# GEOMETRIC ASPECTS OF RIEMANNIAN SUBMANIFOLDS AND THEIR WARPED PRODUCTS

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#### ABSTRACT

The objective of this thesis is to devote a self-contained study of Riemannian submanifolds and their warped products. This is completed in five major footprints: constructing basic lemmas, proving existence, deriving characterizations, and applying geometric inequalities to obtain geometric applications in physical sciences. The whole thesis is divided into seven chapters. The first two chapters are an excursion from the origins of this area of research to the current study. It includes the definitions, basic formulae, and research open problems. It is known that the existence problem is central in the field of Riemannian geometry, particularly in the warped product submanifolds. Moreover, a lot of important results such as preparatory lemmas for subsequent chapters are stated in these two chapters. In this thesis, we have hypothesized Sahin (2009b) open problems in more general settings, and new inequalities are established by means of a new method such as mixed totally geodesic submanifolds with equality cases are discussed in details. In a hope to provide new methods by means of Gauss equation; instead of Codazzi equation, deriving the Chen (2003) type inequalities with slant immersions, and equalities are considered. An advantage has been taken from Nash's embedding theorem to deliberate geometric situations in which the immersion may pass such minimality, totally geodesic, totally umbilical and totally mixed geodesic submanifolds. As applications, we present the non-existence conditions of a warped product submanifold in a different ambient space forms using Green theory on a compact Riemannian manifold with or without boundary in various mathematical and physical terms such as Hessian, Hamiltonian, kinetic energy and Euler-Lagrange equation as well. The rest of this work is devoted to establish a relationship between intrinsic invariant and extrinsic invariants in terms of slant angle and pointwise slant functions. As a consequence, a wide scope of research was presented.

#### ABSTRAK

Objektif utama tesis ini adalah untuk mengkaji secara mendalam dan mandiri submanifold Riemannian dan produk meleding mereka. Ia menghasilkan penyelidikan dalam lima bahagian besar iaitu asas pembinaan lemma, pembuktikan kewujudan, pencarian sifat-sifat dan pengaplikasian tak kesamaan persamaan geometri bagi mendapatkan aplikasi geometri dalam sains fizikal. Tesis ini terbahagi kepada tujuh bab. Dua bab pertama adalah kronologi penyelidikan dari permulaan kajian sub manifold Riemannian hingga kepada kajian semasa. Definasi, formula-formula asas dan pelbagai persoalan yang masih terbuka dalam penyelidikan dalam bidang ini dimuatkan dalam dua bab pertama. Persoalan mengenai masalah kewujudan yang merupakan masalah utama dalam manifold Riemannian; khasnya manifold produk meleding, dibincangkan. Dalam tesis ini, kami juga telah memberi hipotesis Sahin (2009b) bagi masalah terbuka dalam keadaan yang teritlak dan ketak samaan baru telah dibina dengan kaedah baru. Kami juga membincangkan dengan terperinci submanifold geodesi campuran keseluruhan dengan kes ketaksamaan. Dengan penuh harapan untuk mendapatkan kaedah baru dengan menggunakan persamaan Gauss ; selain dari persamaan Codazzi, kami mendapatkan jenis ketaksamaan Chen (2003) dengan tenggelaman condong dan ketaksamaannya. Kami juga telah menggunakan faedah dari Teorem Penerapan Nash untuk menjelaskan situasi geometri yang mana tenggelaman mungkin melalui minimaliti, geodesi keseluruhan, pusat keseluruhan dan sub manifold campuran keseluruhan. Kami juga mempersembahkan keadaan tak kewujudan bagi manifold produk meleding dalam bentuk ruang yang berbeza menggunakan Teorem Green dalam manifold Riemannian yang padat dengan atau tanpa sempadan dalam pelbagai bentuk matematik dan fizikal seperti Hessian, Hamiltonian, tenaga kinetik dan juga dalam bentuk persamaan Euler-Langrangian. Selain dari itu, penyelidikan dalam tesis ini adalah juga membina pertalian antara intrinsik tak berubah dan ektrinsik tak berubah dalam bentuk sudut condong dan fungsi condong titik pintar. Akibatnya, skop penyelidikan yang luas dibentangkan

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## LIST OF SYMBOLS AND ABBREVIATION

- $\nabla(f) \equiv \text{Gradient of } f.$
- $\Delta(f) \equiv$  Laplacian of f.
- $E(f) \equiv$  Kinetic enery function of f.
- $H(df, p) \equiv$  Hamitonian of function of f.
- $H^f \equiv$  Hessian function of f.
- $L_f \equiv$  Langrangian function of f.
- $||h||^2 \equiv$  Squared norm of the second fundamental form.
- $||\vec{H}||^2 \equiv$  Squared norm of the mean curvature vector.
- $A \equiv$  Shape operator.
- $\nabla_X Z \equiv$  Covariant derivative <sup>1</sup> of Z in the direction of X.
- $\delta \equiv$  Delta invariant or (Chen invariant).
- $Ric \equiv Ricci curvature.$
- $\xi \equiv$  Characteristic vector field of almost contact manifolds.
- $\theta \equiv$  Slant angle or pointwise slant function.
- $\tilde{M} \equiv m$ -dimensional Riemannian ambient manifold.
- $M \equiv n$ -dimensional Riemannian submanifold.
- $\Gamma(TM) \equiv n$ -dimensional module of smooth vector fields tangent to *M*.
- $\mathfrak{F}(\tilde{M}) \equiv$  Algebra of smooth functions on  $\tilde{M}$ .
- $\zeta \equiv$  Normal vector field belongs to the normal subbundle *v*.
- $\nabla_X^{\perp} \zeta \equiv$  Normal connection on the normal tangent bundle.

<sup>&</sup>lt;sup>1</sup>Throughout this thesis, the vector fields X, Y are taken to be tangent to the first factor while the vector fields Z, W are considered to be tangent to the second factor of warped product submanifolds, unless otherwise stated.

- $\tilde{K}(X \wedge Y) = \tilde{K}_{XY} \equiv$  Sectional curvature of the plane spanned by the linearly independent vectors *X* and *Y*.
- $\tilde{\tau}(T_x M)$  and  $\tau(T_x M) \equiv$  Scalar curvatures of M at some  $x \in M$  with respect to  $\tilde{M}$  and M, respectively.
- $\mathscr{D} \equiv$  Smooth distribution of vector fields.
- $\{e_1, \cdots, e_n\} \equiv$  Local fields of orthonormal frame of  $\Gamma(T\tilde{M})$ .
- $J \equiv$  Almost Hermitian structure.
- $\phi \equiv (1, 1)$ -tensor field of almost contact structure.
- $\varphi \equiv$  Isometric immersion.
- $R(X,Y;Z,W) \equiv$  Riemannian curvature tensor.
- $\eta \equiv 1$ -form of almost contact structures.
- $\varphi: M = M_1 \times_f M_2 \longrightarrow \tilde{M} \equiv$  Isometric immersion  $\varphi$  from the warped product submanifold  $M = M_1 \times_f M_2$  into the ambient manifold  $\tilde{M}$ .
- $M =_{f_2} M_1 \times_{f_1} M_2 \equiv$  Doubly warped product submanifold.
- *g̃* and *g* ≡ Riemannian metric <sup>1</sup> on the ambient manifold *M̃<sup>m</sup>* and the corresponding induced metric on the Riemannian submanifold *M<sup>n</sup>*, respectively.
- TX(PX) and  $FX \equiv$  Tangential and normal components of JX or  $\phi X$ , respectively.
- $\vec{H}_i \equiv$  Partial mean curvature vectors restricted to  $N_i$  for i = 1, 2.
- $\mathbf{P}_X Y$  and  $\mathbf{Q}_X Y \equiv$  Tangential and normal components of  $(\tilde{\nabla}_X J) Y$  (resp.  $(\tilde{\nabla}_X \phi) Y$ ) in  $\tilde{M}$  (resp.  $\tilde{M}$ ), where  $X, Y \in \Gamma(TM)$ .
- $v \equiv J$ -invariant subbundle of the normal bundle  $T^{\perp}M$ .
- $\mu \equiv \phi$ -invariant subbundle of the normal bundle  $T^{\perp}M$ .

<sup>&</sup>lt;sup>1</sup>For simplicity's sake, we use g to refer for the inner products carried by both  $\tilde{g}$  and g.

- $h_{ij}^r \equiv h(e_i, e_j)$ -component, which is in the direction of the unit normal vector field  $e_r$ .
- $\sum_{1 \le i \ne j \le n} h_{ij}^r \equiv$  Summation of  $h_{ij}^r$ , where *i* and *j* run from 1 to *n* such that  $i \ne j$ .

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#### **CHAPTER 1**

#### **INTRODUCTION**

#### 1.1 BACKGROUND OF THE STUDY

One of the most fascinating topics in modern differential geometry is the theory of submanifolds. Geometric aspects of Riemannian submanifolds of almost Hermitian and almost contact metric manifolds with various structures are very inventive fields of differential geometry. It has shown an increasing development in pure Riemannian geometry. There are several classes of submanifolds in almost Hermitian manifolds and almost contact metric manifolds. Thus, corresponding to holomorphic distribution the concept of holomorphic submanifolds of Kaehler manifolds defined by Yano (1965). In early, Chen & Ogiue (1974), Yano & Kon (1977), and Chen et al. (1977) initiated one the class of submanifold which is called totally real submanifold. Bejancu (1979) introduced a new class of submanifolds known as CR-submanifolds such that the above two classes of submanifolds are generalized and also showed that a CR-submanifold is a CR-product if it is a locally Riemannian product of a holomorphic submanifold and a totally real submanifold. Further, Chen (1990) defined the slant submanifolds in complex manifolds as a natural generalization of invariant and anti-invariant submanifolds. As far as contact geometry is concerned, Latta (1996) extended this study in almost contact geometry and further studied by Cabrerizo et al. (2000).

Papaghiuc (1994) innovated a new class of submanifolds of almost Hermitian manifolds with slant distribution which is called semi-slant submanifold. The semi-slant submanifolds are the natural generalization of CR-submanifolds and also a generalization of slant submanifold, holomorphic and totally real submanifolds under some particular conditions. Cabrerizo et al. (1999) created the study of semi-slant submanifolds of contact manifolds. On the other hand, there is a class of submanifolds whose both complementary orthogonal distributions are slant with different wirtinger angles known as bi-slant submanifolds. It has been introduced and simultaneously given the notion of pseudoslant (hemi-slant) submanifolds. However, pseudo-slant submanifolds were defined by Carriazo (2000) under the name anti-slant submanifolds as a particular class of bi-slant submanifolds.

The notion of pointwise slant submanifolds in almost Hermitian manifolds initiated by Etayo (1998) under the name of quasi-slant submanifolds as a generalization of slant submanifolds and semi-slant submanifolds, that is, a submanifold with a slant function is called a pointwise slant submanifold. Recently, Chen & Garay (2012) studied the geometry of pointwise slant submanifolds in almost Hermitian manifolds and gave the most productive characterizations of it. Then, Sahin (2013) defined the notion of the pointwise semi-slant submanifolds in Kaehler manifold and also provided an example. Later on, K. S. Park (2014) introduced the idea of pointwise slant and semi-slant submanifolds in almost contact metric manifolds, and obtained some classification results with some examples. It means that pointwise semi-slant submanifolds can be considered as a natural generalization of holomorphic submanifolds, totally real submanifolds, CR-submanifolds, slant submanifolds and semi-slant submanifolds in almost Hermitian manifolds with a globally constant slant function. Among some classes were studied in almost contact metric manifolds by several geometers in Uddin & Ozel (2014), Park (2014, 2015); Balgeshir (2016).

The most inventive topics in differential geometry currently are the theory of warped product manifolds. Henceforth, the notion of warped products played some important roles in the theory of general relativity as they have been providing the best mathematical models of our universe for now. That is, the warped product scheme was successfully applied in general relativity and semi-Riemannian geometry in order to build basic cosmological models for the universe. It is well known that the concept of warped product manifolds is widely used in differential geometry and many applications of these warped product manifolds have been found in numerous situations to general relativity theory in physics and black holes Hawking (1974). We also note that a lot of space-time models are examples of warped product manifolds such as Robertson-Walker spacetimes, asymptotically flat space-time, Schwarzschild space-time and Reissner-Nordstrom space-time

which was discussed by Hawking & Ellis (1973), and GFR (1973). The idea of warped product manifolds was produced by Bishop & O'Neill (1969) with manifolds of negative curvature. In an attempt to construct manifolds of negative curvatures, R. L. Bishop and B. O'Neill defined the notion of warped product manifolds homothetically warping the product metric of a product manifold  $B \times F$  on the fiber  $p \times F$  for each  $p \in B$ . This generalized product metric appeared in differential geometric studies in a natural way, making the studies of warped product manifolds inevitable with intrinsic geometric point of view.

In 1954, one of the most important contributions in the field of Riemannian submanifolds theory appeared. It well-known Nash first embedding theorem,  $C^1$  embedding theorem, published by Nash (1954).

**Theorem 1.1.1.** Let (M,g) be a Riemannian manifold and  $\varphi : M^m \to \mathbb{R}^n$  a short  $C^{\infty}$ embedding (or immersion) into Euclidean space  $\mathbb{R}^n$ , where  $n \ge m+1$ . Then, for arbitrary  $\varepsilon > 0$  there is an embedding (or immersion)  $\varphi_{\varepsilon} : M^m \to \mathbb{R}^n$  which is

- (i) in class  $C^1$ ;
- (ii) isometric: for any two vectors  $X, Y \in T_x M$  in the tangent space at  $x \in M$ ,

$$g(X,Y) = \langle d\varphi_{\varepsilon}(X), d\varphi_{\varepsilon}(Y) \rangle;$$

(iii)  $\varepsilon$ -close to  $\varphi$ :

$$|\boldsymbol{\varphi}(x) - \boldsymbol{\varphi}_{\varepsilon}(x)| < \varepsilon \ \forall \ x \in M$$

During this thesis, when we refer to the Nash's embedding theorem, we mean the  $C^k$  embedding theorem of Nash (1956).

**Theorem 1.1.2.** Every compact Riemannian n-manifold can be isometrically embedded in any small portion of a Euclidean N-space  $\mathbb{E}^N$  with  $N = \frac{1}{2}n(3n+11)$ . Every noncompact Riemannian n-manifold can be isometrically embedded in any small portion of a Euclidean m-space  $\mathbb{E}^m$  with  $m = \frac{1}{2}n(n+1)(3n+11)$ .

The study of differential geometry of warped product submanifolds was intensified after that, when S. Nölker (1996) gave a warped product version of Moore's result. Let  $M_0, \dots, M_k$  be Riemannian manifolds,  $M = M_0 \times \dots \times M_k$  their product, and  $\pi_i : M \to M_i$  the canonical projection. If  $\rho_1, \dots, \rho_k : M_0 \to \mathbb{R}_+$  are positive-valued functions, then

$$\langle X,Y\rangle := \langle \pi_{0*}X,\pi_{0*}Y\rangle + \sum_{i=1}^{k} (\rho_i \circ \pi_0)^2 \langle \pi_{i*}X,\pi_{i*}Y\rangle$$

defines a Riemannian metric on M, called a warped product metric. M endowed with this metric is denoted by  $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ .

A warped product immersion is defined as follows: Let  $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$  be a warped product and let  $\varphi_i : N_i \to M_i$ ,  $i = 0, \dots, k$ , be isometric immersions, and define  $f_i := \rho_i \circ \varphi_0 : N_0 \to \mathbb{R}_+$  for  $i = 1, \dots, k$ . Then the map

$$\varphi: N_0 \times_{f_1} N_1 \times \cdots \times_{f_k} N_k \to M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$$

given by  $\varphi(x_0, \dots, x_k) := (\varphi_0(x_0), \varphi_1(x_1), \dots, \varphi_k(x_k))$  is an isometric immersion, which is called a warped product immersion.

Nölker (1996) extended the Moore's result by showing the natural existence of mixed totally geodesic warped product submanifolds in Riemannian space forms  $\tilde{M}^m(c)$  as the following.

**Theorem 1.1.3.** Let  $\varphi : N_0 \times_{f_1} N_1 \times \cdots \times_{f_k} N_k \to \tilde{M}^m(c)$  be an isometric immersion into a Riemannian manifold of constant curvature c. If h is the second fundamental form of  $\varphi$ and  $h(X_i, X_j) = 0$ , for all vector fields  $X_i$  and  $X_j$ , tangent to  $N_i$  and  $N_j$  respectively, with  $i \neq j$ , then, locally,  $\varphi$  is a warped product immersion.

The approach of warped product submanifolds in almost Hermitian and almost contact metric manifolds has been an important field for the few decades. Especially, Chen (2001) inaugurated the notions of CR-warped product submanifolds in Kaehler manifolds. He first proved the nonexistence of warped products of the type  $N_{\perp} \times_f N_T$ . Reversing the factors, he used a result of Hiepko (1979) to give a characterization theorem of the *CR*warped product submanifolds of the type  $N_T \times_f N_{\perp}$  in Kaehler manifolds.

#### **1.2 LITERATURE REVIEW AND SCOPE**

Three decades ago with the appearance of the celebrated Nash embedding theorem, Riemannian manifolds purpose was materialized by Omori (1967). Here, Riemannian manifolds were considered to be Riemannian submanifolds of Euclidean spaces. Inspiration by this fact, B. Y. Chen started research in this area as to review impressibility and nonimpressibility of Riemannian warped product in Riemannian manifolds. Particularly, in Riemannian space forms  $M^m(c)$  for reference Chen (2001)-Chen (2013). As a result, he proposed so many open problems on this topic. Recently, a lot of solutions were provided to his issues by several geometers, many gaps still remain. Moreover, several generalizations are often done to save lots of effort and time for a potential research. Therefore, this gave us a motivation to fill these gaps and prove such generalizations

As far as concern, if tangent space of Riemannian submanifold can be decomposed with holomorphic submanifold and totally real submanifold, then it is called a CR-submanifold. A CR-submanifold is called a CR-product if it is a Riemannian product of holomorphic submanifold and totally real submanifold. It was proved that a CR-submanifold of Kaehler manifold is a CR-product if and only if the tangential endomorphism is parallel, and also it is shown that every CR-product in a complex Hermitian space is a direct product of holomorphic submanifold of linear complex subspace and a totally real submanifold of another linear complex submanifold. It has been seen that there does not exist a warped product of the form  $N_{\perp} \times_f N_T$  in any Kaehler manifold except CR-products, where  $N_{\perp}$ is a totally real submanifold and  $N_T$  is a holomorphic submanifold. By contrast, it was shown in Chen (2001) that there exist many CR-submanifolds which are called warped product submanifolds of the form  $N_T \times_f N_{\perp}$  and such a warped product CR-submanifold simply represent a CR-warped product. The following general inequality of a CR-warped product in a Kaehler manifold was obtained:

$$||h||^2 \le 2p||\nabla \ln f||^2, \tag{1.2.1}$$

where  $\nabla(\ln f)$  is the gradient of  $\ln f$  and h is the second fundamental form. It is a one of the most fundamental relationship. For a survey on warped product submanifolds in different kind of structures, we refer to Chen (2013).

Some relations among the second fundamental form which is an extrinsic invariant, Laplacian of the warping function and constant sectional curvature of a warped product submanifold of a complex, cosymplectic and Sasakian space forms, and its totally geodesic and totally umbilical submanifolds are described from the exploitation of the Gauss equation instead of the Codazzi equation. These relations provide an approach to the classifications of equalities by the following case studied of Atçeken (2011); Atceken (2013); Atçeken (2015). These are exemplified by the classifications of the to-tally geodesic and totally umbilical submanifolds.

By the isometrically embedding theorem of J. F. Nash (1956) we know that every Riemannian manifold can be isometrically immersed into a Euclidean space with sufficiently high dimension. Afterward followed to case study of Nölker (1996), Chen (2003) developed a sharp inequality under the name of another general inequality in a CR-warped product  $N_T^h \times_f N_{\perp}^p$  in a complex space form  $M^m(4c)$  with a holomorphic constant sectional curvature 4c by means of Codazzi equation satisfying the relation

$$||\boldsymbol{\sigma}||^2 \leq 2p \bigg( ||\nabla \ln f||^2 + \Delta(\ln f) + 4hc \bigg),$$

where  $h = \dim_C N_T$ ,  $p = \dim N_{\perp}$  and  $\sigma$  is the second fundamental form. Therefore, it is called Chen's second inequality of the second fundamental form. Inspired by these studies, other geometers Mihai (2004), Munteanu (2005), Arslan et al. (2005), Mustafa et al. (2015) obtained some sharp inequalities for the squared norm of the second fundamental form, which is an extrinsic invariant, in terms of the warping function for the contact CR-warped products isometrically immersed in both a Sasakian space form and a Kenmotsu space form using the same techniques by taking equation of Codazzi. Some classifications of contact CR-warped products in spheres which satisfying the equality cases identically are given.

In more general case, Papaghiuc (1994) found a new class of submanifolds such that CR-submanifolds and slant submanifolds as particular classes of this class, and it is called semi-slant submanifold. Followed the definition of semi-slant submanifold in a Kaehler manifold, Sahin (2006a) proved the following results;

**Theorem 1.2.1.** Sahin (2006a) There does not exist of warped product semi-slant submanifold  $M = N_T \times_f N_{\theta}$  of a Kaehler manifold such that  $N_T$  is a holomorphic submanifold and  $N_{\theta}$  is slant submanifold.

By reversing the two factors  $N_T$  and  $N_{\theta}$  of warped product submanifold, we have

Theorem 1.2.2. Sahin (2006a) There do not exists of semi-slant warped product sub-

manifold of type  $M = N_{\theta} \times_f N_T$  of a Kaehler manifold such that  $N_{\theta}$  is a proper slant submanifold and  $N_T$  is a holomorphic submanifold.

The above two theorems show that warped product semi-slant submanifold failed to generalized of CR-warped product submanifold in a Kaehler manifold in the sense of N. Papaghiuc (1994). For instance, to see in almost contact metric manifolds, the warped product semi-slant submanifold does not admit non-trivial warped product in some ambient manifolds (see Al-Solamy & Khan (2008); K. A. Khan et al. (2008)). Therefore, the class of pointwise slant submanifold becomes very important to study in Riemannian geometry.

After the pointwise slant immersion concept was introduced by Chen & Garay (2012), this imposes significant restrictions on the geometry of its generalization. Therefore, the warped product pointwise semi-slant submanifold might be regarded as the simplest generalization of CR-warped product submanifolds. To the contrary of semi-slant and motivated by the study of pointwise slant submanifold, B. Sahin (2013) introduced the new class of pointwise semi-slant submanifold of a Kaehler manifold and also discussed warped product pointwise semi-slant submanifold of Kaehler manifold. He proved the following theorem on nonexistence of warped product pointwise such that

**Theorem 1.2.3.** There does not exist pointwise semi-slant warped product submanifold of type  $M = N_{\theta} \times_f N_T$  in a Kaehler manifold such that  $N_{\theta}$  is proper pointwise slant submanifold and  $N_T$  is a holomorphic submanifold.

Moreover, Sahin (2013) also proved the existence of warped product pointwise semi-slant submanifold of a Kaehler manifold by reversing the two factors  $N_T$  and  $N_{\theta}$  and established the following inequality theorem,

**Theorem 1.2.4.** Let M be a non-trivial warped product pointwise semi-slant submanifold of the form  $N_T \times_f N_{\theta}$  in a Kaehler manifold such that  $N_T$  is a holomorphic submanifold dimension  $n_1$  and  $N_{\theta}$  is a proper pointwise slant submanifold dimension  $n_2$  of  $\widetilde{M}$ . Then

(i) The second fundamental form h is satisfied the relation

$$||h||^{2} \ge 2n_{2} \left(\csc^{2}\theta + \cot^{2}\theta\right) ||\nabla(\ln f)||^{2}.$$
(1.2.2)

# (ii) If the equality sign hold in (4.2.9), then $N_T$ is totally geodesic in $\widetilde{M}$ and $N_{\theta}$ is totally umbilical submanifold in $\widetilde{M}$ . Moreover, M is minimal submanifold of $\widetilde{M}$ .

It can be easily seen that the Theorem 1.2.4 is generalized to the inequality (1.2.1) as particular case such that the pointwise slant function  $\theta$  becomes globally constant and substitute  $\theta = \frac{\pi}{2}$  in Eqs (1.2.2). It can be concluded that warped product pointwise semislant submanifold  $N_T \times_f N_{\theta}$  of Kaehler manifold succeed to generalize of CR-warped product submanifold of type  $N_T \times_f N_{\perp}$  claimed by Sahin (2013). Meanwhile, the existence of warped product pointwise semi-slant submanifold in cosymplectic, Sasakian and Kenmotsu manifolds of the type  $M = N_T \times_f N_{\theta}$ , in a case, when the structure vector field  $\xi$  is tangent to  $N_T$ , was proved by Park (2014). He derived numerous of examples on the existence of the warped product pointwise semi-slant submanifolds in different ambient manifolds, and constructed many general inequalities for the second fundamental form in terms of warping functions and pointwise slant functions.

Consequently, there was a difficulty in describing the differential geometric properties and to establish a relation between the squared norm of the second fundamental form and warping functions in terms of slant immersions and pointwise slant functions by using Codazzi. Applying a new method under the assumption of the Gauss equation instead of the Codazzi equation, we establish a sharp general inequality for the warped product pointwise semi-slant submanifolds which are isometrically immersed into a complex, cosymplectic and Sasakian space form as a generalization of CR-warped products. To do these, we choose only non-trivial warped product pointwise semi-slant submanifold of the type  $M = N_T \times_f N_{\theta}$  in Kaehler, cosymplectic and Sasakian manifolds, and then obtain some results.

#### **1.3 OBJECTIVE OF THE STUDY**

- (i) To investigate the existence of CR-warped product submanifolds in nearly Sasakian manifolds and T-manifolds in terms of various endomorphisms.
- (ii) To study and estimate the geometric inequalities of warped product pseudo-slant submanifolds in more general settings, as a locally Riemannian product manifold,

or nearly Kenmotsu manifold, or nearly Sasakian or nearly Trans-Sasakian manifolds.

- (iii) To discuss the proper warped product pointwise semi-slant submanifolds in the settings Kaehler, cosymplectic, Sasakian and Kenmotsu manifolds.
- (iv) To find a relation of the squared norm of the second fundamental for warped product pointwise semi-slant in various ambient space forms which is called a Chen type inequality.
- (v) To discuss some applications of the derived inequalities which relate to physical sciences.

#### **1.4 SIGNIFICANCE OF THE STUDY**

It is worthwhile to study the existence or non-existence of the warped products and even doubly warped products in a more general setting of Kaehler, Sasakian, cosymplectic, Trans-Sasakian, nearly cosymplectic, nearly Sasakian, nearly Trans-Sasakian, and nearly Kenmotsu manifolds, etc. Then, we expect that whenever such warped products exist, their warping functions will emerge a solution of a system of partial differential equation. Our purpose is to study the physical significance of theses problems as we believe that most of the physical solutions or phenomenonas can be described by the system of second-degree equations. So that our study may find the useful applications in physics and different branches of engineering and mathematics.

# **1.5 THESIS OUTLINE**

The research has been outlined as follows:

**Chapter 2** presents a literature review on basic concepts keeping in view of the prerequisites of the subsequent chapters. We will give a brief survey of the structures on manifolds, the notion of Riemannian submanifolds of both classes of almost Hermitian and almost contact metric manifolds. We focus on the Levi-Civitas connection and the curvature tensor. This tensor will be gradually used to define many intrinsic invariants necessary for this work, such as sectional, scalar and mean curvatures. On the other hand, the extrinsic geometry will be explored via the second fundamental form of Riemannian submanifolds, for a background of submanifold theory which is demonstrated, including Gauss formula and equation, Weingarten formula and Codazzi equation. Moreover, the warped products are studied form both the manifolds and the submanifold theories.

**Chapter 3** leads us to obtain characterizations under which a contact CR-submanifold of nearly Sasakian and T-manifolds reduces to a contact CR-warped product in the form of tangential endomorphism and normal valued-one form. We also discuss the necessary and sufficient conditions such that both distributions are integrable which are involved in the definition of a contact CR-submanifold. These results are very useful to prove the theorems on the characterizations.

**Chapter 4** can be considered as a slant version of the previous study. This chapter presents some special inequalities which turn out as fundamental existence theorems. A simple characterization theorem is proved for warped product pseudo-slant submanifolds in a locally Riemannian manifold with mixed totally geodesic conditions. Moreover, some examples to ensure the existence of different types of pseudo-slant submanifold and warped product pseud-slant were derived.

Integrability conditions were imposed in this Chapter. Finally, geometric inequalities of mixed totally geodesic warped product pseudo-slant immersed into a nearly Kenmotsu manifold and nearly Sasakian manifolds were constructed.

**Chapter 5** to the contrary of a semi-slant case, a new class of Riemannian submanifolds was introduced which is called pointwise semi-slant submanifold. First, we defined pointwise semi-slant submanifold in Kaehler, cosymplectic and Sasakian manifolds, and then showed that the existence of non-trivial warped product pointwise semi-slant submanifold in such manifolds by their characterization theorems. Moreover, we show the existence of a wide class of warped product submanifolds in the Riemannian manifolds possessing the minimality properties. In addition, some important and interesting results which are used widely to modified the equality case of the inequalities were also presented. Further, an interesting inequality for the second fundamental form in terms of scalar curvature by mean of Gauss equation was obtained, and some applications of them are derived with considering the equality cases. A number of triviality results for the warped product pointwise semi-slant isometrically immersed into complex space forms, or cosymplectic space forms or Sasakian space forms, various physical sciences terms such as Hessian, Hamiltonian and kinetic energy were also presented. These generalize the triviality results for CR-warped products.

**Chapter 6** presents the study of doubly warped products. Some optimal inequalities in terms of the various curvatures and associated with the warping functions for C-totally real doubly warped products isometrically immersed into a locally conformal almost cosymplectic manifold with pointwise  $\varphi$ -sectional curvature c were constructed. The doubly warped product in a locally conformal almost cosymplectic manifold which satisfies the equality case of the inequality was also examined. Finally, the upper and lower bounds of the associated warping functions were given. Moreover, we obtained Chen's inequalities for warped product semi-slant isometrically immersed into Kenmotsu space forms.

Chapter 7 presents the summary of the study and the suggestion for further research.

#### **CHAPTER 2**

#### PRELIMINARIES AND DEFINITIONS

#### 2.1 INTRODUCTION

Definitions, formulas and basic lemmas are explored briefly in this section. In the first subsection, we introduce the notions of Riemannian manifolds, the Levi-Civitas connection and the Riemannian curvature tensor. After that, we discuss the intrinsic geometry of such manifolds. Meaning that, many intrinsic invariants are defined systematically, such as sectional curvature, scalar curvature, Ricci curvature, Riemannian invariants and Chen first invariant. The second subsection is devoted for almost Hermitian and almost contact structures. The geometry of Riemannian submanifolds is discussed in the third subsection. After that, warped products are treated as Riemannian submanifolds. Warped products are defined as Riemannian manifolds in the last subsection.

#### 2.1.1 Manifolds

**Definition 2.1.1.** Let  $\widetilde{M}$  be a topological space. Then the pair  $(U, \phi)$  is called a chart, if  $\phi$ :  $U \to \phi(U) \subset \mathbb{R}^n$  is a homeomorphism of the open set U in  $\widetilde{M}$  onto an open set  $\phi(U) \subset \mathbb{R}^n$ . The coordinate function on U are defined as  $x^j : U \to \mathbb{R}^n$ , and  $\phi(p) = (x^1(p), \dots, x^j(p))$ , presented as  $x^j = u^j o \phi$ , where  $u^j : \mathbb{R}^n \to \mathbb{R}$ ,  $u^j(a_1 \dots, a_n) = a_j$  is the  $j^{th}$  projections and n is called the dimension of coordinate system.

**Definition 2.1.2.** A topological space  $\widetilde{M}$  is called Hausdorff, if for every two different point  $x_1, x_2 \in M$ , there exists two open sets  $U_1, U_2 \subset \widetilde{M}$  such that

$$x_1 \in U_1, x_2 \in U_2, U_1 \cap U_2 = \emptyset.$$

**Definition 2.1.3.** An atlas  $\mathscr{A}$  of dimension *n* associated with the topological space  $\widetilde{M}$  is collection of all charts  $(U_{\alpha}, \phi_{\alpha})$  such that

- (i)  $U_{\alpha} \subset M$ ,  $\cup_{\alpha} U_{\alpha} = M$
- (i) If  $U_{\alpha} \cap U_{\beta} \neq \emptyset$ , the map

$$F_{\alpha\beta} = \phi_{\alpha} \circ \phi_{\beta}^{-1} : \phi_{\beta}(U_{\alpha} \cap U_{\beta}) \to \phi_{\beta}(U_{\alpha} \cap U_{\beta})$$

is smooth.

The two atlas  $\mathscr{A}$  and  $\mathscr{A}'$  are called compatible if their union is again an atlas on topological space  $\widetilde{M}$ . However, the set of all compatible atlas can be organized by inclusion. The maximal element is called the *complete* atlas  $\mathscr{C}$ .

**Definition 2.1.4.** A smooth manifold  $\widetilde{M}$  is a Hausdorff space endowed with a complete atlas. The dimension *n* of an atlas is said to be dimension of the manifold  $\widetilde{M}$ .

*Example* 2.1.1. The space  $\mathbb{R}^n$  is a smooth manifold of dimension *n* defined by only one chart, that is identity map.

**Definition 2.1.5.** A function  $f: \widetilde{M} \to \mathbb{R}$  is called smooth function, if for every  $(U, \phi)$  on  $\widehat{M}$ , the function  $fo\phi^{-1}: \phi(U) \to \mathbb{R}$  is smooth. So, the set of all smooth function defined on manifold  $\widetilde{M}$  is denoted by  $\mathscr{F}(\widetilde{M})$ .

**Definition 2.1.6.** A tangent vector at a point  $p \in \widetilde{M}$  is a map  $X_p : \mathscr{F}(\widetilde{M}) \to \mathbb{R}$  such that

(i)  $X_p(af+bg) = aX_p(f) + bX_p(g), \ \forall a, b \in \mathbb{R} \ and \ \forall f, g \in \mathscr{F}(M)$ 

(ii) 
$$X_p(f.g) = X_p(f)g(p) + X_p(g)f(p), \ \forall f,g \in \mathscr{F}(M).$$

The second equation is called Leibnitz rule.

Moreover, the set of all tangent vector at a point  $p \in \widetilde{M}$  is called tangent space and it is denoted by  $T_p\widetilde{M}$ , and this is a vector space with respect to smooth function of dimension n.

The physical notion of velocity corresponds to the geometrical concept of a vector field. The following definition is that there is a reference system in which n - 1 components of the vector vanish and the  $n^{th}$  components is equal to one, i.e.,

**Definition 2.1.7.** A smooth map  $X : \widetilde{M} \to \bigcup_{p \in \widetilde{M}} T_p \widetilde{M}$  that assigns to each point  $p \in \widetilde{M}$ , then a vector  $X_p$  in  $T_p \widetilde{M}$  is called a vector field.

Similarly, the set of all vector fields on  $\widetilde{M}$  will be denoted by  $\Gamma(T\widetilde{M})$  throughout in this study. In a local coordinate system of vector field is defined as  $X = \sum X^i \frac{\partial}{\partial x_i}$ , where the components  $X^i \in \Gamma(T\widetilde{M})$  are given by  $X^i = X(x_i)_{i=1\cdots n}$ . Now we give the definition of differential map,

**Definition 2.1.8.** A map  $F : \widetilde{M} \to \widetilde{N}$  between two manifolds  $\widetilde{M}$  and  $\widetilde{N}$  is smooth map at point  $p \in M$ , for any charts  $(U, \psi)$  on about p and  $(V, \phi)$  on  $\widetilde{N}$  at point F(p), the application  $\psi \circ F \circ \phi^{-1}$  is smooth from  $\phi(U) \subset \mathbb{R}^m$  and  $\psi(V) \subset \mathbb{R}^n$ .

**Definition 2.1.9.** For every point  $p \in \widetilde{M}$ , the differentiable map dF at p is defined by  $dF_p: T_P\widetilde{M} \to T_{F(p)}\widetilde{N}$  with

$$(dF_p)(V)(f) = V(f \circ F), \ \forall V \in \Gamma(T_p\widetilde{M}), \ f \in \mathscr{F}(\widetilde{N}).$$

$$(2.1.1)$$

Locally, it is given by

$$dF\left(\frac{\partial}{\partial x_j|p}\right) = \sum_{k=1}^n \frac{\partial F^k}{\partial x_j|p} \frac{\partial}{\partial y^k_{|F(p)}},$$
(2.1.2)

where  $F = (F^1 \cdots F^n)$ . The matrix  $(\frac{\partial F^k}{\partial x_j})_{k,j}$  is the Jacobian matrix of F with respect to the chart  $(x^1 \cdots x^m)$  and  $(y^1 \cdots y^n)$  on  $\widetilde{M}$  and  $\widetilde{N}$ , respectively. There is an important operation on vector field is the Lie bracket  $[,]: T\widetilde{M} \times T\widetilde{M} \to T\widetilde{M}$  defined by

$$[X,Y] = XY - YX. (2.1.3)$$

In coordinates system

$$[X,Y] = \sum_{i=1}^{n} \left( \frac{\partial Y^{i}}{\partial x_{j}} X^{j} - \frac{\partial X^{i}}{\partial x_{j}} \right) \frac{\partial}{\partial x_{i}}.$$
(2.1.4)

#### 2.1.2 Riemannian Manifolds

There are manifolds which we may want to measure distance, angles, and lengths of vectors and curves. From the math point of view they represent generalizations of the surfaces more than two dimensions. From the mechanics point of view, they constitute the models for the coordinates spaces of dynamical systems. Their tangent space represent the phase space. The metric they are endowed with allows measuring the energy and constructing Lagrangian on the phase space and Hamiltonian on the cotangent bundle.

This way, Riemannian geometry becomes an element frame and proper environment for doing Classical Mechanics.

**Definition 2.1.10.** A Riemannian metric g on a smooth manifold  $\widetilde{M}$  is a symmetric, positive definite (0,2)-tensor filed.

The above definition means that  $\forall p \in \widetilde{M}, g_p : T_p \widetilde{M} \times T_p \widetilde{M} \to \mathbb{R}$ . Furthermore, in local coordinates

$$g = g_{ij}dx^i \oplus dx^j. \tag{2.1.5}$$

**Definition 2.1.11.** A differential manifold  $\widetilde{M}$  endowed with a Riemannian metric g is said to be a Riemannian manifold and its denoted as  $(\widetilde{M}, g)$ .

The existence of Riemannian metric is given by the following result:

**Theorem 2.1.1.** Let  $\widetilde{M}$  be a smooth manifold. Then there exists at least one Riemannian metric on  $\widetilde{M}$ .

*Proof.* Denote by <,> the Euclidean scalar product on  $\mathbb{R}^n$ , and consider the immersion  $\phi: M \to \mathbb{E}^{2n+1}$  given by the Whitney Imbedding Theorem

$$g(X,Y) = \langle \phi_* X, \phi_* Y \rangle, \quad \forall X, Y \in \Gamma(TM).$$
(2.1.6)

Then (M,g) is a Riemannian manifold.

Similarly, the linear connection is an extension of the directional derivative from the Euclidean case.

**Definition 2.1.12.** A linear connection  $\widetilde{\nabla}$  on a smooth manifold  $\widetilde{M}$  is a map  $\widetilde{\nabla} : T\widetilde{M} \times T\widetilde{M} \to T\widetilde{M}$  with the following properties.

- (i)  $\widetilde{\nabla}_X Y$  is  $\mathscr{F}(\widetilde{M})$  linear in X.
- (ii)  $\widetilde{\nabla}_X Y$  is  $\mathbb{R}$ -linear in Y.

(iii) It satisfies the Leibnitz rule, i.e.,  $\widetilde{\nabla}_X(fY) = X(f)Y + f\widetilde{\nabla}_X Y, \ \forall f \in \mathscr{F}(\widetilde{M})$ 

for any  $X \in \Gamma(T\widetilde{M})$ .

If g is a Riemannian metric tensor, the linear connection  $\widetilde{\nabla}$  is called a metric connection when

$$\widetilde{\nabla}_V g = 0, \ \forall V \in \Gamma(TM).$$
(2.1.7)

It means that

$$Vg(X,Y) = g(\widetilde{\nabla}_V X, Y) + g(\widetilde{\nabla}_V Y, X), \ V, X, Y \in \Gamma(T\widetilde{M}).$$
(2.1.8)

The fascinating fact is that, there is only one metric connection that has zero torsion. This constitutes the geometry of Riemannian manifolds. The following theorem can be considered as definition of the Levi-Civitas connection.

**Theorem 2.1.2.** On a Riemannian manifold there is a unique torsion-free metric connection  $\widetilde{\nabla}$ . Furthermore,  $\widetilde{\nabla}$  is given by the Koszul formula

$$2g(\tilde{\nabla}_{Z}X,Y) = Zg(X,Y) + Xg(Y,Z) + Yg(X,Z) -g(Z,[X,Y]) + g(X,[Y,Z]) + g(Y,[X,Z]),$$
(2.1.9)

for any  $X, Y, Z \in \Gamma(T\widetilde{M})$ .

# 2.2 ALMOST HERMITIAN AND ALMOST CONTACT METRIC MANIFOLDS

In this section, we study the geometrical and topological properties in almost Hermitian and almost contact metric manifolds. Moreover, some very important formulas are given which we will require further study.

#### 2.2.1 Almost Hermitian manifolds

Almost complex structure on a real vector space V is an endomorphism such that

$$J: V \to V,$$

which satisfying  $J^2 = -I_d$ , where,  $J^2 = J \circ J$  and  $I_d$  is identity map  $I : V \to V$ . A real vector space endowed with an almost complex structure is almost complex vector space and denoted by (V,J). Some fundamental remarks and examples on almost complex vector spaces are the following.

*Remark* 2.2.1. Every complex vector space is an almost complex vector space.

*Example* 2.2.1. C itself is a vector space having complex dimension one.

*Remark* 2.2.2. For every integer *n*, the flat space  $\mathbb{R}^{2n}$  admits an almost complex structure. An example for such an almost complex structure is  $(1 \le i, j \le 2n) : J_{ij} = -\delta_{i,j-1}$  for odd  $i, J_{ij} = \delta_{i,j+1}$  for even *i*.

**Remark 2.2.3.** An almost complex vector space can be turned into a complex vector space using J to make V into a complex vector space such that the multiplication by complex numbers extends the multiplication by reals.

*Remark* 2.2.4. In an almost complex vector space (V, J), the vector X and JX are linearly independent.

*Note.* It is clear that from the above study that minimum dimension of an almost complex vector space is 2.

**Definition 2.2.1.** An inner product h on almost complex vector space (V, J) is said to be a Hermitian inner product if the following invariancy condition is satisfied

$$h(JX,JY) = h(X,Y),$$

for all  $X, Y \in V$ .

**Definition 2.2.2.** Let  $\widetilde{M}$  be a smooth manifold. If at each point  $p \in \widetilde{M}$  there exists an endomorphism  $J_P: T_p\widetilde{M} \to T_p\widetilde{M}$  such that  $J^2 = -I_d$  and the map  $J: p \to J_p$  is  $C^{\infty}$ -map, then J is said to be an almost complex structure on  $\widetilde{M}$ , and the manifold  $\widetilde{M}$  endowed with fixed almost complex structure is called almost complex manifold. It is denoted by  $(\widetilde{M}, J)$ .

**Definition 2.2.3.** Let  $(\widetilde{M}, J)$  be an almost complex manifold and g Riemannian metric such that  $g(JX, JY) = g(X, Y), \ \forall X, Y \in \Gamma(T\widetilde{M})$ . That is, g is a Hermitian metric on  $\widetilde{M}$ . In this case, the manifold  $(\widetilde{M}, J, g)$  is an almost Hermitian manifold. A complex manifold endowed with a Hermitian metric is called an almost Hermitian manifold.

**Definition 2.2.4.** Let  $(\widetilde{M}, J, g)$  be an almost Hermitian manifold. Then we define a fundamental 2-form  $\Phi$  on  $\widetilde{M}$  as

$$\Phi(X,Y) = g(X,JY), \ \forall X,Y \in \Gamma(TM).$$

**Definition 2.2.5.** Let  $(\widetilde{M}, J, g)$  be an almost Hermitian manifold. Then the Nijenhuis tensor of J is as follows.

$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y],$$

for any  $X, Y \in \Gamma(T\widetilde{M})$ .

*Remark* 2.2.5. The Nijenhuis tensor of complex structure is trivial, i.e., zero.

*Remark* 2.2.6. The converse of the Remark 2.2.5 also true. Hence an almost complex structure is complex if and only if  $N_J = 0$ .

**Definition 2.2.6.** Let  $\widetilde{\nabla}$  be a connection on an almost complex manifold  $(\widetilde{M}, J)$ . Then J is called parallel on  $\widetilde{M}$  if  $\widetilde{\nabla}_X J = 0, \forall X \in \Gamma(T\widetilde{M})$ .

**Definition 2.2.7.** A p-form  $\omega$  is said to be a closed form if  $d\omega = 0$ . Moreover, a p-form  $\Omega$  is called an exact form if there exists a p-1-form  $\alpha$  such that  $d\alpha = \Omega$ .

**Remark 2.2.7.** Clearly, every exact form is closed, i.e.,  $d\omega = 0$ , for any form  $\omega$ .

**Theorem 2.2.1.** Let  $\widetilde{M}$  be an almost Hermitian manifold with an almost complex structure J and Hermitian metric g. Then the Riemannian connection  $\widetilde{\nabla}$  on g, the fundamental 2-form  $\Phi$  and the Nijenhuis tensor  $N_J$  of J satisfy

$$2g((\widetilde{\nabla}_X J)Y, Z) = 3d\Phi(X, JY, JZ) - 3d\Phi(X, Y, Z) + g([Y, Z], JX).$$
(2.2.1)

*Corollary* 2.2.1. If  $\widetilde{M}$  is an almost Hermitian manifold with fundamental 2-form closed, then the complex structure J is parallel on  $\widetilde{M}$ .

