = v^{\alpha}$, $\alpha > 0$,

(d) the Z_n, K_n, U_n, V_n are mutually independent of each other.

By assuming stationarity, the *L.T.* of the process $\{X_n\}$ is obtained as

$$\begin{aligned} \phi_{X_n}(s) &= E[\exp(-sX_n)] \\ &= E\{\exp[-s(K_n Z_n + V_n Y_{n-1})]\} \\ &= E\{\exp[-s(K_n Z_n)]\}\ E\{\exp[-s(V_n Y_{n-1})]\} \\ &= \left[\theta + (1-\theta)\left(\frac{\lambda}{\lambda+\beta s}\right)\right]\left(\frac{\lambda}{\lambda+s}\right)^{\alpha}. \end{aligned} \tag{3.18}$$

Equation (3.18) verifies the assertion that $\{X_n\}$ of (3.17) is a mixed Gamma process, i.e., the process is the sum of a $G(\lambda, \alpha)$ and a *r.v.* which is zero with probability θ and $Exp(\lambda/\beta)$ with probability $(1-\theta)$.

By inverting the *L.T.* (3.18), the *p.d.f.* of the process is obtained as

$$f_{X_n}(x) = \theta\lambda(\lambda x)^{\alpha-1}\exp(-\lambda x)/\Gamma(\alpha)$$
$$+ (1-\theta)\lambda(\lambda x)^{\alpha}\exp(-\lambda x)\,{}_1F_1[1;\ \alpha+1;\ (\beta-1)\lambda x/\beta]/\beta\Gamma(\alpha+1).$$

The *a.c.f* of $\{X_n\}$ is given by

$$\rho_j = \begin{cases} \left(\dfrac{\alpha}{\alpha+1}\right)\dfrac{\alpha^2 + (1-\theta)(\alpha+1)\beta}{(\alpha+1)[\alpha+(1-\theta^2)\beta^2]} & j = 1 \\ \left(\dfrac{\alpha}{\alpha+1}\right)\rho_{j-1} & j > 1 \end{cases} \tag{3.19}$$

For fitting this model to hydrologic time series, the parameters λ, α, β and θ of the model are estimated by using the method of moments as the method of maximum likelihood is computationally difficult to apply here. In order to preserve the variance s^2, skewness γ, the lag-1 serial correlation r_1 and the lag-2 serial correlation r_2 of the historical data, we shall solve the following set of equations, viz,

$$s^2 = \frac{1}{\lambda^2}\,[\alpha+(1-\theta^2)\beta^2] \tag{3.20}$$

$$\frac{r_2}{r_1} = \frac{\alpha}{\alpha+1} \tag{3.21}$$

$$\frac{r_1^2}{r_2} = \frac{\alpha^2+(1-\theta)(\alpha+1)\beta}{(\alpha+1)[\alpha+(1-\theta^2)\beta^2]} \tag{3.22}$$

$$\gamma = \frac{2[\alpha+(1-\theta^3)\beta^3]}{[\alpha+(1-\theta^2)\beta^2]^{3/2}} \tag{3.23}$$

From (3.21), the value of α can be obtained easily. With α known, the values of β and θ can be evaluated by solving the simultaneous non-linear equations (3.22) and (3.23). The value of λ can then be obtained from (3.20). Finally, the mean $\bar{x}$ of the historical data is preserved by shifting the origin by a magnitude of c, whose value is given by

$$\bar{x} = \frac{1}{\lambda}\ [\alpha+(1-\theta)\beta]\ +\ c. \tag{3.24}$$

This model is fitted to the monthly flows of Perak river (January 1948 to December 1969) and the simulated sequence bears a close resemblance to the historical sequence (Sim, 1987a).

However, there are restrictions in the model that the lag-1 serial correlation r_1 and the sample skewness γ are confined to certain ranges for the solution of the two non-linear simultaneous equations (3.22) and (3.23) to exist.

In view of these difficulties, we shall introduce a new Gamma-like *ARMA*(1, 1) process in the next section, which will avoid the non-linear estimations and yet preserve the sample moments up to the third order.