**Definition 2.2.8.** A Hermitian metric g on an almost Hermitian manifold  $(\widetilde{M}, J, g)$  is called Kaehler metric if the fundamental 2-form is closed on  $\widetilde{M}$ . Further, a complex manifold endowed with a Kaehler metric is said to be a Kaehler manifold.

From the above definition it is concluded that on a Kaehler manifold, J is parallel. Thus the structure equation of Kaehler manifold is characterized as

$$(\widetilde{\nabla}_X J)Y = 0, \ \forall X, Y \in \Gamma(T\widetilde{M}).$$
 (2.2.2)

**Definition 2.2.9.** An almost Hermitian manifold  $(\widetilde{M}, J, g)$ , is said to be a nearly Kaehler manifold if the following satisfies

$$(\overline{\nabla}_X J)X = 0, \tag{2.2.3}$$

which is also equivalently

$$(\widetilde{\nabla}_X J)Y + (\widetilde{\nabla}_Y J)X = 0, \quad \forall X, Y \in \Gamma(T\widetilde{M})$$
(2.2.4)

*Remark* 2.2.8. Every Kaehler manifold is nearly Kaehler. But converse not true.

*Example 2.2.2.* The unit sphere  $S^6$  is a non-Kaehler, nearly Kaehler manifold.

#### 2.2.2 Almost Contact Metric Manifolds

An almost contact manifold is an odd-dimensional manifold  $\widetilde{M}$  which carries a field  $\varphi$  of endomorphisms of tangent space, vector field  $\xi$ , is called *characteristic* or *Reeb vector field* and 1-form  $\eta$  satisfying

$$\varphi^2 = -I + \eta \oplus \xi, \quad \eta(\xi) = 1, \tag{2.2.5}$$

where  $I: T\widetilde{M} \to T\widetilde{M}$  is the identity mapping. Now, from the definition it follows that

$$\varphi \circ \xi = 0 \quad and \quad \eta \circ \varphi = 0, \tag{2.2.6}$$

then the (1,1) tensor field  $\varphi$  has constant rank 2n. An almost contact manifold  $(\widetilde{M}, \varphi, \eta, \xi)$ is said to be *normal* when the tensor field  $N_{\varphi} = [\varphi, \varphi] + 2d\eta \oplus \xi$  vanishes identically, where  $[\varphi, \varphi]$  is the *Nijenhuis tensor* of  $\varphi$ . An almost contact metric structure  $(\varphi, \xi, \eta)$  is said to be a normal in the form of almost complex structure if almost complex structure Jon a product manifold  $\widetilde{M} \times R$  given by

$$J\left(X,\alpha\frac{d}{dt}\right) = \left(\varphi X - \alpha\xi, \eta(X)\frac{d}{dt}\right),\tag{2.2.7}$$

where  $\alpha$  is a smooth function on  $\widetilde{M} \times R$  has no torsion, i.e., *J* is integrable. There always exits a Riemannian metric *g* on an almost contact manifold  $(\widetilde{M}, \varphi, \eta, \xi)$  which is satisfies the following compatibility condition:

(*i*) 
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$
, (*ii*)  $\eta(X) = g(X, \xi)$ , (2.2.8)

for all  $X, Y \in \Gamma(T\widetilde{M})$ . Then this metric *g* is called *compatible metric* and the manifold  $\widetilde{M}$  endowed with almost contact structure  $(\varphi, \eta, \xi, g)$  is called an almost contact metric manifold. As an immediate consequence of almost contact metric structure, i.e,  $g(\varphi X, Y) = -g(X, \varphi Y)$  and the fundamental 2-form  $\Phi$  is defined by  $\Phi(X, Y) = g(X, \varphi Y)$ . Several types of almost contact structures are defined in the literature such as normal almost cosymplectic, cosymplectic, Sasakian, quasi-Sasakian, normal contact, Kenmotsu, Trans-Sasakian. An almost contact metric manifold such that both  $\eta$  and  $\Phi$  are closed is called almost cosymplectic manifold and those for which  $d\eta = \Phi$  are called contact metric manifolds. Finally, a normal almost cosymplectic manifold is called cosymplectic manifold, a globally Riemannian product of line or circle with Kaehler manifold of holomorphic sectional curvature. The cosymplectic manifolds are characterized by tensorial equation, i.e.,

(i) 
$$(\widetilde{\nabla}_X \varphi) Y = 0$$
, and (ii)  $\widetilde{\nabla}_X \xi = 0$ , (2.2.9)

Similarly, a normal contact manifold is called Sasakian manifold and defined by Sasaki (1960). Sasakian manifolds involving the covariant derivative of  $\varphi$  can be expressed as;

(a) 
$$(\widetilde{\nabla}_X \varphi) Y = -g(X, Y)\xi + \eta(Y)X,$$
 (b)  $\widetilde{\nabla}_X \xi = \varphi X,$  (2.2.10)

for any  $X, Y \in \Gamma(T\widetilde{M})$ . It should be noted that both in cosymplectic and Sasakian manifolds  $\xi$  is killing vector field. On the other hand, the Sasakian and cosymplectic manifolds represent the two external cases of the larger class of quasi-Sasakian manifolds (see D. E. Blair (1967)). The structure is said to be a closely cosymplectic, if  $\varphi$  is killing and  $\eta$  closed. The concept of Kenmotsu manifolds were initiated by Kenmotsu (1972). He proved that a warped product  $I \times_f \widetilde{M}$  of Kaehler manifold  $\widetilde{M}$  and an interval I is called Kenmotsu manifold with warping function  $f = e^t$ . Hence, the Kenmotsu manifolds are characterized in terms of the covariant derivative  $\varphi$  and Levi-Civitas connection as

(i) 
$$(\widetilde{\nabla}_X \varphi) Y = g(\varphi X, Y) \xi - \eta(Y) \varphi X$$
, (ii)  $\widetilde{\nabla}_X \xi = X - \eta(X) \xi$ . (2.2.11)

In this case the holomorphic sectional curvature is strictly less than zero. Other than these classes of almost contact metric manifolds. As a generalization of cosymplectic, Sasakian

and Kenmotsu structures of an almost contact metric structure  $(\varphi, \eta, \xi)$  on  $\widetilde{M}$  is called a Trans-Sasakian structure (see D. E. Blair (2010)) if and only if

$$(\widetilde{\nabla}_X \varphi)Y = \alpha \Big( g(X,Y)\xi - \eta(Y)X \Big) + \beta \Big( g(\varphi X,Y)\xi - \eta(Y)\varphi X \Big), \qquad (2.2.12)$$

for any smooth function  $\alpha$  and  $\beta$  defined on  $\widetilde{M}$ . It is called  $(\alpha, \beta)$  type Trans-Sasakian manifold endowed with  $\widetilde{M}$ . Moreover, it can be conclude that Trans-Sasakian manifold include cosymplectic, Sasakian and Kenmotsu manifolds such that  $\alpha$ ,  $\beta = 0$ ,  $\alpha = 1, \beta = 0$  and  $\alpha = 0, \beta = 1$ , respectively. There are many others different structures are appeared as a nearly case. An almost contact metric structure  $(\varphi, \eta, \xi)$  is said to be nearly cosymplectic structure, if  $\varphi$  is killing, then

$$(\widetilde{\nabla}_X \varphi) Y = 0$$
 or equivalently  $(\widetilde{\nabla}_X \varphi) Y + (\widetilde{\nabla}_Y \varphi) X = 0,$  (2.2.13)

for any X, Y are tangent to  $\widetilde{M}$ , where  $\widetilde{\nabla}$  is the Riemannian connection on the metric g on  $\widetilde{M}$ . A manifold  $\widetilde{M}$  endowed with this almost contact metric structure  $(\varphi, \eta, \xi)$  is called nearly cosymplectic manifold.

## **Theorem 2.2.2.** On a nearly cosymplectic manifold $\varphi$ is Killing.

From the above Theorem, one has,

$$g(\tilde{\nabla}_X \xi, Y) + g(\tilde{\nabla}_X \xi, Y) = 0, \qquad (2.2.14)$$

for any vector fields X, Y are tangent to  $\widetilde{M}$ , where  $\widetilde{M}$  is a nearly cosymplectic manifold. Similarly, the notion of nearly Sasakian manifold are studied by D. Blair et al. (1976). The structure equation of nearly Sasakian manifold is defined as in the tensorial form

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X.$$
(2.2.15)

Moreover, class of nearly Kenmotsu manifold is defined by

$$(\widetilde{\nabla}_X \varphi) Y + (\widetilde{\nabla}_Y \varphi) X = -\eta(Y) \varphi X - \eta(X) \varphi Y, \qquad (2.2.16)$$

for any  $X, Y \in \Gamma(T\widetilde{M})$ . Furthermore, Erken et al. (2016) are obtained the following result for nearly Kenmotsu manifold as:

Lemma 2.2.1. On nearly Kenmotsu manifold, we have

$$g(\tilde{\nabla}_X \xi, Y) + g(\tilde{\nabla}_X \xi, Y) = 2g(\varphi X, \varphi Y), \quad \widetilde{\nabla}_X \xi = -\varphi^2 X + HX, \quad (2.2.17)$$

where H is the mean curvature vector of M.

#### 2.3 **RIEMANNIAN SUBMANIFOLDS**

This section is devoted to study Riemannian submanifold of Riemannian manifolds and their basic fundamental properties are discussed.

Let M and  $\widetilde{M}$  be smooth manifolds of dimensions n and m, respectively and  $F: M^n \to \widetilde{M}^m$  be a differentiable map then F is called an immersion if rank(F) = n. A one-to-one immersion is called an imbedding. Suppose M is a subset of  $\widetilde{M}$  and the inclusion map  $i: M \to \widetilde{M}$  is an imbedding. Then M is said to be a submanifold of  $\widetilde{M}$ .

**Remark 2.3.1.** The difference in the dimension, that (m-n) is known as the co-dimension of M in  $\widetilde{M}$ . If the difference is one, then submanifold is called hypersurface.

Let  $(\widetilde{M}, \widetilde{g})$  be a Riemannian manifold and  $M \subset \widetilde{M}$  be a submanifold, then we can pull back Riemannian metric  $\widetilde{g}$  onto M as;

$$g(X,Y) = \widetilde{g}(i_*X,i_*Y), \qquad (2.3.1)$$

for any  $X, Y \in \Gamma(TM)$ . Moreover, the differentiable map  $i_* : TM \to T\widetilde{M}$  preserves the above Riemannian metric which is called an *isometric immersion*. It is easy to check that g is a Riemannian metric on M, we call g as the induced metric on M. Clearly, if  $\widetilde{M}$  is a Riemannian manifold, then any submanifold of  $\widetilde{M}$  is Riemannian under the induced Riemannian metric g.

**Definition 2.3.1.** A submanifold  $M \subset \widetilde{M}$  associate with induced Riemannian metric g is called Riemannian submanifold of Riemannian manifold  $\widetilde{M}$ .

Now we shall identify TM with  $i_*(TM)$  by the isomorphism  $i_*$ . A tangent vector in TM to M shall mean tangent vector which is the image of an element in TM under  $i_*$ . More specific, a  $C^{\infty}$ -cross section of the restriction of  $T\widetilde{M}$  on M shall be called a vector field of  $\widetilde{M}$  of M. Those tangent vectors of  $T\widetilde{M}$ , which are normal to TM form the normal bundle

 $T^{\perp}M$  of M. Hence, for every point  $p \in M$ , the tangent space  $T_{i_*(p)}\widetilde{M}$  of  $\widetilde{M}$  admits the following decomposition as:

$$T_{i_*(p)}\widetilde{M}=TM\oplus T^{\perp}M.$$

The Riemannian connection  $\widetilde{\nabla}$  of  $\widetilde{M}$  induced canonically the connection  $\nabla$  and  $\nabla^{\perp}$  on *TM* and on the normal bundle  $TM^{\perp}$ , respectively governed by the Gauss and Wiengarten formulas are defined

$$\nabla_X Y = \nabla_X Y + h(X, Y), \qquad (2.3.2)$$

$$\widetilde{\nabla}_X N = -A_N + \nabla_X^{\perp} N, \qquad (2.3.3)$$

where X, Y are tangent to M and N is normal to M, respectively. Moreover, h and A are the second fundamental form and Wiengarten operator, respectively. Then they are related as:

$$g(h(X,Y),N) = g(A_N X,Y).$$
 (2.3.4)

**Definition 2.3.2.** If the second fundamental form h = 0 on a submanifold M of a Riemannian manifold  $\widetilde{M}$ , then submanifold M is called totally geodesic submanifold.

**Definition 2.3.3.** A vector field H normal to submanifold M of  $\widetilde{M}$  defined by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

is known as the mean curvature vector, where  $\{e_1 \cdots e_i\}$  is a frame of orthonormal vector fields on *M* and *n* is the dimension of *M*.

Definition 2.3.4. If the second fundamental h is satisfies the following

$$h(X,Y) = g(X,Y)H.$$

Then the submanifold M of  $\widetilde{M}$  is called totally umbilical submanifold.

*Definition* **2.3.5.** A submanifold for which the mean curvature *H* is identically zero, then it is called a minimal submanifold.

Now we define the covariant differentiation of  $\widetilde{\nabla}$  with respect to Riemannian connection
in  $TM \oplus T^{\perp}M$  of the second fundamental form is given by

$$(\widetilde{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y,Z) - h(\nabla_X Z,Y)$$
(2.3.5)

Let  $\widetilde{R}$  and R denote the curvature tensor of Riemannian manifolds  $\widetilde{M}$  and M, respectively. Then the equations of Gauss, Codazzi, and Rici are given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,Z),h(Y,W) + g(h(X,W),h(Y,Z)).$$
(2.3.6)

$$(\widetilde{R}(X,Y)W)^{\perp} = (\widetilde{\nabla}_X h)(Y,Z) - (\widetilde{\nabla}_Y h)(X,Z).$$
(2.3.7)

$$\widetilde{R}(X,Y,N_1,N_2) = R^{\perp}(X,Y,N_1,N_2) - g([A_{N_1},A_{N_2}]X,Y), \qquad (2.3.8)$$

where  $(\widetilde{R}(X,Y)W)^{\perp}$  normal components of  $\widetilde{R}$  and  $R^{\perp}$  in (2.3.8) denote the curvature tensor corresponding to the normal connection  $\nabla^{\perp}$  in  $T^{\perp}M$ , and  $X, Y, Z, W \in \Gamma(TM)$  and  $N_1, N_2 \in \Gamma(T^{\perp}M)$ , respectively. Also, we have

$$||h||^{2} = \sum_{i,j=1}^{n} g(h(e_{i},e_{j}),h(e_{i},e_{j})) \quad and \quad h_{ij}^{r} = g(h(e_{i},e_{j}),e_{r}),$$
(2.3.9)

for  $\{e_i\}_{i=1,\dots,n}$ , and  $\{e_r\}_{r=n+1,\dots,2m}$ , are orthonormal frames tangent to M and normal to M, respectively. The scalar curvature  $\tau$  for a submanifold M of an almost complex manifold  $\widetilde{M}$  is given by

$$\tau(TM) = \sum_{1 \le i \ne j \le n} K(e_i \land e_j).$$
(2.3.10)

At the above equation  $e_i$  and  $e_j$  spanned plane section and its sectional curvature is denoted by  $K(e_i \wedge e_j)$ . Let  $G_r$  be a r-plane section on TM and  $\{e_1, e_2, \dots e_r\}$  be any orthonormal basis of  $G_r$ . Then the scalar curvature  $\tau(G_r)$  of  $G_r$  is defined as:

$$\tau(G_r) = \sum_{1 \le i \ne j \le r} K(e_i \land e_j).$$
(2.3.11)

Now, let f be a differential function defined on M. Thus the gradient  $\vec{\nabla} f$  is given as:

$$g(\vec{\nabla}f, X) = Xf, \text{ and}, \ \vec{\nabla}f = \sum_{i=1}^{n} e_i(f)e_i.$$
 (2.3.12)

Thus, from the above equation, the Hamiltonian in a local orthonormal frame is defined

$$H(\nabla f, x) = \frac{1}{2} \sum_{j=1}^{n} df(e_j)^2 = \frac{1}{2} \sum_{j=1}^{n} e_j(f)^2 = \frac{1}{2} ||\nabla f||^2.$$
(2.3.13)

Moreover, the Laplacian  $\Delta f$  of f is also given by

by

$$\Delta f = \sum_{i=1}^{n} \{ (\nabla_{e_i} e_i) f - e_i(e_i(f)) \} = -\sum_{i=1}^{n} g(\nabla_{e_i} grad f, e_i).$$
(2.3.14)

Similarly, the Hessian of the function f is given by

$$\Delta f = -TraceH^{f} = -\sum_{i=1}^{n} H^{f}(e_{i}, e_{i}), \qquad (2.3.15)$$

where  $H^f$  is Hessian of function f. The compact Riemannian manifold M will be considered as without boundary, i.e.,  $\partial M = \emptyset$ . Thus, following lemma given by Calin & Chang (2006),

*Lemma* 2.3.1. (Hopf lemma) Let *M* be a compact connected Riemannian manifold and *f* be a smooth function on *M* such that  $\Delta f \ge 0$  ( $\Delta f \le 0$ ). Then *f* is a constant function on *M*.

Moreover, for a compact orientate Riemannian manifold *M* without boundary, the concept of integration on manifolds give the following formula,

$$\int_{M} \Delta f \mathrm{d}V = 0, \qquad (2.3.16)$$

such that dV denotes the volume of M (see Yano & Kon (1985)). At the case when M is manifold with boundary, the Hopf lemma becomes the uniqueness theorem for the Dirichlet problem. Thus we have following result,

**Theorem 2.3.1.** Let M be a connected, compact manifold and f a positive differentiable function on M such that  $\Delta f = 0$ , On M and  $f/\partial M = 0$ . Then f = 0, where  $\partial M$  is the boundary of M.

Moreover, let M be a compact Riemannian manifold and f be a positive differentiable function on M. Then the kinetic energy is defined as in Calin & Chang (2006), i.e,

$$E(f) = \frac{1}{2} \int_{M} ||\nabla f||^2 \mathrm{d}V.$$
 (2.3.17)

Since, *M* is compact, then  $0 < E(f) < \infty$ .

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Theorem 2.3.2. The Euler-Lagrange equation for the Lagrangian is

$$\Delta f = 0. \tag{2.3.18}$$

### 2.3.1 Submanifolds of an Almost Hermitian Manifold

Let  $(\widetilde{M}, J)$  be an almost Hermitian manifold with Hermitian metric g and M be a submanifold of  $\widetilde{M}$ . Following Chen (1990) notations, for any  $p \in M$ , we are defining tangential endomorphism  $T : T_pM \to T_pM$  and a normal valued linear map  $F : T_pM \to T^{\perp}M$  which are satisfying the following

$$JX = TX + FX, (2.3.19)$$

for any  $X \in \Gamma(T_p M)$ . Since, *T* is a (1,1) tensor field and *F* is called normal valued 1-form on *M*. Similarly for any normal vector field  $N \in \Gamma(T^{\perp}M)$  is defined by

$$JN = tN + fN, \qquad (2.3.20)$$

where tN and fN are called tangential and normal parts of JN, respectively. Moreover, the covariant derivatives of these tensor fields are defined as:

$$(\widetilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \tag{2.3.21}$$

$$(\widetilde{\nabla}_X F)Y = \nabla_X^{\perp} FY - F\nabla_X Y \tag{2.3.22}$$

$$(\widetilde{\nabla}_X t)N = \nabla_X tN - t\nabla_X^{\perp}N \qquad (2.3.23)$$

$$(\widetilde{\nabla}_X f)N = \nabla_X^{\perp} fN - f \nabla_X^{\perp} N$$
(2.3.24)

Assume that  $\widetilde{M}$  be a Kaehler manifold, thus from (2.2.2), (2.3.19), (2.3.20), and (2.3.21)-(2.3.24), we deduce that

$$(\nabla_U T)V = A_{FV}U + th(U,V), \qquad (2.3.25)$$

$$(\widetilde{\nabla}_U F)V = fh(U,V) - h(U,TV), \qquad (2.3.26)$$

$$(\widetilde{\nabla}_U t)N = A_{fN}U + TA_N X, \qquad (2.3.27)$$

$$(\nabla_U f)N = -h(tN,U) - FA_N U. \qquad (2.3.28)$$

Assume that  $\widetilde{M}$  be a Hermitian manifold and M is a submanifold of  $\widetilde{M}$ , the operation of almost complex structure J on the tangent bundle TM provides the various distribution on M such that

**Definition 2.3.6.** A submanifold M of an almost Hermitian manifold  $\widetilde{M}$ . The distribution  $\mathscr{D}$  is said to be a holomorphic(complex) if  $J\mathscr{D} \subseteq \mathscr{D}$ , i.e,  $J\mathscr{D}_x = \mathscr{D}_x$  for each  $x \in M$ .

**Definition 2.3.7.** A totally real distribution  $\mathscr{D}^{\perp}$  is defined by, i.e.,  $J\mathscr{D}^{\perp} \subseteq T^{\perp}M$ , that is  $J\mathscr{D}_x^{\perp} \subseteq T_x^{\perp}M$ , for every point  $x \in M$ .

A class of submanifolds of an almost Hermitian manifold considered by Bejancu (1979) which gives rise to a single setting to the concept of holomorphic and totally real distributions. These new class is called CR-submanifold which include holomorphic (complex) and totally real submanifolds. He has given the following definition

**Definition 2.3.8.** Let  $\widetilde{M}$  be an almost Hermitian manifold. Then a submanifold M of  $\widetilde{M}$  is called CR-submanifold, if there exists involute distribution  $\mathscr{D} : x \to \mathscr{D}_x \subseteq T_x M, x \in M$  on M and its orthogonal distribution  $\mathscr{D}^{\perp} : x \to \mathscr{D}_x^{\perp} \subseteq T_x^{\perp} M$  such that

- (i)  $TM = \mathscr{D} \oplus \mathscr{D}^{\perp}$ ,
- (ii)  $\mathscr{D}$  is holomorphic, i.e,  $J\mathscr{D} \subseteq \mathscr{D}$ ,
- (iii)  $\mathscr{D}^{\perp}$  is totally real, that is  $J\mathscr{D}^{\perp} \subseteq T^{\perp}M$ .

A CR-submanifold is classified as, if  $\mathscr{D} \neq 0$ , and  $\mathscr{D}^{\perp} \neq 0$ , then *M* is called proper CR-submanifold. *M* is totally real submanifold, if  $\mathscr{D} = 0$ . Similarly, *M* is said to be a holomorphic, if  $\mathscr{D}^{\perp} = 0$ . If the corresponding distribution of CR-submanifold are parallel on *M*, or in other words, both distribution are integrable and their leaves are totally geodesic in *M*, Then a CR-submanifold is called CR-product. The following are simple characterizations of CR-product in a Kaehler manifold which was proved by Chen (1981) such that

**Theorem 2.3.3.** A CR-submanifold of a Kaehler manifold is a CR-product if and only if T is parallel i.e.,

$$\widetilde{\nabla}T = 0.$$

Theorem 2.3.4. A CR-submanifold of a Kaehler manifold is a CR-product if and only if

$$A_{J \mathscr{D}^{\perp}} \mathscr{D} = 0.$$

Assume that  $T_xM - \{0\}$  be a set containing of all non-zero tangent vectors fields of immersion M into  $\widetilde{M}$  at a point  $x \in M$ . Then for each vector  $X \in (T_xM)$  at a point  $x \in M$ , the angle between JX and the tangent space  $T_xM$  is said to be Wirtinger angle of X at  $x \in M$  and its denoted by  $\theta(X)$ . In this case, a submanifold M of  $\widetilde{M}$  is called slant submanifold such that  $\theta$  is a slant angle. It is clear that, the slant submanifolds include totally real and holomorphic (complex) submanifolds. However, Chen (1990) proved the following characterization theorem of slant submanifolds such that

**Theorem 2.3.5.** Let  $\widetilde{M}$  is an almost Hermitian manifold and M be a submanifold of  $\widetilde{M}$ . Thus M is slant if and only if there exists a constant  $\lambda \in [0,1]$  such that

$$T^2 = -\lambda I, \qquad (2.3.29)$$

where  $\lambda = \cos^2 \theta$ , for slant angle  $\theta$  defined on the tangent bundle TM of M.

Hence, the following consequences of the Theorem 2.3.5

$$g(TX,TY) = \cos^2 \theta g(X,Y), \qquad (2.3.30)$$

$$g(FX, FY) = \sin^2 \theta g(X, Y), \qquad (2.3.31)$$

for any  $X, Y \in \Gamma(TM)$ .

There is another concept of submanifolds which is called semi-slant submanifold as natural generalization of slant submanifold, CR-submanifold, holomorphic and antiholomorphic submanifolds in almost Hermitian manifold. The semi-slant submanifolds were studied and defined by Papaghiuc (1994), a natural extension of CR-submanifold of in almost Hermitian manifold. He has given the following definition as:

**Definition 2.3.9.** A submanifold M of  $\widetilde{M}$  is defines as semi-slant submanifold, if there exists two complementary distributions  $\mathcal{D}$  and  $\mathcal{D}^{\theta}$  such that

- (i)  $TM = \mathscr{D} \oplus \mathscr{D}^{\theta}$ .
- (ii)  $\mathscr{D}$  is holomorphic distribution, i.e.,  $J(\mathscr{D}) = \mathscr{D}$ .

(iii)  $\mathscr{D}^{\theta}$  is called slant distribution such that slant angle  $\theta \neq 0, \frac{\pi}{2}$ .

**Remark 2.3.2.** On a semi-slant submanifold, let us consider the dimensions of  $\mathcal{D}$  and  $\mathcal{D}^{\theta}$  by  $d_1$  and  $d_2$ , then M is *holomorphic* if  $d_2 = 0$  and *slant* if  $d_1 = 0$ . Further, if  $\theta = \frac{\pi}{2}$  and  $d_1 = 0$ , then M is represented as anti-holomorphic (totally real) submanifold. Also, M is called a proper semi-slant submanifold, if  $\theta$  is different from 0 and  $\frac{\pi}{2}$ . We can also define that M is proper if  $d_1 \neq 0$  and  $d_2 \neq 0$ .

The orthogonal projections on  $\mathscr{D}$  and  $\mathscr{D}^{\perp}$  are denoted by *B* and *C*, respectively. Then, for any  $U \in \Gamma(TM)$ , we define

$$U = BU + CU, \tag{2.3.32}$$

where  $BU \in \Gamma(\mathscr{D})$  and  $CU \in \Gamma(\mathscr{D}^{\perp})$ . From (2.3.19), (2.3.20) and (2.3.32), we have

$$TU = JBU, \quad FU = JCU. \tag{2.3.33}$$

*Remark* 2.3.3. If v is an invariant normal subspace under almost complex structure J of normal bundle  $T^{\perp}M$ , thus the normal bundle is decomposed as:

$$T^{\perp}M = F\mathscr{D}^{\theta} \oplus \mathcal{V}. \tag{2.3.34}$$

On a semi-slant submanifold M of an almost Hermitian manifold  $\widetilde{M}$ , the following are straightforward observations

$$\begin{array}{l} (i) \ F\mathscr{D} = 0, \qquad (ii) \ T\mathscr{D} = \mathscr{D}, \\ \\ (iii) \ t(T^{\perp}M) \subseteq \mathscr{D}^{\theta}, \quad (iv) \ T\mathscr{D}^{\theta} \subseteq \mathscr{D}^{\theta}. \end{array} \right\}$$

$$(2.3.35)$$

A special class of submanifold which another generalization of CR-submanifold, is the class of pseudo-slant submanifold. Carriazo (2000) extended the concept of anti-slant submanifolds to the Kaehlerian manifold as:

**Definition 2.3.10.** A submanifold M of an almost Hermitian  $\widehat{M}$  is called pseudo-slant submanifold if there exist two orthogonal complimentary distribution  $\mathscr{D}^{\perp}$  and  $\mathscr{D}^{\theta}$  such that  $\mathscr{D}^{\perp}$  is totally real distribution and  $\mathscr{D}^{\theta}$  is slant distribution with slant angle  $\theta$  which satisfies the following

$$TM = \mathscr{D}^{\perp} \oplus \mathscr{D}^{\theta}.$$

In this case, the normal bundle  $T^{\perp}M$  admits the following orthogonal direct sum decomposition

$$TM = J\mathscr{D}^{\perp} \oplus F\mathscr{D}^{\theta} \oplus \mathbf{v},$$

where v is the orthogonal complement of  $J\mathscr{D}^{\perp} \oplus F\mathscr{D}^{\theta}$  in *TM*. Let  $m_1$  and  $m_2$  be the dimensions of distributions  $\mathscr{D}^{\perp}$  and  $\mathscr{D}^{\theta}$ , respectively. Thus if

- (i)  $m_2=0$ , then *M* is totally real submanifold.
- (ii)  $m_1=0$  and  $\theta = 0$ , then M is holomorphic submanifold.
- (iii)  $m_1=0$  and  $\theta \neq 0, \frac{\pi}{2}$ , then *M* is proper slant submanifold.
- (iv)  $\theta = \frac{\pi}{2}$ , then *M* is totally real submanifold.
- (v)  $\theta = 0$ , then *M* is CR-submanifold.
- (vi)  $\theta \neq 0, \frac{\pi}{2}$ , then *M* is proper pseudo-slant submanifold.

*Definition* 2.3.11. A pseudo-slant submanifold of an almost Hermitian manifold is called mixed totally geodesic if and only if the second fundamental form satisfies h(X, Z) = 0, for every  $X \in \Gamma(\mathcal{D}^{\theta})$  and  $Z \in \Gamma(\mathcal{D}^{\perp})$ .

Let us define a new class which is called pointwise slant submanifold. It has been studied in almost Hermitian manifolds by Chen & Garay (2012). They defined these submanifolds as follows:

**Definition 2.3.12.** Assume that the set  $T^*M$  consisting of all non-zero tangent vectors on a submanifold M of an almost Hermitian manifold  $\widetilde{M}$ . Then for any nonzero vector  $X \in$  $\Gamma(T_xM), x \in M$ , the angle  $\theta(X)$  between JX and tangent space  $T_xM$  is called the Wirtinger angle of X. The Wirtinger angle become a real-valued function which is defined on  $T^*M$ such that  $\theta : T^*M \to \mathbb{R}$ , is called the **Wirtinger function(pointwise slant function)**. In this case, the submanifold M of almost Hermitian manifolds  $\widetilde{M}$  with pointwise slant function  $\theta$  is called pointwise slant submanifold.

**Definition 2.3.13.** A point x in a pointwise slant submanifold is called a totally real point if its pointwise slant function  $\theta$  satisfies  $\cos \theta = 0$ , at x. In the same way, a point x is called a complex point if its slant function satisfies  $\sin \theta = 0$  at x. **Definition 2.3.14.** If every point  $x \in M$  of almost Hermitian manifold  $\tilde{M}$  is a totally real point, then pointwise slant submanifold M is called totally real submanifold. Similarly, if every point  $x \in M$  is a complex point, then M is said to be a complex submanifold.

In fact, Chen & Garay (2012) obtained the following characterization theorem:

**Theorem 2.3.6.** Let M be a submanifold of an almost Hermitian manifold M. Then M is pointwise slant if and only if there exists a constant  $\lambda \in [0, 1]$  such that

$$T^2 = -\lambda I. \tag{2.3.36}$$

Furthermore, in such a case,  $\theta$  is real-valued function defined on the tangent bundle TM, then it satisfies that  $\lambda = \cos^2 \theta$ .

Hence, for a pointwise slant submanifold M of an almost Hermitian manifold M, the following relations which are consequences of the Theorem 2.3.6, take place:

$$g(TU, TV) = \cos^2 \theta g(U, V), \qquad (2.3.37)$$

$$g(FU,FV) = \sin^2 \theta g(U,V). \qquad (2.3.38)$$

for any  $U, V \in \Gamma(TM)$ .

Some interesting examples on pointwise slant submanifolds are as:

*Example* **2.3.1.** Every two dimensional submanifold of almost Hermitian manifold is a pointwise slant submanifold.

*Example* **2.3.2.** Every slant (resp, proper slant) submanifold of almost Hermitian manifold is pointwise slant (resp proper pointwise ) submanifold.

The simple characterization on pointwise slant submanifolds is as follows.

**Theorem 2.3.7.** A pointwise slant submanifold M in a Kaehler manifold is slant if and only if the shape operator of M satisfies

$$A_{FX}TX = A_{FTX}X,$$

for any  $X \in \Gamma(TM)$ .

### 2.3.2 Submanifolds of an Almost Contact Metric manifold

Let *M* to be isometrically immersed into an almost contact metric manifolds with induced Riemannian metric *g*. If  $\nabla$  and  $\nabla^{\perp}$  are the induced Riemannian connection on the tangential bundle *TM* and normal bundle  $T^{\perp}$  of *M*, respectively. Then Gauss and Weingarten formulas are defined in (2.3.2) and (2.3.3), respectively. Thus, for any  $U \in \Gamma(TM)$  and  $N \in \Gamma(T^{\perp}M)$ , the tangential and normal components of  $\varphi U$  and  $\varphi N$  are decomposed as:

$$\varphi U = PU + FU, \tag{2.3.39}$$

$$\varphi N = tN + fN, \qquad (2.3.40)$$

respectively. Moreover, the covariant derivatives of P and F has been defined as

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y,$$
 (2.3.41)

$$(\nabla_X F)Y = \nabla_X^{\perp} FY - F \nabla_X Y.$$
(2.3.42)

The covariant derivative of tensor field  $\varphi$  is defined as:

$$(\widetilde{\nabla}_X \varphi Y) = \widetilde{\nabla}_X \varphi Y - \varphi \widetilde{\nabla}_X Y, \qquad (2.3.43)$$

for any  $X, Y \in \Gamma(T\widetilde{M})$  and  $\widetilde{\nabla}$  is the Riemannian connection on  $\widetilde{M}$ . Furthermore, let us denotes tangential and normal parts of  $(\widetilde{\nabla}_U \varphi) V$  by  $\mathbf{P}_U V$  and  $\mathbf{Q}_U V$  respectively. Therefore, making use of (2.3.39), (2.3.40) and (2.3.21)-(2.3.24), we derive

$$\mathbf{P}_U V = (\widetilde{\nabla}_U P) V - A_{FV} U - th(U, V)$$
(2.3.44)

$$\mathbf{Q}_U V = (\widetilde{\nabla}_U F) V + h(U, PV) - fh(U, V).$$
(2.3.45)

Similarly, if we denote the tangential and normal part of  $(\widetilde{\nabla}_U \varphi)N$  by  $\mathbf{P}_U N$  and  $\mathbf{Q}_U N$ respectively, for any  $N \in \Gamma(T^{\perp}M)$ . Then  $\mathbf{P}_U N$  and  $\mathbf{Q}_U N$  are defined as;

$$\mathbf{P}_{U}N = (\widetilde{\nabla}_{U}t)N + PA_{N}X - A_{fN}U, \qquad (2.3.46)$$

$$\mathbf{Q}_U N = (\widetilde{\nabla}_U f) N + h(tN, U) + FA_N U.$$
(2.3.47)

The following properties of  $\mathbf{P}$  and  $\mathbf{Q}$  are straightforward to verify which one enlists here of later use

(i) 
$$\mathbf{P}_{U+V}W = \mathbf{P}_{U}W + \mathbf{P}_{V}W$$
, (ii)  $\mathbf{Q}_{U+V}W = \mathbf{Q}_{U}W + \mathbf{Q}_{V}W$ ,  
(iii)  $\mathbf{P}_{U}(W+Z) = \mathbf{P}_{U}W + \mathbf{P}_{U}Z$ ,  
(iv)  $\mathbf{Q}_{U}(W+Z) = \mathbf{Q}_{U}W + \mathbf{Q}_{U}Z$ ,  
(v)  $g(\mathbf{P}_{U}V,W) = -g(V,\mathbf{P}_{U}W)$ , (vi)  $g(\mathbf{Q}_{U}V,N) = -g(V,\mathbf{P}_{U}N)$ ,  
(vii)  $\mathbf{P}_{U}\varphi V + \mathbf{Q}_{U}\varphi V = -\varphi(\mathbf{P}_{U}V + \mathbf{Q}_{U}V)$ .  
(2.3.48)

Moreover, from (2.2.13), the following property satisfying for nearly cosymplectic manifold, i.e.,

(*i*) 
$$\mathbf{P}_U V + \mathbf{P}_V U = 0$$
, (*ii*)  $\mathbf{Q}_U V + \mathbf{Q}_V U = 0$ . (2.3.49)

for any  $U, V \in \Gamma(T\widetilde{M})$ . Similarly, for nearly Sasakian manifold, one shows that by using (2.3.44) and (2.2.15), i.e.,

(*i*) 
$$\mathbf{P}_U V + \mathbf{P}_V U = 2g(U, V)\xi - \eta(U)V - \eta(V)U$$
, (*ii*)  $\mathbf{Q}_U V + \mathbf{Q}_V U = 0$ . (2.3.50)

Hence, under the action of endomorphism  $\varphi$  on tangent bundle of Riemannian submanifold gives more classifications of submanifolds. Let M be submanifold of an almost contact metric manifold  $(\hat{M}, \varphi, \xi, \eta, g)$ , then M is called *invariant submanifold* if  $\varphi T_x M \subseteq$  $T_x M$ ,  $\forall x \in M$ . Further, M is said to be *anti-invariant submanifold* if  $\varphi T_x M \subseteq T_x^{\perp} M$ ,  $\forall x \in M$ . Similarly, it can be easily seen that a submanifold M of an almost contact metric manifolds  $\hat{M}$  is said to be *invariant(anti-invariant)*, if F (or P) are identically zero in (2.3.39). Now we give definition of contact CR-submanifold which is a generalization of *invariant* and *anti-invariant* submanifolds.

**Definition 2.3.15.** A submanifold *M* tangent to structure vector field  $\xi$  isometrically immersed into almost contact metric manifold  $\widehat{M}$ . Then *M* is said to be contact CR-submanifold if there exists a pair of orthogonal distribution  $\mathscr{D}: x \to \mathscr{D}_x$  and  $\mathscr{D}^{\perp}: x \to \mathscr{D}_x^{\perp}$ ,  $\forall x \in M$  such that

(i) TM = D⊕D<sup>⊥</sup>⊕ < ξ >, where < ξ > is the one dimensional distribution spanned by ξ.

- (ii)  $\mathscr{D}$  is invariant, i.e.,  $\varphi \mathscr{D} = \mathscr{D}$ .
- (iii)  $\mathscr{D}^{\perp}$  is anti-invariant, i.e.,  $\varphi \mathscr{D} \subseteq T^{\perp} M$ .

Anti-invariant and invariant submanifolds are the special case of contact CR-submanifold. If we denote the dimensions of the distributions  $\mathscr{D}$  and  $\mathscr{D}^{\perp}$  by  $m_1$  and  $m_2$ , respectively. Then *M* is called *anti-invariant* (resp. *invariant*) if  $m_2 = 0$ (*resp.m*<sub>1</sub> = 0). Let us denote the orthogonal projections on  $\mathscr{D}$  and  $\mathscr{D}^{\perp}$  by *B* and *C*, respectively. Then, we define

$$U = BU + CU + \eta(U)\xi, \qquad (2.3.51)$$

for any  $U \in \Gamma(TM)$ , where  $BU \in \Gamma(\mathscr{D})$  and  $CU \in \Gamma(\mathscr{D}^{\perp})$ . From (2.3.39), (2.3.40) and (2.3.51), we have

$$PU = \varphi BU, \quad FU = \varphi CU. \tag{2.3.52}$$

It is straightforward to observe that

(i) 
$$PC = 0$$
, (ii)  $FB = 0$ ,  
(iii)  $t(T^{\perp}M) = \mathscr{D}^{\perp}$  (iv)  $f(T^{\perp}M) \subset \mu$ .  
(2.3.53)

There is a class of submanifolds in almost contact metric manifolds. For each non-zero vector *X* tangent *M* at *p*, such that *X* is not proportional to  $\xi_p$ , we denote by  $\theta(X) \in [0, \frac{\pi}{2}]$ , the angle between  $\varphi X$  and  $T_p M$  is said to be a *Wirtinger angle* if the angle of  $\varphi X$  is constant for all  $X \in \Gamma(T_p M - \langle \xi_p \rangle)$ . Thus *M* is called slant submanifold defined by Cabrerizo et al. (2000) and the angle  $\theta$  is a slant angle of *M*. It is obvious that if  $\theta = 0, M$  is an *invariant* and if  $\theta = \frac{\pi}{2}, M$  is an *anti-invariant* submanifold. A slant submanifold *M* is called is proper slant if  $\theta \neq 0, \frac{\pi}{2}$ . We recall the following theorem obtained by Cabrerizo et al. (2000).

**Theorem 2.3.8.** Let M be a submanifold of an almost contact metric manifold  $\hat{M}$ , such that  $\xi$  is tangent to M. Then M is slant if and only if there exists a constant  $0 \le \delta \le 1$  such that

$$P^2 = \delta(-I + \eta \otimes \xi). \tag{2.3.54}$$

Moreover, if  $\theta$  is slant angle, then  $\delta = \cos^2 \theta$ . The following results are straightforward consequence of Theorem 2.3.8, i.e.,

$$g(PU, PV) = \cos^2 \theta \left( g(U, V) - \eta(U) \eta(V) \right), \qquad (2.3.55)$$

$$g(FU,FV) = \sin^2 \theta \left( g(U,V) - \eta(U)\eta(V) \right), \qquad (2.3.56)$$

for any  $U, V \in \Gamma(TM)$ . As a natural generalization of CR-submanifold of almost Hermitian manifolds in terms of slant factor is known semi-slant submanifold which was defined by Papaghiuc (1994). These submanifold was extended in an almost contact metric manifold by Cabrerizo et al. (1999),

**Definition 2.3.16.** A Riemannian submanifold M of almost contact metric manifold M is said to be a semi-slant submanifold if there exist two orthogonal distributions  $\mathcal{D}$  and  $\mathcal{D}^{\theta}$  such that

- (i)  $TM = TM = \mathscr{D} \oplus \mathscr{D}^{\theta} \oplus \langle \xi \rangle$ .
- (ii)  $\mathscr{D}$  is invariant, i.e.,  $\varphi \mathscr{D} = \mathscr{D}$ .
- (iii)  $\mathscr{D}$  is slant distribution with slant angle  $\theta \neq 0, \frac{\pi}{2}$ .

If we denote the dimension of  $\mathcal{D}_j$  by  $d_j$  for j = 1, 2. Then it is clear that contact CRsubmanifolds and slant submanifolds are semi-slant submanifolds with  $\theta = \frac{\pi}{2}$  and  $\mathcal{D}^{\theta} =$ (0), respectively. As a particular class of bi-slant submanifolds are pseudo-slant submanifolds which is initiated by Carriazo (2002). However, the term "anti-slant" seems that there is no slant part, which is not a case, as one can see the following definition:

**Definition 2.3.17.** Let a submanifold M of almost contact metric manifold  $\widetilde{M}$  is said to be a semi-slant submanifold if there exist two complementary distributions  $\mathscr{D}^{\perp}$  and  $\mathscr{D}^{\theta}$  such that

- (i) *TM* admits the orthogonal direct decomposition  $TM = TM = \mathscr{D}^{\perp} \oplus \mathscr{D}^{\theta} \oplus \langle \xi \rangle$ .
- (ii)  $\mathscr{D}$  is slant distribution with slant angle  $\theta \neq 0, \frac{\pi}{2}$ .
- (iii)  $\mathscr{D}^{\perp}$  is anti-invariant, i.e.,  $\varphi \mathscr{D} \subseteq T^{\perp} M$ .

Now we give an example of pseudo-slant submanifold in almost contact metric manifold which defined by Dirik et al. (2016).

*Example* 2.3.3. Let *M* be a submanifold of  $\mathbb{R}^7$  is given by

$$\phi(u,v,s,t,z) = \left(\sqrt{3u}, v, v\sin\theta, v\cos\theta, s\cos t, -s\cos t, 0\right)$$

For the almost complex structure of  $\varphi$  of  $\mathbb{R}^7$  is defined by

$$\varphi(\frac{\partial}{\partial x_i}) = (\frac{\partial}{\partial y_i}), \ \varphi(\frac{\partial}{\partial y_j}) = -(\frac{\partial}{\partial x_j}), \ and \ \varphi(\frac{\partial}{\partial z}) = 0, \ 1 \le i, j \le 3, j \le 3$$

such that  $\xi = \frac{\partial}{\partial z}$ ,  $\eta = dw$ . Therefore, it easy to choose tangent bundle of *M* which is spanned to *M*, i.e.,

$$X_{1} = \sqrt{3}\frac{\partial}{\partial x_{1}}, X_{2} = \frac{\partial}{\partial y_{1}} + \sin\theta \frac{\partial}{\partial x_{2}} + \cos\theta \frac{\partial}{\partial y_{2}}, X_{3} = \cos t \frac{\partial}{\partial x_{3}} - \cos t \frac{\partial}{\partial y_{3}}$$
$$X_{4} = -s \sin t \frac{\partial}{\partial x_{3}} + s \sin t \frac{\partial}{\partial y_{3}}, \quad X_{5} = \frac{\partial}{\partial z}.$$

Therefore, it is easy to see that,  $\mathscr{D}^{\theta} = \{X_1, X_2\}$  is a slant distributions with slant angle  $\theta = 45^{\circ}$ . Moreover, anti-invariant distribution is spanned by  $\mathscr{D}^{\perp} = \{X_3, X_4, X_5\}$ . Hence, *M* is 5–dimensional pseudo-slant submanifold of  $\mathbb{R}^7$  with its usual almost metric structure.

Let us define a new class which is called pointwise slant submanifold. It has been studied in almost contact manifolds by Park (2014). They defined these submanifolds as follows:

**Definition 2.3.18.** Let M be a submanifold of an almost contact metric manifold  $\widehat{M}$ . Then M is called pointwise slant if at any point  $p \in M$  and for each nonzero  $X \in \Gamma(T_pM)$  linearly independent of  $\xi$ , the Wirtinger angle  $\theta(X)$  between  $\varphi X$  and  $T_pM$  only depends on the choice of p and it is independent of the choice of X. On these submanifolds wirtinger angle  $\theta(X)$  can be consider as function which is called a slant function  $\theta: M_p \to R$ .

In fact, Park (2014) obtained the following characterization theorem, i.e.,

**Theorem 2.3.9.** Let M be a submanifold of an almost contact metric manifold  $\tilde{M}$ . Then M is pointwise slant if and only if there exists a constant  $\lambda \in [0,1]$  such that

$$P^2 = -\lambda (I - \eta \otimes \xi). \tag{2.3.57}$$

Furthermore, in such a case,  $\theta$  is real-valued function defined on the tangent bundle  $T^*M = \bigcup_{p \in M} M_p = \bigcup_{p \in M} (X \in T_p M | g(X, \xi(p)) = 0)$ , then it satisfied that  $\lambda = \cos^2 \theta$ .

Hence, for a pointwise slant submanifold M of an almost contact metric manifold  $\widetilde{M}$ , the following relations which are consequences of the Theorem 2.3.9 as:

$$g(PU, PV) = \cos^2 \theta \left( g(U, V) - \eta(U) \eta(V) \right), \qquad (2.3.58)$$

$$g(FU,FV) = \sin^2 \theta \left( g(U,V) - \eta(U)\eta(V) \right), \qquad (2.3.59)$$

for every  $U, V \in \Gamma(TM)$ . The following example classify the concept of pointwise slant submanifold as follows:

*Example* 2.3.4. An almost contact metric manifold  $\widehat{M} = \mathbb{R}^5$  is called cosymplectic manifold with the metric  $g = \sum_{i=1}^2 (dx_i \otimes dx_i + dy_i \otimes dy_i) + dt \otimes dt$ , one form  $\eta = dt$ , the structure field  $\xi = \partial x$  and endomorphism  $\varphi(e_1, e_2, e_3, e_5, t) = (-e_3, e_4, e_1, e_2)$ . Thus, let suppose that the two dimensional submanifold  $M(X,Y) = (X,X,Y\cos\theta,Y\sin\theta,t)$ . It is easily prove that M is a pointwise slant function with slant function  $\phi = \cos^{-1}(\frac{\cos\theta + \sin\theta}{\sqrt{2}})$ , where  $\theta$  is a real value function defined on  $\widetilde{M}$ .

Similarly, pointwise semi-slant submanifolds were also defined and studied by Park (2014) as a natural generalization of CR-submanifolds of almost contact metric manifold in terms of slant function. They defined these submanifolds as follows:

**Definition 2.3.19.** A submanifold M of an almost contact metric manifold  $\widetilde{M}$  is said to be a pointwise semi-slant submanifold if there exists two orthogonal distributions  $\mathcal{D}$  and  $\mathcal{D}^{\theta}$ such that

- (i)  $TM = \mathscr{D} \oplus \mathscr{D}^{\theta} \oplus \langle \xi \rangle$ , where  $\langle \xi(p) \rangle$  is a 1-dimensional distribution spanned by  $\xi(p)$  for each point  $p \in M$ :
- (ii)  $\mathscr{D}$  is invariant, i.e.,  $\varphi(\mathscr{D}) \subseteq \mathscr{D}$ .
- (iii)  $\mathscr{D}^{\theta}$  is pointwise slant distribution with slant function  $\theta: M \to R$ .

Let us denote p and q are the dimensions of *invariant* distribution  $\mathscr{D}$  and *pointwise slant* distribution  $\mathscr{D}^{\theta}$  of pointwise semi-slant submanifold in an almost contact metric manifold  $\widetilde{M}$ . Then, we have the following remarks:

**Remark 2.3.4.** *M* is invariant if q = 0 and pointwise slant if p = 0.

*Remark* 2.3.5. If the slant function  $\theta: M \to R$  is globally constant on M with  $\theta = \frac{\pi}{2}$ , then M is called contact CR-submanifold.

**Remark 2.3.6.** If the slant function  $\theta: M \to (0, \frac{\pi}{2})$ , then *M* is called proper pointwise semi-slant submanifold.

*Remark* 2.3.7. If  $\mu$  is an *invariant* subspace under  $\varphi$  of normal bundle  $T^{\perp}M$ , then in case of semi-slant submanifold, the normal bundle  $T^{\perp}M$  can be decomposed as  $T^{\perp}M = F \mathscr{D}^{\theta} \oplus \mu$ .

For examples and the integrability conditions of distributions involved in the definition pointwise semi-slant submanifold in almost contact metric manifolds, we refer Park (2014) and Park (2015).

### 2.4 WARPED PRODUCT MANIFOLDS

Bishop & O'Neill (1969) initiated the notion of warped product manifolds to construct examples of Riemannian manifolds with negative curvature. They defined these manifolds as:

**Definition 2.4.1.** Let  $(M_1, g_1)$  and  $(M_2, g_2)$  be two Riemannian manifolds and  $f: M_1 \rightarrow (0, \infty)$ , a positive differentiable function on  $M_1$ . Consider the product manifold  $M_1 \times M_2$ with its canonical projections  $\pi_1: M_1 \times M_2 \rightarrow M_1$ ,  $\pi_2: M_1 \times M_2 \rightarrow M_2$  and the projection maps are given by  $\pi_1(p,q) = p$ . and  $\pi_2(p,q) = q$  for every  $t = (p,q) \in M_1 \times M_2$ . The warped product  $M = M_1 \times_f M_2$  is the product manifold  $M_1 \times M_2$  equipped with the Riemannian structure such that

$$||X||^{2} = ||\pi_{1}*(U)||^{2} + f^{2}(\pi_{1}(p))||\pi_{2}*(X)||^{2}.$$
(2.4.1)

for any tangent vector  $U \in \Gamma(T_pM)$ , where \* is the symbol of tangent maps, and we have  $g = g_1 + f^2 g_2$ .

Thus the function f is called the warping function on M. The following lemma which is consequence of warped product manifolds is derived as:

*Lemma* 2.4.1. Let  $M = M_1 \times_f M_2$  be a warped product manifold. Then we have

- (i)  $\nabla_X Y \in \Gamma(TM_1)$ ,
- (ii)  $\nabla_Z X = \nabla_X Z = (X \ln f) Z$ ,
- (iii)  $\nabla_Z W = \nabla'_Z W g(Z, W) \nabla \ln f$ ,

for any  $X, Y \in \Gamma(TM_1)$  and  $Z, W \in \Gamma(TM_2)$ , where  $\nabla$  and  $\nabla'$  denote the Levi-Civitas connections on M and  $M_2$ , respectively, and  $\nabla \ln f$  is the gradient of  $\ln f$  which is defined as  $g(\nabla \ln f, U) = U \ln f$ . A warped product manifold  $M = M_1 \times_f M_2$  is said to be *trivial* if the warping function f is constant.

*Lemma* 2.4.2. If  $M = M_1 \times_f M_2$  be a warped product manifold. Then

- (i)  $M_1$  is totally geodesic submanifold in M,
- (ii)  $M_2$  is totally umbilical submanifold of M.

Assume that  $\phi: M = M_1 \times_f M_2 \to \widetilde{M}$  be an isometric immersion of a warped product  $M_1 \times_f M_2$  into a Riemannian manifold of  $\widetilde{M}$  of constant section curvature *c*. Moreover, let  $n_1$ ,  $n_2$  and *n* be the dimensions of  $M_1$ ,  $M_2$ , and  $M_1 \times_f M_2$ , respectively. Then for unit vectors *X* and *Z* tangent to  $M_1$  and  $M_2$ , respectively, we have

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f} \{ (\nabla_X X) f - X^2 f \}.$$
(2.4.2)

If we consider the local orthonormal frame  $\{e_1, e_2, \ldots e_n\}$  such that  $e_1, e_2, \ldots e_{n_1}$  tangent to  $M_1$  and  $e_{n_1+1}, \ldots e_n$  are tangent to  $M_2$ . Then in view of Gauss equation (2.3.6), we derive

$$\rho(TM) = \widetilde{\rho}(TM) + \sum_{r=1}^{2m} \sum_{1 \le i \ne j \le n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2).$$
(2.4.3)

for each  $j = n_1 + 1, ... n$ . Now, we gives the following formula obtained by Chen (2002) for general warped product submanifold, i.e.,

$$\sum_{i=1}^{n_1} \sum_{j=n_1+1}^n K(e_i \wedge e_j) = \frac{n_2 \nabla f}{f}.$$
(2.4.4)

Similarly, from units vector fields X and Z are tangent to  $M_1$  and  $M_2$ , respectively, thus

$$K(X \wedge Z) = g(R(X,Z)X,Z) = (\nabla_X X) \ln fg(Z,Z) - g(\nabla_X ((X \ln f)Z),Z)$$
  
=  $(\nabla_X X) \ln fg(Z,Z) - g(\nabla_X (X \ln f)Z + (X \ln f)\nabla_X Z,Z)$   
=  $(\nabla_X X) \ln fg(Z,Z) - (X \ln f)^2 - X(X \ln f).$  (2.4.5)

Let  $\{e_1, \dots, e_n\}$  be an orthonormal frame for  $M^n$ , then taking summing up over the vector fields such that

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} K(e_i \wedge e_j) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \left( (\nabla_{e_i} e_i) \ln f - e_i (e_i \ln f) - (e_i \ln f)^2 \right),$$

which implies that

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} K(e_i \wedge e_j) = n_2 \Big( \Delta(\ln f) - ||\nabla(\ln f)||^2 \Big).$$
(2.4.6)

But it was proved in (2.4.4) that for arbitrary warped product submanifold such that

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} K(e_i \wedge e_j) = \frac{n_2 \Delta(f)}{f}.$$
(2.4.7)

Thus from (2.4.6) and (2.4.7), we get

$$\frac{\Delta f}{f} = \Delta(\ln f) - ||\nabla(\ln f)||^2.$$
(2.4.8)

*Example* 2.4.1. A surface revolution is a warped product with leaves as the different position of the rotated curve and fivers as the circle of revolution. If M is obtained by revolving a plane curve C about an axis in  $\mathbb{R}^3$  and  $f: C \to \mathbb{R}^+$ , gives distance to the axis, then the surface  $M = C \times_f S^1(i)$  is a warped product manifold.

*Example* **2.4.2.** The standard space-time models of the universe are warped products as the simplest models of neighbourhoods of stars and black holes.

Similarly, O'NeilO'neill (1983) gave the following lemma for Ricci curvature:

*Lemma* 2.4.3. The Ricci curvature Ric of warped product  $M = M_1 \times_f M_2$  with  $n_2 = dimM_2$ 

- (i)  $Ric(X,Y) = Ric^{M_1}(X,Y) \frac{n_1}{f}H^f$ ,
- (ii) Ric(X,Z) = 0,

(iii) 
$$Ric(Z,W) = Ric^{M_2}(Z,W) - g(Z,W) \left(\frac{(n_2-1)}{f} ||\nabla^{M_2}(f)||^2 - \frac{\Delta f}{f}\right)$$

for any  $X, Y \in \Gamma(TM_1)$  and  $Z, W \in \Gamma(TM_2)$ .

Now we recall the following definition of S. Hiepko (1979).

**Definition 2.4.2.** Let  $\mathscr{D}_1$  be a vector sub bundle in the tangent bundle of Riemannian manifold M and  $\mathscr{D}_2$  be its normal bundle. Suppose that the two distributions are integrable, then we denote the integral manifolds of  $\mathscr{D}_1$  and  $\mathscr{D}_2$  by  $M_1$  and  $M_2$ , respectively. Thus M is locally isometric to warped product  $M = M_1 \times_f M_2$ , if the integral manifold  $M_1$  is totally geodesic and the integral manifold  $M_2$  is an extrinsic sphere, i.e.,  $M_2$  is totally umbilical submanifold with parallel mean curvature vector.

To follows the above definition, we can give the following definition such as:

**Definition 2.4.3.** Assume that M be a any Riemannian submanifold of a Riemannian manifold  $\widetilde{M}$ , then we say that M is a locally warped product manifold submanifold of  $\widetilde{M}$  if  $\mathscr{D}_1$  defines a totally geodesic foliation on M and  $\mathscr{D}_2$  defines a spherical foliation on M, that is each leaf of  $\mathscr{D}_2$  is totally umbilical with parallel mean curvature vector field in M.

### **CHAPTER 3**

# GEOMETRY OF CR-WARPED PRODUCT SUBMANIFOLDS AND THEIR CHARACTERIZATIONS

### 3.1 INTRODUCTION

The purpose of this Chapter is to study CR-warped product submanifold of nearly Sasakian manifolds and *T*-manifolds. In the first section, we characterize CR-submanifolds to be CR-warped products in terms of tensor fields *P* and *F*. For a arbitrary submanifold *M* of an almost contact metric manifold  $(\tilde{M}, \varphi, \xi, \eta, g)$ . We decompose  $\varphi U$  into tangential and normal components as  $\varphi U = PU + FU$ , for any vector field *U* tangent to the submanifold *M*. Many author's including Chen (1981) described geometric properties of submanifolds in terms of tensor fields *P* and *F*. Later, such characterizations were extended for warped products in almost Hermitian as well as almost contact settings by Al-Luhaibi et al. (2009); Chen (2001); V. Khan et al. (2009); V. A. Khan & Shuaib (2014); Munteanu (2005).

In the second section, we study CR-warped product submanifolds of T-manifolds. We prove that the CR-warped product submanifolds with invariant fiber are trivial warped products, and provide a characterization theorem of CR-warped products with anti-invariant fiber of T-manifolds. Moreover, we develop an inequality of CR-warped product submanifolds for the second fundamental form in terms of warping function with considered the equality cases, and prove some results on minimality. Also we find a necessary and sufficient condition for compact oriented CR-warped products turn into CR-products of T-space forms. Further, we derive the second inequality for the second fundamental form with constant curvature by means of Gauss equation instead of Codazzi equation in the sense of Chen (2003).

## 3.2 CHARACTERIZATIONS OF CR-WARPED PRODUCTS IN A NEARLY SASAKIAN MANIFOLD

### 3.2.1 CR-submanifolds of nearly Sasakian manifolds

In this subsection, we obtain some integrability conditions for distributions involved in the definition of CR-submanifold in a nearly Sasakian manifold which are useful to further study.

*Lemma* 3.2.1. Let *M* be a CR-submanifold of a nearly Sasakian manifold  $\widetilde{M}$ . Then  $\mathscr{D} \oplus \xi$  is integrable if and only if the following satisfies

$$2g(\nabla_X Y, Z) = g(A_{\varphi Z} X, \varphi Y) + g(A_{\varphi Z} Y, \varphi X), \qquad (3.2.1)$$

for any  $X, Y \in \Gamma(\mathscr{D} \oplus \langle \xi \rangle)$  and  $Z \in \Gamma(\mathscr{D}^{\perp})$ .

*Proof.* Using (2.2.8) and (2.3.2), for any  $X, Y \in \Gamma(\mathscr{D} \oplus \langle \xi \rangle)$  and  $Z \in \Gamma(\mathscr{D}^{\perp})$ , we obtain  $g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(\varphi \widetilde{\nabla}_X Y, \varphi Z) + \eta(\widetilde{\nabla}_X Y)\eta(Z)$ . Since  $\eta(Z) = 0$ , we get  $g(\nabla_X Y, Z) = g(\varphi \widetilde{\nabla}_X Y, \varphi Z)$ . Then from (2.3.43), we obtain  $g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X \varphi Y, \varphi Z) - g((\widetilde{\nabla}_X \varphi)Y, \varphi Z)$ . Using (2.3.2) and structure equation (2.2.15), we get

$$g(\nabla_X Y, Z) = g(h(X, \varphi Y), \varphi Z) + g((\widetilde{\nabla}_Y \varphi) X, \varphi Z) - 2g(X, Y)g(\xi, \varphi Z)$$
$$+ \eta(X)g(Y, \varphi Z) + \eta(Y)g(X, \varphi Z).$$

Again using (2.3.2), (2.3.43) and (2.2.8), we obtain

$$g(\nabla_X Y, Z) = g(h(X, \varphi Y), \varphi Z) + g(h(Y, \varphi X), \varphi Z) - g(\nabla_Y X, Z)$$

Follows the property of Lie bracket, and the relation between the second fundamental form and shape operator (2.3.4), we get the desired result.

*Lemma* 3.2.2. The anti-invariant distribution  $\mathscr{D}^{\perp}$  of a CR-submanifold M of a nearly Sasakian manifold  $\widetilde{M}$  defines a totally geodesic foliation if and only if

$$g(h(Z, \varphi X), \varphi W) + g(h(W, \varphi X), \varphi Z) = 0,$$

for any  $X \in \Gamma(\mathscr{D} \oplus \langle \xi \rangle)$  and  $Z, W \in \Gamma(\mathscr{D}^{\perp})$ .

*Proof.* Taking account of (2.2.8) and (2.3.2), we have  $g(\nabla_Z W, X) = g(\phi \overline{\nabla}_Z W, \phi X)$ . Then (2.3.43), implies that  $g(\nabla_Z W, X) = g(\widetilde{\nabla}_Z \phi W, \phi X) - g((\widetilde{\nabla}_Z \phi)W, \phi X)$ . Thus (2.3.3) and (2.2.15), gives  $g(\nabla_Z W, X) = -g(A_{\phi W}Z, \phi X) + g((\widetilde{\nabla}_W \phi)Z, \phi X)$ . Again using (2.3.43) and (2.3.3), we arrive at  $2g(\nabla_Z W, X) = -g(A_{\phi W}Z, \phi X) - g(A_{\phi Z}W, \phi X) + g([Z, W], X)$ . Which proves our assertion and this completes the proof of the lemma.

### 3.2.2 CR-warped products of a nearly Sasakian manifold

Before going to prove the characterization theorem, we need to discuss some consequences of CR-warped product submanifolds in a nearly Sasakian manifold. In this way, from (2.3.44) and (2.3.50), implies that

$$(\widetilde{\nabla}_X P)Y + (\widetilde{\nabla}_Y P)X = 2th(X,Y) + 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X.$$
(3.2.2)

for any  $X, Y \in \Gamma(TM_T)$ . As  $M_T$  is being totally geodesic in M, so that  $(\widetilde{\nabla}_X P)Y$  completely lies on  $M_T$  and its the second fundamental form should be zero. Thus comparing the components tangent to  $M_T$  from formula (3.2.2), we obtain th(X,Y) = 0 which implies that  $h(X,Y) \in \mu$  and then we get the following

$$(\widetilde{\nabla}_X P)Y + (\widetilde{\nabla}_Y P)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X.$$
(3.2.3)

If we setting  $Y = \xi$  in (3.2.3), we derive

$$(\widetilde{\nabla}_X P)\xi + (\widetilde{\nabla}_\xi P)X = 2g(X,\xi)\xi - \eta(X)\xi - \eta(\xi)X$$
  
=  $2\eta(X)\xi - \eta(X)\xi - X.$ 

From (2.3.41), it can be easily seen that

$$(\widetilde{\nabla}_{\xi} P)X = -(\widetilde{\nabla}_{X} P)\xi + \eta(X)\xi - X$$
  
=  $P\nabla_{X}\xi + \eta(X)\xi - X.$  (3.2.4)

Similarly, for this direction, we prove the following lemma.

Lemma 3.2.3. Let  $M = M_T \times_f M_{\perp}$  be a CR-warped product submanifold of a nearly

Sasakian manifold  $\widetilde{M}$ . Then

$$(\widetilde{\nabla}_Z P)X = (PX\ln f)Z, \qquad (3.2.5)$$

$$(\nabla_U P)Z = g(CU, Z)P\nabla \ln f, \qquad (3.2.6)$$

for each  $U \in \Gamma(TM)$ ,  $X \in \Gamma(TM_T)$  and  $Z \in \Gamma(TM_{\perp})$ .

*Proof.* From Lemma 2.4.1(ii), we have

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z,$$

for any  $X \in \Gamma(TM_T)$  and  $Z \in \Gamma(TM_{\perp})$ . Using the above equation in (2.3.41), we obtain

$$(
abla_Z P)X = 
abla_Z PX - P
abla_Z X$$
  
=  $(PX \ln f)Z - (X \ln f)PZ$   
=  $(PX \ln f)Z$ ,

which is first part (3.2.5) of lemma. On the other hand, from (2.3.21), for each  $Z \in \Gamma(TM_{\perp})$  and  $U \in \Gamma(TM)$ , we derive

$$(\widetilde{\nabla}_U P)Z = \nabla_U PZ - P\nabla_U Z$$
$$= -P\nabla_U Z.$$

Applying the orthogonal projections property (2.3.51), the above equation can be modified as:

$$(\nabla_U P)Z = -P\nabla_{BU}Z - P\nabla_{CU}Z - \eta(U)P\nabla_{\mathcal{E}}Z.$$

By Lemma 2.4.1(ii) and PZ = 0, we obtain

$$(\widetilde{\nabla}_U P)Z = (BU\ln f)PZ - P\nabla_{CU}Z - \eta(U)(\xi\ln f)PZ$$
$$= -P\nabla_{CU}Z.$$

Taking account of the property (iii) of Lemma 2.4.1, gives

$$(\widetilde{\nabla}_U P)Z = -P\nabla'_{CU}Z + g(CU, Z)P\nabla \ln f$$
$$=g(CU, Z)P\nabla \ln f,$$

which is the second part of lemma. This completes the proof of lemma.

The characterization in terms of  $\nabla P$ .

**Theorem 3.2.1.** Let M be a CR-submanifold of a nearly Sasakian manifold  $\widetilde{M}$  with both distributions are integrable. Then M is locally isometric to a CR-warped product of type  $M = M_T \times_f M_{\perp}$  if and only if

$$(\widetilde{\nabla}_U P)U = (PBU\lambda)CU + ||BU||^2\xi + ||CU||^2P\nabla\lambda - \eta(U)BU, \qquad (3.2.7)$$

or equivalently

$$(\widetilde{\nabla}_{U}P)V + (\widetilde{\nabla}_{V}P)U = (PBV\lambda)CU + (PBU\lambda)CV + 2g(BU, BV)\xi + 2g(CU, CV)P\nabla\lambda - \eta(U)BV - \eta(V)BU, \qquad (3.2.8)$$

for each  $U, V \in \Gamma(TM)$  and a  $C^{\infty}$ -function  $\lambda$  on M with  $Z\lambda = 0$  for each  $Z \in \Gamma(\mathscr{D}^{\perp})$ .

*Proof.* Assume that M be a CR-warped product submanifold of  $\widetilde{M}$  in a nearly Sasakian manifold. Applying property (2.3.51) in  $(\widetilde{\nabla}_U P)U$ , we derive

$$\begin{split} (\widetilde{\nabla}_U P)U = (\widetilde{\nabla}_{BU} P)BU + (\widetilde{\nabla}_{CU} P)BU + \eta(U)(\widetilde{\nabla}_{\xi} P)BU \\ + (\widetilde{\nabla}_U P)CU + \eta(U)(\widetilde{\nabla}_U P)\xi. \end{split}$$

From (3.2.2), (3.2.3) for contact CR-warped product and Lemma (3.2.3), we obtain

$$(\widetilde{
abla}_U P)U = ||BU||^2 \xi - \eta (BU)BU + (PBU\ln f)CU + ||CU||^2 P \nabla \ln f$$
  
  $+ \eta (U)P \nabla_{BU} \xi - \eta (U)P \nabla_{BU} \xi.$ 

Since  $\ln f = \lambda$ , we get

$$(\widetilde{
abla}_U P)U = (PBU\lambda)CU + ||BU||^2\xi + ||CU||^2P\nabla\lambda - \eta(U)BU,$$

which is desire result (3.2.7). Furthermore, replacing U by U + V in the above equation and using linearity of Riemannian metric we get required result (3.2.8).

*Conversely*, suppose that *M* is a CR-submanifold of  $\widetilde{M}$  with both distributions are integrable on *M* such that (3.2.8) holds. Thus CX = 0, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ , then

using in (3.2.8), we obtain

$$(\widetilde{\nabla}_X P)Y + (\widetilde{\nabla}_Y P)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X.$$
(3.2.9)

Since  $\widetilde{M}$  is a nearly Sasakian manifold, then from (2.3.41) and (2.2.15), we derive

$$(\widetilde{\nabla}_X P)Y + (\widetilde{\nabla}_Y P)X = 2th(X,Y) + 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X.$$
(3.2.10)

Thus from (3.2.9) and (3.2.10), we obtain that th(X,Y) = 0. This means that  $h(X,Y) \in \mu$ . Also, we assumed that  $\mathscr{D} \oplus \langle \xi \rangle$  is integrable, then from Lemma (3.2.1), it implies that  $g(\nabla_X Y, Z) = 0$  for each  $Z \in \Gamma(\mathscr{D}^{\perp})$ . Which means that  $\nabla_X Y \in \Gamma(\mathscr{D} \oplus \langle \xi \rangle)$ , i.e., the leaves of a distribution  $\mathscr{D} \oplus \langle \xi \rangle$  are totally geodesic in M. On the other hand, from (3.2.8), one derives

$$(\widetilde{\nabla}_{Z}P)W + (\widetilde{\nabla}_{W}P)Z = 2g(Z,W)P\nabla\lambda, \qquad (3.2.11)$$

for any  $Z, W \in \Gamma(\mathscr{D}^{\perp})$ . From (2.3.41) and (2.3.50), we arrive at

$$(\widetilde{\nabla}_Z P)W + (\widetilde{\nabla}_W P)Z = A_{FW}Z + A_{FZ}W + 2th(Z,W) + 2g(Z,W)\xi.$$
(3.2.12)

From (3.2.11) and (3.2.12), it follows that

$$A_{FW}Z + A_{FZ}W = 2g(Z, W)P\nabla\lambda - 2th(Z, W) - 2g(Z, W)\xi.$$
 (3.2.13)

Taking inner product in (3.2.13) with  $\varphi X$  and using (2.3.4), (2.3.39), we obtain

$$g(h(Z, \varphi X), \varphi W) + g(h(W, \varphi X), \varphi Z) = 2g(Z, W)g(P\nabla\lambda, \varphi X).$$

From (2.3.2), we find that

$$g(\widetilde{\nabla}_{Z}\varphi X,\varphi W) + g(\widetilde{\nabla}_{W}\varphi X,\varphi Z) = 2g(Z,W)g(P\nabla\lambda,\varphi X)$$

Using the orthogonality of vector fields, we derive

$$g(\widetilde{\nabla}_Z \varphi W, \varphi X) + g(\widetilde{\nabla}_W \phi Z, \varphi X) = -2g(Z, W)g(P \nabla \lambda, \varphi X).$$

Then property of the covariant derivative (2.3.43), we find that

$$-2g(Z,W)g(\varphi\nabla\mu,\varphi X) = g((\widetilde{\nabla}_{Z}\varphi)W + (\widetilde{\nabla}_{W}\varphi)Z,\varphi X) + g(\varphi\widetilde{\nabla}_{Z}W,\varphi X) + g(\varphi\widetilde{\nabla}_{W}Z,\varphi X).$$
(3.2.14)

Applying nearly Sasakian structure (2.2.15) in the first term of right hand side of equation (3.2.14) and then using (2.2.8), one shows that

$$g(\widetilde{\nabla}_{Z}W,X)+g(\widetilde{\nabla}_{W}Z,X)=-2g(Z,W)g(\nabla\lambda,X)-2g(Z,W)\eta(\nabla\lambda)\eta(X),$$

which implies that

$$g(\nabla_Z W + \nabla_W Z, X) = -2g(Z, W)g(\nabla\lambda, X) - 2g(Z, W)(\xi \ln f)\eta(X).$$

As  $\mathscr{D}^{\perp}$  assumed to be integrable and  $\xi \ln f = 0$ , we obtain

$$g(\nabla_Z W, X) = -g(Z, W)g(\nabla \mu, X). \tag{3.2.15}$$

Let  $M_{\perp}$  be a leaf of  $\mathscr{D}^{\perp}$  and  $h^{\perp}$  is the second fundamental form of immersion of  $M_{\perp}$  into M, Thus from (2.3.2) find that

$$g(h^{\perp}(Z,W),X) = g(\nabla_Z W,X),$$
 (3.2.16)

for any  $X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ . From (3.2.15) and (3.2.16), it can be easily seen that

$$g(h^{\perp}(Z,W),X) = -g(Z,W)g(\nabla\lambda,X),$$

which means that

$$h^{\perp}(Z,W) = -g(Z,W)\nabla\lambda.$$

From the above equation, we conclude that  $M_{\perp}$  is totally umbilical in M with mean curvature vector  $H = -\nabla \lambda$ . Now we shall prove that H is parallel corresponding to the normal

connection  $\nabla'$  of  $M_{\perp}$  in M. For this consider  $Y \in \Gamma(\mathscr{D} \oplus \langle \xi \rangle)$  and  $Z \in \Gamma(\mathscr{D}^{\perp})$ ,

$$g(\nabla'_{Z}\nabla\lambda, Y) = g(\nabla_{Z}\nabla\lambda, Y)$$
$$= Zg(\nabla\lambda, Y) - g(\nabla\lambda, \nabla_{Z}Y)$$
$$= Z(Y(\lambda)) - g(\nabla\lambda, [Z, Y]) - g(\nabla\lambda, \nabla_{Y}Z)$$
$$= Y(Z\lambda) + g(\nabla_{Y}\nabla\lambda, Z)$$
$$= 0.$$

Since  $Z(\lambda) = 0$  for all  $Z \in \Gamma(\mathscr{D}^{\perp})$  and thus  $\nabla_Y \nabla \lambda \in \Gamma(\mathscr{D} \oplus \langle \xi \rangle)$ . That is, the leaves of  $\mathscr{D}^{\perp}$  are *extrinsic* sphere in *M*. Hence, follows the Definition 2.4.3, we conclude that *M* is a warped product submanifold. This completes the proof of the theorem.

*Lemma* 3.2.4. Let  $M = M_T \times_f M_{\perp}$  be a CR-warped product submanifold of a nearly Sasakian manifold  $\widetilde{M}$ . Then

(i)  $g((\widetilde{\nabla}_X F)Y, \varphi W) = 0$ ,

(ii) 
$$g((\widetilde{\nabla}_Z F)X, \varphi W) = -(X \ln f)g(Z, W),$$

(iii) 
$$g((\widetilde{\nabla}_{\xi}F)Z, \varphi W) = 0$$

for all  $X, Y \in \Gamma(TM_T)$  and  $Z, W \in \Gamma(TM_{\perp})$ .

*Proof.* Assume that M be a CR-warped product submanifold of a nearly Sasakian manifold  $\widetilde{M}$ , we have

$$g((\nabla_X F)Y, \varphi W) = -g(F\nabla_X Y, \varphi W)$$
$$= -g(\nabla_X Y, W),$$

for any  $X, Y \in \Gamma(TM_T)$  and  $W \in \Gamma(TM_{\perp})$ . The first result directly follows that  $M_T$  is totally geodesic in M. From (2.3.42), we have

$$((\nabla_Z F)X, \varphi W) = -g(F\nabla_Z X, \varphi W).$$

By Lemma (2.4.1)(ii), we obtain the second result. Now from (2.3.45), we have

$$g((\widetilde{\nabla}_{\xi}F)Z, \varphi W) = g(\mathbf{Q}_{\xi}Z + fh(\xi, Z), \varphi W)$$
$$= g(\mathbf{Q}_{\xi}Z, \varphi W),$$

for any  $Z, W \in \Gamma(TM_{\perp})$ . Then property (2.3.48)(v)-(vii), we get

$$g((\widetilde{\nabla}_{\xi}F)Z, \boldsymbol{\varphi}W) = g(\boldsymbol{\varphi}\xi, \mathbf{P}_{Z}W) = 0,$$

which is the last result of lemma. This completes the proof of lemma.

**Theorem 3.2.2.** Let M be a CR-submanifold of a nearly Sasakian manifold  $\widetilde{M}$  with invariant and anti-invariant distributions are integrable. Then M is CR-warped product if and only if

$$g((\widetilde{\nabla}_{U}F)V + (\widetilde{\nabla}_{V}F)U, \varphi W) = g(\mathbf{Q}_{BU}CV, \varphi W) + g(\mathbf{Q}_{BV}CU, \varphi W) - (BU\lambda)g(CV, W) - (BV\lambda)g(CU, W), \quad (3.2.17)$$

for each  $U, V \in \Gamma(TM)$  and  $W \in \Gamma(TM_{\perp})$ , where  $\lambda$  is a  $C^{\infty}$ -function on M satisfying  $Z\lambda = 0$ , for each  $Z \in \Gamma(TM_{\perp})$ .

*Proof.* Suppose that  $M = M_T \times_f M_\perp$  be a CR-warped product submanifold of a nearly Sasakian manifold  $\widetilde{M}$ . Then virtue (2.3.51), gives

$$g((\widetilde{\nabla}_{U}F)V, \varphi W) = g((\widetilde{\nabla}_{BU}F)BV, \varphi W) + g((\widetilde{\nabla}_{CU}F)BV, \varphi W)$$
$$+ \eta(U)g((\widetilde{\nabla}_{\xi}F)BV, \varphi W) + g((\widetilde{\nabla}_{BU}F)CV, \varphi W)$$
$$+ g((\widetilde{\nabla}_{CU}F)CV, \varphi W) + \eta(U)g((\widetilde{\nabla}_{\xi}F)CV, \varphi W)$$
$$+ \eta(V)g((\widetilde{\nabla}_{U}F)\xi, \varphi W).$$

Using (2.3.45) and Lemma 3.2.4, we get

$$g((\widetilde{\nabla}_U F)V, \varphi W) = g(\mathbf{Q}_{CU}CV, \varphi W) + g(\mathbf{Q}_{BU}CV, \varphi W) - (BV\lambda)g(CU, W).$$
(3.2.18)

By interchanging U and V, we derive

$$g((\widetilde{\nabla}_{V}F)U, \varphi W) = g(\mathbf{Q}_{CV}CU, \varphi W) + g(\mathbf{Q}_{BV}CU, \varphi W) - (BU\lambda)g(CV, W).$$
(3.2.19)

Therefore, from (3.2.18), (3.2.19) and (2.3.50)(ii) we get the desire result (3.2.17). Now for converse part, let M be a CR-submanifold of a nearly Sasakian manifold  $\widetilde{M}$  with integrable distributions  $\mathscr{D}$  and  $\mathscr{D}^{\perp}$ . Then using the fact that CX = CY = 0 in (3.2.17), i.e.,

$$g((\widetilde{\nabla}_X F)Y + (\widetilde{\nabla}_Y F)X, \varphi W) = 0, \qquad (3.2.20)$$

for any  $X, Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ . From (2.3.42), we get

$$2g(fh(X,Y),\varphi W) - g(h(X,PY) + h(Y,PX),\varphi W) = 0,$$

which implies that

$$g(h(X, PY), \varphi W) + g(h(Y, PX), \varphi W) = 0$$

Taking account of (2.3.2) and (2.3.39), we derive

$$g(\widetilde{\nabla}_X \phi Y, \phi W) + g(\widetilde{\nabla}_Y \phi X, \phi W) = 0.$$

From the covariant derivative property (2.3.43) and (2.3.2), it is easily seen that

$$g((\widetilde{\nabla}_X \boldsymbol{\varphi})Y + (\widetilde{\nabla}_Y \boldsymbol{\varphi})X, \boldsymbol{\varphi}W) + g(\nabla_X Y + \nabla_Y X, W) = 0.$$

Using tensorial equation (2.2.15), we obtain

$$g(\nabla_X Y + \nabla_Y X, W) = 0,$$

which means that  $\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ . As  $\mathcal{D} \oplus \langle \xi \rangle$  is assumed to be a integrable, then it implies  $\nabla_X Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ . By definition the leaves of  $\mathcal{D} \oplus \langle \xi \rangle$  are totally geodesic in *M*. Similarly, follows from (3.2.17), we derive

$$g((\widetilde{\nabla}_X F)Z, \varphi W) + g((\widetilde{\nabla}_Z F)X, \varphi W) = g(\mathbf{Q}_X Z, \varphi W) - (X\lambda)g(Z, W),$$

for any  $X \in \Gamma(\mathscr{D} \oplus \langle \xi \rangle)$  and  $Z, W \in \Gamma(\mathscr{D}^{\perp})$ . Taking of help from (2.3.45) and (2.3.42), we obtain

$$g(\mathbf{Q}_X Z + fh(X, Z), \varphi W) - g(F \nabla_Z X, \varphi W) = g(\mathbf{Q}_X Z, \varphi W) - (X \lambda)g(Z, W),$$

which implies that

$$g(F\nabla_Z X, \varphi W) = (X\lambda)g(Z, W).$$

Simplification gives

$$g(\nabla_Z W, X) = -(X\lambda)g(Z, W). \tag{3.2.21}$$

As  $\mathscr{D}^{\perp}$  assumed to be a integrable and then let  $M_{\perp}$  be a leaf of  $\mathscr{D}^{\perp}$ . If  $\nabla'$  denotes a induced Riemannian connection on  $M_{\perp}$  and  $h^{\perp}$  be the second fundamental form of an immersion  $M_{\perp}$  into M. Thus in view of (3.2.21) and (2.3.2), it can be verified as:

$$g(h^{\perp}(Z,W),X) = -(X\lambda)g(Z,W).$$

The property of gradient function, gives

$$g(h^{\perp}(Z,W),X) = -g(Z,W)g(\nabla\lambda,X).$$

It follows that

$$h^{\perp}(Z,W) = -g(Z,W)\nabla\lambda.$$

This means that  $M_{\perp}$  is totally umbilical in M with mean curvature vector  $H = -\nabla \mu$ . Now we can easily prove that H is parallel corresponding to the normal connection  $\mathcal{D}$  of  $M_{\perp}$  in M as the same way of the Theorem (3.2.1), this means that the leaves of  $\mathcal{D}^{\perp}$  are *extrinsic sphere* in M. By Definition 2.4.3, M is a warped product submanifold. This completes the proof of theorem.

## 3.3 GEOMETRY OF CR-WARPED PRODUCT SUBMANIFOLDS OF T-MANIFOLDS

Firstly, we need to define some preliminaries formulas and definitions for T-manifolds which are different from other ambient manifolds due to T-structures.

### 3.3.1 Motivations

Let  $\widetilde{M}$  be a (2n+s)-dimensional differentiable manifold of class  $C^{\infty}$  endowed with a  $\varphi$ structure of rank 2*n*. According to D. E. Blair (1976), the  $\varphi$ -structure  $\varphi$  is said to be a *complemented frame* if there exist structure vector fields  $\xi^1, \xi^2, \dots, \xi^s$  and its dual 1-forms

 $\eta_1, \eta_2, \cdots \eta_s$  such that

$$\varphi^2 X = -X + \sum_{p=1}^s \eta_p(X) \otimes \xi^p \tag{3.3.1}$$

$$\eta_p(\xi^q) = \delta_p^{\ q}, \quad \varphi(\xi^p) = 0, \quad \eta_p \circ \varphi = 0, \tag{3.3.2}$$

where  $\delta_p^q$  denotes the Frolicker delta and  $p, q = 1, 2, \dots s$ . A manifold  $\widetilde{M}$  is said to have a *metric*  $\varphi$ -structure if there exists a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{p,q=1}^{s} \eta_p(X) \eta_q(Y), \qquad (3.3.3)$$

for any  $X, Y \in \Gamma(T\widetilde{M})$ . In this case

$$g(\varphi X, Y) = -g(X, \varphi Y). \tag{3.3.4}$$

The fundamental 2-form F on  $\widetilde{M}$  is given by

$$\Phi(X,Y) = g(X,\varphi Y). \tag{3.3.5}$$

An almost contact metric manifold  $\widetilde{M}$  is said to be a *K-manifold* if the fundamental 2-form is closed and the metric  $\varphi$ -structure is normal by follows to D. E. Blair (1976), that is

$$S_{\varphi}(X,Y) + 2\sum_{p=1}^{s} d\eta_{p}(X,Y)\xi^{p} = 0, \qquad (3.3.6)$$

where  $S_{\varphi}(X,Y)$  denotes the Nijenhuis tensor with respect to the tensor field  $\varphi$ . A Kmanifold with  $d\eta_p = 0$  for each  $p = 1, 2, \dots s$ , is said to be a *T*-manifold and is defined by D. E. Blair (1970). Let  $\widetilde{\nabla}$  be the Levi-Civitas connection with respect to the metric tensor g on a *T*-manifold  $\widetilde{M}$ , then we have

(a) 
$$(\widetilde{\nabla}_X \varphi) Y = 0$$
, (b)  $\widetilde{\nabla}_X \xi^p = 0$  (3.3.7)

for each  $X, Y \in \Gamma(T\widetilde{M})$ , where  $T\widetilde{M}$  denotes the tangent bundle of  $\widetilde{M}$  (see D. E. Blair (1976)).

Assume that M to be an isometrically immersed into almost contact metric manifold  $\widetilde{M}$  with induced Riemannian metric g. Then Gauss and Weingarten formulas are given by

(2.3.2) and (2.3.3), respectively. From equation (3.3.7)(b) and (2.3.2), we obtain

(a) 
$$\nabla_X \xi^p = 0$$
, (b)  $h(X, \xi^p) = 0$ , (3.3.8)

for each  $X \in \Gamma(TM)$  and  $p = 1, 2, \dots s$ . For any submanifolds tangent to the structure vector fields  $\xi_1, \xi_2, \dots, \xi_s$ , we define CR-submanifold as follows:

**Definition 3.3.1.** A submanifold M tangent to  $\xi_1, \xi_2, \dots, \xi_s$  is called a *contact CR-submanifold* if it admits a pair of differentiable distributions  $\mathscr{D}$  and  $\mathscr{D}^{\perp}$  such that  $\mathscr{D}$  is invariant and its orthogonal complementary distribution  $\mathscr{D}^{\perp}$  is anti-invariant such that  $TM = \mathscr{D} \oplus \mathscr{D}^{\perp} \oplus \langle \xi_s \rangle$  with  $\varphi(\mathscr{D}_p) \subseteq \mathscr{D}_p$  and  $\varphi(\mathscr{D}_p^{\perp}) \subset T_p^{\perp}M$ , for every  $p \in M$ .

In particular, the contact CR-submanifold *M* tangent to  $\xi_1, \xi_2, \dots, \xi_s$ , is *invariant* if  $\mathscr{D}^{\perp} = \{0\}$  and *anti-invariant* if  $\mathscr{D} = \{0\}$ . Let *M* be an *m*-dimensional CR-submanifold of a *T*-manifold  $\widetilde{M}$ . Then,  $F(T_pM)$  is a subspace of  $T_p^{\perp}M$ . Then for every  $p \in M$ , there exists an invariant subspace  $\mu_p$  of  $T_p\widetilde{M}$  such that

$$T_p M = T_p M \oplus F(T_p M) \oplus \mu_p.$$
(3.3.9)

### 3.3.2 Geometry of warped product CR-submanifolds

Throughout this subsection, M assume to be a CR-warped product submanifold of a T-manifold. If  $M_T$  and  $M_{\perp}$  are invariant and anti-invariant submanifolds of a T-manifold  $\widetilde{M}$  and then their warped product may given by one of the following forms:

- (i)  $M = M_{\perp} \times {}_{f}M_{T}$ ,
- (ii)  $M = M_T \times {}_f M_\perp$ .

The structure vector fields  $\xi^p$  either tangent to the invariant submanifold  $M_T$  or tangent to the anti-invariant submanifold  $M_{\perp}$  for all  $p = 1, 2, \dots s$ . We start with the case when  $\xi^p$  tangent to  $M_{\perp}$ , for all  $p = 1, 2, \dots s$ .

**Theorem 3.3.1.** Every proper warped product CR-submanifold of type  $M_{\perp} \times_f M_T$  where  $M_{\perp}$  is an anti-invariant submanifold and  $M_T$  is an invariant submanifold of a T-manifold  $\widetilde{M}$  such that  $\xi^p$  tangent to  $M_{\perp}$  for all  $p = 1, 2, \cdots$  s is a simply Riemannian product manifold.

*Proof.* Assume that  $M = M_{\perp} \times_f M_T$  be a warped product CR-submanifold of a T-manifold  $\widetilde{M}$  with  $\xi^p \in TM_{\perp}$  for all  $p = 1, 2, \dots s$ , then for any  $X \in TM_T$  and  $Z \in TM_{\perp}$ , in particular,  $Z = \xi^p$  in (3.3.7)(b), then

$$\nabla_X \xi^p = (\xi^p \ln f) X.$$

From (3.3.8), we obtain

$$\xi^p \ln f = 0$$
, for all  $p = 1, 2, \dots s$ . (3.3.10)

Now taking the product with X in (ii) of Lemma 2.4.1, we get

$$g(\nabla_X Z, X) = (Z \ln f) ||X||^2.$$
(3.3.11)

On the other hand, by (3.3.4) and (2.3.2) we have

$$g(\nabla_X Z, X) = g(\widetilde{\nabla}_X Z, X) = g(\varphi \widetilde{\nabla}_X Z, \varphi X) + \sum_{p,q=1}^{s} \eta_p(\nabla_X Z) \eta_q(X).$$

As we considered that  $\xi^p$  tangent to  $M_{\perp}$  for all  $p = 1, 2, \dots s$ , we get

$$g(\nabla_X Z, X) = g(\varphi \overline{\nabla}_X Z, \varphi X).$$

Then from tensorial equation (3.3.7) of T-manifold, we obtain

$$g(\nabla_X Z, X) = g(\widetilde{\nabla}_X \varphi Z, \varphi X).$$

Thus from (2.3.3), we get

$$g(\nabla_X Z, X) = -g(A_{FZ} X, \varphi X) = -g(h(X, \varphi X), FZ).$$
(3.3.12)

The above equation takes the form with the account of equation (3.3.11),

$$(Z\ln f) \|X\|^2 = -g(h(X, \varphi X), FZ).$$
(3.3.13)

Replacing X by  $\varphi X$  in (3.3.13) and then using (3.3.1), (3.3.4), and the fact that  $\xi^p$  tangent to  $M_{\perp}$  for all  $p = 1, 2, \dots, s$ , we get

$$(Z\ln f) \|X\|^2 = g(h(X, \varphi X), FZ).$$
(3.3.14)

From equations (3.3.13) and (3.3.14), we obtain that

$$(Z\ln f) \|X\|^2 = 0. \tag{3.3.15}$$

Thus from equations (3.3.10) and (3.3.15) it follows that f is constant on  $M_{\perp}$ . This completes the proof of the theorem.

Now the other case i.e.,  $M_T \times_f M_{\perp}$  with  $\xi^p \in TM_T$  for all  $p = 1, 2, \dots, s$ , is deal with the following result.