3.4.2 A New Gamma-Like ARMA(1, 1) Process

This new Gamma-like mixed autoregressive-moving average process is given by the following stochastic difference equations,

$$X_n = \nu Y_{n-1} + B_n G_n$$
$$Y_n = p_2 * Y_{n-1} + (1-p_2)G_n \tag{3.25}$$

where

(a) the G_n's and B_n's are as defined in (3.13), i.e., the G_n's are *i.i.d. r.v.'s* $G(k\bar{p}_1,\ \theta^{-1})$ and B_n's are *i.i.d. r.v.'s* $B(kp_1,\ k(1-2p_1))$, and

(b) the operator '* ' is defined as in Section 3.2.2, i.e.,

$$p_2 * Y = \sum_{i=0}^{N(Y)} W_i$$

where W_i's are *i.i.d. r.v.'s* $Exp(\alpha)$ with $\alpha^{-1} = \theta(1-p_2)$, and for fixed y, $N(y)$ is a Poisson *r.v.* with parameter $\lambda = \alpha p_2,\quad 0 \le p_2 < 1.$

It is trivial that $(1-p_2)G_n$ is a $G(k\bar{p}_1, [\theta(1-p_2)]^{-1})$ and by taking $\alpha = [\theta(1-p_2)]^{-1}$ in the results obtained in Section 3.2.2, the Y_n's are verified as $G(k\bar{p}_1, \theta^{-1})$; the (B_nG_n)'s are verified as $G(kp_1, \theta^{-1})$ by using Lemma 3.1. Therefore, the *L.T.* of X_n is given as

$$\begin{aligned}\phi_{X_n}(s) &= E[\exp(-sX_n)] \\ &= E[\exp(-s\nu Y_{n-1} - sB_nG_n)] \\ &= \left(1+\theta s\right)^{-kp_1}\left(1+\nu\theta s\right)^{-k\bar{p}_1}. \end{aligned} \tag{3.26}$$

By inverting the *L.T.* (3.26), we obtain the *p.d.f.* of X_n as

$$f_{X_n}(x) = \frac{x^{k-1}\, e^{-x/\theta}\, \nu^{-k\bar{p}_1}}{\Gamma(k)\theta^k}\; {}_1F_1\left(kp_1;\; k;\; \frac{(\nu-1)x}{\nu\theta}\right).$$

From the correlation properties as shown in (3.9) of the *GAR*(1) process, the autocovariance function of X_n and X_{n+j} is obtained as

$$\begin{aligned}\mathrm{Cov}(X_n, X_{n+j}) &= \mathrm{Cov}(\nu y_{n-1} + B_nG_n,\ \nu y_{n+j-1} + B_{n+j}G_{n+j}) \\ &= \mathrm{Cov}(\nu y_{n-1}, \nu y_{n+j-1}) + \mathrm{Cov}(\nu y_{n+j-1}, B_nG_n) \\ &= \nu^2 p_2^j\ \mathrm{Var}(y_{n-1}) + \nu\bar{p}_2 p_2^{j-1}\ \mathrm{Var}(G_n)E(B_n) \\ &= k\theta^2\nu(\nu p_2\bar{p}_1 + p_1\bar{p}_2)p_2^{j-1},\end{aligned}$$

hence, the *a.c.f.* of $\{X_n\}$ is given by

$$\rho_j = \begin{cases} \dfrac{\nu(\nu p_2\bar{p}_1 + p_1\bar{p}_2)}{p_1 + \nu^2\bar{p}_1} & j = 1, \\ p_2^{j-1}\rho_{j-1} & j > 1. \end{cases} \tag{3.27}$$

The joint *L.T.* of X_n and X_{n+1} is established as follow :

$$\phi_2(s_1,\ s_2) = E[\exp(-s_1X_{n+1}-s_2X_n)$$

$$= E\{\exp[-s_1(\nu y_n + B_{n+1}G_{n+1}) - s_2(\nu y_{n-1} + B_nG_n)]\}$$

$$= E[\exp(-s_1BG)]\ E\{\exp[-s_1\nu(p_2 * y_{n-1}) - s_2\nu y_{n-1}]\}$$

$$\cdot E[\exp(-s_1\nu\bar{p}_2G_n - s_2B_nG_n)$$

$$= \left(1+\theta s_1\right)^{-kp_1} \left(\frac{\alpha + \nu s_1}{\alpha + \nu s_1 + \alpha\theta s_2}\right)^{kp_1}$$

$$\cdot\left(\frac{\alpha^2\bar{p}_2}{\alpha^2\bar{p}_2 + \nu\alpha(s_1+s_2) + \nu^2 s_1 s_2}\right)^{k\bar{p}_1} \tag{3.28}$$

where the third expectation is evaluated by using Theorem 3.1.