*Lemma* 3.3.1. Assume that  $M = M_T \times_f M_\perp$  be a non-trivial CR-warped product submanifold of T – manifold. Then

$$g(\widetilde{\nabla}_Z W, \varphi X) = -(\varphi X \ln f)g(Z, W), \qquad (3.3.16)$$

$$g(h(X,Z), \varphi W) = -(\varphi X \ln f)g(Z,W),$$
 (3.3.17)

for any  $X \in \Gamma(\mathscr{D} \oplus \xi)$  and  $Z, W \in \Gamma(\mathscr{D}^{\perp})$ .

*Proof.* Let  $M = M_T \times_f M_\perp$  be a CR-warped product submanifold with the structure vector fields  $\xi^p$  tangent to  $M_T$  for all  $p = 1, 2, \dots, s$ , then, we have

$$g(\widetilde{\nabla}_Z W, \varphi X) = g(\nabla_Z W, \varphi X) = -g(\nabla_Z \varphi X, W),$$

for any  $X \in \Gamma(TM_T)$  and  $Z, W \in \Gamma(TM_{\perp})$ . Thus from (3.3.4), we obtain

$$g(\widetilde{\nabla}_Z W, \varphi X) = -(\varphi X \ln f)g(Z, W)$$

which is the result (3.3.16) of lemma. On the other hand, with help from (3.3.4), we have

$$g(\widetilde{\nabla}_Z W, \varphi X) = -g(\varphi \widetilde{\nabla}_Z W, X).$$

As  $\widetilde{M}$  is a *T*-manifold, then

$$g(\widetilde{\nabla}_Z W, \varphi X) = -g(\widetilde{\nabla}_Z \varphi W, X).$$

Using (2.3.3), we get

$$g(\nabla_Z W, \varphi X) = g(A_{\varphi W} Z, X) = g(h(X, Z), FW).$$
 (3.3.18)

Then from (3.3.16) and (3.3.18), we obtain

$$g(h(X,Z),\varphi W) = -(\varphi X \ln f)g(Z,W).$$

This is the final result of lemma. This completes the proof of the lemma.

**Theorem 3.3.2.** Every proper CR-submanifold M of a T-manifold  $\widetilde{M}$  is locally a contact CR-warped product if and only if

$$A_{\varphi Z}X = -(\varphi X\lambda)Z, \qquad (3.3.19)$$

for any  $X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ , and  $Z \in \Gamma(\mathcal{D}^{\perp})$ . Moreover, for some function  $\lambda$  on M satisfying  $W(\lambda) = 0$  for each  $W \in \Gamma(\mathcal{D}^{\perp})$ .

*Proof.* Let us consider that  $M = M_{\perp} \times {}_{f}M_{T}$  is a CR-warped product submanifold of a *T*-manifolds. Then first part directly follows from (3.3.17) of Lemma 3.3.2 with setting  $\lambda = \ln f$ .

Conversely, suppose that M is a CR-submanifold of  $\widetilde{M}$  and satisfying the condition (3.3.19), then

$$g(h(X,Y),\varphi Z) = g(A_{\varphi Z}X,Y) = -(\varphi X\mu)g(Y,Z) = 0.$$

Using (3.3.4) and the fact that M is a T-manifold, we obtain

$$g(\widetilde{\nabla}_X Y, \varphi Z) = -g(\varphi \widetilde{\nabla}_X Y, Z) = -g(\widetilde{\nabla}_X \varphi Y, Z) = 0.$$

This implies that

$$g(\nabla_X Y, Z) = 0.$$

This means that  $\mathscr{D} \oplus \langle \xi \rangle$  is integrable and its leaves are totally geodesic in M. If  $M_{\perp}$  be a leaf of  $\mathscr{D}^{\perp}$  and  $h^{\perp}$  be the second fundamental form of the immersion of  $M_{\perp}$  into M, then for any  $Z, W \in \Gamma(\mathscr{D}^{\perp})$ , we have

$$g(h^{\perp}(Z,W),\varphi X) = g(\nabla_W Z,\varphi X) = g(\widetilde{\nabla}_W Z,\varphi X) = -g(\varphi \widetilde{\nabla}_W Z,X).$$

Using the characteristic equation of T-manifold, we get

$$g(h^{\perp}(Z,W),\varphi X) = -g(\widetilde{\nabla}_W \varphi Z, X) = g(A_{\varphi Z}W, X).$$

Then from the relation between shape operator and the second fundamental form, we obtain

$$g(h^{\perp}(Z,W),\varphi X) = g(h(X,W),\varphi Z)$$
  
or  
 $g(h^{\perp}(Z,W),\varphi X) = g(A_{\varphi Z}X,W).$ 

Thus from (3.3.19), we obtain

$$h^{\perp}(Z,W) = -g(Z,W)\nabla\lambda, \qquad (3.3.20)$$

which implies that  $M_{\perp}$  is totally umbilical in M with the mean curvature vector  $H = -\nabla \lambda$ . Moreover, as  $Z\lambda = 0$  for all  $Z \in \Gamma(\mathscr{D}^{\perp})$  that is, the mean curvature is parallel on  $M_{\perp}$ , this shows that  $M_{\perp}$  is extrinsic sphere. Applying the Definition 2.4.3, we obtain that M is locally a CR-warped product submanifold  $M_T \times {}_f M_{\perp}$  with the warping function  $f = e^{\lambda}$ of a T-manifold  $\widetilde{M}$ . Hence, the theorem is proved.

**Proposition 3.3.1.** Let  $M = M_T \times {}_f M_{\perp}$  be a CR-warped product submanifold of *T*-manifolds such that the structure vector field  $\xi^p$  is tangent to  $N_T$  for all  $p = 1, 2 \cdots s$ . Then

$$g(h(\varphi X, Z), \varphi W) = (X \ln f)g(Z.W), \qquad (3.3.21)$$

$$g(h(\varphi X, Y), \varphi Z) = g(h(X, Y), \varphi Z) = 0,$$
 (3.3.22)

$$g(h(X,X),\tau) + g(h(\varphi X,\varphi X),\tau) = 0,$$
 (3.3.23)

for any  $X, Y \in \Gamma(TM_T)$  and  $Z, W \in \Gamma(TM_{\perp})$ . Moreover,  $\tau \in \Gamma(\mu)$ .

*Proof.* From (2.3.2), (2.3.39), (3.3.7)(a) and Lemma 2.4.1(ii), we derive

$$g(h(\varphi X, Z), \varphi W) = g(\widetilde{\nabla}_Z \varphi X, \varphi W) = g(\varphi \widetilde{\nabla}_Z X, \varphi W) = (X \ln f)g(Z.W),$$

which is the result (3.3.21). On the other parts, from (3.3.3), (3.3.4) and (3.3.7)(a), we obtain

$$g(h(\varphi X,Y),\varphi Z) = g(\widetilde{\nabla}_Y \varphi X,\varphi Z) = g(\widetilde{\nabla}_Y X,Z) - \sum_{p=1}^s \eta^p(\widetilde{\nabla}_Y X)\eta^p(Z).$$

Thus from the facts that  $\xi$  is tangent  $M_T$  and  $M_T$  is totally geodesic in M, and then using

(2.3.2), we get the required result (3.3.22). In similar way, we obtain (3.3.23). This completes the proof of the proposition.

**Proposition 3.3.2.** Assume that  $M = M_T \times_f M_\perp$  be a CR-warped product submanifold of *T*-manifolds such that  $N_T$  is invariant submanifolds tangent to  $\xi^1, .\xi^2, ..., \xi^s$ . Then

$$||h(X,Z)||^{2} = g(h(\varphi X,Z),\varphi h(X,Z)) + (\varphi X \ln f)^{2}||Z||^{2}$$

for any  $X, Y \in \Gamma(TM_T)$  and  $Z, W \in \Gamma(TM_\perp)$ .

*Proof.* By the definition of the norm and (3.3.3), we derive

$$||h(X,Z)||^2 = g(\varphi h(X,Z), \varphi h(X,Z)) + \sum_{p=1}^{s} \eta^{2p}(h(X,Z)).$$

Using (2.3.2), (3.3.7)(a) and Lemma 2.4.1(ii), we obtain

$$||h(X,Z)||^2 = g(\widetilde{\nabla}_Z \varphi X, \varphi h(X,Z)) - (\varphi X \ln f)g(\varphi h(X,Z),Z).$$

Again from (2.3.2) and Lemma 2.4.1(ii), we arrive at

$$||h(X,Z)||^2 = g(h(\varphi X,Z),\varphi h(X,Z)) - (\varphi X \ln f)g(h(X,Z),\varphi Z).$$

Thus using (3.3.17) in the last term of right hand side of the above equation, we get required result. This completes the proof of the proposition.

**Theorem 3.3.3.** Let  $\widetilde{M}$  be a (2m+s)-dimensional T-manifold and  $M = M_T \times_f M_{\perp}$  be a ndimensional CR-warped product submanifold of  $\widetilde{M}$  such that  $M_T$  is  $(2\alpha+s)$ -dimensional invariant submanifold tangent to  $\xi^p$ . Then

(i) The squared norm of the second fundamental form is given by

$$||h||^2 \ge 2\beta ||\nabla \ln f||^2, \tag{3.3.24}$$

where  $\beta$  is dimension of anti-invariant submanifold  $M_{\perp}$  and  $p = 1, 2, \dots s$ .

(ii) The equality holds in (3.3.24), then  $M_T$  is totally geodesic and  $M_{\perp}$  is totally umbilical submanifolds of  $\widetilde{M}$ , respectively. Moreover, M is minimal submanifold of  $\widetilde{M}$ .
*Proof.* Suppose that  $M = M_T \times {}_f M_{\perp}$  be a  $(n = 2\alpha + \beta + s)$ -dimensional CR-warped product submanifold in an 2m + s-dimensional *T*-manifold  $\widetilde{M}$ , where  $M_T$  is a  $2\alpha + s$ dimensional invariant submanifold tangent to  $\xi^p$  for all  $p = 1, 2, \dots s$  and  $M_{\perp}$  is antiinvariant submanifold of dimension  $\beta$ . Then we consider that  $\{e_1 = \tilde{e}_1, \dots e_{\alpha} = \tilde{e}_{\alpha}, e_{\alpha+1} = \phi \tilde{e}_1, \dots e_{2\alpha} = \phi \tilde{e}_{\alpha}, e_{2\alpha+1} = \xi^1, \dots e_{2\alpha+s} = \xi^p\}$  and  $\{e_{2\alpha+s+1} = \bar{e}_1, \dots e_{2\alpha+s+\beta} = \bar{e}_{\beta}\}$  are orthonormal frames for integral manifolds  $M_T$  and  $M_{\perp}$  of  $\mathscr{D}$  and  $\mathscr{D}^{\perp}$ , respectively. Moreover, the orthonormal frames for the normal sub-bundle  $\phi \mathscr{D}^{\perp}$  and  $\mu$  are  $\{e_{n+1} = \dot{e}_1 = \phi \bar{e}_1, \dots e_{n+\beta} = \dot{e}_{\beta} = \phi \bar{e}_{\beta}\}$  and  $\{e_{n+\beta+1}, \dots e_{2m+s}\}$ , respectively. Thus the definition of the second fundamental form, we have

$$|h||^{2} = ||h(\mathscr{D}, \mathscr{D})||^{2} + ||h(\mathscr{D}^{\perp}, \mathscr{D}^{\perp})||^{2} + 2||h(\mathscr{D}, \mathscr{D}^{\perp})||^{2}.$$

The above equation can be expressed as

$$||h||^{2} = \sum_{r=n+1}^{2m+s} \sum_{i,j=1}^{2\alpha+s} g(h(e_{i},e_{j}),e_{r})^{2} + \sum_{r=n+1}^{2m+s} \sum_{i,j=1}^{\beta} g(h(e_{i},e_{j}),e_{r})^{2} + 2\sum_{r=n+1}^{2m+s} \sum_{i=1}^{2\alpha+s} \sum_{j=1}^{\beta} g(h(e_{i},e_{j}),e_{r})^{2}.$$
(3.3.25)

Leaving all terms except third term and using adapted frame in (3.3.25), then (3.3.25) takes the new form

$$||h||^{2} \ge 2 \sum_{r=n+1}^{n+\beta} \sum_{i=1}^{2\alpha} \sum_{j=1}^{\beta} g(h(e_{i},e_{j}),e_{r})^{2} + 2 \sum_{r=n+1}^{n+\beta} \sum_{j=1}^{s} \sum_{j=1}^{\beta} g(h(\xi^{p},e_{j}),e_{r})^{2} + 2 \sum_{r=n+\beta+1}^{2m+s} \sum_{i=1}^{2\alpha+s} \sum_{j=1}^{\beta} g(h(e_{i},e_{j}),e_{r})^{2}.$$
(3.3.26)

The second term of the right hand side in the above equation is identically zero by using the fact that  $h(\xi^p, Z) = 0$ , for all  $p = 1, 2, \dots s$ . Now consider only first term and using the adapted frame, we derive

$$||h||^{2} \geq 2\sum_{r,j=1}^{\beta} \sum_{i=1}^{\alpha} g(h(\tilde{e}_{i},\bar{e}_{j}),\varphi\bar{e}_{r})^{2} + 2\sum_{r,j=1}^{\beta} \sum_{i=1}^{\alpha} g(h(\varphi\tilde{e}_{i},\bar{e}_{j}),\varphi\bar{e}_{r})^{2}.$$

Thus from (3.3.17) and (3.3.21), we obtain

$$||h||^{2} \ge 2\sum_{r,j=1}^{\beta} \sum_{i=1}^{\alpha} (\varphi \tilde{e}_{i} \ln f)^{2} g(\bar{e}_{r}, \bar{e}_{j})^{2} + 2\sum_{r,j=1}^{\beta} \sum_{i=1}^{\alpha} (\tilde{e}_{i} \ln f)^{2} g(\bar{e}_{r}, \bar{e}_{j})^{2}.$$
(3.3.27)

Adding and subtracting the same terms  $\xi^p \ln f$  in (3.3.27) and from (3.3.10), for all p =

 $1, 2, \cdots s$ , we get

$$||h||^2 \ge 2\beta \sum_{i=1}^{2\alpha+s} (e_i \ln f)^2.$$

Hence, the inequality (3.3.24) holds. If the equality holds in (3.3.24), then by leaving the terms in (3.3.25), we get

$$h(\mathscr{D}, \mathscr{D}) = 0, \ h(\mathscr{D}^{\perp}, \mathscr{D}^{\perp}) = 0, \tag{3.3.28}$$

for  $\mathscr{D} = \mathscr{D}^T \oplus \xi$ . Also from (3.3.26), we obtain

$$h(\mathscr{D}, \mathscr{D}^{\perp}) \subseteq \varphi \mathscr{D}^{\perp}. \tag{3.3.29}$$

Then from (3.3.28) and (3.3.29), it is easy to conclude that  $M_T$  is totally geodesic in  $\widetilde{M}$  due to totally geodesic in M and  $M_{\perp}$  is totally umbilical submanifold of  $\widetilde{M}$ . Furthermore,  $M_{\perp}$  is totally umbilical, then we can write h(Z,W) = g(Z,W)H for  $Z, W \in \Gamma(\mathcal{D}^{\perp})$ . Thus from (3.3.28), we get g(Z,W)H = 0, which implies that H = 0, its means that  $N_{\perp}$  is minimal submanifold. This completes the proof of the theorem.

## 3.3.3 Compact CR-warped product submanifolds in *T*-space forms

If  $\widetilde{M}$  has constant  $\varphi$ -sectional curvature c. Then  $\widetilde{M}$  is called T-space form and denoted by  $\widetilde{M}(c)$ . Then the Riemannian curvature tensor  $\widetilde{R}$  on  $\widetilde{M}(c)$  is defined as:

$$\begin{split} (\widetilde{R}(X,Y)Z,W) &= \frac{c}{4} \Biggl\{ g(Y,W)g(X,Z) - g(X,W)g(Y,Z) \\ &- g(X,Z) \sum_{p,q=1}^{s} \eta^{p}(Y)\eta^{q}(W) - g(Y,W) \sum_{p,q=1}^{s} \eta^{q}(X)\eta^{q}(Z) \\ &+ g(X,W) \sum_{p,q=1}^{s} \eta^{p}(Y)\eta^{q}(Z) + g(Y,Z) \sum_{p,q=1}^{s} \eta^{p}(X)\eta^{q}(W) \\ &+ \Bigl( \sum_{p,q=1}^{s} \eta^{p}(Y)\eta^{q}(W) \Bigr) \Bigl( \sum_{p,q=1}^{s} \eta^{p}(X)\eta^{q}(Z) \Bigr) \\ &- \Bigl( \sum_{p,q=1}^{s} \eta^{p}(Y)\eta^{q}(Z) \Bigr) \Bigl( \sum_{p,q=1}^{s} \eta^{p}(X)\eta^{q}(W) \Bigr) \\ &+ g(\varphi Y,X)g(\varphi Z,W) - g(\varphi X,Z)g(\varphi Y,W) \\ &+ 2g(X,\varphi Y)g(\varphi Z,W) \Biggr\}, \end{split}$$
(3.3.30)

for any  $X, Y, Z, W \in \Gamma(T\widetilde{M}(c))$  which given in Aktan et al. (2008). If we set s = 0, then T-space form is generalized to complex space form which defined by Chen (2003). Similarly, if s = 1 the above formula generalize the Riemannian curvature tensor for cosymplectic space forms which is presented by Atçeken (2011).

**Proposition 3.3.3.** On a CR-warped product submanifold  $M = M_T \times_f M_{\perp}$  of *T*-manifold  $\widetilde{M}$ . Then

$$g(\widetilde{R}(X,\varphi X)Z,\varphi Z) + 2||h(X,Z)||^2 = \left(H^{\ln f}(X,X) + H^{\ln f}(\varphi X,\varphi X) + 2(\varphi X \ln f)^2\right)||Z||^2,$$

for any  $X \in \Gamma(TM_T)$  and  $Z \in \Gamma(TM_{\perp})$ .

*Proof.* Making use of Codazzi equation (2.3.7) and Riemannian curvature tensor  $\widetilde{R}$ , we obtain

$$g(\widetilde{R}(X,\varphi X)Z,\varphi Z) = g((\widetilde{\nabla}_X h)(\varphi X,Z),\varphi Z) - g((\widetilde{\nabla}_{\varphi X} h)(\varphi X,Z),\varphi Z),$$

for  $X \in \Gamma(TM_T)$  and  $Z \in \Gamma(TM_{\perp})$ . Thus from the covariant derivative of the second fundamental form (2.3.5), (2.3.3) and Lemma 2.4.1(ii), we derive

$$g(\widetilde{R}(X,\varphi X)Z,\varphi Z) = g(\widetilde{\nabla}_X h(\varphi X,Z),\varphi Z) - g(h(\nabla_X \varphi X,Z),\varphi Z) - (X \ln f)g(h(\varphi X,Z),\varphi Z) - g(\widetilde{\nabla}_{\varphi X} h(X,Z),\varphi Z) + g(h(\nabla_{\varphi X} X,Z),\varphi Z) + (\varphi X \ln f)g(h(X,Z),\varphi Z).$$

In the right hand side the above equation, the first and fourth terms follow the property of the derivative of vector fields. Hence, we arrive at

$$\begin{split} g(\widetilde{R}(X,\varphi X)Z,\varphi Z) =& Xg(h(\varphi X,Z),\varphi Z) - g(h(\varphi X,Z),\widetilde{\nabla}_X \varphi Z) \\ &- g(h(\nabla_X \varphi X,Z),\varphi Z) - (X\ln f)g(h(\varphi X,Z),\varphi Z) \\ &+ g(h(\nabla_{\varphi X} X,Z),\varphi Z) + (\varphi X\ln f)g(h(X,Z),\varphi Z) \\ &- \varphi Xg(h(X,Z),\varphi Z) + g(h(X,Z),\widetilde{\nabla}_{\varphi X} \varphi Z). \end{split}$$

Thus from (3.3.21) Lemma 2.4.1(ii) and (3.3.7)(a), we obtain

$$\begin{split} g(\widetilde{R}(X,\varphi X)Z,\varphi Z) =& X(X\ln f)||Z||^2 + 2(X\ln f)^2||Z||^2 \\ &-g(h(\varphi X,Z),\varphi \widetilde{\nabla}_X Z) - g(h(\nabla_X \varphi X,Z),\varphi Z) \\ &-(X\ln f)^2||Z||^2 + g(h(\nabla_{\varphi X} X,Z),\varphi Z) \\ &-(\varphi X\ln f)^2||Z||^2 + \varphi X(\varphi X\ln f)||Z||^2 \\ &+ 2(\varphi X\ln f)^2||Z||^2 + g(h(X,Z),\varphi \widetilde{\nabla}_{\varphi X} Z). \end{split}$$

Since  $\nabla_{\varphi X} X, \nabla_X \varphi X \in \Gamma(TM_T)$  and the fact that  $M_T$  is totally geodesic in M. Thus from (2.3.2) and (3.3.17), we get

$$\begin{split} g(\widetilde{R}(X,\varphi X)Z,\varphi Z) = & X(X\ln f)||Z||^{2} + (X\ln f)^{2}||Z||^{2} + \varphi X(\varphi X\ln f)||Z||^{2} \\ & -g(h(\varphi X,Z),\varphi \nabla_{X}Z) - 2g(h(\varphi X,Z),\varphi h(X,Z)) \\ & + (\varphi \nabla_{X}\varphi X\ln f)||Z||^{2} - (\varphi \nabla_{\varphi X}X\ln f)||Z||^{2} \\ & + (\varphi X\ln f)^{2}||Z||^{2} + g(h(X,Z),\varphi \nabla_{\varphi X}Z). \end{split}$$

Using (3.3.17), Lemma 2.4.1(ii) and Proposition 3.3.2, we arrive at

$$g(\widetilde{R}(X,\varphi X)Z,\varphi Z) = X(X\ln f)||Z||^{2} + (X\ln f)^{2}||Z||^{2} + \varphi X(\varphi X\ln f)||Z||^{2} - (X\ln f)^{2}||Z||^{2} - 2||h(X,Z)||^{2} + 2(\varphi X\ln f)^{2}||Z||^{2} + (\varphi \nabla_{X}\varphi X\ln f)||Z||^{2} - (\varphi \nabla_{\varphi X} X\ln f)||Z||^{2} + (\varphi X\ln f)^{2}||Z||^{2} - (\varphi X\ln f)^{2}||Z||^{2},$$

which implies that

$$g(\widetilde{R}(X,\varphi X)Z,\varphi Z) = X(X\ln f)||Z||^{2} + \varphi X(\varphi X\ln f)||Z||^{2} - 2||h(X,Z)||^{2} + 2(\varphi X\ln f)^{2}||Z||^{2} - \nabla_{X} X\ln f||Z||^{2} - \nabla_{\varphi X} \varphi X\ln f||Z||^{2}.$$

Thus from the definition of Hessian form, we get the required result. This completes the proof of the proposition.

**Proposition 3.3.4.** Let  $\widetilde{M}(c)$  be a *T*-space form and  $M = M_T \times {}_f M_{\perp}$  be a CR-warped

product submanifold of  $\widetilde{M}(c)$ . Then

$$||h(X,Z)||^{2} = \frac{1}{2} \left( H^{\ln f}(X,X) + H^{\ln f}(\varphi X,\varphi X) + (\varphi X \ln f)^{2} + \frac{c}{4} ||X||^{2} \right) ||Z||^{2},$$

for any  $X \in \Gamma(TM_T)$  and  $Z \in \Gamma(TM_{\perp})$ .

*Proof.* Taking account that the ambient manifold is a *T*-space form, then from (2.3.16) and the fact that  $\xi^p$  tangent to  $M_T$  for all  $p = 1, 2, \dots s$ , we obtain

$$g(\widetilde{R}(X,\varphi X)Z,\varphi Z)=\frac{c}{4}||X||^2||Z||^2.$$

Hence, using Proposition 3.3.3 in the above relation, we get the required result.

**Theorem 3.3.4.** Let  $M = M_T \times_f M_\perp$  be a compact CR-warped product submanifold in a T-space form  $\widetilde{M}(c)$  such that  $M_T$  is invariant submanifold tangent to  $\xi^2, \xi^2, \dots, \xi^s$  of dimension  $n_1 = 2\alpha + s$  and  $M_\perp$  is anti-invariant  $n_2 = \beta$ -dimensional submanifold. Then M is a CR-product if and only if

$$\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h_{\mu}(e_i, e_j)||^2 = \frac{\alpha.\beta.c}{4},$$
(3.3.31)

where  $h_{\mu}$  denote the components of h in  $\Gamma(\mu)$ .

*Proof.* From (2.3.14), (2.3.15) and using adapted frame of CR-warped product submanifolds, we obtain

$$\Delta(\ln f) = -\sum_{i=1}^{2\alpha} H^{\ln f}(e_i, e_i) - \sum_{j=1}^{\beta} H^{\ln f}(e_j, e_j) - \sum_{p=1}^{s} g(\nabla_{\xi^p} grad \ln f, \xi^p)$$

Since ambient space  $\widetilde{M}$  is *T*-manifold and the fact that  $grad \ln f \in \Gamma(TM_T)$ , which means that  $g(\nabla_{\xi^p} grad \ln f, \xi^p) = 0$  for all  $p = 1, 2, \dots s$ , we have

$$\Delta(\ln f) = -\sum_{i=1}^{\alpha} H^{\ln f}(e_i, e_i) - \sum_{i=1}^{\alpha} H^{\ln f}(\varphi e_i, \varphi e_i) - \sum_{j=1}^{\beta} g(\nabla_{e_j} \nabla \ln f, e_j).$$

Now taking account that M is compact orientable submanifold, from (2.3.15), we get

$$\begin{split} \int_{M} \left( \sum_{i=1}^{\alpha} \{ H^{\ln f}(e_{i}, e_{i}) + H^{\ln f}(\varphi e_{i}, \varphi e_{i}) \} + \sum_{j=1}^{\beta} e_{j}g(\nabla \ln f, e_{j}) \right) dV \\ &= \int_{M} \left( \sum_{j=1}^{\beta} g(\nabla \ln f, \nabla_{e_{j}}e_{j}) \right) dV, \end{split}$$

where dV is a volume element over integration of compact submanifolds M. Thus from gradient function of  $\ln f$ , we derive

$$\int_M \left( \sum_{i=1}^{\alpha} \{ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \} + \sum_{j=1}^{\beta} \left( e_j(e_j \ln f) - \nabla_{e_j} e_j \ln f \right) \right) dV = 0.$$

Using Lemma 2.4.1(ii) and the fact that  $\nabla \ln f \in \Gamma(TM_T)$ , we obtain

$$\int_{M} \left( \sum_{i=1}^{\alpha} \{ H^{\ln f}(e_{i}, e_{i}) + H^{\ln f}(\varphi e_{i}, \varphi e_{i}) \} + \beta ||\nabla \ln f||^{2} \right) dV = 0.$$
(3.3.32)

On the other hand, let  $X = e_i \& Z = e_j$  for  $1 \le i \le \alpha \& 1 \le j \le \beta$ , the second fundamental form can be expressed as

$$h(e_i,e_j) = \sum_{r=1}^{\beta} g(h(e_i,e_j),\varphi e_j)\varphi e_j + \sum_{r=\beta+1}^{2m+s} g(h(e_i,e_j),\tau e_r)\tau e_r,$$

where  $\tau \in \Gamma(\mu)$ . Taking summation over  $\alpha$  and  $\beta$ , we obtain

$$\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h(e_i, e_j)||^2 = \sum_{i=1}^{\alpha} \sum_{r, j=1}^{\beta} g(h(e_i, e_j), \varphi e_r)^2 + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \sum_{r=1}^{2m+s} g(h(e_i, e_j), \tau e_r)^2.$$

Using (3.3.17) in the first term of right hand side of the above equation, we get

$$\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h(e_i, e_j)||^2 = \beta \sum_{i=1}^{\alpha} (\varphi e_i \ln f)^2 + \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h_{\mu}(e_i, e_j)||^2.$$
(3.3.33)

Summing up Proposition 3.3.4, we derive

$$\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h(e_i, e_j)||^2 = \frac{\beta}{2} \sum_{i=1}^{\alpha} \{ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \} + \beta \sum_{i=1}^{\alpha} (\varphi e_i \ln f)^2 + \frac{c.\alpha.\beta}{4}.$$
(3.3.34)

From (3.3.33) and (3.3.34), we obtain

$$2\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}||h_{\mu}(e_{i},e_{j})||^{2} = \sum_{i=1}^{\alpha}\left(H^{\ln f}(e_{i},e_{i}) + H^{\ln f}(\varphi e_{i},\varphi e_{i})\right) + \frac{c.\alpha.\beta}{2}.$$
 (3.3.35)

Finally, using the above equation in (3.3.32), we find that

$$\int_{M} \left( \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} ||h_{\mu}(e_{i}, e_{j})||^{2} - \frac{c \cdot \alpha \cdot \beta}{4} + \frac{\beta}{2} ||\nabla \ln f||^{2} \right) dV = 0.$$
(3.3.36)

Suppose that *M* be a compact CR-warped product submanifold with condition (3.3.31) holds. Then from (3.3.36), we get  $||\nabla \ln f||^2 = 0$ , which implies that *grad*  $\ln f = 0$ , its means that the warping function *f* is constant on *M*. Hence by hypothesis of warped product manifolds, *M* is simply CR-product of invariant and anti-invariant submanifolds  $M_T$  and  $M_{\perp}$  respectively. *Conversely*, let *M* is a compact CR-product in *T*-space form  $\widetilde{M}(c)$ . Then the warping function *f* must be constant on *M*, i.e.,  $\nabla \ln f = 0$ . Thus equality (3.3.36) implies the equality (3.3.31). This completes the proof of the theorem.

**Corollary 3.3.1.** Let  $M = M_T \times_f M_\perp$  be a compact CR-warped product submanifold in a T-space form  $\widetilde{M}(c)$  such that  $M_T$  is invariant submanifold tangent to  $\xi^2, \xi^2, \dots, \xi^s$  of dimension  $n_1 = 2\alpha + s$  and  $M_\perp$  is anti-invariant  $n_2$ -dimensional submanifold. Then M is a CR-product if and only if

$$H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) = 0, \qquad (3.3.37)$$

where *H* is a Hessian form and  $1 \le i \le \alpha$ .

*Proof.* The proof of the above corollary follows from (3.3.32).

## **3.3.4** Chen type inequality of CR-warped products in *T*-space forms

We have proved the inequality by Codazzi equation in the previous section. By using Gauss equation instead of Codazzi, we construct an inequality of CR-warped product submanifolds in T-space forms for the second fundamental form in terms of warping function by using Gauss equation. Some other inequalities are also generalized from these inequality. The equality cases are considered and some applications are derived. In the following section we shall investigate warped product CR-submanifolds of a T-manifold<sup>1</sup>.

<sup>&</sup>lt;sup>1</sup>For the simplicity to understand the dimensions, we use indices for each submanifolds

We proved the following results, i.e,

**Theorem 3.3.5.** On a CR-warped product submanifold  $M^n = M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  of a Tmanifold  $\widetilde{M}^{2m+s}$  such that  $M_T^{n_1+s}$  is invariant submanifold of dimension  $n_1 + s$  tangent to  $\xi^1, \xi^2, \dots \xi^s$ . Then  $M_T^{n_1+s}$  is  $\varphi$ -minimal submanifold of  $\widetilde{M}^{2m+s}$ .

*Proof.* Suppose that  $M^n = M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  be a  $n = (n_1 + n_2 + s)$ -dimensional CR-warped product submanifold into (2m + s)-dimensional *T*-manifold  $\widetilde{M}$ , where  $M_T^{n_1+s}$  is a  $2\delta + s$ dimensional invariant submanifold tangent to  $\xi^p$  for all  $p = 1, 2, \dots s$  and  $M_{\perp}^{n_2}$  is antiinvariant submanifold of dimension  $n_2$ . Then, we choose that  $\{e_1 = \tilde{e}_1, \dots e_{\delta} = \tilde{e}_{\delta}, e_{\delta+1} = \phi \tilde{e}_1, \dots e_{2\delta} = \phi \tilde{e}_{\delta}, e_{2\delta+1} = \xi^1, \dots e_{2\delta+s} = \xi^p\}$  and  $\{e_{2\delta+s+1} = \bar{e}_1, \dots e_{2\delta+s+n_2} = \bar{e}_{n_2}\}$ are orthonormal frames for integral manifolds  $M_T^{n_1+s}$  and  $M_{\perp}^{n_2}$  of  $\mathscr{D}$  and  $\mathscr{D}^{\perp}$ , respectively. Moreover, the orthonormal frames for the normal sub-bundle  $\phi \mathscr{D}^{\perp}$  and  $\mu$  are  $\{e_{n+1} = \dot{e}_1 = \phi \bar{e}_1, \dots e_{n+n_2} = \dot{e}_{n_2} = \phi \bar{e}_{n_2}\}$  and  $\{e_{n+n_2+1}, \dots e_{2m+s}\}$ , respectively. The mean curvature vector of  $M_T^{n_1+s}$  is defined as:

$$||H_T||^2 = \frac{1}{(n_1+s)^2} \sum_{r=n_1+s+1}^{2m+s} \left(h_{11}^r + h_{22}^r + \dots + h_{n_1+s,n_1+s}^r\right)^2,$$

where  $n_1 + s$  is the dimension of  $M_T^{n_1+s}$  such that  $n_1 + s = 2\delta + s$ . Thus from adapted frame, we can express the above equation as:

$$||H_T||^2 = \frac{1}{(n_1 + s)^2} \sum_{r=n_1 + s+1}^{2m+s} \left( h_{11}^r + \dots + h_{\delta\delta}^r + h_{\delta+1\delta+1}^r + \dots + h_{2\delta2\delta}^r + h_{\xi^1\xi^1}^r + \dots + h_{\xi^s\xi^s}^r \right)^2.$$

Thus from the fact  $h_{\xi^s\xi^s}^r = 0$ , for all  $p = 1, 2, \dots s$ , we derive

$$||H_T||^2 = \frac{1}{(n_1 + s)^2} \sum_{r=n_1 + s+1}^{2m+s} \left( h_{11}^r + \dots + h_{\delta\delta}^r + h_{\delta+1\delta+1}^r + \dots + h_{2\delta2\delta}^r \right)^2.$$
(3.3.38)

Hence, there are two cases, i.e.,  $e_r \in \Gamma(\varphi \mathscr{D}^{\perp})$  or  $e_r \in \Gamma(\mu)$ . If  $e_r \in \Gamma(\varphi \mathscr{D}^{\perp})$ , then (3.3.38) can be written as:

$$||H_T||^2 = \frac{1}{(n_1 + s)^2} \sum_{r=1}^{n_2} \left( g(h(\tilde{e}_1, \tilde{e}_1), \varphi \bar{e}_r) + \dots + g(h(\tilde{e}_{\delta}, \tilde{e}_{\delta}), \varphi \bar{e}_r) \right)$$
(3.3.39)

$$+g(h(\tilde{e}_{\delta+1},\tilde{e}_{\delta+1}),\varphi\bar{e}_r)+\dots+g(h(\tilde{e}_{2\delta},\tilde{e}_{2\delta}),\varphi\bar{e}_r)\bigg)^{-}.$$
(3.3.40)

Thus from equation (3.3.22), we get  $H_T = 0$ , i.e.,  $M_T^{n_1+s}$  is minimal submanifold. On the other case, if  $e_r \in \Gamma(\mu)$ , then (3.3.38) can be written by using virtue (3.3.23)

$$||H_{T}||^{2} = \frac{1}{n_{1}^{2}} \sum_{r=n+s+n_{2}+1}^{2m+s} \left( g(h(\tilde{e}_{1},\tilde{e}_{1}),e_{r}) + \dots + g(h(\tilde{e}_{\delta},\tilde{e}_{\delta}),e_{r}) - g(h(\tilde{e}_{1},\tilde{e}_{1}),e_{r}) - \dots - g(h(\tilde{e}_{\delta},\tilde{e}_{\delta}),e_{r}) \right)^{2}, \quad (3.3.41)$$

which implies that  $H_T = 0$ . Hence in both the cases it easy to conclude that  $M_T^{n_1+s}$  is  $\varphi$ -minimal submanifold of  $\widetilde{M}$ . This completes the proof of theorem.

Hence, using the Theorem 3.3.5, we immediately obtain the following theorem, i.e.,

**Theorem 3.3.6.** Assume that  $\phi: M^n = M_T^{n_1+s} \times_f M_{\perp}^{n_2} \longrightarrow \widetilde{M}^{2m+s}$  be an isometric immersion of an n-dimensional contact CR-warped product submanifold  $M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  in T-manifold  $\widetilde{M}^{2m+s}$ . Thus

(i) The squared norm of the second fundamental form of  $M^n$  is satisfies

$$||h||^{2} \ge 2\left(n_{2}||\nabla \ln f||^{2} + \widetilde{\tau}(TM) - \widetilde{\tau}(TM_{T}) - \widetilde{\tau}(TM_{\perp}) - n_{2}\Delta(\ln f)\right), \quad (3.3.42)$$

where  $n_2$  is the dimension of anti-invariant submanifold  $M_{\perp}^{n_2}$  and  $\Delta$  is the Laplacian operator of  $M_T^{n_1+s}$ .

(ii) The equality holds in (3.3.42) if and only if  $M_T^{n_1+s}$  is totally geodesic and  $M_{\perp}^{n_2}$  is totally umbilical in  $\widetilde{M}^{2m+s}$ . Moreover  $M^n$  is minimal submanifold of  $\widetilde{M}^{2m+s}$ .

*Proof.* We are leaving the proof of the above Theorem due to similarity of the proof of Theorem 4.4 in Mustafa et al. (2015) as base manifold is T-manifold instead of Kenmotsu manifolds.

As a direct application of the Theorem 3.3.6, we prove the following theorem which is important part of this paper, i.e.,

**Theorem 3.3.7.** Let  $\phi: M^{n+s} = M_T^{n_1+s} \times_f M_{\perp}^{n_2} \longrightarrow \widetilde{M}^{2m+s}(c)$  be an isometric immersion of an (n+s)-dimensional CR-warped product submanifold  $M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  into T-space form  $\widetilde{M}^{2m+s}(c)$  such that c such that  $\xi^1, ... \xi^2, ..., \xi^s$  are tangent to  $M_T^{n_1+s}$ . Then (i) The second fundamental form is given by

$$||h||^{2} \ge 2n_{2} \left( ||\nabla \ln f||^{2} + \frac{c}{4} (n_{1} + s + 1) - \Delta(\ln f) \right),$$
(3.3.43)

where  $n_i = dim M_i$ , i = 1, 2 and  $\Delta$  is the Laplacian operator on  $M_T^{n_1+s}$ .

(ii) The equality holds in (3.3.43) if and only if  $M_T^{n_1+s}$  is totally geodesic and  $M_{\perp}^{n_2}$  is totally umbilical in  $\widetilde{M}^{2m+s}(c)$ . Moreover  $M^n$  is minimal submanifold of  $\widetilde{M}^{2m+s}(c)$ .

*Proof.* Let us substituting  $X = W = e_i$  and  $Y = Z = e_j$  in the equation (3.3.30) and change some indices, we get

$$(\widetilde{R}(e_{i},e_{j})e_{j},e_{i}) = \frac{c}{4} \left( g(e_{j},e_{i})g(e_{i},e_{j}) - g(e_{i},e_{i})g(e_{j},e_{j}) - g(e_{j},e_{i})\sum_{\alpha,\beta=1}^{s} \eta^{\alpha}(e_{i})\eta^{\beta}(e_{i}) - g(e_{j},e_{i})\sum_{\alpha,\beta=1}^{s} \eta^{\alpha}(e_{i})\eta^{\beta}(e_{j}) + g(e_{i},e_{i})\sum_{\alpha,\beta=1}^{s} \eta^{\alpha}(e_{j})\eta^{\beta}(e_{j}) + g(e_{j},e_{j})\sum_{\alpha,\beta=1}^{s} \eta^{\alpha}(e_{i})\eta^{\beta}(e_{i}) + \left(\sum_{\alpha,\beta=1}^{s} \eta^{\alpha}(e_{j})\eta^{\beta}(e_{i})\right)\left(\sum_{\alpha,\beta=1}^{s} \eta^{\alpha}(e_{i})\eta^{\beta}(e_{j})\right) - \left(\sum_{\alpha,\beta=1}^{s} \eta^{\alpha}(e_{j})\eta^{\beta}(e_{j})\right)\left(\sum_{\alpha,\beta=1}^{s} \eta^{\alpha}(e_{i})\eta^{\beta}(e_{i})\right) + g(\varphi e_{j},e_{i})g(\varphi e_{j},e_{i}) - g(\varphi e_{i},e_{j})g(\varphi e_{j},e_{i}) - g(\varphi e_{j},e_{j})g(\varphi e_{j},e_{i}) \right).$$

$$(3.3.44)$$

Taking summation over the basis vector fields of *TM* such that  $1 \le i \ne j \le n$  and using virtue (2.3.10) in last equation, then it is easy to obtain that

$$2\tilde{\tau}(TM) = \frac{c}{4}n(n-1) + \frac{3c}{4}||P||^2 - \frac{c}{2}(n-s).$$
(3.3.45)

But  $||P||^2 = \sum_{i,j=1}^{n} g(Pe_i, e_j) = n - s$ , we get

$$2\tilde{\tau}(TM) = \frac{c}{4}n(n-1) + \frac{c}{4}(n-s).$$
(3.3.46)

On the other hand, by helping the frame fields of  $TM_{\perp}^{n_2}$ , we derive

$$2\tilde{\tau}(TM_{\perp}) = \frac{c}{4} \sum_{n_1+1 \le i < j \le n} \left( g(e_j, e_i)g(e_i, e_j) - g(e_i, e_i)g(e_j, e_j) \right)$$
$$= \frac{c}{4} n_2(n_2 - 1).$$
(3.3.47)

By hypothesis, for an  $n_1$ -dimensional invariant submanifold such that  $n_3 = (n_1 + s)$  with  $\xi^p$  are tangent to  $TM_T$ , ones derive  $||P||^2 = (n_3 - s)$ , we find that

$$2\tilde{\tau}(TM_{\perp}) = \frac{c}{4} (n_3(n_3 - 1) + (n_3 - s)).$$
(3.3.48)

Therefore using (3.3.46), (3.3.47) and (3.3.48) in (3.3.42), we easily obtain the inequality (3.3.43). Moreover, the equality cases hold as usual as the second case of Theorem 3.3.6. It complete proof of theorem.

*Remark* **3.3.1.** If we assume that s = 1 in Theorem 3.3.7, then Theorem 3.3.7 generalizes to the inequality of contact CR-warped product in cosymplectic space form (see Theorem 1.2 in Uddin & Alqahtani (2016)).

*Remark* 3.3.2. Let us considered that s = 0 in Theorem 3.3.7. Then it is generalized to the result of CR-warped product into complex space form.

## 3.3.5 Applications

**Theorem 3.3.8.** Let  $M^n = M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  be a compact orientable CR-warped product into T-space form  $\widetilde{M}^{2m+s}(c)$ . Then  $M^n$  is CR-product if and only if

$$||h||^2 \ge n_2 \frac{c}{2} (n_1 + s + 1)$$
 (3.3.49)

where  $n_1 + s = \dim M_T$  and  $n_2 = \dim M_{\perp}$ .

*Proof.* From the Theorem 3.3.7, we get

$$||h||^2 \ge n_2 \frac{c}{2} (n_1 + s + 1) - 2n_2 \Delta(\ln f) + 2n_2 ||\nabla \ln f||^2$$

and

$$2n_2||\nabla \ln f||^2 + n_2 \frac{c}{2}(n_1 + s + 1) - ||h||^2 \le 2n_2 \Delta(\ln f).$$
(3.3.50)

From the integration theory on compact orientable Riemannian manifold  $M^n$  without

boundary, we obtain

$$\int_{M} \left( n_2 \left( \frac{c}{4} \left( n_1 + s + 1 \right) + 2n_2 ||\nabla \ln f||^2 - ||h||^2 \right) \mathrm{d}V \le 2n_2 \int_{M} \Delta(\ln f) \mathrm{d}V = 0. \quad (3.3.51)$$

Now, if

$$||h||^2 \ge n_2 \frac{c}{2} (n_1 + s + 1),$$

Then, from (3.3.51), we find

$$\int_M (||\nabla \ln f||^2) dV \le 0,$$

which is impossible for a positive integrable function, and hence  $\nabla \ln f = 0$ , i.e., f is a constant function on M. Thus the definition of warped product manifold,  $M^n$  is trivial. The converse part is straightforward.

**Corollary 3.3.2.** Assume that  $M^n = M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  be a CR-warped product in T-space form  $\widetilde{M}^{2m+s}(c)$ . Let  $M_T^{n_1+s}$  is compact invariant submanifold and  $\lambda$  be non-zero eigenvalue of the Laplacian on  $M_T^{n_1+s}$ . Then

$$\int_{M_T^{n_1+s}} ||h||^2 dV_T \ge \int_{M_T^{n_1+s}} n_2 \Big(\frac{c}{2} (n_1+s+1)\Big) dV_T + 2n_2 \lambda \int_{M_T^{n_1+s}} (\ln f)^2 dV_T. \quad (3.3.52)$$

Proof. Thus using the minimum principle property, we obtain

$$\int_{M_T^{n_1+s}} ||\nabla \ln f||^2 dV_T \ge \lambda \int_{M_T^{n_1+s}} (\ln f)^2 dV_T.$$
(3.3.53)

From (3.3.43) and (3.3.53) we get required the result (3.3.52). It completes proof of corollary

**Corollary 3.3.3.** If the warping function  $\ln f$  of a CR-warped product  $M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  satisfies  $\Delta \ln f \leq 0$  at a some point on  $M_T^{n_1+s}$ , then  $M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  cannot be realized as non-trivial CR-warped product in T-space form  $\widetilde{M}^{2m+s}(c)$ .

**Corollary 3.3.4.** Every CR-warped product  $M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  with harmonic warping function cannot be realized as non-trivial CR-warped product in T-space form  $\widetilde{M}^{2m+s}(c)$ such that  $c \leq 0$ .

**Corollary 3.3.5.** Every CR-warped product  $M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  cannot be realized as nontrivial CR-warped product in T-space form  $\widetilde{M}^{2m+s}(c)$  with  $c \leq 0$ . **Corollary 3.3.6.** Let  $M^n = M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  be a CR-warped product in T-space form  $\widetilde{M}^{2m+s}(c)$ . Assume that  $M_T^{n_1+s}$  is compact invariant submanifold and  $\lambda$  be non-zero eigenvalue of the Laplacian on  $M_T^{n_1+s}$ . Then it cannot realized a non-trivial CR-warped product in T-space form with  $c \leq 0$ .

**Theorem 3.3.9.** Let  $M^n = M_T^{n_1+s} \times_f M_{\perp}^{n_2}$  be a connected, compact CR-warped product into T-space form  $\widetilde{M}^{2m+s}(c)$ . Then  $M^n$  is CR-product if and only if

$$\sum_{i=1}^{n_1+s} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = n_2 \frac{c}{4} (n_1 + s + 1).$$
(3.3.54)

*Proof.* Assume that the equality holds in the inequality (3.3.43), we have

$$n_2 \frac{c}{2} (n_1 + s + 1) - 2n_2 \Delta(\ln f) + 2n_2 ||\nabla \ln f||^2 = ||h||^2.$$

By the definition of the components  $\mathscr{D}$  and  $\mathscr{D}^{\perp}$ . The above equation can be expressed as:

$$n_{2}\frac{c}{2}(n_{1}+s+1) - 2n_{2}\Delta(\ln f) + 2n_{2}||\nabla \ln f||^{2} = ||h(\mathscr{D},\mathscr{D})||^{2} + ||h(\mathscr{D}^{\perp},\mathscr{D}^{\perp})||^{2} + 2||h(\mathscr{D},\mathscr{D}^{\perp})||^{2}.$$
(3.3.55)

We take the help from the orthonormal frame which is defined in the Theorem 3.3.5. Then following these orthonormal frame and taken the help from (2.3.9), the equation (3.3.55) takes the new from

$$2n_{2}||\nabla \ln f||^{2} - 2n_{2}\Delta(\ln f) = \sum_{r=1}^{2m+s} \sum_{i,j=1}^{2\alpha+s} g(h(e_{i}, e_{j}), e_{r})^{2} + \sum_{r=1}^{2m} \sum_{i,j=1}^{\beta} g(h(e_{i}^{*}, e_{j}^{*}), e_{r})^{2} + 2\sum_{r=1}^{2m+s} \sum_{i=1}^{2\alpha} \sum_{j=1}^{\beta} g(h(e_{i}, e_{j}^{*}), e_{r})^{2} - n_{2} \frac{c}{2} (n_{1} + s + 1).$$

$$(3.3.56)$$

The first term should be zero right hand side of the above equation by fact that  $M_T^{n_1+s}$  is totally geodesic in  $\tilde{M}^{2m+s}(c)$  and the second fundamental form corresponding to  $M_{\perp}^{n_2}$  also should be zero as  $M_{\perp}^{n_2}$  is totally umbilical such that  $M^n$  is minimal submanifolds of  $\tilde{M}^{2m+s}(c)$ . Hence using the equations (3.3.17) and (3.3.21) in the last of right hand side

of the (3.3.56), we get

$$2n_2||\nabla \ln f||^2 - 2n_2\Delta(\ln f) = 2n_2||\nabla(\ln f)||^2 + \sum_{i=1}^{n_1+s} \sum_{j=1}^{n_2} ||h_\mu(e_i, e_j^*)||^2 - n_2\frac{c}{2}(n_1+s+1),$$

which implies that

$$\Delta(\ln f) + n_2 \frac{c}{2} (n_1 + s + 1) = \sum_{i=1}^{n_1 + s} \sum_{j=1}^{n_2} ||h_\mu(e_i, e_j^*)||^2.$$
(3.3.57)

As we look that the equality (3.3.54) if and only if the following condition hold from the equation (3.3.57) such that  $\Delta(\ln f) = 0$ . However,  $M^n$  is connected, compact warped product submanifold, thus follows the Theorem 2.3.1, we find that f is constant function on M. This means that  $M^n$  is simply a Riemannian product manifold or CR-products. This completes the proof of the theorem.

## **CHAPTER 4**

## PSEUDO-SLANT SUBMANIFOLDS AND THEIR WARPED PRODUCTS

### 4.1 INTRODUCTION

In the following,  $M = M_{\theta} \times_f M_{\perp}$  is called proper warped product pseudo-slant if  $M_{\perp}$  and  $M_{\theta}$  are anti-invariant and proper slant submanifolds of  $\widetilde{M}$ , respectively. Recently, Atceken (2008), has given an example for the existence of such warped products. Afterward, B. Sahin (2009b) established a general inequality for warped product pseudo-slant(as the name hemi-slant) isometrically immersed in a Kaehler manifold for mixed totally geodesic. Later on, S. Uddin et al. (2011) obtained some inequalities of warped product submanifolds in different structures.

In this Chapter, we study the warped product submanifolds of the form  $M = M_{\theta} \times_f M_{\perp}$  (we call such warped products as pseudo-slant warped products) of a locally product Riemannian manifold  $\tilde{M}$  and a nearly Kenmotsu manifold follow by case study of Sahin (2009b). Initially, we give some preparatory results for later use and provide to examples of such immersions, then obtain a characterization for warped products. Also, we establish a relationship between the squared norm of the second fundamental form and the warping function. Furthermore, the equality case of the inequality is considered.

# 4.2 WARPED PRODUCT PSEUD0-SLANT SUBMANIFOLDS OF LOCALLY RIEMANNIAN MANIFOLDS

It can be easily seen that, Sahin (2006b) studied slant submanifolds of locally product Riemannian manifolds, and this idea extended for semi-slant submanifolds of locally product Riemannian manifolds Li & Liu (2005) which are generalizations of semi-invariant submanifolds studied by Atçeken (2007). They obtained some characterizations results on under lying submanifolds of locally product Riemannian manifolds.

On the other hand, the warped product submanifolds in a locally product Rieman-

nian manifold were studied by Taştan (2015); Sahin (2006c). Recently, Sahin (2009a) and Atceken (2008) studied warped product semi-slant submanifolds of a locally product Riemannian manifold. They proved that the warped products of the form  $M_T \times_f M_{\theta}$  and  $M_{\perp} \times_f M_{\theta}$  do not exist in a locally product Riemannian manifold  $\widetilde{M}$ , where  $M_T$ ,  $M_{\perp}$  and  $M_{\theta}$  are invariant, anti-invariant and proper slant submanifolds of  $\widetilde{M}$ , respectively. They provided some examples of warped product submanifolds of the form  $M_{\theta} \times_f M_T$  and  $M_{\theta} \times_f M_{\perp}$  on their existence. For the survey of such warped product submanifolds, we also refer to Chen (2011, 2013).

### 4.2.1 Motivations

Let  $\widetilde{M}$  be an *n*-dimensional Riemannian manifold with a tensor field of the type (1,1) such that

$$F^2 = I(F \neq \pm I), \tag{4.2.1}$$

where *I* denotes the identity transformation. Then we say that  $\widetilde{M}$  is an almost product manifold with almost product structure *F*. If an almost product manifold  $\widetilde{M}$  admits a Riemannian metric *g* such that

$$g(FU, FV) = g(U, V), \qquad g(FU, V) = g(U, FV),$$
 (4.2.2)

for any vector fields U and V on  $\widetilde{M}$ , then  $\widetilde{M}$  is called an almost product Riemannian manifold. Let  $\widetilde{\nabla}$  denotes the Levi Civitas connection on  $\widetilde{M}$  with respect to g. If  $(\widetilde{\nabla}_U F)V =$ 0, for all  $U, V \in \Gamma(T\widetilde{M})$ , where  $\Gamma(T\widetilde{M})$  denotes the set of all vector fields of  $\widetilde{M}$ , then  $(\widetilde{M}, g)$ is called a locally product Riemannian manifold with Riemannian metric g in sense of Bejancu (1984).

Let *M* be a submanifold of a locally product Riemannian manifold  $\widetilde{M}$  with induced Riemannian metric *g* and if  $\nabla$  and  $\nabla^{\perp}$  are the induced Riemannian connections on the tangent bundle *TM* and the normal bundle  $T^{\perp}M$  of *M*, respectively then, for any  $X \in$  $\Gamma(TM)$  and  $N \in \Gamma(T^{\perp}M)$ , we write

(i) 
$$FU = PU + \omega U$$
, (ii)  $FN = tN + fN$ , (4.2.3)

where PU(tN) and  $\omega U(fN)$  are the tangential and normal components of FU(FN), re-

spectively. The covariant derivatives of the endomorphisms F, T and  $\omega$  are defined respectively as

$$(\widetilde{\nabla}_U F)V = \widetilde{\nabla}_U FV - F\widetilde{\nabla}_U V, \ \forall U, V \in \Gamma(T\widetilde{M})$$
(4.2.4)

$$(\widetilde{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \ \forall U, V \in \Gamma(TM)$$
(4.2.5)

$$(\widetilde{\nabla}_U \omega)V = \nabla_U^{\perp} \omega V - \omega \nabla_U V \quad \forall \ U, V \in \Gamma(TM).$$
(4.2.6)

Let M be a submanifold of a locally product Riemannian manifold  $\widetilde{M}$ , then for each non zero vector U tangent to M at a point  $p \in M$ , define an angle  $\theta(U)$  between FU and the tangent space  $T_pM$  known as Wirtinger angle of U in M. If the angle  $\theta(U)$  is constant which is independent of the choice of  $U \in T_pM$  and  $p \in M$ , then M is said be a *slant submanifold* of  $\widetilde{M}$ . Invariant and anti-invariant submanifolds are slant submanifolds with slant angle  $\theta = 0$  and  $\theta = \pi/2$ , respectively. M is *proper slant* if it is neither invariant nor anti-invariant. The normal bundle  $T^{\perp}M$  of a slant submanifold M, is decomposed as

$$T^{\perp}M = \omega(TM) \oplus \mu. \tag{4.2.7}$$

where  $\mu$  is the invariant normal sub-bundle with respect to *F* orthogonal to  $\omega(TM)$ . We recall the following result for a slant submanifold of a locally product Riemannian manifold Sahin (2006b).

**Theorem 4.2.1.** Let M be a submanifold of a locally product Riemannian manifold  $\tilde{M}$ . Then M is slant if and only if there exists a constant  $\delta \in [0,1]$  such that  $P^2 = \delta I$ . In this case,  $\theta$  is slant angle of M and satisfies  $\delta = \cos^2 \theta$ .

The following relations are the consequences of the above theorem

$$g(PU, PV) = \cos^2 \theta g(U, V), \qquad (4.2.8)$$

$$g(\omega U, \omega V) = \sin^2 \theta g(U, V), \qquad (4.2.9)$$

for any  $U, V \in \Gamma(TM)$ . Also, for a slant submanifold, (4.2.3)(i)-(ii) and Theorem 4.2.1 yield

$$t\omega U = \sin^2 \theta U, \quad \omega P U = -f\omega U. \tag{4.2.10}$$

#### 4.2.2 Pseudo-slant submanifolds in a locally Riemannian product manifold

Semi-slant submanifolds of locally product Riemannian manifolds were studied by Li & Liu (2005). They defined these submanifolds as follows:

**Definition 4.2.1.** A submanifold M of a locally product Riemannian manifold M is a semi-slant submanifold, if there exist two orthogonal distributions  $\mathcal{D}$  and  $\mathcal{D}^{\theta}$  such that

- (i)  $TM = \mathscr{D} \oplus \mathscr{D}^{\theta}$ ,
- (ii) the distribution  $\mathscr{D}$  is invariant, i.e.  $F(\mathscr{D}) = \mathscr{D}$ ,
- (iii) the distribution  $\mathscr{D}^{\theta}$  is slant with slant angle  $\theta \neq 0, \frac{\pi}{2}$ .

On the similar line, we define pseudo-slant submanifolds as follows:

**Definition 4.2.2.** Let M be a submanifold of a locally product Riemannian manifold  $\widetilde{M}$  with a pair of orthogonal distributions  $\mathscr{D}^{\perp}$  and  $\mathscr{D}^{\theta}$ , then M is said to be a *pseudo-slant* submanifold of  $\widetilde{M}$  if

- (i)  $TM = \mathscr{D}^{\perp} \oplus \mathscr{D}^{\theta}$
- (ii) the distribution  $\mathscr{D}^{\perp}$  is anti-invariant under *F*, i.e.,  $F(\mathscr{D}^{\perp}) \subset T^{\perp}M$
- (iii)  $\mathscr{D}^{\theta}$  is a slant distribution with slant angle  $\theta \neq 0, \frac{\pi}{2}$ .

Let us denote by  $m_1$  and  $m_2$ , the dimensions of  $\mathscr{D}^{\perp}$  and  $\mathscr{D}^{\theta}$ , then *M* is *anti-invariant* if  $m_2 = 0$  and *proper slant* if  $m_1 = 0$ . It is *proper pseudo-slant*, if the slant angle is different from 0 and  $\pi/2$  and  $m_1 \neq 0$ .

Moreover, if  $\mu$  is an *invariant* normal sub-bundle under F of the normal bundle  $T^{\perp}M$ , then in case of pseudo-slant submanifold, the normal bundle  $T^{\perp}M$  can be decomposed as  $T^{\perp}M = F \mathscr{D}^{\perp} \oplus \omega \mathscr{D}^{\theta} \oplus \mu$ .

A pseudo-slant submanifold of a locally product Riemannian manifold is said to be *mixed totally geodesic* if h(X,Z) = 0, for any  $X \in \Gamma(\mathcal{D}^{\theta})$  and  $Z \in \Gamma(\mathcal{D}^{\perp})$ .

First, we give the following example of a proper pseudo-slant submanifold.

*Example* **4.2.1.** Consider a submanifold *M* of  $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$  with coordinates  $(x_1, x_2, y_1, y_2)$  and the product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \ F\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial y_i}, \ i = 1, 2.$$

For any  $\theta \in (0, \frac{\pi}{4})$ , consider a submanifold *M* into  $\mathbb{R}^4$  which is given by the immersion

$$f(u,v) = (u\cos\theta, v, u\sin\theta, v), \quad u, v \neq 0.$$

Then, the tangent space TM of M is spanned by the following vector fields

$$e_1 = \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial y_1}, \qquad e_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}.$$

With respect to the product Riemannian structure F, we find

$$Fe_1 = \cos\theta \frac{\partial}{\partial x_1} - \sin\theta \frac{\partial}{\partial y_1}, \qquad Fe_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2}$$

It is easy to see that  $Fe_2$  is orthogonal to TM, thus the anti-invariant distribution is  $\mathscr{D}^{\perp} =$ span $\{e_2\}$  and  $\mathscr{D}^{\theta_1} =$  span $\{e_1\}$  is the slant distribution with slant angle  $\theta_1 = \arccos\left(\frac{g(Fe_1,e_1)}{\|Fe_1\|\|e_1\|}\right) =$  $2\theta$  and hence M is a proper pseudo-slant submanifold with slant angle  $\theta_1 = 2\theta$ .

The detailed study of pseudo-slant submanifolds of a locally product Riemannian manifold is given by Taştan & Özdemir (2015) under the name of hemi-slant submanifolds. Now, we have the following result for later use.

**Proposition 4.2.1.** On a pseudo-slant submanifold M of a locally product Riemannian manifold  $\widetilde{M}$ , we have

$$g(\nabla_X Y, Z) = \sec^2 \theta \left( g(A_{FZ} PY, X) + g(A_{\omega PY} Z, X) \right)$$

for any  $X, Y \in \Gamma(\mathscr{D}^{\theta})$  and  $Z \in \Gamma(\mathscr{D}^{\perp})$ .

*Proof.* For any  $X, Y \in \Gamma(\mathscr{D}^{\theta})$  and  $Z \in \Gamma(\mathscr{D}^{\perp})$ , we have

$$g(\nabla_X Y, Z) = g(\nabla_X Y, Z)$$
$$= g(F\widetilde{\nabla}_X Y, FZ)$$
$$= g(\widetilde{\nabla}_X FY, FZ) - g((\widetilde{\nabla}_X F)Y, FZ)$$

Using (4.2.3)(i) and the characteristic of locally product Riemannian structure, we get

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X PY, FZ) + g(\widetilde{\nabla}_X \omega Y, FZ)$$
$$= g(h(X, PY), FZ) + g(F\widetilde{\nabla}_X \omega Y, Z)$$
$$= g(A_{FZ} PY, X) + g(\widetilde{\nabla}_X F \omega Y, Z) - g((\widetilde{\nabla}_X F) \omega Y, Z).$$

Again, using (4.2.3)(ii) and the locally product Riemannian structure, we derive

$$g(\nabla_X Y, Z) = g(A_{FZ}PY, X) + g(\nabla_X t \omega Y, Z) + g(\nabla_X f \omega Y, Z).$$

Then from (4.2.10), we obtain

$$g(\nabla_X Y, Z) = g(A_{FZ}PY, X) + \sin^2 \theta g(\widetilde{\nabla}_X Y, Z) - g(\widetilde{\nabla}_X \omega PY, Z).$$

Using (2.3.2), and (2.3.3), we arrive at

$$\cos^2 \theta g(\nabla_X Y, Z) = g(A_{FZ} PY, X) + g(A_{\omega PY} X, Z),$$

thus the assertion follows from the last relation.

## 4.2.3 Warped product submanifolds in a locally Riemannian product manifold

In this section, we study the warped products of slant and anti-invariant submanifolds of a locally product Riemannian manifold. In order to study of warped product submanifolds, we follows the brief introduction of warped product manifolds which were first introduced by Bishop & O'Neill (1969) in Section 2.4.

To proceed the study, we will consider the warped product pseudo-slant submanifolds of the form  $M = M_{\theta} \times_f M_{\perp}$ , where  $M_{\theta}$  and  $M_{\perp}$  are proper slant and anti-invariant submanifolds of a locally product Riemannian manifold  $\widetilde{M}$ , respectively. For a proper warped product pseudo-slant submanifold  $M = M_{\theta} \times_f M_{\perp}$ , we have the following useful lemmas.

*Lemma* 4.2.1. On a warped product pseudo-slant submanifold  $M = M_{\theta} \times_f M_{\perp}$  of a locally product Riemannian manifold  $\widetilde{M}$ , we have

- (i)  $g(h(X,Y),FZ) = -g(h(X,Z),\omega Y),$
- (ii)  $g(h(PX,Y),FZ) = -g(h(PX,Z),\omega Y),$

for any  $X, Y \in \Gamma(TM_{\theta})$  and  $Z \in \Gamma(TM_{\perp})$ .

*Proof.* For any  $X \in \Gamma(TM_{\theta})$  and  $Z \in \Gamma(TM_{\perp})$ , we have

$$g(h(X,Y),FZ) = g(\nabla_X Y,FZ).$$

Then from (4.2.2) and (4.2.3)(i), we obtain

$$g(h(X,Y),FZ) = g(\widetilde{\nabla}_X PY,Z) + g(\widetilde{\nabla}_X \omega X,Z).$$

Since,  $M_{\theta}$  is totally geodesic in M, using this fact in the above relation then from (2.3.3), we get

$$g(h(X,Y),FZ) = -g(A_{\omega Y}X,Z) = -g(h(X,Z),\omega Y),$$

which is (i). If we interchange X by PX in (i), we can get (ii). Thus, the lemma is proved.

*Lemma* 4.2.2. Let  $M = M_{\theta} \times_f M_{\perp}$  be a proper warped product pseudo-slant submanifold of a locally product Riemannian manifold  $\widetilde{M}$ . Then

- (i)  $g(h(Z,W),\omega X) = -g(h(X,Z),FW) + (PX\ln f)g(Z,W),$
- (ii)  $g(h(Z,W), \omega PX) = -g(h(PX,Z), FW) + \cos^2 \theta(X \ln f)g(Z,W),$

for any  $X, Y \in \Gamma(TM_{\theta})$  and  $Z, W \in \Gamma(TM_{\perp})$ .

*Proof.* For any  $X \in \Gamma(TM_{\theta})$  and  $Z, W \in \Gamma(TM_{\perp})$ , we have

$$g(h(Z,W),\omega X) = g(\widetilde{\nabla}_Z W,\omega X).$$

From (4.2.3)(i), we get

$$g(h(Z,W),\omega X) = g(\widetilde{\nabla}_Z W, FX) - g(\widetilde{\nabla}_Z W, PX)$$
$$= g(F\widetilde{\nabla}_Z W, X) + g(\widetilde{\nabla}_Z PX, W).$$

By Lemma 2.4.1 (ii), we derive

$$g(h(Z,W),\omega X) = g(\nabla_Z FW, X) + (PX\ln f)g(Z,W).$$

Using (2.3.3), we get

$$g(h(Z,W),\omega X) = -g(A_{FW}Z,X) + (PX\ln f)g(Z,W)$$
$$= -g(h(X,Z),FW) + (PX\ln f)g(Z,W)$$

which proves (i). If we interchange X by PX in the above relation and using Theorem 4.2.1, we can easily get the second part, which proves the lemma completely.

Now, we construct the following example of a proper warped product pseudo-slant submanifold in a locally product Riemannian manifold.

*Example* 4.2.2. Consider a submanifold *M* of  $\mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4$  with coordinates  $(x_1, x_2, x_3, y_1, y_2, y_3, y_4)$  and the product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, F\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial y_j}, i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4$$

Let us consider the immersion f of M into  $\mathbb{R}^7$  a follows

$$f(u,\varphi) = (u\cos\varphi, u\sin\varphi, 2u, \sqrt{2u}, -u, u\sin\varphi, u\cos\varphi), \quad \varphi \neq 0 \quad u \neq 0.$$

Then the tangent space TM of M is spanned by the following vector fields

$$Z_1 = \cos\varphi \frac{\partial}{\partial x_1} + \sin\varphi \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} + \sqrt{2}\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} + \sin\varphi \frac{\partial}{\partial y_3} + \cos\varphi \frac{\partial}{\partial y_4}$$

and

$$Z_2 = -u\sin\varphi\frac{\partial}{\partial x_1} + u\cos\varphi\frac{\partial}{\partial x_2} + u\cos\varphi\frac{\partial}{\partial y_3} - u\sin\varphi\frac{\partial}{\partial y_4}.$$

Then with respect to the product Riemannian structure F, we get

$$FZ_1 = \cos\varphi \frac{\partial}{\partial x_1} + \sin\varphi \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} - \sqrt{2}\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} - \sin\varphi \frac{\partial}{\partial y_3} - \cos\varphi \frac{\partial}{\partial y_4}$$

and

$$FZ_2 = -u\sin\varphi\frac{\partial}{\partial x_1} + u\cos\varphi\frac{\partial}{\partial x_2} - u\cos\varphi\frac{\partial}{\partial y_3} + u\sin\varphi\frac{\partial}{\partial y_4}$$

Then, it is easy to see that  $FZ_2$  is orthogonal to TM, thus the anti-invariant distribution is  $\mathscr{D}^{\perp} = \operatorname{span}\{Z_2\}$  and the slant distribution is  $\mathscr{D}^{\theta} = \operatorname{span}\{Z_1\}$  with slant angle  $\theta = \operatorname{arccos}\left(\frac{g(FZ_1,Z_1)}{\|FZ_1\|\|Z_1\|}\right) = \operatorname{arccos}\left(\frac{1}{9}\right)$ . It is easy to see that both the distributions are integrable. If the integral manifolds of  $\mathscr{D}^{\perp}$  and  $\mathscr{D}^{\theta}$  are denoted by  $M_{\perp}$  and  $M_{\theta}$ , respectively, then the induced metric tensor  $g_M$  on M is given by

$$g_M = 9du^2 + 2u^2 d\phi^2 = g_{M_\theta} + (\sqrt{2}u)^2 g_{M_\perp}.$$

Thus *M* is a warped product submanifold of the form  $M = M_{\theta} \times_{f_1} M_{\perp}$  with warping function  $f_1 = \sqrt{2}u$ .

**Theorem 4.2.2.** Let M be a pseudo-slant submanifold of a locally product Riemannian manifold  $\widetilde{M}$ . Then M is locally a mixed totally geodesic warped product submanifold if and only if

(i) 
$$A_{FZ}X = 0$$
, (ii)  $A_{\omega PX}Z = \cos^2\theta(X\lambda)Z$ , (4.2.11)

for each  $X \in \Gamma(TM_{\theta})$  and a  $C^{\infty}$ -function  $\lambda$  on M with  $Z\lambda = 0$ , for each  $Z \in \Gamma(\mathscr{D}^{\perp})$ .

*Proof.* If *M* is a mixed totally geodesic warped product submanifold of a locally product Riemannian manifold  $\widetilde{M}$ , then for any  $X \in \Gamma(TM_{\theta})$  and  $Z, W \in \Gamma(TM_{\perp})$ , we have  $g(A_{FZ}X,W) = g(h(X,W),FZ) = 0$ , i.e.,  $A_{FZ}X$  has no component in  $TM_{\perp}$ . Also, from Lemma 4.2.1,  $g(A_{FZ}X,Y) = 0$ , i.e.,  $A_{FZ}X$  has no component in  $TM_{\theta}$ . Therefore it follows that  $A_{FZ}X = 0$ , which is first part of (4.2.11). Similarly,  $g(A_{\omega PX}Z,Y) = g(h(Y,Z), \omega PX) =$ 0, i.e.,  $A_{\omega PX}Z$  has no component in  $TM_{\theta}$  for any  $X, Y \in \Gamma(TM_{\theta})$  and  $Z \in \Gamma(TM_{\perp})$ . Therefore, the second part of (4.2.11) follows from Lemma 4.2.2 (ii).

Conversely, let M be a proper pseudo-slant submanifold of a locally product Riemannian manifold  $\widetilde{M}$  such that (4.2.11) holds. Then by Proposition 4.2.1 and the relation (4.2.11),we find  $g(\nabla_X Y, Z) = 0$ , which means that the leaves of  $\mathscr{D}^{\theta}$  are totally geodesic in M. On the other hand, for any  $X \in \Gamma(\mathscr{D}^{\theta})$  and  $Z, W \in \Gamma(D^{\perp})$  we have

$$g([Z,W],PX) = g(\nabla_Z W,PX) - g(\nabla_W Z,PX).$$

From (4.2.3)(i), we get

$$g([Z,W],PX) = g(\widetilde{\nabla}_Z W,FX) - g(\widetilde{\nabla}_Z W,\omega X) - g(\widetilde{\nabla}_W Z,FX) + g(\widetilde{\nabla}_Z W,\omega X).$$

Using (4.2.2), we obtain

$$g([Z,W],PX) = g(\widetilde{\nabla}_Z FW,X) - g(\widetilde{\nabla}_Z \omega X,W) - g(\widetilde{\nabla}_W FZ,X) + g(\widetilde{\nabla}_W \omega X,Z).$$

Thus by (2.3.3), we derive

$$g([Z,W],PX) = -g(A_{FW}X,Z) + g(A_{\omega X}Z,W) + g(A_{FZ}X,W) - g(A_{\omega X}W,Z)$$

The first and third terms of right hand side are identically zero by using (4.2.11)(i) and the second and fourth terms can be evaluated from (4.2.11)(i) by interchanging *X* by *PX*, as follows

$$g([Z,W],PX) = (PX\lambda)g(Z,W) - (PX\lambda)g(W,Z) = 0$$

which means that, the distribution  $\mathscr{D}^{\perp}$  is integrable, thus if we consider  $M_{\perp}$  be a leaf of  $\mathscr{D}^{\perp}$  in M and  $h^{\perp}$  be a second fundamental form of  $M_{\perp}$  in M, then for any  $Z, W \in \Gamma(D^{\perp})$ , we have

$$g(A_{FW}PX,Z) = g(h(Z,PX),FW) = g(\widetilde{\nabla}_Z PX,FW).$$

Then by (4.2.2), we derive

$$g(A_{FW}PX,Z) = g(F\nabla_Z PX,W).$$

From (4.2.4) and the structure equation of a locally product Riemannian manifold, we obtain

$$g(A_{FW}PX,Z) = g(\widetilde{\nabla}_Z FPX,W).$$

Using (4.2.3)(i), we get

$$g(A_{FW}PX,Z) = g(\widetilde{\nabla}_Z P^2 X, W) + g(\widetilde{\nabla}_Z \omega P X, W).$$

Thus by Theorem 4.2.1 and (2.3.3), we derive

$$g(A_{FW}PX,Z) = \cos^2 \theta g(\widetilde{\nabla}_Z X, W) - g(A_{\omega PX}Z, W)$$
$$= -\cos^2 \theta g(\widetilde{\nabla}_Z W, X) - g(A_{\omega PX}Z, W).$$

As  $\mathscr{D}^{\perp}$  is integrable thus on using (2.3.3), we get

$$g(A_{FW}PX,Z) = -\cos^2\theta g(h^{\perp}(Z,W),X) - g(A_{\omega PX}Z,W).$$

From (4.2.11)(i) and (4.2.11)(ii), we obtain

$$\cos^2\theta g(h^{\perp}(Z,W),X) + \cos^2\theta(X\lambda)g(Z,W) = 0.$$

Since *M* is proper slant thus, we find

$$g(h^{\perp}(Z,W),X) = -X(\lambda)g(Z,W)$$
  
=  $-g(Z,W)g(\nabla\lambda,X)$ 

which means  $h^{\perp}(Z,W) = -g(Z,W)\nabla\lambda$  that is  $M_{\perp}$  is totally umbilical in M with the mean curvature vector  $H^{\perp} = -\nabla\lambda$ . We can easily see that  $H^{\perp}$  is a parallel mean curvature vector corresponding to the normal connection of  $M_{\perp}$  into M. Hence by Definition 2.4.3, M is a warped product pseudo-slant submanifold, which proves the theorem completely.

### 4.2.4 Inequality for warped products in a locally Riemannian product manifold

In this subsection, we obtain a sharp estimation between the squared norm of the second fundamental form of the warped product immersion and the warping function. In order to obtain the relation for the squared norm of the second fundamental form, we construct the following orthonormal frame fields for a proper warped product pseudo-slant submanifold of the form  $M = M_{\theta} \times_f M_{\perp}$ .

*Frame* 4.2.1. Let  $M = M_{\theta} \times_{f} M_{\perp}$  be a *m*-dimensional warped product pseudo-slant submanifold of an *n*-dimensional locally product Riemannian manifold  $\widetilde{M}$  such that  $p = \dim M_{\theta}$  and  $q = \dim M_{\perp}$ , where  $M_{\theta}$  and  $M_{\perp}$  are proper slant and anti-invariant submanifolds of  $\widetilde{M}$ , respectively. Denote the tangent bundles of  $M_{\theta}$  and  $M_{\perp}$  by  $\mathscr{D}^{\theta}$  and  $\mathscr{D}^{\perp}$ , respectively. Consider the orthonormal frame fields  $\{e_{1}, \dots, e_{q}\}$  and  $\{e_{q+1} = e_{1}^{*} =$  $\sec \theta P e_{1}^{*}, \dots, e_{m} = e_{p}^{*} = \sec \theta P e_{p}^{*}\}$  of  $\mathscr{D}^{\perp}$  and  $\mathscr{D}^{\theta}$ , respectively. Then the orthonormal frame fields of the normal sub-bundles  $F \mathscr{D}^{\perp}$ ,  $\omega \mathscr{D}^{\theta}$  and  $\mu$ , respectively are  $\{e_{m+1} =$  $F e_{1}, \dots, e_{m+q} = F e_{q}\}$ ,  $\{e_{m+q+1} = \widetilde{e}_{1} = \csc \theta \omega e_{1}^{*}, \dots, e_{m+p+q} = \widetilde{e}_{p} = \csc \theta \omega e_{p}^{*}\}$ and  $\{e_{2m+1}, \dots, e_{n}\}$ .

**Theorem 4.2.3.** Let  $M = M_{\theta} \times_f M_{\perp}$  be a mixed totally geodesic warped product pseudoslant submanifold of a locally product Riemannian manifold  $\widetilde{M}$  such that  $M_{\perp}$  is an antiinvariant submanifold and  $M_{\theta}$  is a proper slant submanifold of  $\widetilde{M}$ . Then: (i) The squared norm of the second fundamental form h of M satisfies

$$||h||^2 \ge q \cot^2 \theta ||\nabla^{\theta} \ln f||^2$$

where  $q = \dim M_{\perp}$  and  $\nabla^{\theta} \ln f$  is the gradient of  $\ln f$  along  $M_{\theta}$ .

(ii) If the equality sign of above holds identically, then  $M_{\theta}$  is totally geodesic and  $M_{\perp}$  is totally umbilical in  $\widetilde{M}$ .

Proof. From the definition, we have

$$||h||^{2} = ||h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta})||^{2} + ||h(\mathscr{D}^{\perp}, \mathscr{D}^{\perp})||^{2} + 2||h(\mathscr{D}^{\theta}, \mathscr{D}^{\perp})||^{2}.$$

Since *M* is mixed totally geodesic, then the second term in the above relation is identically zero, thus we find

$$||h||^{2} = ||h(\mathscr{D}^{\perp}, \mathscr{D}^{\perp})||^{2} + ||h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta})||^{2}.$$

Then from (2.3.9), we obtain

$$||h||^{2} = \sum_{r=m+1}^{n} \sum_{i,j=1}^{q} g(h(e_{i},e_{j}),e_{r})^{2} + \sum_{r=m+1}^{n} \sum_{i,j=1}^{p} g(h(e_{i}^{*},e_{j}^{*}),e_{r})^{2}.$$

The above relation can be expressed in terms of the components of  $F \mathscr{D}^{\perp}$ ,  $\omega \mathscr{D}^{\theta}$  and  $\mu$  as follows

$$||h||^{2} = \sum_{r=1}^{q} \sum_{i,j=1}^{q} g(h(e_{i},e_{j}),Fe_{r})^{2} + \sum_{r=1}^{p} \sum_{i,j=1}^{q} g(h(e_{i},e_{j}),\csc\theta\,\omega e_{r}^{*})^{2} + \sum_{r=2m+1}^{n} \sum_{i,j=1}^{q} g(h(e_{i},e_{j}),e_{r})^{2} + \sum_{r=1}^{q} \sum_{i,j=1}^{p} g(h(e_{i}^{*},e_{j}^{*}),Fe_{r})^{2} + \sum_{r=1}^{p} \sum_{i,j=1}^{p} g(h(e_{i}^{*},e_{j}^{*}),\widetilde{e}_{r})^{2} + \sum_{r=2m+1}^{n} \sum_{i,j=1}^{p} g(h(e_{i}^{*},e_{j}^{*}),e_{r})^{2}.$$
(4.2.12)

As we have not found any relation for  $g(h(e_i, e_j), Fe_r)$ , for any  $i, j, r = 1, \dots, q$  and  $g(h(e_i^*, e_j^*), \tilde{e}_r)$ , for any  $i, j, r = i, \dots, p$ , therefore we shall leave these terms in the above relation (4.2.12). Also, the third and sixth terms have  $\mu$ -components therefore we also leave these two terms, then by using Lemma 4.2.1 and Lemma 4.2.2, we derive

$$||h||^2 \ge \csc^2 \theta \sum_{i,j=1}^q \sum_{r=1}^p (Pe_r^* \ln f)^2 g(e_i, e_j)^2.$$

From the assumed orthonormal frame fields of  $\mathscr{D}^{\theta}$ , we have  $Pe_r^* = \cos \theta e_r^*$ , for r =

 $1, \cdots, p$ . Using this fact, we find

$$||h||^{2} \ge \cot^{2} \theta \sum_{i,j=1}^{q} \sum_{r=1}^{p} (e_{r}^{*} \ln f)^{2} g(e_{i}, e_{j})^{2} = q \cot^{2} \theta \|\nabla^{\theta} \ln f\|^{2}$$

which is inequality (i). If the equality holds in (i), then from the remaining fifth and sixth terms of (4.2.12), we obtain the following conditions,

$$h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta}) \perp \omega \mathscr{D}^{\theta}, \quad h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta}) \perp \mu,$$
 (4.2.13)

which means that

$$h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta}) \subset F \mathscr{D}^{\perp}.$$
(4.2.14)

Also, from Lemma 4.2.1, we get

$$h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta}) \perp F \mathscr{D}^{\perp}.$$
(4.2.15)

Then from (4.2.14) and (4.2.15), we get

$$h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta}) = 0. \tag{4.2.16}$$

Then  $M_{\theta}$  is totally geodesic in  $\widetilde{M}$ , by using the fact that  $M_{\theta}$  is totally geodesic in M with (4.2.16). Also, from the remaining first and third terms in (4.2.12), we have

$$h(\mathscr{D}^{\perp},\mathscr{D}^{\perp}) \perp F \mathscr{D}^{\perp}, \quad h(\mathscr{D}^{\perp},\mathscr{D}^{\perp}) \perp \mu \Rightarrow h(\mathscr{D}^{\perp},\mathscr{D}^{\perp}) \subset \omega \mathscr{D}^{\theta}.$$
 (4.2.17)

From Lemma 4.2.2 for a mixed totally geodesic warped product submanifold we have

$$g(h(Z,W),\omega PX) = \cos^2 \theta(X \ln f)g(Z,W). \tag{4.2.18}$$

Hence,  $M_{\perp}$  is totally umbilical in  $\widetilde{M}$  by using the fact that  $M_{\perp}$  is totally umbilical by M Bishop & O'Neill (1969); Chen (2001) with (4.2.17) and (4.2.18), thus the proof is complete.

# 4.3 SOME INEQUALITIES OF WARPED PRODUCT PSEUDO-SLANT SUB-MANIFOLDS OF NEARLY KENMOTSU MANIFOLDS

## 4.3.1 Motivations

In the present section, we extend the previous idea into a nearly Kenmotsu manifold and derive the geometric inequalities of non-trivial warped product pseudo-slant submanifolds which are the natural extensions of CR-warped product submanifolds. As we know that every CR-warped product submanifold is a non-trivial warped product pseudo-slant submanifold of the forms  $M_{\perp} \times_f M_{\theta}$  and  $M_{\theta} \times_f M_{\perp}$  with slant angle  $\theta = 0$ . But such warped product pseudo-slant submanifold.

## 4.3.2 Warped product pseudo-slant in a nearly Kenmotsu manifold

We derive the following lemmas to obtain the main inequalities of this section by using the properties of nearly Kenmotsu manifold and warped product manifolds.

*Lemma* 4.3.1. Let  $M = M_{\perp} \times_f M_{\theta}$  is a warped product pseudo-slant submanifold of a nearly Kenmotsu manifold  $\widetilde{M}$  such that the structure vector field  $\xi$  is tangent to  $M_{\perp}$ . Then

$$g(h(X, PX), \varphi Z) = g(h(X, Z), FPX) + \frac{1}{3} \left( \eta(Z) - (Z \ln f) \right) \cos^2 \theta ||X||^2, \quad (4.3.1)$$

for any  $X \in \Gamma(TM_{\theta})$  and  $Z \in \Gamma(TM_{\perp})$ .

*Proof.* Assume that  $M = M_{\perp} \times_f M_{\theta}$  be a warped product pseudo-slant submanifold of a nearly Kenmotsu manifold  $\widetilde{M}$ . Then (2.3.2) gives

$$g(h(X, PX), \varphi Z) = -g(\varphi \widetilde{\nabla}_X PX, Z).$$

From the covariant derivative of  $\varphi$ , we simplifies as:

$$g(h(X, PX), \varphi Z) = g((\widetilde{\nabla}_X \varphi) PX, Z) - g(\widetilde{\nabla}_X \varphi PX, Z).$$

Using the structure equation (2.2.16) and Theorem 2.3.8, we derive

$$g(h(X, PX), \varphi Z) = -g((\widetilde{\nabla}_{PX}\varphi)X, Z) - \cos^2\theta g(\widetilde{\nabla}_X Z, X)$$
$$+g(h(X, Z), FPX).$$

Lemma 2.4.1(ii) and from (2.3.43), we obtain

$$g(h(X, PX), \varphi Z) = -g(\widetilde{\nabla}_{PX}\varphi X, Z) - g(\widetilde{\nabla}_{PX}X, \varphi Z)$$
$$+g(h(X, Z), FPX) - \cos^2 \theta(Z \ln f) ||X||^2$$

Using the relations (2.3.3) and (2.3.39), we arrive at

$$2g(h(X, PX), \varphi Z) = g(\nabla_{PX}Z, PX) - g(\widetilde{\nabla}_{PX}FX, Z) + g(h(X, Z), FPX) - \cos^2\theta(Z\ln f)||X||^2.$$

Finally, using Lemma 4.3.1(ii) and relation (2.3.55), we get

$$2g(h(X, PX), \varphi Z) = g(h(PX, Z), FX)) + g(h(X, Z), FPX).$$
(4.3.2)

Moreover, we have

$$g(h(X,Z),FPX) = g(\widetilde{\nabla}_Z X,FPX),$$

for any  $X \in \Gamma(TM_{\theta})$  and  $Z \in \Gamma(TM_{\perp})$ . Since, from the fact that  $\xi$  is tangent to  $M_{\perp}$  and (2.3.2), (2.3.39), we obtain

$$g(h(X,Z),FPX) = -g(\varphi \widetilde{\nabla}_Z X,PX) + \cos^2 \theta g(\nabla_Z X,X)$$

Then from definition of the covariant derivative of  $\varphi$  and Lemma 4.3.1(ii), we can derive

$$g(h(X,Z),FPX) = g((\widetilde{\nabla}_Z \varphi)X,PX) - g(\widetilde{\nabla}_Z \phi X,PX) + \cos^2 \theta(Z \ln f) ||X||^2.$$

On nearly Kenmotsu manifold (2.2.16) and (2.3.55), one shows that

$$g(h(X,Z),FPX) = -g((\widetilde{\nabla}_X \varphi)Z,PX) - \eta(Z)\cos^2 \theta ||X||^2 - g(\nabla_Z PX,PX)$$
$$-g(\widetilde{\nabla}_Z FX,PX) + \cos^2 \theta(X\ln f) ||Z||^2.$$

Using Lemma 4.3.1(ii) and relations (2.3.3), (2.3.55), we arrive at

$$g(h(X,Z),FPX) = -g(\widetilde{\nabla}_X \varphi Z,PX) - g(\widetilde{\nabla}_X Z,\varphi PX)$$
$$-\eta(Z)\cos^2\theta||X||^2 + g(h(Z,PX),FX)$$

The Weingarten formula (2.3.3) and the Theorem 2.3.8, for a slant submanifold, give us

$$g(h(X,Z),FPX) = g(A_{\varphi Z}X,PX) + \cos^2 \theta g(\nabla_X Z,X) - \eta(Z)\cos^2 \theta ||X||^2$$
$$-g(\widetilde{\nabla}_X Z,FPX) + g(h(Z,PX),FX).$$

From Lemma 4.3.1(ii) and (2.3.2), we derive

$$g(h(X, PX), \varphi Z) = 2g(h(X, Z), FPX) - \left((Z \ln f) - \eta(Z)\right) \cos^2 \theta ||X||^2 - g(h(Z, PX), FX).$$
(4.3.3)

Thus from the (4.3.2) and (4.3.3), it follows that

$$g(h(X, PX), \varphi Z) - g(h(X, Z), FPX) = \frac{1}{3} \left( \eta(Z) - (Z \ln f) \right) \cos^2 \theta ||X||^2$$

which is final result. This completes proof of lemma.

Lemma 4.3.2. Let  $M = M_{\perp} \times_f M_{\theta}$  be a warped product pseudo-slant submanifold of a nearly Kenmotsu manifold M. Thus

(i)  $g(h(X,X), \varphi Z) = g(h(Z,X), FX)$ , (ii)  $g(h(PX,PX), \varphi Z) = g(h(Z,PX), FX)$ for any  $X \in \Gamma(TM_{\theta})$  and  $Z \in \Gamma(TM_{\perp})$ .

(ii) 
$$g(h(PX,PX),\varphi Z) = g(h(Z,PX),FPX),$$

Proof. On warped product pseudo-slant submanifold, we have

$$g(h(X,X),\varphi Z) = g(\widetilde{\nabla}_X X,\varphi Z) = -g(\varphi \widetilde{\nabla}_X X,Z)$$
$$= g(\widetilde{\nabla}_X \varphi) X,Z - g(\widetilde{\nabla}_X \varphi X,Z),$$

for any  $X \in \Gamma(TM_{\theta})$  and  $Z \in \Gamma(TM_{\perp})$ . Then from (2.2.16) and (2.2.8), we obtain

$$g(h(X,X),\varphi Z) = g(\widetilde{\nabla}_X Z, PX) - g(\widetilde{\nabla}_X FX, Z).$$

Thus from Lemma 4.3.1(ii) and (2.3.2), the above equation can be written as:

$$g(h(X,X),\varphi Z) = (Z\ln f)g(X,PX) + g(A_{FX}X,Z)$$

As X and PX are orthogonal vector fields. Then

$$g(h(X,X),\varphi Z) = g(h(X,Z),FX),$$
 (4.3.4)

which is the first result of lemma. Interchanging X by PX in (4.3.4), we get the last result of lemma. This completes proof of the lemma.

*Lemma* 4.3.3. On a non-trivial warped product pseudo-slant  $M = M_{\theta} \times_f M_{\perp}$  of a nearly Kenmotsu manifold  $\widetilde{M}$ . Then

(i) 
$$g(h(Z,Z), FPX) = g(h(Z,PX), \varphi Z) + (\eta(X) - (X \ln f)) \cos^2 \theta ||Z||^2,$$
  
(ii)  $g(h(Z,Z), FX) = g(h(Z,X), \varphi Z) - (PX \ln f) ||Z||^2,$ 

for any  $X \in \Gamma(TM_{\theta})$  and  $Z \in \Gamma(TM_{\perp})$ , where the structure vector field  $\xi$  is tangent to  $M_{\theta}$ .