Since $\phi_2(s_1,\ s_2) \neq \phi_2(s_2,\ s_1)$, it is obvious that the proposed Gamma-like $ARMA(1,\ 1)$ process is time-irreversible, therefore the process cannot be transformed to Gaussian process.

By inverting the joint $L.T.$ (3.28), the joint $p.d.f.$ of X_n and X_{n+1}, for $\nu = 1$, is obtain as

$$f(x,\ y) = \left(\alpha^{k+k\bar{p}_1}\right)\left(\bar{p}_2^{\ k+kp_1}\right)\frac{x^{k-1}e^{-\alpha x-\alpha y}}{\Gamma(k)}\left(\frac{1}{\bar{p}_2x}\right)^{kp_1}$$

$$\sum_{n=0}^{\infty}\frac{(k\bar{p}_1)_n(\alpha^2p_2x)^n}{(k)_n\ n!}\ \frac{\Gamma(k+n)}{\Gamma(kp_1)\Gamma(k\bar{p}_1+n)\Gamma(k\bar{p}_1+n)}$$

$$\cdot\int_0^{\min(y,\ \bar{p}_2x)}\exp\left(\frac{\alpha}{1-p_2}u\right)\left(u\right)^{kp_1-1}\left(1-\frac{u}{\bar{p}_2x}\right)^{k\bar{p}_1+n-1}$$

$$\left(y-u\right)^{k\bar{p}_1+n-1}\ {}_1F_1(kp_1;\ k\bar{p}_1+n;\ \alpha p_2(y-u))\ du.$$

The conditional expectation of X_{n+1} given $X_n = x$ is given by

$E(X_{n+1} | X_n = x)$

$$= k\theta[\nu p_2(1-2p_1) + p_1)]$$

$$+ \frac{\left[{}_1F_1(kp_1;\ k+1;\ z)p_2\bar{p}_1 + {}_1F_1(kp_1+1;\ k+1;\ z)\nu\bar{p}_2 p_1\right]x}{{}_1F_1(kp_1;\ k;\ z)},$$

where $z = \dfrac{(\nu-1)x}{\nu\theta}$.

Figure 3.5 is a graphical representation of the above conditional expectation.

In order to compare the performance of this model to that of the *MGARMA*(1, 1) model of Sim (1987a), we have fitted this model to the monthly flows of Perak river and the computational results are discussed in Chapter Four.

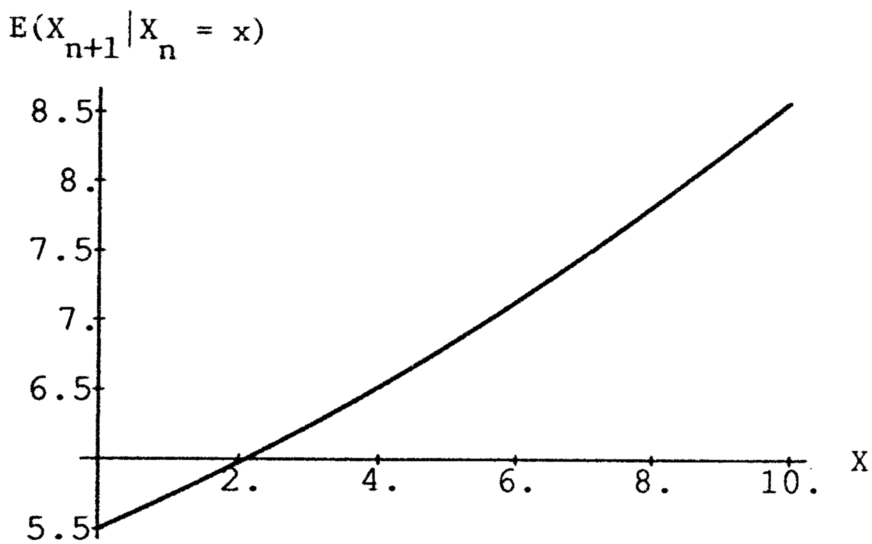

Figure 3.5 Plot Of The Conditional Expectation Of The New Gamma-Like ARMA(1,1) Process