*Proof.* From (2.3.2), and (2.3.39), we find

$$g(h(Z,Z),FPX) = g(\widetilde{\nabla}_Z Z,FPX) = g(\widetilde{\nabla}_Z Z,\phi PX) - g(\widetilde{\nabla}_Z Z,P^2X).$$

Follows the Theorem 2.3.8, we obtain

$$g(h(Z,Z),FPX) = -g(\varphi \widetilde{\nabla}_Z Z,PX) + \cos^2 \theta \left( g(\widetilde{\nabla}_Z Z,X) - \eta(X)g(\widetilde{\nabla}_Z Z,\xi) \right).$$

Using the property of Riemannian connection and the covariant derivative of an endomorphism, we derive

$$g(h(Z,Z), FPX) = g((\widetilde{\nabla}_{Z} \varphi)Z, PX) - g(\widetilde{\nabla}_{Z} \varphi Z, PX) - \cos^{2} \theta g(\nabla_{Z} X, Z)$$
$$+ \cos^{2} \theta \eta(X) g(\widetilde{\nabla}_{Z} \xi, Z).$$

In a nearly Kenmotsu manifold  $\xi \ln f = 1$ , and Lemma 4.3.1(ii), we arrive at

$$g(h(Z,Z),FPX) = g(h(Z,PX),\phi Z) + \left(\eta(X) - (X\ln f)\right)\cos^2\theta||Z||^2, \quad (4.3.5)$$

which is the first part of lemma. The second part of the lemma easily obtain by interchanging X by PX in (4.3.5). This completes proof of the lemma.

## **4.3.3** Inequality for warped products of the forms $M_{\perp} \times_f M_{\theta}$

In this subsection, we obtain geometric inequalities of warped product pseudo-slant submanifold in terms of the second fundamental form such that  $\xi$  is normal to the fiber with case of mixed totally geodesic submanifold. First of all, we define an orthonormal frame for later use.

*Frame* 4.3.1. Assume that  $M = M_{\perp} \times_f M_{\theta}$  be a *m*-dimensional warped product pseudoslant submanifold of a 2n + 1-dimensional nearly Kenmotsu manifold  $\widetilde{M}$  with  $M_{\theta}$  of dimension  $d_1 = 2\beta$  and  $M_{\perp}$  of dimension  $d_2 = \alpha + 1$ , where  $M_{\theta}$  and  $M_{\perp}$  are the integral manifolds of  $\mathscr{D}^{\theta}$  and  $\mathscr{D}^{\perp}$ , respectively. Then we consider that  $\{e_1, e_2, \cdots e_{\alpha}, e_{d_2=\alpha+1} =$  $\xi$ ,  $\}$  and  $\{e_{\alpha+2} = e_1^*, \cdots e_{\alpha+\beta+1} = e_{\beta}^*, e_{\alpha+\beta+2} = e_{\beta+1}^* = \sec \theta P e_1^*, \cdots e_{\alpha+1+2\beta} = e_{2\beta}^* =$  $\sec \theta P e_{\beta}^*\}$  are orthonormal frames of  $\mathscr{D}^{\perp}$  and  $\mathscr{D}^{\theta}$ , respectively. Thus the orthonormal frames of the normal sub-bundles  $\varphi \mathscr{D}^{\perp}$ ,  $F \mathscr{D}^{\theta}$  and  $\mu$ , respectively are  $\{e_{m+1} = \bar{e}_1 =$  $\varphi e_1, \cdots e_{m+\alpha} = \bar{e}_{\alpha} = \varphi e_{\alpha}\}$ ,  $\{e_{m+\alpha+1} = \bar{e}_{\alpha+1} = \tilde{e}_1 = \csc \theta F e_1^*, \cdots e_{m+\alpha+\beta} = \bar{e}_{\alpha+\beta} = \tilde{e}_{\beta} =$  $\csc \theta F e_{\beta}^*, e_{m+\alpha+\beta+1} = \bar{e}_{\alpha+\beta+1} = \tilde{e}_{\beta+1} = \csc \theta \sec \theta F P e_1^*, \cdots e_{m+\alpha+2\beta} = \bar{e}_{\alpha+2\beta} = \tilde{e}_{2\beta} =$  $\csc \theta \sec \theta F P e_{\beta}^*\}$  and  $\{e_{2m-1} = \bar{e}_m, \cdots e_{2n+1} = \bar{e}_{2(n-m+1)}\}$ .

Thus, we are going to prove our main theorem by using above orthonormal frame

**Theorem 4.3.1.** Let  $M = M_{\perp} \times_f M_{\theta}$  be a m-dimensional mixed totally geodesic warped product pseudo-slant submanifold of a (2n + 1)-dimensional nearly Kenmotsu manifold  $\widetilde{M}$  such that  $\xi \in \Gamma(TM_{\perp})$ , where  $M_{\perp}$  is an anti-invariant submanifold of dimension  $d_2$ and  $M_{\theta}$  is a proper slant submanifold of dimension  $d_1$  of  $\widetilde{M}$ . Then

(i) The squared norm of the second fundamental form of M is given by

$$||h||^{2} \ge \frac{2\beta}{9} \cos^{2} \theta \left( ||\nabla^{\perp} \ln f||^{2} - 1 \right).$$
(4.3.6)

(ii) If the equality holds in (4.3.6), then  $M_{\perp}$  is totally geodesic and  $M_{\theta}$  is totally umbilical submanifolds into  $\widetilde{M}$ . *Proof.* The squared norm of the second fundamental form is defined as:

$$||h||^{2} = ||h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta})||^{2} + ||h(\mathscr{D}^{\perp}, \mathscr{D}^{\perp})||^{2} + 2||h(\mathscr{D}^{\theta}, \mathscr{D}^{\perp})||^{2}$$

As we have considered M to be a mixed totally geodesic, then

$$||h||^{2} = ||h(\mathscr{D}^{\perp}, \mathscr{D}^{\perp})||^{2} + ||h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta})||^{2}.$$

$$(4.3.7)$$

Follows the relation (2.3.9) and leaving the first term, we obtain

$$||h||^2 \ge \sum_{l=m+1}^{2n+1} \sum_{i,j=1}^{2\beta} g(h(e_i^*, e_j^*), e_l)^2.$$

The above equation can be expressed as in the components of  $\varphi \mathscr{D}^{\perp}$ ,  $F \mathscr{D}^{\theta}$  and  $\mu$ , i.e.,

$$||h||^{2} \geq \sum_{l=1}^{\alpha} \sum_{i,j=1}^{2\beta} g(h(e_{i}^{*}, e_{j}^{*}), \bar{e}_{l})^{2} + \sum_{l=\alpha+1}^{2\beta+\alpha} \sum_{i,j=1}^{2\beta} g(h(e_{i}^{*}, e_{j}^{*}), \bar{e}_{l})^{2} + \sum_{l=m}^{2(n-m+1)} \sum_{i,j=1}^{2\beta} g(h(e_{i}^{*}, e_{j}^{*}), \bar{e}_{l})^{2}.$$

$$(4.3.8)$$

Leaving all the terms except first, we get

$$||h||^2 \ge \sum_{l=1}^{\alpha} \sum_{i,j=1}^{2\beta} g(h(e_i^*, e_j^*), \bar{e}_l)^2.$$

Using the components  $\mathcal{D}^{\theta}$  which are defined in orthonormal Frame 4.3.1, we derive

$$\begin{split} ||h||^{2} &\geq \sum_{i=1}^{\alpha} \sum_{r,k=1}^{\beta} g(h(e_{r}^{*},e_{k}^{*}),\bar{e}_{i})^{2} + \sec^{2}\theta \sum_{i=1}^{\alpha} \sum_{r,k=1}^{\beta} g(h(Pe_{r}^{*},e_{k}^{*}),\bar{e}_{i})^{2} \\ &+ \sec^{2} \sum_{i=1}^{\alpha} \sum_{r,k=1}^{\beta} g(h(e_{r}^{*},Pe_{k}^{*}),\bar{e}_{i})^{2} + \sec^{4}\theta \sum_{i=1}^{p} \sum_{r,k=1}^{\beta} g(h(Pe_{r}^{*},Pe_{k}^{*}),\bar{e}_{i})^{2} \end{split}$$

Then for a mixed totally geodesic warped product, the first and last terms of the right hand side in the above equation vanishes identically by using Lemma 4.3.2. Then

$$||h||^2 \ge 2\sec^2\theta \sum_{i=1}^{\alpha} \sum_{r=1}^{\beta} g(h(Pe_r^*, e_r^*), \bar{e}_i)^2.$$

Thus from Lemma 4.3.1, for a mixed totally geodesic warped product pseudo-slant and using the fact that  $\eta(e_i) = 0$ ,  $i = 1, 2, \dots (d_2 - 1)$ , for an orthonormal frame 4.3.1, we

arrive at

$$||h||^{2} \ge \frac{2}{9}\cos^{2}\theta \sum_{i=1}^{\alpha} \sum_{r=1}^{\beta} (\bar{e}_{i}\ln f)^{2}g(e_{r}^{*}, e_{r}^{*})^{2}.$$
(4.3.9)

Now we adding and subtracting the same term  $\xi \ln f$  in (4.3.9), thus simplification gives

$$||h||^{2} \geq \frac{2}{9}\cos^{2}\theta \sum_{i=1}^{\alpha+1} \sum_{r=1}^{\beta} \left(\bar{e}_{j}\ln f\right)^{2} g(e_{r}^{*}, e_{r}^{*})^{2} - \frac{2}{9}\cos^{2}\theta \sum_{r=1}^{\beta} \left(\xi\ln f\right)^{2} g(e_{r}^{*}, e_{r}^{*})^{2}.$$
 (4.3.10)

It well know that  $\xi \ln f = 1$ , for a warped product submanifold in a nearly Kenmotsu manifold. The equation (4.3.10) simplifies as

$$||h||^2 \geq \frac{2\beta}{9}\cos^2\theta \Big(||\nabla^{\perp}\ln f||^2 - 1\Big),$$

which is the inequality (4.3.6). If the equality sign holds in (4.3.6), we obtain the following conditions from leaving terms in (4.3.7), i.e.,

$$h(\mathscr{D}^{\perp},\mathscr{D}^{\perp})=0,$$

which means that,  $M_{\perp}$  is totally geodesic in  $\widetilde{M}$  due to totally geodesic in M. Similarly, from leaving the second and third terms in (4.3.8), we derive

$$g(h(\mathscr{D}^{\theta},\mathscr{D}^{\theta}),F\mathscr{D}^{\theta})=0, \ g(h(\mathscr{D}^{\theta},\mathscr{D}^{\theta}),\mu)=0,$$

which implies that

$$h(\mathscr{D}^{\theta},\mathscr{D}^{\theta})\perp F\mathscr{D}^{\theta}, \ h(\mathscr{D}^{\theta},\mathscr{D}^{\theta})\perp\mu\Longrightarrow h(\mathscr{D}^{\theta},\mathscr{D}^{\theta})\subset\phi\mathscr{D}^{\perp}.$$

From the above conditions and Lemma 4.3.1 which show that  $M_{\theta}$  is a totally umbilical into  $\widetilde{M}$  due to totally umbilical in M. This completes proof of the theorem.

# 4.3.4 Inequality for a warped product pseudo-slant submanifold of the form $M_{\theta} \times_f M_{\perp}$

We will use the same orthonormal frame which defined as Frame 4.3.1 by taking  $\xi$  tangent to slant submanifold instead of anti-invariant submanifold.

**Theorem 4.3.2.** Let  $M = M_{\theta} \times_f M_{\perp}$  be a m-dimensional mixed totally geodesic warped product pseudo-slant submanifold of a (2n + 1)-dimensional nearly Kenmotsu manifold  $\widetilde{M}$  such that  $\xi \in \Gamma(TM_{\theta})$ , where  $M_{\perp}$  is an anti-invariant submanifold of dimension  $d_2 = \alpha$ and  $M_{\theta}$  is a proper slant submanifold of dimension  $d_1 = 2\beta + 1$  of  $\widetilde{M}$ . Then

(i) The squared norm of the second fundamental form of M is given by

$$||h||^{2} \ge \alpha \cot^{2} \theta \left( ||\nabla^{\theta} \ln f||^{2} - 1 \right).$$

$$(4.3.11)$$

(ii) If equality holds identically in (4.3.11), then  $M_{\theta}$  is totally geodesic submanifold and  $M_{\perp}$  is a totally umbilical submanifold of  $\widetilde{M}$ , respectively.

*Proof.* Follows the definition of the second fundamental form with linearity of Hermitian metric such that

$$||h||^{2} = ||h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta})||^{2} + ||h(\mathscr{D}^{\perp}, \mathscr{D}^{\perp})||^{2} + 2||h(\mathscr{D}^{\theta}, \mathscr{D}^{\perp})||^{2}$$

As the hypothesis of the Theorem, M to be a mixed totally geodesic, we obtain

$$||h||^{2} = ||h(\mathscr{D}^{\perp}, \mathscr{D}^{\perp})||^{2} + ||h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta})||^{2}.$$

$$(4.3.12)$$

The relation (2.3.9), gives

$$||h||^2 \ge \sum_{l=m+1}^{2n+1} \sum_{r,k=1}^{lpha} g(h(e_r,e_k),e_l)^2.$$

The above equation can be expressed in the components of  $\varphi \mathscr{D}^{\perp}$ ,  $F \mathscr{D}^{\theta}$  and  $\mu$  defined in orthonormal frame 4.3.1, and considering  $\xi$  tangent to  $M_{\theta}$ , i.e.,

$$||h||^{2} \geq \sum_{l=1}^{\alpha} \sum_{r,k=1}^{\alpha+1} g(h(e_{r},e_{k}),\bar{e}_{l})^{2} + \sum_{l=\alpha+1}^{2\beta+\alpha} \sum_{r,k=1}^{\alpha} g(h(e_{r},e_{k}),\bar{e}_{l})^{2} + \sum_{l=m}^{2(n-m+1)} \sum_{r,k=1}^{\alpha} g(h(e_{r},e_{k}),\bar{e}_{l})^{2}.$$
(4.3.13)

Leaving all the terms except the second term, then

$$||h||^2 \ge \sum_{l=1}^{2\beta} \sum_{r,k=1}^{\alpha} g(h(e_r,e_k),\tilde{e}_l)^2$$

Using the components of  $F \mathscr{D}^{\theta}$  for adapted Frame 4.3.1, we derive

$$\begin{split} ||h||^2 \geq & \csc^2 \theta \sum_{j=1}^{\beta} \sum_{r=1}^{\alpha} g(h(e_r, e_r), Fe_j^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{j=1}^{\beta} \sum_{r=1}^{\alpha} g(h(e_r, e_r), FPe_j^*)^2. \end{split}$$

By hypothesis of the theorem *M* to be a mixed totally geodesic warped product. Hence, form Lemma 4.3.3 and the fact that  $\eta(e_j) = 0$ ,  $1 \le j \le d_1 - 1$  for orthonormal vector fields, we arrive at

$$\begin{split} ||h||^{2} \geq & \csc^{2} \theta \sum_{j=1}^{\beta} \sum_{r=1}^{\alpha} (Pe_{j}^{*} \ln f)^{2} g(e_{r}, e_{r}) \\ &+ \csc^{2} \theta \cos^{2} \theta \sum_{j=1}^{\beta} \sum_{r=1}^{\alpha} (e_{j}^{*} \ln f)^{2} g(e_{r}, e_{r}). \end{split}$$

Simplification gives

$$||h||^2 \ge \alpha \csc^2 \theta \sum_{j=1}^{\beta} (Pe_j^* \ln f)^2 + \alpha \cot^2 \theta \sum_{j=1}^{\beta} (e_j^* \ln f)^2.$$

Adding and subtracting the same terms in the above equation, we derive

$$||h||^{2} \ge \alpha \csc^{2} \theta \sum_{j=1}^{2\beta+1} (Pe_{j}^{*} \ln f)^{2} - \alpha \csc^{2} \theta \sum_{j=1}^{\beta} (Pe_{j}^{*} \ln f)^{2} - \alpha \csc^{2} \theta (\xi \ln f)^{2} + \alpha \cot^{2} \theta \sum_{j=1}^{\beta} \sum_{r=1}^{\alpha} (e_{j}^{*} \ln f)^{2}.$$

Since,  $\xi \ln f = 1$ , for nearly Kenmotsu manifold, we obtain

$$||h||^{2} \ge \alpha \csc^{2} \theta ||P\nabla^{\theta} \ln f||^{2} + \alpha \cot^{2} \theta \sum_{j=1}^{\beta} (e_{j}^{*} \ln f)^{2}$$
$$\alpha \csc^{2} \theta - \alpha \csc^{2} \theta \sum_{j=1}^{\beta} \beta (e_{j+\beta}^{*} P\nabla^{\theta} \ln f)^{2}.$$

After some simplifications. Applying the property (2.3.55) in the above equation, we get

$$\begin{split} ||h||^2 &\geq \alpha \cot^2 \theta ||\nabla^{\theta} \ln f||^2 - \alpha \cot^2 \theta + \alpha \cot^2 \theta \sum_{j=1}^{\beta} (e_j^* \ln f)^2 \\ &- \alpha \csc^2 \theta \sec^2 \theta \sum_{j=1}^{\beta} g(Pe_j^*, P\nabla^{\theta} \ln f)^2. \end{split}$$
From (2.3.55), it is easily seen that

$$\begin{split} ||h||^2 &\geq \alpha \cot^2 \theta ||\nabla^{\theta} \ln f||^2 + \alpha \cot^2 \theta \sum_{j=1}^a (e_j^* \ln f)^2 \\ &- \alpha \cot^2 \theta \sum_{j=1}^\beta (e_j^* \ln f)^2 - \alpha \cot^2 \theta, \end{split}$$

which implies that

$$||h||^2 \ge \alpha \cot^2 \theta \Big( ||\nabla^{\theta} \ln f||^2 - 1 \Big).$$

This is the inequality (4.3.11). If the equality holds in (4.3.11) identically, then from the leaving terms in (4.3.12) and (4.3.13), we obtain the following conditions, such as:

$$||h(\mathscr{D},\mathscr{D})||^2 = 0, \quad g(h(\mathscr{D}^{\perp},\mathscr{D}^{\perp}), \varphi \mathscr{D}^{\perp}) = 0$$

and

$$g(h(\mathscr{D}^{\perp},\mathscr{D}^{\perp}),\mu)=0,$$

where  $\mathscr{D} = \mathscr{D}^{\theta} \oplus \xi$ . It means that  $M_{\theta}$  is totally geodesic in  $\widetilde{M}$  and  $h(\mathscr{D}^{\perp}, \mathscr{D}^{\perp}) \subset F \mathscr{D}^{\theta}$ . Now from Lemma 4.3.3, for a mixed totally geodesic warped product, we have

$$g(h(Z,W),FX) = (PX\ln f)g(Z,W),$$

for  $Z, W \in \Gamma(TM_{\perp})$  and  $X \in \Gamma(TM_{\theta})$ . This imply that  $M_{\perp}$  is totally umbilical in  $\widetilde{M}$ . This completes the proof of the theorem.

**Theorem 4.3.3.** Let  $M = M_{\theta} \times_f M_{\perp}$  be a m-dimensional mixed totally geodesic warped product pseudo-slant submanifold of a (2n+1)-dimensional nearly Sasakian manifold  $\widetilde{M}$ such that  $\xi \in \Gamma(TM_{\theta})$ . Then the squared norm of the second fundamental form of M is given by

$$||h||^2 \ge \alpha \cot^2 \theta \left( ||\nabla^{\theta} \ln f||^2 \right), \tag{4.3.14}$$

where  $\alpha$  is dimension of  $M_{\perp}$  is an anti-invariant submanifold. If equality holds identically in (4.3.11), then  $M_{\theta}$  is totally geodesic submanifold and  $M_{\perp}$  is a totally umbilical submanifold of  $\widetilde{M}$ , respectively.

## **CHAPTER 5**

# POINTWISE SEMI-SLANT SUBMANIFOLDS AND THEIR WARPED PRODUCTS

#### 5.1 INTRODUCTION

It is well known from Yano & Kon (1985) that the integration of the Laplacian of a smooth function defined on a compact orientable Riemannian manifold without boundary vanishes with respect to the volume element. As we have seen that, B. Sahin (2013), studied the warped product pointwise semi-slant submanifolds in Kaehler manifolds and later studied in cosymplectic, Kenmotsu and Sasakian manifolds by Park (2014). They have obtained a lot of examples on the existence of warped product pointwise semi-slant in Kaehler, cosymplectic, Kenmotsu and Sasakian manifolds, and derived general inequalities for the second fundamental form involving the warping functions and pointwise slant functions.

In this chapter, we find out the some potential applications of this notion, and study the concept of warped product pointwise semi-slant submanifolds in Kaehler, cosymplectic and Sasakian manifolds as a generalization of CR-warped product submanifolds. Then, we prove the existence of warped product pointwise semi-slant submanifolds by their characterizations in terms of Weingarten operator, tensor fields, and give some examples supporting to this idea. Moreover, some characterizations results are generalized to CR-warped product submanifold of this type in different structures.

Further, we obtain some interesting inequalities in terms of the second fundamental form and the scalar curvature using Gauss equation in the place of Codazzi equation and then, derive some applications of it with considering the equality cases. We provide many triviality results for the warped product pointwise semi-slant isometrically immersed into complex space form, cosymplectic space forms and Sasakian space form in various mathematical and physical terms such as Hessian, Hamiltonian and kinetic energy. However these triviality results are generalized the study of CR-warped product submanifolds as well.

# 5.2 GEOMETRY OF WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS OF KAEHLER MANIFOLDS

#### 5.2.1 Pointwise semi-slant submanifold in a Kaehler manifold

The concept of semi-slant submanifolds were defined and studied by Papaghiuc (1994) as natural extension of CR-submanifolds of almost Hermitian manifolds in terms of slant immersion. Similarly, in terms of pointwise slant function, the pointwise semi-slant continued into Kaehler manifolds by Sahin (2013). He defined these submanifolds as follows:

**Definition 5.2.1.** Let M be a submanifold of Kaehler manifold M is said to be a pointwise semi-slant submanifold if there exists two orthogonal distributions  $\mathcal{D}$  and  $\mathcal{D}^{\theta}$  such that

- (i)  $TM = \mathscr{D} \oplus \mathscr{D}^{\theta}$ ,
- (ii)  $\mathscr{D}$  is holomorphic (complex), i.e.,  $J(\mathscr{D}) \subseteq \mathscr{D}$ ,
- (iii)  $\mathscr{D}^{\theta}$  is pointwise slant distribution with slant function  $\theta: TM \to \mathbb{R}$ .

Let  $d_1$  and  $d_2$  be dimensions of *complex* distribution  $\mathscr{D}$  and *pointwise slant* distribution  $\mathscr{D}^{\theta}$  of pointwise semi-slant submanifold in a Kaehler manifold  $\widetilde{M}$ . Then, we have the following remarks:

*Remark* 5.2.1. *M* is holomorphic if  $d_2 = 0$  and pointwise slant if  $d_1 = 0$ .

*Remark* 5.2.2. If we consider the slant function  $\theta : M \to R$  is globally constant on M and  $\theta = \frac{\pi}{2}$ , then M is called CR-submanifold.

**Remark 5.2.3.** If the slant function  $\theta: M \to (0, \frac{\pi}{2})$ , then *M* is called proper pointwise semi-slant submanifold. It is called proper semi-slant if pointwise slant function is globally constant.

*Remark* 5.2.4. If *v* is an *invariant* subspace under *J* of normal bundle  $T^{\perp}M$ , then in the case of semi-slant submanifold, the normal bundle  $T^{\perp}M$  can be decomposed as  $T^{\perp}M = F \mathscr{D}^{\theta} \oplus v$ .

For the integrability conditions of distributions involved in the definition pointwise semislant submanifolds and its examples, we refer to Sahin (2013). Now we obtain the following theorem which is important for characterization theorem:

**Theorem 5.2.1.** Let M be a pointwise semi-slant submanifold M of a Kaehler manifold  $\widetilde{M}$ . Then the distribution  $\mathcal{D}$  is defined as a totally geodesic foliation if and only if

$$h(X,JY) \in \Gamma(v),$$

for any  $X, Y \in \Gamma(\mathscr{D})$ .

*Proof.* Let  $X, Y \in \Gamma(\mathscr{D})$  and  $Z \in \Gamma(\mathscr{D}^{\theta})$ , we have

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(J\widetilde{\nabla}_X Y, JZ).$$

Using (2.3.19), we obtain

$$g(\nabla_X Y, Z) = g(\nabla_X JY, TZ) + g(\nabla_X JY, FZ) - g((\nabla_X J)Y, JZ).$$

From (2.2.2), (2.3.2) and the definition of totally geodesic foliation, we arrive at the final result. This completes the proof of the theorem.

### 5.2.2 Warped product pointwise semi-slant submanifold in a Kaehler manifold

There are two cases for defining to the warped product pointwise semi-slant submanifolds in a Kaehler manifold, i.e.,

- (i)  $M_{\theta} \times_f M_T$ ,
- (ii)  $M_T \times_f M_{\theta}$ .

The following result shows the non-existence of the first type warped product which was proved by Sahin (2013),

**Theorem 5.2.2.** There do not exist a proper warped product pointwise semi-slant submanifold  $M = M_{\theta} \times_f M_T$  in a Kaehler manifold  $\widetilde{M}$  such that  $M_{\theta}$  is a proper pointwise slant submanifold and  $M_T$  is a holomorphic submanifold of  $\widetilde{M}$ .

Moreover, for the second type of warped product, Sahin (2013) obtained the following lemma.

*Lemma* 5.2.1. Let  $M = M_T \times_f M_\theta$  be a warped product pointwise semi-slant submanifold in a Kaehler manifold  $\widetilde{M}$ . Then

$$g(h(X,Z),FTW) = -(JX\ln f)g(Z,TW) - (X\ln f)\cos^2\theta g(Z,W),$$
 (5.2.1)

$$g(h(Z,JX),FW) = (X\ln f)g(Z,W) + (JX\ln f)(Z,TW),$$
(5.2.2)

$$g(h(X,Y),FZ) = 0,$$
 (5.2.3)

for any  $X \in \Gamma(TM_T)$  and  $Z, W \in \Gamma(TM_\theta)$ , where  $M_T$  and  $M_\theta$  are holomorphic and pointwise slant submanifolds of  $\widetilde{M}$ , respectively.

Also, he gave the following characterization theorem.

**Theorem 5.2.3.** Let M be a pointwise semi-slant submanifold of Kaehler manifolds  $\widetilde{M}$ . Then M is locally a non-trivial warped product submanifolds of the form  $M = M_T \times_f M_{\theta}$ such that  $M_T$  is a holomorphic submanifold and  $M_{\theta}$  is a pointwise slant submanifold if and only if

$$A_{FTZ}X - A_{FZ}JX = -(1 + \cos^2\theta)(X\lambda)Z,$$

where  $\lambda$  is a function such that  $Z\lambda = 0$  for any  $Z \in \Gamma(TM_T)$ .

Motivated by above study and from V. A. Khan & Khan (2014), we derive the following important lemmas to prove the characterization theorems.

*Lemma* 5.2.2. On a non-trivial warped product pointwise semi-slant submanifold  $M = M_T \times_f M_{\theta}$  in a Kaehler manifold  $\widetilde{M}$ , we have

$$(\widetilde{\nabla}_X T)Z = 0, \tag{5.2.4}$$

$$(\nabla_Z T)X = (JX\ln f)Z - (X\ln f)TZ, \qquad (5.2.5)$$

$$(\nabla_{TZ}T)X = (JX\ln f)TZ + \cos^2\theta(X\ln f)Z, \qquad (5.2.6)$$

for any  $X \in \Gamma(TM_T)$  and  $Z \in \Gamma(TM_\theta)$ .

Proof. From (2.3.21) and Lemma 2.4.1(ii), we derive

$$(\widetilde{\nabla}_X T)Z = \nabla_X TZ - T\nabla_X Z$$
  
=  $(X \ln f)TZ - (X \ln f)TZ$   
= 0,

for any  $X \in \Gamma(TM_T)$  and  $Z \in \Gamma(TM_{\theta})$ . Again from (2.3.21) and Lemma 2.4.1(ii), we obtain

$$(\widetilde{\nabla}_Z T)X = \nabla_Z TX - T\nabla_Z X$$
  
=  $(JX\ln f)Z - (X\ln f)TZ$ ,

which is (5.2.5) of lemma. If we replace Z by TZ in (5.2.5) and using Theorem 2.3.6, we get the last result of lemma. This completes proof of the lemma.

Similarly, we prove the next result;

*Lemma* 5.2.3. Assume that  $M = M_T \times_f M_\theta$  be a warped product pointwise semi-slant submanifold in a Kaehler manifold  $\widetilde{M}$ . Then

$$(\widetilde{\nabla}_U T)X = (JX\ln f)CU - (X\ln f)TCU, \qquad (5.2.7)$$

$$(\widetilde{\nabla}_U T)Z = g(CU, Z)J\nabla \ln f - g(CU, TZ)\nabla \ln f, \qquad (5.2.8)$$

$$(\widetilde{\nabla}_U T)TZ = g(CU, TZ)J\nabla\ln f + \cos^2\theta g(CU, Z)\nabla\ln f, \qquad (5.2.9)$$

for any  $U \in \Gamma(TM)$ ,  $X \in \Gamma(TM_T)$  and  $Z, W \in \Gamma(TM_{\theta})$ .

*Proof.* Thus from using (2.3.25), it follows that

$$(\widetilde{\nabla}_X T)Y = th(X,Y),$$

for  $X, Y \in \Gamma(TM_T)$ . As the hypothesis of warped product submanifold,  $M_T$  is being totally geodesic in M, this means that the term  $(\widetilde{\nabla}_X T)Y$  lies in  $M_T$  and its second fundamental form should be identically zero. Equating the components along  $M_T$  in the last equation, we get th(X,Y) = 0, which implies that  $h(X,Y) \in \Gamma(v)$ . Thus the last equation becomes

$$(\widetilde{\nabla}_X T)Y = 0. \tag{5.2.10}$$

Now apply (2.3.32) into  $(\widetilde{\nabla}_U T)X$  to derive

$$(\widetilde{\nabla}_U T)X = (\widetilde{\nabla}_{BU}T)X + (\widetilde{\nabla}_{CU}T)X,$$

for  $U \in \Gamma(TM)$ . The first part of right hand side in the above equation should be zero by virtue (5.2.10) and the second part of the above equation follows by (5.2.5). Again taking help from (2.3.32), we find

$$(\widetilde{\nabla}_U T)Z = (\widetilde{\nabla}_{BU}T)Z + (\widetilde{\nabla}_{CU}T)Z, \qquad (5.2.11)$$

for  $Z \in \Gamma(TM_{\theta})$  and  $U \in \Gamma(TM)$ . Taking the inner product with  $X \in \Gamma(TM_T)$  in (5.2.11) and using (2.3.21), we obtain

$$g((\nabla_U T)Z,X) = g(\nabla_{CU}TZ,X) - g(T\nabla_{CU}Z,X)$$
$$= g(\nabla_{CU}Z,JX) - g(\nabla_{CU}X,TZ)$$
$$= -g(\nabla_{CU}JX,Z) - g(\nabla_{CU}X,TZ).$$

From Lemma 2.4.1(ii), we get

$$\begin{split} g((\widetilde{\nabla}_U T)Z,X) &= -(JX\ln f)g(CU,Z) - (X\ln f)g(CU,TZ) \\ &= g(CU,Z)g(J\nabla\ln f,X) - g(CU,TZ)g(\nabla\ln f,X), \end{split}$$

which implies that

$$(\nabla_U T)Z = g(CU, Z)J\nabla \ln f - g(CU, TZ)\nabla \ln f.$$

This implies (5.2.8) of lemma. Replacing Z by TZ in the result (5.2.8) and using Theorem 2.3.6 for pointwise slant submanifold  $M_{\theta}$ . The above equation takes the form

$$(\widetilde{\nabla}_U T)TZ = g(CU, TZ)J\nabla \ln f + \cos^2 \theta g(CU, Z)\nabla \ln f,$$

which is the last result of lemma. This completes proof of the lemma.

Now we give the characterization theorems of this Chapter in term of  $\nabla T$ ,

**Theorem 5.2.4.** Let M be a pointwise semi-slant submanifold of a Kaehler manifold  $\tilde{M}$  whose pointwise slant distribution is integrable. Then M is locally a warped product

submanifold of type  $M = M_T \times_f M_{\theta}$  if and only if

$$(\widetilde{\nabla}_U T)V = (JBV\lambda)CU - (BV\lambda)TCU + g(CU, CV)J\nabla\lambda - g(CU, TCV)\nabla\lambda, \qquad (5.2.12)$$

for every  $U, V \in \Gamma(TM)$  and a  $C^{\infty}$ -function  $\lambda$  on M with  $Z\lambda = 0$ , for each  $Z \in \Gamma(\mathcal{D}^{\theta})$ .

*Proof.* Assume that M be a warped product pointwise semi-slant submanifold in a Kaehler manifold  $\widetilde{M}$ . Then using (2.3.32), it can be easily derived that

$$(\widetilde{\nabla}_U T)V = (\widetilde{\nabla}_U T)BV + (\widetilde{\nabla}_U T)CV,$$

for  $U, V \in \Gamma(TM)$ . Thus the first part directly follows to (5.2.7)-(5.2.9) of Lemma 5.2.3 and setting  $\ln f = \lambda$  in the above equation. Let us prove the converse part, consider that M is a pointwise semi-slant submanifold of a Kaehler manifold  $\widetilde{M}$  such that the given condition (5.2.12) holds. It is easy to obtain the following condition such as:

$$(\widetilde{\nabla}_X T)Y = 0. \tag{5.2.13}$$

By setting U = X and V = Y in (5.2.12), for each  $X, Y \in \Gamma(\mathcal{D})$ . Taking the inner product in (5.2.13) with  $TZ \in \Gamma(\mathcal{D}^{\theta})$  and using (2.3.21), we derive

$$g(\widetilde{\nabla}_X JY, TZ) = g(T\nabla_X Y, TZ).$$

As TZ and JY are orthogonal, then from the property of Riemannian connection, we derive

$$g(\widetilde{\nabla}_X TZ, JY) = -g(\nabla_X Y, T^2Z).$$

From the covariant derivative of an almost complex structure J and Theorem 2.3.6, it is easily seen that

$$g((\widetilde{\nabla}_X J)TZ, Y) - g(\widetilde{\nabla}_X JTZ, Y) = \cos^2 \theta g(\nabla_X Y, Z).$$

Thus using the tensorial equation of Kaehler manifold and (2.3.19), we arrive at

$$g(\widetilde{\nabla}_X T^2 Z, Y) + g(\widetilde{\nabla}_X F T Z, Y) = \cos^2 \theta g(\nabla_X Y, Z).$$

Using Theorem 2.3.6, in the first part of the above equation for pointwise slant function

 $\theta$  and also from (2.3.2), we obtain

$$\sin 2\theta X(\theta)g(Z,Y) - \cos^2\theta g(\nabla_X Z,Y) = g(h(X,Y),FTZ) + \cos^2\theta g(\nabla_X Y,Z),$$

which implies that

$$g(h(X,Y),FTZ)=0.$$

The above relation indicate that  $h(X,Y) \in \Gamma(v)$  for all  $X,Y \in \Gamma(\mathcal{D})$ . Hence, from Theorem 5.2.1, it shows that the distribution  $\mathcal{D}$  defines a totally geodesic foliations and its leaves are totally geodesic in M. Furthermore, by setting U = Z and V = W in (5.2.12), we derive

$$(\widetilde{\nabla}_Z T)W = g(Z, W)J\nabla\lambda + g(TZ, W)\nabla\lambda,$$

for  $Z, W \in \Gamma(\mathcal{D}^{\theta})$ . Taking the inner product in the above equation with  $X \in \Gamma(\mathcal{D})$  and using (2.3.19), we obtain

$$g(\nabla_Z TW, X) - g(T\nabla_Z W, X) = -(X\lambda)g(Z, TW) - (JX\lambda)g(Z, W).$$

By hypothesis of the theorem, as we have considered that the pointwise slant distribution is integrable. It is obvious,  $M_{\theta}$  be a leaf of  $\mathscr{D}^{\theta}$  in M and  $h^{\theta}$  be the second fundamental form of  $M_{\theta}$  in M. Then

$$g(h^{\theta}(Z,TW),X) + g(h^{\theta}(Z,W),JX) = -(X\lambda)g(Z,TW) - (JX\lambda)g(Z,W).$$
(5.2.14)

Replacing W by TW and X by JX in (5.2.14) and from the Theorem 2.3.6, we derive

$$-\cos^{2}\theta g(h^{\theta}(Z,W),JX) - g(h^{\theta}(Z,TW),X) = \cos^{2}\theta(JX\lambda)g(Z,W) + (X\lambda)g(Z,TW).$$
(5.2.15)

Thus from (5.2.14) and (5.2.15), it follows that

$$\sin^2\theta g(h^{\theta}(Z,W),JX) = -\sin^2\theta(JX\lambda)g(Z,W).$$

which implies that

$$g(h^{\theta}(Z,W),JX) = -(JX\lambda)g(Z,W).$$

The property of gradient function provide that

$$h^{\theta}(Z,W) = -g(Z,W)\nabla\lambda,$$

From the above relation, we conclude that  $M_{\theta}$  is totally umbilical in M such that  $H^{\theta} = -\nabla \lambda$  is the mean curvature vector of  $M_{\theta}$ . Now, we can easily show that the mean curvature vector  $H^{\theta}$  is parallel corresponding to the normal connection  $\nabla'$  of  $M_{\theta}$  in M. This means that  $M_{\theta}$  is an extrinsic spheres in M. From result of Hiepko (1979), M is called a warped product submanifold of integral manifolds  $M_T$  and  $M_{\theta}$  of  $\mathcal{D}$  and  $\mathcal{D}^{\theta}$ , respectively. This completes proof of the theorem.

The following proposition provides another simple characterization of warped product pointwise semi-slant immersions in terms of  $\nabla F$ .

**Theorem 5.2.5.** A pointwise semi-slant submanifold M of a Kaehler manifold  $\widetilde{M}$  with integrable distribution  $\mathscr{D}^{\theta}$  is a locally warped product submanifold of the form  $M = M_T \times_f M_{\theta}$  if and only if

$$(\nabla_U F)V = fh(U, BV) - h(U, TCV) - (BV\lambda)FCU, \qquad (5.2.16)$$

for every  $U, V \in \Gamma(TM)$  and a  $C^{\infty}$ -function  $\lambda$  on M with  $Z\lambda = 0$ , for each  $Z \in \Gamma(\mathcal{D}^{\theta})$ .

*Proof.* Assume that M is a warped product pointwise semi-slant submanifold in a Kaehler manifold  $\widetilde{M}$ . Then using (2.3.32) in  $(\widetilde{\nabla}_U F)X$ , we derive

$$(\widetilde{\nabla}_U F)X = (\widetilde{\nabla}_{BU}F)X + (\widetilde{\nabla}_{CU}F)X,$$

for each  $U \in \Gamma(TM)$  and  $X \in \Gamma(TM_T)$ . The first term of the above equation is identically zero by using the fact that  $M_T$  is totally geodesic on M and last term follows from (2.3.22). Thus from Lemma 2.4.1(ii), we obtain

$$(\widetilde{\nabla}_U F)X = -F\nabla_{CU}X - (X\ln f)FCU.$$
(5.2.17)

From (2.3.45), we derive

$$(\widetilde{\nabla}_U F)Z = fh(U,Z) - h(U,TZ), \qquad (5.2.18)$$

for any  $Z \in \Gamma(TM_{\theta})$ . Furthermore, again taking account of (2.3.32), it is easily seen that

$$(\widetilde{\nabla}_U F)V = (\widetilde{\nabla}_U F)BV + (\widetilde{\nabla}_U F)CV.$$
(5.2.19)

Hence, using (5.2.17), (5.2.18) in (5.2.19), we get required result (5.2.16).

Conversely, let us assume that M is a pointwise semi-slant submanifold of a Kaehler manifold  $\widetilde{M}$  with integrable distribution  $\mathscr{D}^{\theta}$  and (5.2.16) holds. Then, it follows from (5.2.16), i.e.,  $-F\nabla_X Y = 0$ , for any X and Y are tangent to holomorphic distribution  $\mathscr{D}$ , which implies that  $\nabla_X Y \in \Gamma(\mathscr{D})$ , thus the leaves of  $\mathscr{D}$  are totally geodesic in M. On the other hand, the pointwise slant distribution  $\mathscr{D}^{\theta}$  is assumed to be integrable. Then we can consider  $M_{\theta}$  to be a leaf of  $\mathscr{D}^{\theta}$  and  $h^{\theta}$  to be the second fundamental form of immersion Minto  $\widetilde{M}$ . Thus replacing U = Z and V = X, in (5.2.16), for any  $Z \in \Gamma(\mathscr{D}^{\theta})$  and  $X \in \Gamma(\mathscr{D})$ and using the fact that CX = 0, we derive

$$(\widetilde{\nabla}_Z F)X = -(X\lambda)FZ.$$

Taking inner product with *FW* for  $W \in \Gamma(\mathscr{D}^{\theta})$  and using relation (2.3.31), then it follows as

$$g((\widetilde{\nabla}_Z F)X, FW) = -\sin^2 \theta(X\lambda)g(Z, W).$$
(5.2.20)

Apply (2.3.38) in left hand side of (5.2.20), we obtain

$$g(-F\nabla_Z X, FW) = -\sin^2 \theta(X\lambda)g(Z, W).$$

Then by virtue (2.3.31) and definition of gradient of  $\ln f$ , we arrive at

$$-\sin^2\theta g(\nabla_Z X, W) = -\sin^2\theta g(\nabla\lambda, X)g(Z, W),$$

which implies that

$$h^{\theta}(Z,W) = -g(Z,W)\nabla\lambda. \tag{5.2.21}$$

The meaning of the equation (5.2.21) shows that the leaf  $M_{\theta}$  (of  $\mathscr{D}^{\theta}$ ) is totally umbilical in M such that  $H^{\theta} = -\nabla \lambda$ , is the mean curvature vector of  $M_{\theta}$ . Moreover, the condition  $Z\lambda = 0$ , for any  $Z \in \Gamma(\mathscr{D}^{\theta})$  implies that the leaves of  $\mathscr{D}^{\theta}$  are extrinsic spheres in M, i.e., the integral manifold  $M_{\theta}$  of  $\mathscr{D}^{\theta}$  is totally umbilical and its mean curvature vector is non zero and parallel along  $M_{\theta}$ . Thus immeditely follows to Definition 2.4.3, i.e.,  $M = M_T \times_f M_{\theta}$  is a locally warped product submanifold, where  $M_T$  is an integral manifold of  $\mathscr{D}$  and f is a warping function. This completes proof of the theorem.

**Remark 5.2.5.** As an immediate consequences of Theorem 5.2.4 and Theorem 5.2.5 by using Remark 5.2.2, then TCU = TCV = 0, for totally real submanifold  $M_{\perp}$ . Thus a warped product pointwise semi-slant submanifold  $M = M_T \times_f M_{\theta}$  is turn into CR-warped product submanifold in a Kaehler manifold of the type  $M = M_T \times_f M_{\perp}$  such that  $M_T$  and  $M_{\perp}$  are invariant and anti-invariant submanifolds, respectively. In other words, Theorem 5.2.4 and Theorem 5.2.5 generalize the non-trivial characterization for CR-warped products which were proved by V. Khan et al. (2009), i.e,

**Theorem 5.2.6.** V. Khan et al. (2009) Let  $\widetilde{M}$  be a Kaehler manifold and CR-submanifold M of  $\widetilde{M}$  is locally a CR-warped products if and only if

$$(\widetilde{\nabla}_U T)V = (JBV\lambda)CU + g(CU, CV)J\nabla\lambda,$$

for each  $U, V \in \Gamma(TM)$  and a  $C^{\infty}$ -function  $\lambda$  on M with  $Z\lambda = 0$ , for each  $Z \in \Gamma(\mathcal{D}^{\perp})$ . Furthermore, B and C are orthogonal projections on  $\mathcal{D}$  and  $\mathcal{D}^{\perp}$ , respectively.

**Theorem 5.2.7.** A CR-submanifold M of a Kaehler manifold  $\widetilde{M}$  is CR-warped product submanifold of the form  $M = M_T \times_f M_\perp$  if and only if

$$(\widetilde{\nabla}_U F)V = fh(U, BV) - (BV\lambda)FCU,$$

for every  $U, V \in \Gamma(TM)$  and a  $C^{\infty}$ -function  $\lambda$  on M with  $Z\lambda = 0$ , for each  $Z \in \Gamma(\mathscr{D}^{\perp})$ .

*Note.* In this sense, the warped product pointwise semi-slant submanifolds are natural generalization of CR-warped product in Kaehler manifold.

# 5.2.3 Some inequalities of warped product pointwise semi-slant submanifold in Kaehler manifold

In view of Chen (2003) studied, we construct some geometric properties of the mean curvature for the warped product pointwise semi-slant submanifolds and using these result

to derive a general inequality. A similar inequality has been obtained for the squared norm of the second fundamental form for contact CR-warped product submanifolds in Kenmotsu manifolds by Mustafa et al. (2015). We are able to obtain such an inequality in a Kaehler manifold due to the warped product exists only in the case when fiber is pointwise slant submanifold. In this sequel, we study some geometric properties about submanifolds. Now we are ready to prove the general inequality. For this, we need to define an orthonormal frame and present some preparatory lemmas.

Let  $M = M_T \times_f M_\theta$  be an  $n = (n_1 + n_2)$ -dimensional warped product pointwise semislant submanifold of 2m-dimensional Kaehler manifold  $\widetilde{M}$  with  $M_T$  of dimension  $n_1 = 2d_1$ and  $M_\theta$  of dimension  $n_2 = 2d_2$ , where  $M_\theta$  and  $M_T$  are integral manifolds of  $\mathscr{D}^\theta$  and  $\mathscr{D}$ , respectively. Hence, we consider that  $\{e_1, e_2, \cdots e_{d_1}, e_{d_1+1} = Je_1, \cdots e_{2d_1} = Je_{d_1}\}$  and  $\{e_{2d_1+1} = e_1^*, \cdots e_{2d_1+d_2} = e_{d_2}^*, e_{2d_1+d_2+1} = e_{d_2+1}^* = \sec \theta Te_1^*, \cdots e_{n_1+n_2} = e_{n_2}^* = \sec \theta Te_{d_2}^*\}$ are orthonormal frames of  $TM_T$  and  $TM_\theta$  respectively. Thus the orthonormal frames of the normal sub bundles,  $F \mathscr{D}^\theta$  and  $\nu$  respectively are,  $\{e_{n+1} = \bar{e}_1 = \csc \theta Fe_1^*, \cdots e_{n+d_2} =$  $\bar{e}_{d_2} = \csc \theta Fe_1^*, e_{n+d_2+1} = \bar{e}_{d_2+1} = \csc \theta \sec \theta FTe_1^*, \cdots e_{n+2d_2} = \bar{e}_{2d_2} = \csc \theta \sec \theta FTe_{d_2}^*\}$ and  $\{e_{n+2d_2+1}, \cdots e_{2m}\}$ .