Appendix 3A

PROOF OF THEOREM 3.1

$$E\{\exp[-(s_1+s_2B)G]\} = E_B[\phi_G(s_1+s_2B)]$$

$$= E_B\left[\left(\frac{\theta}{\theta+s_1+s_2B}\right)^a\right]$$

$$= E_B\left\{\frac{\theta}{\theta+s_1}\cdot\frac{1}{1+\frac{s_2}{\theta+s_1}B}\right\}^a$$

$$= \left(\frac{\theta}{\theta+s_1}\right)^a \sum_{\ell=0}^{\infty} \frac{(a)_\ell}{\ell!} \left(\frac{-s_2}{\theta+s_1}\right)^\ell E(B)^\ell$$

since

$$E(B^\ell) = \frac{B(b+\ell,\ c)}{B(b,\ c)} = \frac{\Gamma(b+\ell)\Gamma(b+c)}{\Gamma(b)\Gamma(b+c+\ell)} = \frac{(b)_\ell}{(b+c)_\ell}$$

thus,

$$E\{\exp[-(s_1+s_2B)G]\} = \left(\frac{\theta}{\theta+s_1}\right)^a \sum_{\ell=0}^{\infty} \frac{(a)_\ell(b)_\ell}{(b+c)_\ell\ \ell!} \left(\frac{-s_2}{\theta+s_1}\right)^\ell$$

$$= \left(\frac{\theta}{\theta+s_1}\right)^a {}_2F_1(a,\ b;\ b+c;\ \frac{-s_2}{\theta+s_1}).$$

Appendix 3B

PROOF OF Theorem 3.2

(a) The proof for (a) is trivial.

(b) From the definition of the operator '* ', we have

$$E[(p * Y)|Y = y]$$

$$= E\left(\sum_{i=1}^{N(Y)} w_i | Y = y\right)$$

$$= \sum_{n=1}^{\infty} E\left(\sum_{i=1}^{n} w_i\right) \Pr[N(y) = n]$$

$$= \sum_{n=1}^{\infty} \frac{n}{\alpha} \Pr[N(y) = n] = \frac{1}{\alpha} E[N(y)] = py.$$

Hence,

$$E(p * Y) = E\{E[(p * Y)|Y = y]\} = pE(Y).$$

(c) From (b), we have

$$\begin{aligned} E(p_1 * (p_2 * Y)) &= p_1 E(p_2 * Y) \\ &= p_1 p_2 E(Y) \\ &= E(p_1 p_2 * Y) \end{aligned}$$

(d) From (b), we have

$$\begin{aligned} E(p * (X + Y)) = pE(X + Y) &= pE(X) + pE(Y) \\ &= E(p * X) + E(p * Y) \\ &= E[(p * X) + (p * Y)] \end{aligned}$$

(e) When $j = 1$, the left-hand side of (e) can be obtained as

$$E\{\exp[-s(p * Y)]\} = E\{E[\exp(-s(p * Y))|Y]\}$$

$$= E\left[\sum_{n=0}^{\infty}\left(\frac{\alpha}{\alpha+1}\right)^{n} \frac{(\lambda Y)^{n}}{n!} \exp(-\lambda Y)\right]$$

$$= E\left[\exp\left(\frac{-p\alpha s}{\alpha+s}Y\right)\right]$$

$$= \left(\frac{\alpha(1-p)+(1-p)s}{\alpha(1-p)+s}\right)^{\nu}$$

which is the same as the right-hand side of (e) when $j = 1$.

Suppose the result holds true for $(j-1)$, we have

$$E\Big(\exp[-s(\underbrace{p * p * \ldots * p * p}_{\leftarrow\ j \text{ times}\ \rightarrow} * Y)]\Big)$$

$$= E\left\{E\Big(\exp[-s(\underbrace{p * p * \ldots * p * p}_{\leftarrow\ j \text{ times}\ \rightarrow} * Y)]\Big) \Big| (\underbrace{p * p * \ldots * p}_{\leftarrow\ (j-1) \text{ times}\ \rightarrow} * Y)\right\}$$

$$= E\left\{\exp\left[\frac{-p\alpha s}{\alpha+s}\right](\underbrace{p * p * \ldots * p}_{\leftarrow\ (j-1) \text{ times}\ \rightarrow} * Y)\right\}$$

$$= \left(\frac{\alpha(1-p)+(1-p^{j})s}{\alpha(1-p)+s}\right)^{\nu}.$$

Hence, by mathematical induction, the assertion is true for all $j \geq 1$.