*Lemma* 5.2.4. Let M be a non-trivial warped product pointwise semi-slant submanifold of a Kaehler manifold  $\widetilde{M}$ . Then

$$g(h(X,X),FZ) = g(h(X,X),FTZ) = 0,$$
 (5.2.22)

$$g(h(JX,JX),FZ) = g(h(JX,JX),FTZ) = 0,$$
 (5.2.23)

$$g(h(X,X),\xi) = -g(h(JX,JX),\xi), \qquad (5.2.24)$$

for any  $X \in \Gamma(TM_T)$ ,  $Z \in \Gamma(TM_{\theta})$  and  $\xi \in \Gamma(\nu)$ .

*Proof.* From relation (2.3.2), we have

$$g(h(X,X),FTZ) = g(\widetilde{\nabla}_X X,FTZ) = -g(\widetilde{\nabla}_X FTZ,X).$$

Thus from relation (2.3.19) and the covariant derivative of almost complex structure J, we obtain

$$g(h(X,X),FTZ) = g(\widetilde{\nabla}_X TZ,JX) + g((\widetilde{\nabla}_X J)TZ,X) + g(\widetilde{\nabla}_X T^2Z,X),$$

Using the structure equation (2.2.2) and Theorem (2.3.6) for pointwise semi-slant submanifold, we get

$$g(h(X,X),FTZ) = -g(\nabla_X JX,TZ) + \sin 2\theta X(\theta)g(Z,X) - \cos^2 \theta g(\nabla_X X,Z).$$

Since,  $M_T$  is totally geodesic in M with these fact, we get result (5.2.22). For other part, interchanging Z by TZ and X by JX in the equation (5.2.22), we get the required result (5.2.23). Now for (5.2.24), from Kaehler manifold (2.2.2), we have  $\widetilde{\nabla}_X JX = J\widetilde{\nabla}_X X$ , by Gauss formula (2.3.2), this relation reduced to

$$\nabla_X JX + h(JX,X) = J\nabla_X X + Jh(X,X).$$

Taking the inner product with  $J\xi$  in the above equation by considering  $\xi \in \Gamma(v)$ , we obtain

$$g(h(JX,X),J\xi) = g(h(X,X),\xi).$$
 (5.2.25)

Interchanging X by JX in (5.2.22) and the fact v is an invariant normal bundle of  $T^{\perp}M$ under an almost complex structure J, we get

$$-g(h(X,JX),J\xi) = g(h(JX,JX),\xi).$$
(5.2.26)

From (5.2.25) and (5.2.26), implies to equation (5.2.24). This completes proof of lemma.

*Lemma* 5.2.5. Let  $\phi : M = M_T \times_f M_\theta \longrightarrow \widetilde{M}$  be an isometrically immersion of a warped product pointwise semi-slant submanifold into Kaehler manifold  $\widetilde{M}$ . Then the squared norm of mean curvature of *M* is given by

$$||H||^{2} = \frac{1}{n^{2}} \sum_{r=n+1}^{2m} \left( h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \right)^{2},$$

where *H* is the mean curvature vector. Moreover,  $n_1$ ,  $n_2$ , *n* and 2m are dimensions of  $M_T$ ,  $M_{\theta}$ ,  $M_T \times_f M_{\theta}$  and  $\widetilde{M}$ , respectively.

Proof. From the definition of the mean curvature vector, we have

$$||H||^{2} = \frac{1}{n^{2}} \sum_{r=n+1}^{2m} \left( h_{11}^{r} + \dots + h_{nn}^{r} \right)^{2},$$

Thus, from consideration of dimension  $n = n_1 + n_2$  of  $M_T \times_f M_{\theta}$  such that  $n_1$  and  $n_2$  are dimensions of  $M_T$  and  $M_{\theta}$ , respectively, we arrive at

$$||H||^{2} = \frac{1}{n^{2}} \sum_{r=n+1}^{2m} \left( h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \right)^{2}.$$

Using the frame of  $\mathscr{D}$  and coefficient of  $n_1$  in right hand side of the above equation, we get

$$\left(h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r}\right)^{2} = \left(h_{11}^{r} + \dots + h_{d_{1}d_{1}}^{r} + h_{d_{1}+1d_{1}+1}^{r} + \dots + h_{2d_{1}2d_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r}\right)^{2} \dots (5.2.27)$$

From the relation  $h_{ij}^r = g(h(e_i, e_j), e_r)$ , for  $1 \le i, j \le n$  and  $n+1 \le r \le 2m$  and orthonormal frame components for  $\mathscr{D}$ , the equation (5.2.27) take the form

$$\begin{pmatrix} h_{11}^{r} + \dots + h_{n_{1}+1}^{r} + h_{n_{1}+1}^{r} + \dots + h_{nn}^{r} \end{pmatrix}^{2} = \begin{pmatrix} g(h(e_{1}, e_{1}), e_{r}) + \dots \\ + g(h(e_{d_{1}}, e_{d_{1}}), e_{r}) \\ + g(h(Je_{d_{1}}, Je_{d_{1}}), e_{r}) \\ + \dots + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \end{pmatrix}^{2}.$$

$$(5.2.28)$$

Thus  $e_r$  belong to normal bundle  $T^{\perp}M$  for ever  $r \in \{n+1\cdots 2m\}$ , it means that there are two cases such that  $e_r$  belongs to  $F(TM_{\theta})$  or v.<sup>1</sup>

<sup>&</sup>lt;sup>1</sup>Due to length of equation, the cases are defined in the next page

*Case* 5.2.1. If  $e_r \in \Gamma(F \mathscr{D}^{\theta})$ , then from using of normal components for pointwise slant distribution  $\mathscr{D}^{\theta}$  which is defined in the frame. Then equation (5.2.28) can be written as:

$$\begin{pmatrix} h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \end{pmatrix}^{2} \\ = \begin{cases} \csc \theta_{g}(h(e_{1}, e_{1}), Fe_{1}^{*}) + \dots \\ + \csc \theta_{g}(h(e_{d_{1}}, e_{d_{1}}), Fe_{d_{2}}^{*}) + \csc \theta \sec \theta_{g}(h(e_{1}, e_{1}), FTe_{1}^{*}) \\ \dots + \csc \theta \sec \theta_{g}(h(e_{d_{1}}, e_{d_{1}}), FTe_{d_{2}}^{*}) + \csc \theta_{g}(h(Je_{1}, Je_{1}), Fe_{1}^{*}) \\ + \csc \theta_{g}(h(Je_{d_{1}}, Je_{d_{1}}), Fe_{d_{2}}^{*}) + \csc \theta \sec \theta_{g}(h(Je_{1}, Je_{1}), FTe_{1}^{*}) \\ + \dots + \csc \theta \sec \theta_{g}(h(Je_{d_{1}}, Je_{d_{1}}), Fe_{d_{2}}^{*}) + \csc \theta \sec \theta_{g}(h(Je_{1}, Je_{1}), FTe_{1}^{*}) \\ + \dots + \csc \theta \sec \theta_{g}(h(Je_{d_{1}}, Je_{d_{1}}), FTe_{d_{2}}^{*}) \\ + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \end{pmatrix}^{2}.$$

Now from virtues (5.2.22) and (5.2.23) of Lemma 5.2.4, we get

$$(h_{11}^r + \dots + h_{n_1n_1}^r + h_{n_1+1n_1+1}^r + \dots + h_{nn}^r)^2 = (h_{n_1+1n_1+1}^r + \dots + h_{nn}^r)^2.$$
(5.2.29)

*Case* 5.2.2. If  $e_r \in \Gamma(v)$ , then from relation (5.2.24) of Lemma 5.2.4, the equation (5.2.28) simplifies as:

$$\begin{pmatrix} h_{11}^r + \dots + h_{n_1n_1}^r + h_{n_1+1n_1+1}^r + \dots + h_{nn}^r \end{pmatrix}^2 = \begin{cases} g(h(e_1, e_1), e_r) + \dots + g(h(e_{d_1}, e_{d_1}), e_r) \\ - g(h(e_1, e_1), e_r) \dots - g(h(e_{d_1}, e_{d_1}), e_r) \\ + \dots + h_{n_1+1n_1+1}^r + \dots + h_{nn}^r \end{cases}^2,$$

which implies that

$$\left(h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r}\right)^{2} = \left(h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r}\right)^{2}.$$
 (5.2.30)

From (5.2.29) and (5.2.30) for every normal vector  $e_r$  belong to the normal bundle  $T^{\perp}M$  and taking the summing up, we can deduce that

$$\sum_{r=n+1}^{2m} \left( h_{11}^r + \dots + h_{n_1n_1}^r + h_{n_1+1n_1+1}^r + \dots + h_{nn}^r \right)^2 = \sum_{r=n+1}^{2m} \left( h_{n_1+1n_1+1}^r + \dots + h_{nn}^r \right)^2.$$

Hence, the above relation proves our assertion. This completes proof of the lemma.

In point of view of Lemma 5.2.5, we may give the following theorem.

**Theorem 5.2.8.** Let  $\phi : M = M_T \times_f M_\theta \longrightarrow \widetilde{M}$  be an isometrically immersed from a warped product pointwise semi-slant submanifold  $M_T \times_f M_\theta$  into Kaehler manifold  $\widetilde{M}$ . Then  $M_T$  is minimal submanifold of  $\widetilde{M}$ .

*Note.* It is easily derived from the Lemma 5.2.5 such that the complex (holomorphic) submanifold is a  $M_T$ -minimal submanifold of warped product pointwise semi-slant submanifold in a Kaehler manifold. We are going to derive next theorem by using the minimality of base manifold of warped product submanifold, and minimality is the necessary condition to obtain the following inequality.

**Theorem 5.2.9.** Let  $\phi : M = M_T \times_f M_\theta \longrightarrow \widetilde{M}$  be an isometrically immersion of an *n*-dimensional warped product pointwise semi-slant submanifold  $M_T \times_f M_\theta$  into 2*m*-dimensional Kaehler manifold  $\widetilde{M}$ . Then

(i) The squared norm of the second fundamental form of M is given by

$$||h||^{2} \ge 2\left(n_{2}||\nabla(\ln f)||^{2} + \widetilde{\tau}(TM) - \widetilde{\tau}(TM_{T}) - \widetilde{\tau}(TM_{\theta}) - n_{2}\Delta(\ln f)\right), \quad (5.2.31)$$

where  $n_2$  is the dimension of pointwise slant submanifold  $M_{\theta}$ .

(ii) The equality holds in the above inequality if and only if  $M_T$  is totally geodesic and  $M_{\theta}$  is totally umbilical submanifolds of  $\widetilde{M}$ . Moreover, M is minimal submanifold of  $\widetilde{M}$ .

*Proof.* Putting  $X = W = e_i$ , and  $Y = Z = e_j$  in Gauss equation (2.3.6), we obtain

$$\widetilde{R}(e_i, e_j, e_j, e_i) = R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_j, e_i)) - g(h(e_i, e_e), h(e_j, e_j)).$$
(5.2.32)

Over  $1 \le i, j \le n(i \ne j)$ , taking summation up in (5.2.32) and using (2.3.10), we derive

$$2\widetilde{\tau}(TM) = 2\rho(TM) - n^2||H||^2 + ||h||^2.$$

Then from (2.4.3), we derive

$$||h||^{2} = n^{2}||H||^{2} + 2\tilde{\tau}(TM) - 2\sum_{i=1}^{n_{1}}\sum_{j=n_{1}+1}^{n}K(e_{i}\wedge e_{j}) - 2\rho(TM_{T}) - 2\tau(TM_{\theta}). \quad (5.2.33)$$

The fourth and fifth terms of the equation (5.2.33) can be obtained by using (2.4.3), thus

$$||h||^{2} = n^{2} ||H||^{2} + 2\tilde{\tau}(TM) - 2\sum_{i=1}^{n_{1}} \sum_{j=n_{1}+1}^{n} K(e_{i} \wedge e_{j})$$
  
$$- 2\tilde{\tau}(TM_{T}) - 2\sum_{r=n+1}^{2m} \sum_{1 \le i \ne t \le n_{1}} \left(h_{ii}^{r}h_{tt}^{r} - (h_{it}^{r})^{2}\right)$$
  
$$- 2\tilde{\tau}(TM_{\theta}) - 2\sum_{r=n+1}^{2m} \sum_{n_{1}+1 \le j \ne l \le n} \left(h_{jj}^{r}h_{ll}^{r} - (h_{jl}^{r})^{2}\right).$$
(5.2.34)

Now we use the formula (2.4.4), for general warped product submanifold in (5.2.34). Then (5.2.34) implies that

$$||h||^{2} = n^{2} ||H||^{2} + 2\tilde{\tau}(TM) - 2\frac{n_{2}\Delta f}{f} - 2\tilde{\rho}(TM_{\theta})$$
$$- 2\tilde{\tau}(TM_{T}) - 2\sum_{r=n+1}^{2m} \sum_{1 \le i \ne t \le n_{1}} \left(h_{ii}^{r}h_{tt}^{r} - (h_{it}^{r})^{2}\right)$$
$$- 2\sum_{r=n+1}^{2m} \sum_{n_{1}+1 \le j \ne l \le n} \left(h_{jj}^{r}h_{ll}^{r} - (h_{jl}^{r})^{2}\right).$$

After adding and subtracting the same terms in the above equation, we find that

$$\begin{split} ||h||^{2} &= n^{2} ||H||^{2} + 2\widetilde{\rho}(TM) - 2\frac{n_{2}\Delta f}{f} - 2\widetilde{\rho}(TM_{\theta}) - 2\widetilde{\rho}(TM_{T}) \\ &- 2\sum_{r=n+1}^{2m} \sum_{1 \leq i \neq t \leq n_{1}} \left( h_{ii}^{r} h_{tt}^{r} - (h_{it}^{r})^{2} \right) - \sum_{r=n+1}^{2m} \left( (h_{11}^{r})^{2} + \dots + (h_{nn}^{r})^{2} \right) \\ &+ \sum_{r=n+1}^{2m} \left( (h_{11}^{r})^{2} + \dots + (h_{nn}^{r})^{2} \right) \\ &- 2\sum_{r=n+1}^{2m} \sum_{n_{1}+1 \leq j \neq l \leq n} \left( h_{jj}^{r} h_{ll}^{r} - (h_{jl}^{r})^{2} \right). \end{split}$$

The above equation is equivalent to the new form

$$||h||^{2} = n^{2} ||H||^{2} + 2\widetilde{\rho}(TM) - 2\frac{n_{2}\Delta f}{f} - 2\widetilde{\rho}(TM_{\theta}) - 2\widetilde{\rho}(TM_{T})$$
  
+  $2\sum_{r=n+1}^{2m} \sum_{i,t=1}^{n_{1}} (h_{it}^{r})^{2} - \sum_{r=n+1}^{2m} \left(h_{11}^{r} + \dots + h_{nn}^{r}\right)^{2}$   
-  $2\sum_{r=n+1}^{2m} \sum_{n_{1}+1 \leq j \neq l \leq n} \left(h_{jj}^{r}h_{ll}^{r} - (h_{jl}^{r})^{2}\right).$  (5.2.35)

Again adding and subtracting the same terms for last term in (5.2.35). Then the equation

## (5.2.35) modified as;

$$|h||^{2} = n^{2} ||H||^{2} + 2\tilde{\tau}(TM) - 2\frac{n_{2}\Delta f}{f} - 2\tilde{\tau}(TM_{\theta}) - 2\tilde{\tau}(TM_{T})$$

$$+ 2\sum_{r=n+1}^{2m} \sum_{i,t=1}^{n_{1}} (h_{it}^{r})^{2} - \sum_{r=n+1}^{2m} (h_{11}^{r} + \dots + h_{nn}^{r})^{2}$$

$$- \sum_{r=n+1}^{2m} \left( (h_{n_{1}+1n_{1}+1}^{r})^{2} + \dots + (h_{nn}^{r})^{2} \right)$$

$$- 2\sum_{r=n+1}^{2m} \sum_{n_{1}+1 \le j \ne l \le n} \left( h_{jj}^{r} h_{ll}^{r} - (h_{jl}^{r})^{2} \right)$$

$$+ \sum_{r=n+1}^{2m} \left( (h_{n_{1}+1n_{1}+1}^{r})^{2} + \dots + (h_{nn}^{r})^{2} \right).$$
(5.2.36)

Thus using Lemma 5.2.5 in (5.2.36). The equation (5.2.36) changes into a new form

$$||h||^{2} = 2\tilde{\tau}(TM) - 2\frac{n_{2}\Delta f}{f} - 2\tilde{\tau}(TM_{\theta}) - 2\tilde{\tau}(TM_{T}) + 2\sum_{r=n+1}^{2m} \sum_{i,t=1}^{n_{1}} (h_{it}^{r})^{2} + 2\sum_{r=n+1}^{2m} \sum_{j,l=n_{1}+1}^{n} (h_{jl}^{r})^{2}.$$
(5.2.37)

The equation (5.2.37) implies the inequality (5.2.31) by using relation (2.4.8). The equality sign in (5.2.31) holds if and only if

$$\sum_{r=n+1}^{2m} \sum_{i,j=1}^{n_1} (g(h(e_i, e_j), e_r))^2 = ||h(\mathscr{D}, \mathscr{D})||^2 = 0,$$
(5.2.38)

$$\sum_{r=n+1}^{2m} \sum_{s,t=n_1+1}^{n_1} (g(h(e_s,e_t),e_r))^2 = ||h(\mathscr{D}^{\theta},\mathscr{D}^{\theta})||^2 = 0.$$
(5.2.39)

As the fact that  $M_T$  is totally geodesic in M, from (5.2.22), (5.2.23) and (5.2.38), it implies that  $M_T$  is totally geodesic in  $\widetilde{M}$ . On the other hand, (5.2.39) implies that h vanishes on  $\mathscr{D}^{\theta}$ . Moreover,  $\mathscr{D}^{\theta}$  is a spherical distribution in M, then it follows that  $M_{\theta}$  is totally umbilical in  $\widetilde{M}$ . This complete proof of the theorem.

**Remark 5.2.6.** It was difficult to obtain the inequality for the second fundamental form by means Codazzi equation which relate to constant holomorphic sectional curvature c and its warping functions with in the fact of pointwise slant immersions as case study of Chen (2003). Therefore, the Theorem 5.2.9 is very useful to construct Chen's type inequalities in terms of pointwise slant immersions.

#### 5.2.4 Applications of Theorem 5.2.9 to complex space forms

We give some interesting applications of the previous Theorem 5.2.9.

**Theorem 5.2.10.** Assume that  $\phi: M = M_T \times_f M_\theta \longrightarrow \widetilde{M}(c)$  is an isometrically immersion of an n-dimensional warped product pointwise semi-slant submanifold  $M_T \times_f M_\theta$  into 2*m*-dimensional complex space form  $\widetilde{M}(c)$  with constant holomorphic sectional curvature c. Then

(i) The squared norm of the second fundamental form of M is given by

$$||h||^{2} \ge 2n_{2} \left( ||\nabla \ln f||^{2} + \frac{n_{1}c}{4} - \Delta(\ln f) \right),$$
 (5.2.40)

where  $n_2$  is the dimension of pointwise slant submanifold  $M_{\theta}$ .

(ii) The equality holds in the above inequality if and only if  $M_T$  is totally geodesic and  $M_{\theta}$  is totally umbilical submanifolds of  $\widetilde{M}(c)$ . Moreover, M is minimal submanifold in  $\widetilde{M}(c)$ .

*Proof.* The Riemannian curvature of complex space form with constant sectional curvature *c* is given by

$$\widetilde{R}(X,Y,Z,W) = \frac{c}{4} \left( g(Y,Z)g(X,W) - g(Y,W)g(X,Z) + g(X,JZ)g(JY,W) - g(Y,JZ)g(JX,W) + 2g(X,JY)g(JZ,W) \right),$$
(5.2.41)

for any  $X, Y, Z, W \in \Gamma(T\widetilde{M})$  (see Chen (2003)). Now substituting  $X = W = e_i$  and  $Y = Z = e_j$  in (5.2.41), we get

$$\widetilde{R}(e_i, e_j, e_j, e_i) = \frac{c}{4} \left( g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)g(e_i, e_j) + g(e_i, Je_j)g(je_j, e_i) - g(e_i, Je_i)g(e_j, Je_j) + 2g^2(Je_j, e_i) \right).$$
(5.2.42)

Taking summation up in (5.2.42) over the basis vector of *TM* such that  $1 \le i \ne j \le n$ , it is easy to obtain that

$$2\tilde{\tau}(TM) = \frac{c}{4} \left( n(n-1) + 3 \sum_{1 \le i \ne j \le n} g^2(Te_i, e_j) \right).$$
(5.2.43)

Let *M* be a proper pointwise semi-slant submanifold of complex space form  $\widetilde{M}(c)$ . Thus we set the following frame, i.e.,

$$e_{1}, e_{2} = Je_{1}, \cdots, e_{2d_{1}-1}, e_{2d_{1}} = Je_{2d_{1}-1}$$
$$e_{2d_{1}+1}, e_{2d_{1}+2} = \sec \theta T e_{2d_{1}+1}, \cdots, \cdots e_{2d_{1}+2d_{2}-1}, e_{2d_{1}+2d_{2}} = \sec \theta T e_{2d_{1}+2d_{2}-1}, e_{2d_{1}+2d_{2}-1}$$

Obviously, we derive from the above frame

$$g^{2}(Je_{i}, e_{i+1}) = 1, \text{ for } i \in \{1, \dots 2d_{1} - 1\}$$
$$= \cos^{2} \theta \text{ for } i \in \{2d_{1} + 1, \dots 2d_{1} + 2d_{2} - 1\}.$$

Thus, it is easy to find that

$$\sum_{i,j=1}^{n} g^2(Te_i, e_j) = 2(d_1 + d_2 \cos \theta).$$
(5.2.44)

From (5.2.43) and (5.2.44), it follows that

$$2\tilde{\tau}(TM) = \frac{c}{4}n(n-1) + \frac{3c}{2}(d_1 + d_2\cos\theta).$$
(5.2.45)

Similarly, for  $TM_T$ , we derive

$$2\tilde{\tau}(TM_T) = \frac{c}{4} \Big( n_1(n_1 - 1) + 3n_1 \Big) = \frac{c}{4} \Big( n_1(n_1 + 2) \Big).$$
(5.2.46)

Now using fact that  $||T||^2 = n_2 \cos^2 \theta$ , for pointwise slant submanifold  $TM_{\theta}$ , we derive

$$2\tilde{\tau}(TM_{\theta}) = \frac{c}{4} \Big( n_2(n_2 - 1) + 3n_2\cos^2\theta \Big) = \frac{c}{4} \Big( n_2^2 + n_2(3\cos^2\theta - 1) \Big).$$
(5.2.47)

Therefore using (5.2.45), (5.2.46) and (5.2.47) in (5.2.31), we get the required result. The equalities can be proved as Theorem 5.2.9 (ii). This completes proof of theorem.

# 5.2.5 Applications to compact orientable warped product pointwise semi-slant submanifold

We consider compact orientable Riemannian manifolds without boundary in this subsection. Using integration theory on manifolds, we obtain some characterizations. **Theorem 5.2.11.** Let  $M = M_T \times_f M_{\theta}$  be a compact warped product pointwise semi-slant submanifold of complex space form  $\widetilde{M}(c)$ . Then M is a Riemannian product if and only if

$$||h||^2 \ge \frac{n_1 n_2 c}{2},\tag{5.2.48}$$

where  $n_1$  and  $n_2$  are dimensions of  $M_T$  invariant submanifold and  $M_{\theta}$  proper pointwise slant submanifold, respectively.

*Proof.* Let us assume that, the inequality holds in Theorem 5.2.10, we get

$$\frac{n_1 n_2 c}{2} + n_2 ||\nabla \ln f||^2 - ||h||^2 \le n_2 \Delta(\ln f).$$
(5.2.49)

From the integration theory on manifolds, i.e., compact orientable Riemannian manifold without boundary on M, we obtain by using (2.3.16)

$$\int_{M} \left( \frac{n_1 n_2 c}{2} + n_2 ||\nabla \ln f||^2 - ||h||^2 \right) dV \le n_2 \int_{M} \Delta(\ln f) dV = 0.$$
 (5.2.50)

If  $||h||^2 \ge \frac{n_1 n_2 c}{2}$ , then

$$\int_M (||\nabla \ln f||^2) dV \le 0.$$

which is impossible for a positive integrable function and hence  $\nabla \ln f = 0$ , i.e., f is a constant function on M. Thus M becomes a Riemannian product. The converse part is straightforward. This completes the proof of the theorem.

**Theorem 5.2.12.** Let  $\phi$  is a  $M_{\theta}$ -minimal isometric immersion of a warped product pointwise semi-slant submanifold  $M_T \times_f M_{\theta}$  into Kaehler manifold  $\widetilde{M}$ . If  $N_{\theta}$  is totally umbilical in  $\widetilde{M}$ , then  $\phi$  is  $M_{\theta}$ -totally geodesic.

*Proof.* Let us assume that the second fundamental forms of M and  $\widetilde{M}$  are denoted by  $h^*$  and  $\widetilde{h}$ , respectively, we defines the following

$$h(Z,W) + h^*(Z,W) = h(Z,W).$$
 (5.2.51)

for any vector fields Z and W are tangent to  $M_{\theta}$ . Thus from the above hypothesis and the definition of warped product submanifold show that  $M_{\theta}$  is totally umbilical in  $\widetilde{M}$  due to the totally umbilical in M. Then follows to the Lemma 2.4.1(iii), the equation (5.2.51)

can be written as

$$h(Z,W) = g(Z,W) (\xi + \nabla(\ln f)).$$
 (5.2.52)

where the vector field  $\xi$  is normal to  $\Gamma(TM_{\theta})$  and  $\xi \in \Gamma(TM)$ . Assume that  $\{e_1^*, \dots, e_{n_2}^*\}$  be an orthonormal frame of the pointwise slant submanifold  $N_{\theta}$ , then taking summation over the vector fields of  $M_{\theta}$  in the equation (5.2.52), we get

$$\sum_{i,j=1}^{n_2} h(e_i^*, e_j^*) = \left(\xi + \nabla(\ln f)\right) \sum_{i,j=1}^{n_2} g(e_i^*, e_j^*).$$
(5.2.53)

The left hand side of the above equation identically vanishes due to the  $\mathcal{D}^{\theta}$ -minimality of  $\phi$  such that  $\sum_{i,j=1}^{n_2} h(e_i^*, e_j^*) = 0$ . Then the equation (5.2.53) takes the form

$$n_2(\boldsymbol{\xi} + \nabla(\ln f)) = 0.$$

It implies that fact  $M_{\theta}$  is nonempty such as:

$$\boldsymbol{\xi} = -\nabla(\ln f). \tag{5.2.54}$$

Thus from (5.2.52) and (5.2.54), it follows that h(Z,W) = 0, for every  $Z, W \in \Gamma(TM_{\theta})$ . This means that  $\phi$  is  $M_{\theta}$ -totally geodesic. This completes the proof of the theorem.

**Theorem 5.2.13.** Let  $M = M_T \times_f M_{\theta}$  be a compact orientable proper warped product pointwise semi-slant submanifold in a complex space form  $\widetilde{M}(c)$ . Then M is Riemannian product of  $M_T$  and  $M_{\theta}$  if and only if

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_v(e_i, e_j)||^2 = \frac{n_1 n_2 c}{4},$$
(5.2.55)

where  $h_{v}$  is a component of h in  $\Gamma(v)$ .

*Proof.* Suppose that the equality sign holds in (5.2.40) and linearity of second fundamental form in almost Hermitian metric, then we have

$$||h||^{2} = ||h(\mathscr{D}, \mathscr{D})||^{2} + ||h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta})||^{2} + 2||h(\mathscr{D}, \mathscr{D}^{\theta})||^{2} = \frac{n_{1}n_{2}c}{2} + 2n_{2}||\nabla \ln f||^{2} - 2n_{2}\Delta(\ln f).$$

Following the equality case of the inequality (5.2.40) implies that  $M_T$  is totally geodesic in  $\widetilde{M}$ , and this means that  $h(e_i, e_j) = 0$ , for any  $1 \le i, j \le 2d_1$ . Also  $M_{\theta}$  is totally umbilical

and it can be written as  $h(e_t^*, e_s^*) = g(e_t^*, e_s^*)H$ , for any  $1 \le t, s \le 2d_2$ . Since, *M* is minimal submanifold of  $\widetilde{M}$ , then its mean curvature vector *H* should be identically zero, i.e, H = 0, hence  $h(e_t^*, e_s^*) = 0$ , for every  $1 \le t, s \le 2d_2$  by using the Theorem 5.2.12. Applying these facts in the last equation, we get

$$\frac{n_1 n_2 c}{4} + n_2 ||\nabla \ln f||^2 = n_2 \Delta(\ln f) + ||h(\mathscr{D}, \mathscr{D}^{\theta})||^2.$$
(5.2.56)

As we assumed that M is compact orientable submanifold, i.e., M is closed and bounded. Hence, integrating the equation (5.2.56) over the volume element dV of M and using (2.3.16), we find

$$\int_{M} \left(\frac{n_1 n_2 c}{4}\right) \mathrm{d}\mathbf{V} + n_2 \int_{M} \left(||\nabla \ln f||^2\right) \mathrm{d}\mathbf{V} = \int_{M} \left(||h(\mathscr{D}, \mathscr{D}^{\theta})||^2\right) \mathrm{d}\mathbf{V}.$$
(5.2.57)

Let us define  $X = e_i$  and  $Z = e_j$  for  $1 \le i \le n_1$  and  $1 \le j \le n_2$  respectively, we have

$$h(e_i, e_j) = \sum_{r=n+1}^{n+n_2} g(h(e_i, e_j), e_r) e_r + \sum_{r=n+n_2+1}^{2m} g(h(e_i, e_j), e_r) e_r$$

The first term in the right hand side of the above equation is  $F \mathscr{D}^{\theta}$ -component and the second term is *v*-component. Taking summation over the vector fields on  $M_T$  and  $M_{\theta}$  and using adapted frame fields, we get

$$\begin{split} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(h(e_i, e_j), h(e_i, e_j)) &= \csc^2 \theta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(e_i, e_j^*), Fe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(e_i, Te_j^*), Fe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(Je_i, e_j^*), FTe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(Je_i, e_j^*), FTe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(Je_i, Te_j^*), FTe_k^*)^2 \end{split}$$

$$+ \csc^{2} \theta \sec^{2} \theta \sum_{i=1}^{d_{1}} \sum_{j,k=1}^{d_{2}} g(h(Je_{i}, Te_{j}^{*}), Fe_{k}^{*})^{2}$$
  
+  $\csc^{2} \theta \sum_{i=1}^{d_{1}} \sum_{j,k=1}^{d_{2}} g(h(Je_{i}, e_{j}^{*}), Fe_{k}^{*})^{2}$   
+  $\csc^{2} \theta \sec^{4} \theta \sum_{i=1}^{d_{1}} \sum_{j,k=1}^{d_{2}} g(h(e_{i}, Te_{j}^{*}), FTe_{r}^{*})^{2}$   
+  $\sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} \sum_{r=n+n_{2}+1}^{2m} g(h(e_{i}, e_{j}), e_{r})^{2}.$ 

Then using Lemma 5.2.1, we derive

$$||h(\mathscr{D}, \mathscr{D}^{\theta})||^{2} = n_{2}(1 + 2\cot^{2}\theta)||\nabla \ln f||^{2} + \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} ||h_{\nu}(e_{i}, e_{j})||^{2}.$$
(5.2.58)

Then from (5.2.57) and (5.2.58), it follows that

$$\int_{M} \left( \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} ||h_{v}(e_{i}, e_{j})||^{2} + 2n_{2} \cot^{2} \theta ||\nabla \ln f||^{2} \right) \mathrm{dV} = \int_{M} \left( \frac{n_{1}n_{2}c}{4} \right) \mathrm{dV}.$$
(5.2.59)

If (5.2.55) holds identically, then from (5.2.59), we either *f* is constant on *M* or  $\cot \theta = 0$ , i.e., *M* is a proper pointwise semi-slant submanifold. Converse part follow immediately from (5.2.59) Hence, the theorem is proved completely.

Now we give direct consequences of the inequality (5.2.40) by using Remark 2.3.5, then the Theorem 5.2.11 and Theorem 5.2.13 are generalized to the results for CR-warped products into complex space forms  $\widetilde{M}(c)$  which follows:

**Corollary 5.2.1.** Let  $M = M_T \times_f M_{\perp}$  be a compact orientate CR-warped product submanifold into complex space form  $\widetilde{M}(c)$ . Then M is CR-product if and only if

$$||h||^2 \ge \frac{n_2 n_1 c}{2},$$

where  $n_1$  and  $n_2$  are dimensions of  $M_T$  and  $M_{\perp}$ , respectively.

**Corollary 5.2.2.** Assume that  $M = M_T \times_f M_\perp$  be a compact orientable CR-warped product submanifold in complex space form  $\widetilde{M}(c)$  such that  $M_T$  is holomorphic and  $M_\perp$  is totally real submanifold in  $\widetilde{M}(c)$ . Then M is simply a Riemannian product if and only if

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mathcal{V}}(e_i, e_j)||^2 = \frac{n_2 n_1 c}{4}.$$

Base on the Laplacian property of positive differential function defined on any compact Riemannian manifold, we obtain a corollary with help of Eqs (5.2.40),

**Corollary 5.2.3.** Assume that  $M = M_T \times_f M_{\theta}$  be a warped product pointwise semi-slant submanifold in a complex space form  $\widetilde{M}(c)$ . Let  $M_T$  is compact invariant submanifold and  $\lambda$  be non-zero eigenvalue of the Laplacian on  $M_T$ . Then

$$\int_{M_T} ||h||^2 \mathrm{d} \mathbf{V}_T \ge \int_{M_T} \left(\frac{n_1 n_2 c}{2}\right) \mathrm{d} \mathbf{V}_T + 2n_2 \lambda \int_{M_T} (\ln f)^2 \mathrm{d} \mathbf{V}_T.$$
(5.2.60)

- (*i*)  $\Delta \ln f = \lambda \ln f$ .
- (ii) In the warped product pointwise semi-slant submanifold both  $M_T$  and  $M_{\theta}$  are totally geodesic.

Proof. Taking account of the minimum principle property, we obtain

$$\int_{M_T} ||\nabla \ln f||^2 \mathrm{d} \mathbf{V}_T \ge \lambda \int_{M_T} (\ln f)^2 \mathrm{d} \mathbf{V}_T.$$
(5.2.61)

From (5.2.40) and (5.2.61) we get required the result (5.2.60). This completes the proof of corollary.

Corollary 5.2.4. Assume that  $M = M = M_T \times_f M_{\theta}$  be a warped product pointwise semislant submanifold in a complex space form  $\widetilde{M}(c)$ . Let  $M_T$  is compact invariant submanifold and  $\lambda$  be non-zero eigenvalue of the Laplacian on  $M_T$ . Then

$$\int_{M_T} \left( \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 \right) \mathrm{d} \mathbf{V}_T \ge \int_{M_T} \left( \frac{c}{4} n_2 n_1 \right) \mathrm{d} \mathbf{V}_T - 2n_2 \lambda \int_{M_T} \left( \cot^2 \theta (\ln f)^2 \right) \mathrm{d} \mathbf{V}_T.$$
(5.2.62)

- (*i*)  $\Delta \ln f = \lambda \ln f$ .
- (ii) In the warped product pointwise semi-slant submanifold both  $M_T$  and  $M_{\theta}$  are totally geodesic.

*Proof.* The proof of the above theorem follows from Eqs.(5.2.58) and the minimum principal properties.

**Corollary 5.2.5.** Let  $\ln f$  is a harmonic function on  $M_T$ . Then there does not exist any warped product pointwise semi-slant  $M_T \times_f M_{\theta}$  into complex space form  $\widetilde{M}(c)$  with  $c \leq 0$ .

*Proof.* Let us consider that there exists a warped product pointwise semi-slant submanifold  $M = M_T \times_f M_{\theta}$  in complex space form  $\widetilde{M}(c)$  with  $\ln f$  is a harmonic function  $M_T$ . Thus, the inequality (5.2.40) gives c > 0. This completes the proof of the corollary.

**Corollary 5.2.6.** There does not exist a warped product pointwise semi-slant submanifold  $M_T \times_f M_{\theta}$  into complex space form  $\widetilde{M}(c)$  with  $c \leq 0$  such that  $\ln f$  be a positive eigenfunction of the Laplacian on  $M_T$  corresponding to an eigenvalue  $\lambda \geq 0$ .

## 5.2.6 Applications to Hessian of warping functions

Throughout of these study, as new results, we try to find some fundamental applications of derived inequality in terms of Hessian of positive differentiable function by taking both inequality and equality cases. We derive some necessary and sufficient conditions under which a warped product pointwise semi-slant isometrically immersed into complex space form to be a Riemannian product manifold or trivial warped product.

**Theorem 5.2.14.** Let  $\phi : M = M_T \times_f M_{\theta}$  be an isometric immersion of a warped product pointwise semi-slant into a complex space form  $\widetilde{M}(c)$ . If the following inequality holds

$$||h||^{2} \ge 2n_{2} \left( \frac{n_{1}c}{4} + \sum_{i=1}^{d_{1}} \left( H^{\ln f}(e_{i}, e_{i}) + H^{\ln f}(Je_{i}, Je_{i}) \right) \right),$$
(5.2.63)

where  $H^{\ln f}$  is Hessian of warping function  $\ln f$ , then M is a trivial warped product pointwise semi-slant submanifold.

*Proof.* In view of (2.3.14), it can be expanded as:

$$\Delta(\ln f) = -\sum_{i=1}^{n} g\left(\nabla_{e_i} grad \ln f, e_i\right)$$
$$= -\sum_{i=1}^{2d_1} g\left(\nabla_{e_i} grad \ln f, e_i\right) + \sum_{j=1}^{2d_2} g\left(\nabla_{e_j} grad \ln f, e_j\right),$$

for every  $1 \le i \le n_1$  and  $2 \le n_2 \le n_2$ . More simplification by using orthonormal components corresponding to  $M_T$  and  $M_{\theta}$  which defined in the previous study, respectively, we

have

$$\Delta(\ln f) = -\sum_{i=1}^{d_1} g\left(\nabla_{e_i} grad \ln f, e_i\right) - \sum_{i=1}^{d_1} g\left(\nabla_{Je_i} grad \ln f, Je_i\right)$$
$$-\sum_{j=1}^{d_2} g\left(\nabla_{e_j} grad \ln f, e_j\right)$$
$$-\sec^2 \theta \sum_{i=1}^{d_2} g\left(\nabla_{Te_j} grad \ln f, Te_j\right).$$

Taking account that  $\nabla$  is Levi-Civitas connection on *M* and (2.3.15), we derive

$$\Delta(\ln f) = -\sum_{i=1}^{d_1} \left( H^{\ln f}(e_i, e_i) + H^{\ln f}(Je_i, Je_i) \right)$$
$$-\sum_{j=1}^{d_2} \left( e_j g(\operatorname{grad} \ln f, e_j) - g(\nabla_{e_j} e_j, \operatorname{grad} \ln f) \right).$$
$$-\operatorname{sec}^2 \theta \sum_{j=1}^{d_2} \left( Te_j g(\operatorname{grad} \ln f, Te_j) - g(\nabla_{Te_j} Te_j, \operatorname{grad} \ln f) \right).$$

Using the gradient property of function (2.3.12), we arrive at

$$\begin{split} \Delta(\ln f) &= -\sum_{i=1}^{d_1} \left( H^{\ln f}(e_i, e_i) + H^{\ln f}(Je_i, Je_i) \right) \\ &- \sum_{j=1}^{d_2} \left( e_j(e_j \ln f) - (\nabla_{e_j} e_j \ln f) \right) \\ &- \sec^2 \theta \sum_{j=1}^{d_2} \left( Te_j(Te_j \ln f) - (\nabla_{Te_j} Te_j \ln f) \right). \end{split}$$

Thus more simplification gives,

$$\begin{split} \Delta(\ln f) &= -\sum_{i=1}^{d_1} \left( H^{\ln f}(e_i, e_i) + H^{\ln f}(Je_i, Je_i) \right) \\ &- \sum_{j=1}^{d_2} \left( e_j \Big( \frac{g(gradf, e_j)}{f} \Big) - \frac{1}{f} g(\nabla_{e_j} e_j, gradf) \Big) \\ &- \sec^2 \theta \sum_{j=1}^{d_2} \left( Te_j \Big( \frac{g(gradf, Te_j)}{f} \Big) - \frac{1}{f} g(\nabla_{Te_j} Te_j, gradf) \Big). \end{split}$$

By hypothesis of warped product pointwise semi-slant submanifold,  $M_T$  defines as totally

geodesic in *M*. It implies that  $grad f \in \Gamma(TM_T)$ , and Lemma 2.4.1(ii), we obtain

$$\begin{split} \Delta(\ln f) &= -\sum_{i=1}^{d_1} \left( H^{\ln f}(e_i, e_i) + H^{\ln f}(Je_i, Je_i) \right) \\ &- \sum_{j=1}^{d_2} \left( g(e_j, e_j) ||\nabla \ln f||^2 + \sec^2 \theta g(Te_j, Te_j) ||\nabla \ln f||^2 \right). \end{split}$$

Finally from (2.3.37), we obtain

$$\Delta(\ln f) + \sum_{i=1}^{d_1} \left( H^{\ln f}(e_i, e_i) + H^{\ln f}(Je_i, Je_i) \right) + 2n_2 ||\nabla \ln f||^2 = 0.$$
 (5.2.64)

Thus (5.2.10) and (5.2.64), follows

$$||h||^{2} \ge 2n_{2} \left( \frac{c}{4} n_{1} + (n_{2} + 2) ||\nabla \ln f||^{2} + \sum_{i=1}^{d_{1}} \left( H^{\ln f}(e_{i}, e_{i}) + H^{\ln f}(Je_{i}, Je_{i}) \right) \right).$$
(5.2.65)

If the inequality (5.3.38) holds, then (5.2.65) implies that  $||\nabla \ln f||^2 \leq 0$ , which is impossible by definition of norm of a positive function. Thus it can be concluded that  $grad \ln f = 0$ , this means that f is a constant function on M. Hence M becomes a trivial warped product pointwise semi-slant submanifold. This completes proof of theorem.

**Theorem 5.2.15.** Let  $\phi: M = M_T \times_f M_{\theta}$  be an isometric immersion of a warped product pointwise semi-slant  $M_T \times_f M_{\theta}$  into complex space form  $\widetilde{M}(c)$ . Assume that the following equality satisfies for warped product submanifold M

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_v(e_i, e_j)||^2 = n_2 \left( \sum_{i=1}^{d_1} \left( H^{\ln f}(e_i, e_i) + H^{\ln f}(Je_i, Je_i) \right) + \frac{n_1 c}{4} \right).$$
(5.2.66)

Then at least one of the statement is true for M.

- (i) The non-trivial warped product pointwise semi-slant  $M_T \times_f M_{\theta}$  is a trivial warped product (or M is simply Riemannian product)
- (ii) The pointwise slant function satisfies  $\theta = \cot^{-1}(\sqrt{n_2})$  of warped product pointwise semi-slant submanifold  $M_T \times_f M_{\theta}$  into complex space form  $\widetilde{M}(c)$ .

*Proof.* Assume that the equality holds in the inequality (5.2.40), then we have

$$\frac{n_1 n_2 c}{2} - 2n_2 \Delta(\ln f) + 2n_2 ||\nabla \ln f||^2 = ||h||^2.$$

By the definition of the components  $\mathscr{D}$  and  $\mathscr{D}^{\theta}$ , the above equation can be expressed as:

$$\frac{n_1 n_2 c}{2} - 2n_2 \Delta(\ln f) + 2n_2 ||\nabla \ln f||^2 = ||h(\mathscr{D}, \mathscr{D})||^2 + ||h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta})||^2 + 2||h(\mathscr{D}, \mathscr{D}^{\theta})||^2.$$
(5.2.67)

Taking account of (5.2.58) for the equality case of the inequality, we find that

$$\frac{n_1 n_2 c}{4} + n_2 ||\nabla \ln f||^2 = n_2 (1 + 2\cot^2 \theta) ||\nabla \ln f||^2 + \sum_{i=1}^{2d_1} \sum_{j=1}^{2d_2} ||h_v(e_i, e_j^*)||^2 + n_2 \Delta(\ln f),$$

which implies that

$$\frac{n_1 n_2 c}{4} = n_2 \Delta(\ln f) + 2n_2 \cot^2 \theta ||\nabla \ln f||^2 + \sum_{i=1}^{2d_1} \sum_{j=1}^{2d_2} ||h_v(e_i, e_j^*)||^2.$$
(5.2.68)

From the relations (5.2.68), and (5.2.64), we derive

$$\frac{1}{n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_v(e_i, e_j)||^2 = \sum_{i=1}^{d_1} \left[ H^{\ln f}(e_i, e_i) + H^{\ln f}(Je_i, Je_i) \right] \\ + 2 \left( n_2 - \cot^2 \theta \right) ||\nabla \ln f||^2 + \frac{n_1 c}{4}.$$
(5.2.69)

By the hypothesis of the theorem, if (5.2.66) holds, then (5.2.69) indicate that  $(n_2 - \cot^2 \theta) ||\nabla \ln f||^2 = 0$ , which implies that either  $||\nabla \ln f||^2 = 0$  or  $(n_2 - \cot^2 \theta) = 0$ . For the first case if  $||\nabla \ln f||^2 = 0$ , then f is a constant function on M, i.e., M is simply a Riemannian product of  $M_T$  and  $M_{\theta}$  (M is trivial) which proves (i). Similarly, for the condition  $(n_2 - \cot^2 \theta) = 0$ , it proves the second statement (ii) of the Theorem. This completes the proof of theorem.

**Theorem 5.2.16.** There does not exist warped product pointwise semi-slant  $M = M_T \times_f M_{\theta}$  into complex space form  $\widetilde{M}(c)$  with  $c \leq 0$  such that  $M_T$  is a compact holomorphic submanifold and  $M_{\theta}$  is pointwise slant submanifold of  $\widetilde{M}(c)$ .

*Proof.* Assume there exists a warped product pointwise semi-slant  $M = M = M_T \times_f M_{\theta}$ in a complex space form  $\widetilde{M}(c)$  with  $c \leq 0$  such that  $M_T$  is compact. Then the function  $\ln f$ has an absolute maximum at some point  $p \in M_T$ . At this critical point, the Hessian  $H^{\ln f}$ is non-positive definite. Thus (5.2.63) leads to a contradiction. This completes the proof of theorem.

# 5.2.7 Applications to kinetic energy functions, Hamiltonian and Euler-Lagrangian Equations

This subsection is devoted to presentation of the essential results of research on connected, compact Riemannian manifolds with nonempty boundary, i.e.,  $\partial M \neq \emptyset$ . As applications of Hopf lemma. Of particular interest is the new information on the non-existence of connected, compact warped product pointwise semi-slant in complex space form with the terms of kinetic energy, Hamiltonian of warping functions. and Euler Lagrange equation.

**Theorem 5.2.17.** Assume that  $\phi : M = M_T \times_f M_{\theta}$  is an isometric immersion of a warped product pointwise semi-slant into complex space form  $\widetilde{M}(c)$ . A connected, compact warped product  $M_T \times_f M_{\theta}$  is simply Riemannian product of  $M_T$  and  $M_{\theta}$  if and only if the kinetic energy function satisfied

$$E(\ln f) = \frac{1}{4n_2} \tan^2 \theta \left[ \int_M \left( \frac{n_2 n_1 c}{4} - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_V(e_i, e_j)||^2 \right) dV - 8n_2 \int_M \left( \csc^2 \theta \cot \theta \left( \frac{d\theta}{dV} \right) E(\ln f) \right) dV \right], \quad (5.2.70)$$

where  $E(\ln f)$  represent the kinetic energy of the warping function  $\ln f$  and dV is volume element on M.

*Proof.* From the Eqs. (5.2.58) for the equality case of the inequality (5.2.40) such that

$$\frac{c}{4}n_2n_1 - n_2\Delta(\ln f) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}||h_v(e_i, e_j)||^2 + 2n_2\cot^2\theta||\nabla\ln f||^2.$$
(5.2.71)

Taking integration along M over the volume element dV with nonempty boundary in the above equation, we can easily find that

$$\int_{M} \left(\frac{c}{4} n_{2} n_{1}\right) dV = \int_{M} \left(\sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} ||h_{v}(e_{i}, e_{j})||^{2}\right) dV + n_{2} \int_{M} \left(\Delta(\ln f)\right) dV + 2n_{2} \int_{M} \left(\cot^{2} \theta\left(||\nabla \ln f||^{2}\right)\right) dV.$$
(5.2.72)

Using the property of partial integration of last term in above theorem due to nonconstant

slant function  $\theta$ , i.e.,

$$\begin{split} \int_{M} \left(\frac{c}{4}n_{2}n_{1}\right) \mathrm{d}V &= \int_{M} \left(\sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} ||h_{V}(e_{i},e_{j})||^{2}\right) \mathrm{d}V \\ &+ n_{2} \int_{M} \left(\Delta(\ln f)\right) \mathrm{d}V \\ &+ 2n_{2} \cot^{2} \theta \int_{M} \left(||\nabla \ln f||^{2}\right) \mathrm{d}V \\ &+ 4n_{2} \int_{M} \left(\csc^{3} \theta \cos \theta \left(\frac{d\theta}{\mathrm{d}V}\right) \left(\int_{M} ||\nabla \ln f||^{2}\right) \mathrm{d}V\right) \mathrm{d}V. \end{split}$$

Then (2.3.17) and (5.2.72), it follows that

$$\int_{M} \left(\frac{c}{4}n_{1}\right) dV = \frac{1}{n_{2}} \int_{M} \left(\sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} ||h_{\mu}(e_{i}, e_{j})||^{2}\right) dV$$
$$+ \int_{M} \Delta(\ln f) dV$$
$$+ 4\cot^{2} \theta E(\ln f)$$
$$+ 8 \int_{M} \left(\csc^{3} \theta \cos \theta \left(\frac{d\theta}{dV}\right) E(\ln f)\right) dV.$$
(5.2.73)

If equality (5.2.70) satisfies if and only if that, we get the following condition from (5.2.73). i.e.,

$$\int_M \Delta(\ln f) \mathrm{d}V = 0 \ on \ M,$$

which implies that

$$\Delta(\ln f) = 0. \tag{5.2.74}$$

As we have assumed that, M is connected, compact warped product pointwise semi-slant submanifold, then (5.2.74) and the Theorem 2.3.1, implies that  $\ln f = 0 \Longrightarrow f = 1$ , which means that f is constant on M. Hence, the theorem is proved completely.

Similarly, we derive some characterization in terms of Hamiltonian such that;

**Theorem 5.2.18.** An isometric immersion  $\phi : M = M_T \times_f M_{\theta}$  of a non-trivial connected, compact warped product pointwise semi-slant submanifold  $M_T \times_f M_{\theta}$  into complex space form  $\widetilde{M}(c)$  is called trivial warped product if and only if it satisfies

$$H(\mathrm{d}(\ln f), p) = \frac{1}{4n_2} \tan^2 \theta \left( \frac{n_2 n_1 c}{4} - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\nu}(e_i, e_j)||^2 \right),$$
(5.2.75)

where  $H(d(\ln f), p)$  is the Hamiltonian of warping function  $\ln f$  at point  $p \in M$ .

Proof. Let the equality holds in (5.2.40), then using (2.3.13) in (5.2.58), we derive

$$\frac{n_2 n_1 c}{4} - 4n_2 \cot^2 \theta H (d(\ln f), p)$$
  
=  $n_2 \Delta(\ln f) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_v(e_i, e_j)||^2.$  (5.2.76)

Since, the equation (5.2.75) implies  $\iff \Delta(\ln f) = 0$  on *M*, thus follow the Theorem 2.3.1, then *M* is a trivial warped product submanifold. This completes the proof of theorem.

**Theorem 5.2.19.** Assume that  $\phi : M = M_T \times_f M_{\theta}$  is an isometric immersion of a compact warped product pointwise semi-slant into complex space form  $\widetilde{M}(c)$ . Let the warping function satisfies the Euler-Lagrange Equation. Then, the necessary condition of M to be a trivial warped product, i.e.,

$$||h||^2 \ge \frac{n_2 n_1 c}{2}.$$
(5.2.77)

*Proof.* If the warping function satisfies the condition of Euler-Lagrange equation, then from Theorem 2.3.1, we obtain

$$\Delta(\ln f) = 0, \tag{5.2.78}$$

Thus from (5.2.40) and (5.2.78), we derive

$$||h||^{2} \ge \frac{n_{2}n_{1}c}{2} + n_{2}||\nabla \ln f||^{2}.$$
(5.2.79)

Suppose the inequality (5.2.77) hold, then (5.2.79) implies that the warping function must be constant on *M*. This completes the proof of the theorem.

**Theorem 5.2.20.** Assume that  $\phi : M = M_T \times_f M_{\theta}$  be an isometric immersion of a compact warped product pointwise semi-slant into complex space form  $\widetilde{M}(c)$ . Let the warping function satisfies the Euler-Lagrange Equation. Then the necessary and sufficient condition the warped product  $M_T \times_f M_{\theta}$  is trivial, i.e.,

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\nu}(e_i, e_j)||^2 = \frac{n_2 n_1 c}{4},$$
(5.2.80)

*Proof.* The proof of the above theorem same as the Theorem 5.2.19 by using (5.2.80), (5.2.68) and Theorem 2.3.1. This completes the proof of the theorem.

# 5.3 GEOMETRIC APPROACH OF WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

## 5.3.1 Pointwise semi-slant submanifold of cosymplectic manifold

As we have given the definition of pointwise semi-slant submanifold of almost contact metric manifold in the second Chapter (see Definition 2.3.19). In this sequence, we obtain some integrability theorem for later use in characterization results.

**Theorem 5.3.1.** Assume that M be a pointwise semi-slant submanifold of a cosymplectic manifold  $\widetilde{M}$ . Then the distribution  $\mathscr{D} \oplus \langle \xi \rangle$  is define as a totally geodesic foliations if and only if,

$$h(X, \varphi Y) \in \Gamma(\mu),$$

for any  $X, Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ .

*Proof.* Let  $X, Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$  and  $Z \in \Gamma(\mathcal{D}^{\theta})$ , we have  $g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(\overline{\nabla}_X Y, \varphi Z)$ . Using (2.3.39) and (2.3.2), we obtain

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X \varphi Y, PZ) + g(\widetilde{\nabla}_X \varphi Y, FZ) - g((\widetilde{\nabla}_X \varphi)Y, \varphi Z).$$

From (2.2.9), (2.3.40) and the definition of totally geodesic foliation, we arrives the final result.

**Theorem 5.3.2.** On a pointwise semi-slant submanifold M in a cosymplectic manifold  $\tilde{M}$ , the distribution  $\mathcal{D}^{\theta}$  defines totally geodesic foliation if and only if

$$g(h(X,Z),FW) = -\sec^2\theta g(h(Z,\varphi X),FPW),$$

for all  $Z \in \Gamma(\mathscr{D}^{\theta})$  and  $X \in \Gamma(\mathscr{D} \oplus \langle \xi \rangle)$ .

*Proof.* Taking account of (2.2.9)(ii) and fact that  $\xi$  is tangent to  $M_T$ , we obtain

$$g(\nabla_Z W, X) = g(\varphi \widetilde{\nabla}_Z W, \varphi X).$$

For any  $Z \in \Gamma(\mathscr{D}^{\theta})$  and  $X \in \Gamma(\mathscr{D} \oplus \langle \xi \rangle)$ . For a tensorial equation (2.2.9)(i) cosym-

plectic manifolds, we have  $\varphi \widetilde{\nabla}_Z W = \widetilde{\nabla}_Z \varphi W$ , and using (2.3.39), we derive

$$g(\nabla_Z W, X) = g(\widetilde{\nabla}_Z PW, \varphi X) + g(\widetilde{\nabla}_Z FW, \varphi X).$$

Thus from relation (2.3.3) and again using the structure equation (2.2.9)(i) of cosymplectic manifold, then

$$g(\nabla_Z W, X) = g(\widetilde{\nabla}_Z P^2 W, X) + g(h(X, Z), FPW) - g(h(Z, \varphi X), FW).$$

With the help of the Theorem 2.3.9 for pointwise slant submanifold, we obtain

$$g(\nabla_Z W, X) = \sin 2\theta Z(\theta) g(W, X) - \cos^2 \theta g(\nabla_Z W, X)$$
$$+ g(h(X, Z), FPW) - g(h(Z, \varphi X), FW),$$

which implies that

$$\sin^2 \theta g(\nabla_Z W, X) = g(h(X, Z), FPW) - g(h(Z, \varphi X), FW).$$

Hence, interchanging *W* by *PW* in the above equation, we get required result. This compete the proof of the theorem.

## 5.3.2 Warped product pointwise semi-slant submanifold in cosymplectic manifolds

Follows the definition of pointwise semi-slant submanifold, in such case we define two type of warped product pointwise semi-slant submanifold such that

- (i)  $M_{\theta} \times_f M_T$
- (ii)  $M_T \times_f M_{\theta}$

For the first opportunity if the structure vector field  $\xi$  tangent to  $M_{\theta}$ , Park (2014) proved the following non-existence Theorem. i.e.,

**Theorem 5.3.3.** There do not exist proper warped product pointwise semi-slant submanifolds  $M = M_{\theta} \times_f M_T$  in a cosymplectic manifold  $\widetilde{M}$  such that  $M_{\theta}$  is a proper pointwise slant submanifold with tangent to  $\xi$  and  $M_T$  is an invariant submanifold.

Now for the second type of warped product submanifold, we have the lemma, that is,

*Lemma* 5.3.1. Let  $M = M_T \times_f M_\theta$  be a warped product pointwise semi-slant submanifold of a cosymplectic manifold  $\widetilde{M}$  such that  $\xi$  is tangent to  $M_T$ , where  $M_T$  and  $M_\theta$  are invariant and proper pointwise slant submanifolds of  $\widetilde{M}$ , respectively. Thus

$$g(h(X,Z),FPW) = -\left(\varphi X \ln f\right)g(Z,PW) - \cos^2\theta(X \ln f)g(Z,W), \tag{5.3.1}$$

$$g(h(Z,\varphi X),FW) = (X\ln f)g(Z,W) - (\varphi X\ln f)g(Z,PW), \qquad (5.3.2)$$

for any  $X \in \Gamma(TM_T)$  and  $Z, W \in \Gamma(TM_\theta)$ .

We define an example with help of Sahin (2013) which shows the existence of warped product pointwise semi-slant in cosymplectic manifold

*Example* 5.3.1. Consider  $\widetilde{M}^{11} = \mathbb{C}^5 \times \mathbb{R}$  be a Riemannian product of Euclidean space  $\mathbb{C}^5$  with line  $\mathbb{R}$  such that the structure vector field  $\xi = \frac{\partial}{\partial t}$ , 1-form  $\eta = dt$  and metric  $g = g_1 + dt^2$ , where  $g_1$  is the metric on Euclidean space. Then  $(\widetilde{M}^{11}, \varphi, \xi, \eta, g)$  is called cosymplectic manifold. Let  $\phi : M^5 \to \widetilde{M}^{11}$  be a pointwise semi-slant submanifold such that 0 < u, v < 1 and  $0 \le \theta_1, \theta_2 < \frac{\pi}{2}$  is define as

$$e_1 = u\cos\theta_1, e_2 = v\cos\theta_1, e_3 = u\cos\theta_2, e_4 = v\cos\theta_2, e_5 = u\sin\theta_1$$
  
 $e_6 = v\sin\theta_1, e_7 = u\sin\theta_2, e_8 = v\sin\theta_2, e_9 = \theta_1, e_{10} = \theta_2, e_{11} = 0.$ 

The tangent space TM is spanned by the vector fields  $X_1, X_2, X_3, X_4, X_5$  such that

$$X_{1} = \sin \theta_{2} \frac{\partial}{\partial e_{8}} + \sin \theta_{1} \frac{\partial}{\partial e_{6}} + \cos \theta_{2} \frac{\partial}{\partial e_{4}} + \cos \theta_{1} \frac{\partial}{\partial e_{2}},$$

$$X_{2} = \sin \theta_{2} \frac{\partial}{\partial e_{7}} + \sin \theta_{1} \frac{\partial}{\partial e_{5}} + \cos \theta_{2} \frac{\partial}{\partial e_{3}} + \cos \theta_{1} \frac{\partial}{\partial e_{1}},$$

$$X_{3} = \xi = \frac{\partial}{\partial t},$$

$$X_{4} = \frac{\partial}{\partial e_{9}} + v \cos \theta_{1} \frac{\partial}{\partial e_{6}} + u \cos \theta_{1} \frac{\partial}{\partial e_{5}} - v \sin \theta_{1} \frac{\partial}{\partial e_{2}} - u \sin \theta_{1} \frac{\partial}{\partial e_{1}},$$

$$X_{5} = \frac{\partial}{\partial e_{10}} + v \cos \theta_{2} \frac{\partial}{\partial e_{7}} + u \cos \theta_{2} \frac{\partial}{\partial e_{8}} - v \sin \theta_{2} \frac{\partial}{\partial e_{4}} - u \sin \theta_{2} \frac{\partial}{\partial e_{3}}.$$

Then  $\mathscr{D}^{\theta} = span\{X_4, X_5\}$  is pointwise slant distribution with slant function  $\cos^{-1}(\frac{1}{u^2+v^2+1})$ and invariant distribution  $\mathscr{D} = span\{X_1, X_2, X_3\}$ . Thus  $M^5 = M_T \times_f M_{\theta}$  be non-trivial warped product pointwise semi-slant submanifold of  $M^{11}$  with warping function  $f = \sqrt{u^2 + v^2 + 1}$ .
We are going to prove characterization theorem which in an important result of this subsection.

**Theorem 5.3.4.** Every proper pointwise semi-slant submanifold M in a cosymplectic manifold  $\widetilde{M}$  is locally a warped product submanifold of the from  $M = M_T \times_f M_{\theta}$  if and only if

$$A_{FZ}\varphi X - A_{FPZ}X = (1 + \cos^2\theta)(X\lambda)Z, \qquad (5.3.3)$$

for any  $X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$  and  $Z \in \Gamma(\mathcal{D}^{\theta})$ . Moreover,  $C^{\infty}$ -function  $\lambda$  on M with  $Z\lambda = 0$ , for each  $Z \in \Gamma(\mathcal{D}^{\theta})$ .

*Proof.* Let us assume that M be a warped product pointwise semi-slant submanifold of a cosymplectic manifold  $\widetilde{M}$ . Then, the Eqs (5.3.3) directly follows by Lemma 5.3.1 and setting  $\ln f = \lambda$ .

**Conversely.** Suppose that *M* be a pointwise semi-slant submanifold of a cosymplectic manifold  $\widetilde{M}$  such that the given condition (5.3.3) holds. Then from (2.2.8), (2.2.9)(i) and (2.3.39), we derive  $g(\nabla_X Y, Z) = -g(\widetilde{\nabla}_X Z, Y) = -(\varphi \widetilde{\nabla}_X Z, \varphi Y) = -g(\widetilde{\nabla}_X PZ, \varphi Y) - g(\widetilde{\nabla}_X FZ, \varphi Y)$ , for any  $X, Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$  and  $Z \in \Gamma(\mathcal{D}^{\theta})$ . We derive the following by using (2.3.43), (2.2.9)(i), (2.3.39), (2.3.3), and Theorem 2.3.9, i.e.,

$$g(\nabla_X Y, Z) = \sin(2\theta)X(\theta)g(Z, Y) - \cos^2\theta g(\widetilde{\nabla}_X Z, Y) + g(A_{FZ}X, \varphi Y) + g(\widetilde{\nabla}_X FPZ, Y).$$

Again using (2.3.3), we obtain

$$\sin^2 \theta g(\nabla_X Y, Z) = g(A_{FZ} \varphi X - A_{FPZ} X, Y).$$

Thus (5.3.3), implies that  $\sin^2 \theta_g(\nabla_X Y, Z) = 0$ . As M be a proper pointwise semi-slant submanifold which implies that  $g(\nabla_X Y, Z) = 0$ . It means that  $\nabla_X Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$  for all  $X, Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ . Hence, the distribution  $\mathcal{D} \oplus \langle \xi \rangle$  defines a totally geodesic foliations and its leaves are totally geodesic in M. Moreover, pointwise slant distribution is concerned, i.e., from (2.2.8), (2.2.9) and (2.3.39), we obtain

$$g(\nabla_Z W, X) = (\varphi \widetilde{\nabla}_Z W, \varphi X) = g(\widetilde{\nabla}_Z P W, \varphi X) + g(\widetilde{\nabla}_Z F W, \varphi X),$$

for any  $X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$  and  $Z, W \in \Gamma(\mathcal{D}^{\theta})$ . Taking account of (2.2.9)(i), (2.3.39) and (2.3.3), it follows that

$$g(\nabla_Z W, X) = -g(\widetilde{\nabla}_Z P^2 W, X) - g(\widetilde{\nabla}_Z F P W, X) - g(A_{FZ} Z, \varphi X).$$

Theorem 2.3.9 and virtue (2.3.3) imply that

$$g(\nabla_Z W, X) = -\sin 2\theta Z(\theta)g(W, X) + \cos^2 \theta g(\nabla_Z W, X)$$
$$+ g(A_{FPW}Z, X) - g(A_{FZ}Z, \varphi X)$$

The simplification gives

$$\sin^2 \theta g(\nabla_Z W, X) = g(A_{FPW}X - A_{FZ}\varphi X, Z).$$
(5.3.4)

Interchanging Z and W in (5.3.4), we get

$$\sin^2 \theta g(\nabla_W Z, X) = g(A_{FPZ} X - A_{FZ} \varphi X, W).$$
(5.3.5)

Hence, from (5.3.4)-(5.3.5) and then, using (5.3.3), we find that  $\sin^2 \theta g([Z,W],X) = 0$ . It indicate that  $[Z,W] \in \Gamma(\mathcal{D}^{\theta})$ , for proper pointwise semi-slant submanifolds. It easily to see that the pointwise slant distribution  $\mathcal{D}^{\theta}$  is integrable. Thus, we consider  $M_{\theta}$  be a leaf of  $\mathcal{D}^{\theta}$  in M and  $h^{\theta}$  is the second fundamental form of  $M_{\theta}$  in M. Then (2.3.2), (5.3.3) and (5.3.4) implies that

$$\sin^2 \theta g(h^{\theta}(Z,W),X) = -(1+\cos^2 \theta)(X\lambda)g(Z,W).$$

which means that

$$g(h^{\theta}(Z,W),X) = -(\csc^2\theta + \cot^2\theta)(X\lambda)g(Z,W).$$

Finally, the property of gradient of function gives

$$h^{\theta}(Z,W) = -(\csc^2\theta + \cot^2\theta)g(Z,W)\nabla\lambda.$$

This means that  $M_{\theta}$  is totally umbilical in M with mean curvature vector  $H^{\theta} = -(\csc^2 \theta + \cot^2 \theta) \nabla \lambda$ . Further, we will prove that the mean curvature  $H^{\theta}$  is parallel along the normal connection  $\nabla^{\theta}$  of  $M_{\theta}$  into M. For this object, we choose  $X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$  and  $Z \in$ 

 $\Gamma(\mathscr{D}^{\theta})$ , i.e.,

$$g(\nabla_{Z}^{\theta}H^{\theta},X) = -(\csc^{2}\theta + \cot^{2}\theta)g(\nabla_{Z}^{\theta}grad\lambda,Y)$$

$$= -(\csc^{2}\theta + \cot^{2}\theta)g(\nabla_{Z}grad\lambda,Y)$$

$$= -(\csc^{2}\theta + \cot^{2}\theta)g(Zg(grad\lambda,X))$$

$$+(\csc^{2}\theta + \cot^{2}\theta)g(grad\lambda,\nabla_{Z}X)$$

$$= -(\csc^{2}\theta + \cot^{2}\theta)(Z(X\lambda))$$

$$+(\csc^{2}\theta + \cot^{2}\theta)g([X,Z],grad\lambda)$$

$$-(\csc^{2}\theta + \cot^{2}\theta)g(\nabla_{X}Z,grad\lambda)$$

$$= -(\csc^{2}\theta + \cot^{2}\theta)(X(Z\lambda))$$

$$-(\csc^{2}\theta + \cot^{2}\theta)g(\nabla_{X}grad\lambda,Z).$$

From the hypothesis of the Theorem is that  $Z\lambda = 0$ , for each  $Z \in \Gamma(\mathcal{D}^{\theta})$  and  $grad\lambda$  lies in  $\mathcal{D} \oplus \langle \xi \rangle$ , thus last equation becomes

$$g(\nabla_Z^{\theta} H^{\theta}, X) = 0, \qquad (5.3.6)$$

which means that  $\nabla^{\theta} H^{\theta} \in \Gamma(\mathscr{D}^{\theta})$ . This implies that the mean curvature  $H^{\theta}$  of  $M_{\theta}$  is parallel. Hence, the condition of spherical is satisfied. Thus the Definition 2.4.3 gives that, M is locally a warped product submanifold of integral manifolds  $M_T$  and  $M_{\theta}$  of  $\mathscr{D} \oplus \langle \xi \rangle$  and  $\mathscr{D}^{\theta}$ , respectively. This complete the proof of the Theorem.

## **5.3.3** Inequalities for warped product pointwise semi-slant in a cosymplectic manifold

In in the direction of cosymplectic space form, we describe geometric properties of the mean curvature of a warped product pointwise semi-slant submanifolds and using this properties to derive a general inequality which represent an intrinsic invariant which is called Chen invariant by using Gauss equation instead of Codazzi equation. Actually, we study some geometric aspect of these submanifolds. We are required to define a frame and some important lemmas.

*Frame* 5.3.1. Let  $M = M_T \times_f M_{\theta}$  be an  $n = n_1 + n_2$ -dimensional warped product point-

wise semi-slant submanifold of 2m + 1-dimensional cosymplectic manifold  $\widetilde{M}$  with  $M_T$ of dimension  $n_1 = 2d_1 + 1$  and  $M_{\theta}$  of dimension  $n_2 = 2d_2$ , where  $M_{\theta}$  and  $M_T$  are integral manifolds of  $\mathscr{D}^{\theta}$  and  $\mathscr{D} \oplus \langle \xi \rangle$ , respectively. Then, we assume that  $\{e_1, e_2, \cdots e_{d_1}, e_{d_1+1} = \varphi e_1, \cdots e_{2d_1} = \varphi e_{d_1}, e_{2d_1+1} = \xi$ ,  $\}$  and  $\{e_{2d_1+2} = e_1^*, \cdots e_{2d_1+1+d_2} = e_{d_2}^*, e_{2d_1+d_2+2} = e_{d_2+1}^* = \sec \theta P e_1^*, \cdots e_{n_1+n_2} = e_{n_2}^* = \sec \theta P e_{d_2}^*\}$  are orthonormal frames of  $\mathscr{D} \oplus \langle \xi \rangle$  and  $\mathscr{D}^{\theta}$ , respectively. Similarly the orthonormal frames of the normal sub-bundles,  $F \mathscr{D}^{\theta}$  and  $\mu$ , respectively are,  $\{e_{n+1} = \bar{e}_1 = \csc \theta F e_1^*, \cdots e_{n+d_2} = \bar{e}_{d_2} = \csc \theta F e_1^*, e_{n+d_2+1} = \bar{e}_{d_2+1} = \csc \theta \sec \theta F P e_1^*, \cdots e_{n+2d_2} = \bar{e}_{2d_2} = \csc \theta \sec \theta F P e_{d_2}^*\}$ and  $\{e_{n+2d_2+1}, \cdots e_{2m+1}\}$ .

*Lemma* 5.3.2. On a non-trivial warped product pointwise semi-slant submanifold M in a cosymplectic manifold  $\widetilde{M}$ . Then

$$g(h(X,X),FZ) = g(h(X,X),FPZ) = 0,$$
 (5.3.7)

$$g(h(\varphi X, \varphi X), FZ) = g(h(\varphi X, \varphi X), FPZ) = 0, \qquad (5.3.8)$$

$$g(h(X,X),\tau) + g(h(\varphi X,\varphi X),\tau) = 0,$$
 (5.3.9)

for any  $X, Y \in \Gamma(TM_T), Z \in \Gamma(TM_\theta)$  and  $\tau \in \Gamma(\mu)$ .

*Proof.* From the Gauss formula (2.3.2), one derives  $g(h(X,X), FPZ) = g(\widetilde{\nabla}_X X, FPZ) = -g(\widetilde{\nabla}_X FPZ, X)$ . Then the covariant derivative of endomorphism  $\varphi$  and (2.2.9)(i), gives

$$g(h(X,X),FPZ) = g((\widetilde{\nabla}_X PZ,\varphi X) + g((\widetilde{\nabla}_X \varphi)PZ,X) + g(\widetilde{\nabla}_X P^2Z,X),$$

Taking account of cosymplectic manifold (2.2.9) and the Theorem 2.3.9, for pointwise slant submanifold, one derives

$$g(h(X,X),FPZ) = g(\widetilde{\nabla}_X \varphi X,PZ) + \sin 2\theta X(\theta)g(Z,X) - \cos^2 \theta g(\widetilde{\nabla}_X X,Z).$$
(5.3.10)

As,  $M_T$  being a totally geodesic submanifold in M invariantly, then, we get required result (5.3.7) from (5.3.10). On the other part, we interchanging Z by PZ and X by  $\varphi X$  in equation (5.3.10), we find the second result (5.3.8). For the last result, by cosymplectic manifold, we have  $\widetilde{\nabla}_X \varphi X = \varphi \widetilde{\nabla}_X X$ , these relation reduced to

$$\nabla_X \varphi X + h(\varphi X, X) = \varphi \nabla_X X + \varphi h(X, X).$$

Taking the inner product with  $\varphi \tau$  in above equation for any  $\tau \in \Gamma(\mu)$ , we obtain

$$g(h(\varphi X, X), \varphi \tau) = g(h(X, X), \tau).$$
(5.3.11)

Interchanging *X* by  $\varphi X$  in the above equation and making use of (2.2.5) and the fact that,  $\mu$  is an  $\varphi$ -invariant normal bundle of  $T^{\perp}M$ , we get

$$-g(h(X,\varphi X),\varphi \tau) = g(h(\varphi X,\varphi X),\tau).$$
(5.3.12)

Thus, (5.3.11) and (5.3.12) implies (5.3.9). It completes proof of lemma.

*Lemma* 5.3.3. Let  $\phi : M = M_T \times_f M_\theta \to \widetilde{M}$  be an isometrically immersion from warped product pointwise semi-slant  $M_T \times_f M_\theta$  into cosymplectic manifold  $\widehat{M}$  such that  $M_T$  is an invariant submanifold tangent to  $\xi$  of  $\widetilde{M}$ . Then the squared mean curvature of M is given by

$$||H||^{2} = \frac{1}{n^{2}} \sum_{r=n+1}^{2m+1} \left( h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \right)^{2}, \qquad (5.3.13)$$

where  $n_1$ ,  $n_2$ , n and 2m + 1 are dimensions of  $M_T$ ,  $M_{\theta}$ ,  $M_T \times_f M_{\theta}$  and  $\widetilde{M}$ , respectively.

Proof. Taking account that the definition mean curvature vector is defines as

$$||H||^{2} = \frac{1}{n^{2}} \sum_{r=n+1}^{2m+1} \left(h_{11}^{r} + \dots + h_{nn}^{r}\right)^{2},$$

Thus from consideration of dimension  $n = n_1 + n_2$  of  $M_T \times_f M_{\theta}$  such that  $n_1$  and  $n_2$  are dimensions of  $M_T$  and  $M_{\theta}$ , respectively, thus, it can be expanded as

$$||H||^{2} = \frac{1}{n^{2}} \sum_{r=n+1}^{2m+1} \left( h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \right)^{2}$$

Using the components of  $\mathscr{D}^{\theta}$  in right hand side of the above equation, we derive

$$\begin{pmatrix} h_{11}^r + \dots + h_{n_1n_1}^r + h_{n_1+1n_1+1}^r + \dots + h_{nn}^r \end{pmatrix}^2$$

$$= \begin{pmatrix} h_{11}^r + \dots + h_{d_1d_1}^r + h_{d_1+1d_1+1}^r + \dots + h_{2d_12d_1}^r + h_{\xi\xi}^r + h_{n_1+1n_1+1}^r + \dots + h_{nn}^r \end{pmatrix}^2.$$

Since,  $h(\xi, \xi) = 0$  and from the relation  $h_{ij}^r = g(h(e_i, e_j), e_r)$ , for  $1 \le i, j \le n$  and  $n+1 \le r \le 2m+1$ . Taking account of frame 5.3.1 for components of  $\mathcal{D}$ , the above equation take

the form

$$\begin{pmatrix} h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \end{pmatrix}^{2} \\ = \begin{cases} g(h(e_{1}, e_{1}), e_{r}) + \dots + g(h(e_{d_{1}}, e_{d_{1}}), e_{r}) + g(h(\varphi e_{1}, \varphi e_{1}), e_{r}) \\ + \dots + g(h(\varphi e_{d_{1}}, \varphi e_{d_{1}}), e_{r}) + \dots + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \end{cases}^{2} \end{cases}$$
(5.3.14)

It is well known that  $e_r$  belong to normal bundle  $T^{\perp}M$  for ever  $r \in \{n+1\cdots 2m+1\}$ , so there are two cases such that  $e_r$  belong to  $F(TM_{\theta})$  or  $\mu$ .

**Case 1:** If  $e_r \in \Gamma(F \mathscr{D}^{\theta})$ , then, using the normal components for pointwise slant distribution  $\mathscr{D}^{\theta}$  which is defined in the orthonormal frame 5.3.1. Then equation (5.3.14) can be modified as;

$$\begin{pmatrix} h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \end{pmatrix}^{2} \\ = \begin{cases} \csc \theta g(h(e_{1}, e_{1}), Fe_{1}^{*}) + \dots + \csc \theta g(h(e_{d_{1}}, e_{d_{1}}), Fe_{d_{2}}^{*}) \\ + \csc \theta \sec \theta g(h(e_{1}, e_{1}), FPe_{1}^{*}) + \dots + \csc \theta \sec \theta g(h(e_{d_{1}}, e_{d_{1}}), FPe_{d_{2}}^{*}) \\ + \csc \theta g(h(\varphi e_{1}, \varphi e_{1}), Fe_{1}^{*}) + \dots + \csc \theta g(h(\varphi e_{d_{1}}, \varphi e_{d_{1}}), Fe_{d_{2}}^{*}) \\ + \csc \theta \sec \theta g(h(\varphi e_{1}, \varphi e_{1}), FPe_{1}^{*}) + \dots + \csc \theta \sec \theta g(h(\varphi e_{d_{1}}, \varphi e_{d_{1}}), FPe_{d_{2}}^{*}) \\ + \csc \theta \sec \theta g(h(\varphi e_{1}, \varphi e_{1}), FPe_{1}^{*}) + \dots + \csc \theta \sec \theta g(h(\varphi e_{d_{1}}, \varphi e_{d_{1}}), FPe_{d_{2}}^{*}) \\ + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \\ \end{pmatrix}^{2}.$$

Finally, from virtues (5.3.7) and (5.3.8), we get

$$\left(h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r}\right)^{2} = \left(h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r}\right)^{2}.$$
 (5.3.15)

**Case 2:** If  $e_r \in \Gamma(\mu)$ , Thus from (5.3.9), the equation (5.3.14) becomes

$$\begin{pmatrix} h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \end{pmatrix}^{2} \\ = \begin{cases} g(h(e_{1}, e_{1}), e_{r}) + \dots + g(h(e_{d_{1}}, e_{d_{1}}), e_{r}) - g(h(e_{1}, e_{1}), e_{r}) - g(h(e_{d_{1}}, e_{d_{1}}), e_{r}) + \dots + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r} \end{cases}^{2},$$

which implies that

$$\left(h_{11}^{r} + \dots + h_{n_{1}n_{1}}^{r} + h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r}\right)^{2} = \left(h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r}\right)^{2}.$$
 (5.3.16)

From (5.3.15) and (5.3.16), for every normal vector field  $e_r$  belongs to the normal bundle  $T^{\perp}M$ , and by taking summing up, we can deduce that

$$\sum_{r=n+1}^{2m+1} \left( h_{11}^r + \dots + h_{n_1n_1}^r + h_{n_1+1n_1+1}^r + \dots + h_{nn}^r \right)^2 = \sum_{r=n+1}^{2m+1} \left( h_{n_1+1n_1+1}^r + \dots + h_{nn}^r \right)^2.$$

Therefore, the above relation proves our assertion. It completes proof of lemma.

**Theorem 5.3.5.** Let  $\phi : M = M_T \times_f M_\theta \to \widetilde{M}$  be an isometric immersion from an ndimensional warped product pointwise semi-slant  $M_T \times_f M_\theta$  into 2m + 1-dimensional cosymplectic manifold  $\widetilde{M}$ . Then

(i) The squared norm of the second fundamental form of M is given by

$$||h||^{2} \ge 2\left(\widetilde{\tau}(TM) - \widetilde{\tau}(TM_{T}) - \widetilde{\tau}(TM_{\theta}) - \frac{n_{2}\Delta f}{f}\right), \qquad (5.3.17)$$

where  $n_2$  is the dimension of pointwise slant submanifold  $M_{\theta}$  into  $\widetilde{M}$ .

(ii) The equality sign holds in (5.3.17) if and only if  $M_T$  is totally geodesic and  $M_{\theta}$  is totally umbilical submanifolds in  $\widetilde{M}$ , respectively. Moreover, M is minimal submanifold of  $\widetilde{M}$ .

*Proof.* We skip the proof of this theorem as it is similar to the proof of the Theorem 5.2.9 for warped product pointwise semi-slant submanifolds in a Kaehler manifold.

The following corollary is generalized by using the Remark 2.3.5 and the Theorem 5.3.5 for contact CR-warped product submanifold such as:

**Corollary 5.3.1.** Let  $\phi : M = M_T \times_f M_\perp \to \widetilde{M}$  be an isometrically immersion of an *n*dimensional contact CR-warped product submanifold M into a 2m+1-dimensional cosymplectic manifold  $\widetilde{M}$ . Then

(i) The squared norm of the second fundamental form of M is given by

$$||h||^{2} \ge 2\left(\widetilde{\rho}(TM) - \widetilde{\rho}(TM_{T}) - \widetilde{\rho}(TM_{\perp}) - \frac{n_{2}\Delta f}{f}\right), \qquad (5.3.18)$$

where  $n_2$  is the dimension of anti-invariant submanifold  $M_{\perp}$ .

(ii) The equality sign holds in (5.3.18) if and only if  $M_T$  is totally geodesic and  $M_{\perp}$  is totally umbilical submanifolds in  $\widetilde{M}$ . Moreover, M is minimal submanifold of  $\widetilde{M}$ .

#### 5.3.4 Applications of derived inequality to cosymplectic space forms

**Theorem 5.3.6.** Assume that  $\phi: M = M_T \times_f M_\theta \to \widetilde{M}(c)$  be an isometrically immersion of a warped product pointwise semi-slant submanifold into cosymplectic space form  $\widetilde{M}(c)$ . Then

(i) The second fundamental form is defined as

$$||h||^{2} \ge 2n_{2} \left(\frac{c}{4}(n_{1}-1) - \frac{\Delta f}{f}\right),$$
  
or,  

$$||h||^{2} \ge 2n_{2} \left(||\nabla \ln f||^{2} + \left(\frac{c}{4}(n_{1}-1) - \Delta(\ln f)\right),$$
(5.3.19)

where  $n_1 = \dim M_T$  and  $n_2 = \dim M_{\theta}$ .

(ii) The equality sign holds in (5.3.19) if and only if  $M_T$  is totally geodesic and  $M_{\theta}$  is totally umbilical submanifolds of  $\widetilde{M}(c)$ , respectively. Moreover, M is minimal submanifold in  $\widehat{M}(c)$ .

*Proof.* A cosymplectic manifold is said to be a cosymplectic space form with constant  $\varphi$ -sectional curvature *c*. Then the Riemannian curvature tensor  $\widetilde{R}$  for cosymplectic space form is given by

$$\widetilde{R}(X,Y,Z,W) = \frac{c}{4} \left( g(Y,Z)g(X,W) - g(X,Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + \eta(X)\eta(Z)g(Y,W) + \eta(W)\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\eta(W) + g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) + 2g(X,\varphi Y)g(\varphi Z,W) \right),$$
(5.3.20)

for any  $X, Y, Z, W \in \Gamma(T\widetilde{M})$ . Putting  $X = W = e_i$ , and  $Y = Z = e_j$  in (5.3.20), we derive

$$\widetilde{R}(e_{i},e_{j},e_{j},e_{i}) = \frac{c}{4} \left( g(e_{i},e_{i})g(e_{j},e_{j}) - g(e_{i},e_{j})g(e_{j},e_{i}) - \eta(e_{j})\eta(e_{j})g(e_{i},e_{i}) \right. \\ \left. + \eta(e_{i})\eta(e_{j})g(e_{i},e_{j}) + \eta(e_{i}) \left( \eta(e_{j})g(e_{i},e_{j}) - \eta(e_{i})g(e_{j},e_{j}) \right) \right. \\ \left. + g(\varphi e_{j},e_{j})g(\varphi e_{i},e_{i}) - g(\varphi e_{i},e_{j})g(\varphi e_{j},e_{i}) \right.$$

$$\left. + 2g^{2}(e_{i},\varphi e_{j}) \right).$$
(5.3.21)

Taking summing up in (5.3.21) over the orthonormal vector fields of  $TM_T$ , we obtain

$$2\widetilde{\tau}(TM_T) = \frac{c}{4} \left[ n_1(n_1 - 1) - 2(n_1 - 1) + 3||P||^2 \right].$$

Since,  $\xi(p)$  is tangent to  $n_1$ -dimensional invariant submanifold  $M_T$ , then, we find  $||P||^2 = n_1 - 1$ . Thus,

$$2\tilde{\tau}(TM_T) = \frac{c}{4}(n_1^2 - 1).$$
 (5.3.22)

Again taking summing up in (5.3.21) over the orthonormal frame corresponding to M, i.e.,

$$2\widetilde{\tau}(TM) = \frac{c}{4} \Big( n(n-1) - 2(n-1) + 3\sum_{i,j=1}^{n} g^2(\varphi e_i, e_j) \Big).$$
(5.3.23)

Let M be a proper pointwise semi-slant submanifold of cosymplectic space form  $\widetilde{M}(c)$ . Thus, we define the following frame, i.e.,

$$e_{1}, e_{2} = \varphi e_{1}, \cdots, e_{2d_{1}-1}, e_{2d_{1}} = \varphi e_{2d_{1}-1},$$

$$e_{2d_{1}+1}, e_{2d_{1}+2} = \sec \theta P e_{2d_{1}+1}, \cdots$$

$$e_{2d_{1}-1}, e_{2d_{1}} = \sec \theta P e_{2d_{1}-1}, \cdots,$$

$$\cdots e_{2d_{1}+2d_{2}-1}, e_{2d_{1}+2d_{2}} = \sec \theta P e_{2d_{1}-1},$$

$$e_{2d_{1}+2d_{2}}, e_{2d_{1}+2d_{2}+1} = \xi.$$

With the help of the above orthonormal frame, we easily derive that

$$g^{2}(\varphi e_{i}, e_{i+1}) = 1, \text{ for } i \in \{1, \dots 2d_{1} - 1\}$$
$$= \cos^{2} \theta \text{ for } i \in \{2d_{1} + 1, \dots 2d_{1} + 2d_{2} - 1\}.$$

It follows that

$$\sum_{i,j=1}^{n} g^{2}(\varphi e_{i}, e_{j}) = 2(d_{1} + d_{2}\cos^{2}\theta).$$
 (5.3.24)

From (5.3.23) and (5.3.24), it implies that

$$2\widetilde{\tau}(TM) = \frac{c}{4}n(n-1) + \frac{c}{4}\left(6(d_1 + d_2\cos^2\theta) - 2(n-1)\right).$$
 (5.3.25)

Similarly, for pointwise slant submanifold  $M_{\theta}$ , we obtain  $||P||^2 = n_2 \cos^2 \theta$ , then

$$2\tilde{\tau}(TM_{\theta}) = \frac{c}{4} \Big( n_2(n_2 - 1) + 3n_2 \cos^2 \theta \Big).$$
 (5.3.26)

Hence, substituting the values (5.3.26), (5.3.25), and (5.3.22) in Theorem 5.3.5, we derive the required result (5.3.19), and the equality case follow as usual of the Theorem 5.3.5(ii). This completes the proof of the theorem.

**Remark 5.3.1.** By assuming that  $\theta = \frac{\pi}{2}$  with globally constant pointwise slant function  $\theta$  in Theorem 5.3.9. Then, Theorem 5.3.9 generalizes to contact CR-warped products in cosymplectic space forms which obtained by Uddin & Alqahtani (2016),

**Corollary 5.3.2.** Assume that  $\phi: M = M_T \times_f M_\perp \to \widetilde{M}(c)$  be an isometrically immersion of a warped product pointwise semi-slant submanifold into cosymplectic space form  $\widetilde{M}(c)$ . Then

(i) The second fundamental form of M is defined as

$$||h||^2 \ge 2n_2 \left(\frac{c}{4}(n_1 - 1) - \frac{\Delta f}{f}\right),$$
 (5.3.27)

(ii) The equality sign holds in (5.3.27) if and only if  $M_T$  is totally geodesic and  $M_{\perp}$  is totally umbilical submanifolds of  $\widetilde{M}(c)$ , respectively. Moreover, M is minimal submanifold in  $\widehat{M}(c)$ .

#### 5.3.5 Applications as a compact orientable Riemannian manifold

For simplicity of presentation, we always consider that M is a compact orientable Riemannian manifold without boundary in this subsection. We now describe briefly the methods to prove the triviality of warped product pointwise semi-slant submanifolds in both cases inequality and equality. **Theorem 5.3.7.** Let  $\phi$  is a  $\mathscr{D}^{\theta}$ -minimal isometric immersion of a warped product pointwise semi-slant submanifold  $M_T \times_f M_{\theta}$  into cosymplectic manifold  $\widetilde{M}$ . If  $M_{\theta}$  is totally umbilical in  $\widetilde{M}$ , then  $\phi$  is  $M_{\theta}$ -totally geodesic.

*Proof.* The proof is similar as the proof of the Theorem 5.2.12.

**Theorem 5.3.8.** On a compact orientate warped product pointwise semi-slant submanifold  $M = M_T \times_f M_{\theta}$  in a cosymplectic space form  $\widetilde{M}(c)$  with the following inequality holds, i.e.,

$$||h||^2 \ge \frac{c}{2}n_2(n_1 - 1) \tag{5.3.28}$$

where  $n_1$  and  $n_2$  are dimensions of  $M_T$  and  $M_{\theta}$ , respectively. Then M is a Riemannian product manifold.

*Proof.* Let us consider that, the inequality holds in Theorem 5.3.6 and the fact that  $\frac{\Delta f}{f} = \Delta(\ln f) - ||\nabla \ln f||^2$ , we obtain

$$\left(\frac{c}{2}n_2(n_1-1)+2n_2||\nabla \ln f||^2-||h||^2\right) \le 2n_2\Delta(\ln f).$$

From the integration theory on manifolds, i.e., compact orientate Riemannian manifold without boundary on M, we derive

$$\int_{M} \left( \frac{c}{2} n_2(n_1 - 1) + 2n_2 ||\nabla \ln f||^2 - ||h||^2 \right) dV \le \int_{M} \Delta(\ln f) dV = 0$$
 (5.3.29)

Assume that the inequality (5.3.28) holds, then, from (5.3.29), we find that

$$\int_M (||\nabla \ln f||^2) dV \le 0.$$

Since, integration always be positive for positive functions. Hence, we derive  $||\nabla \ln f||^2 \le 0$ , but  $||\nabla \ln f||^2 \ge 0$ , which implies that  $\nabla \ln f = 0$ , i.e., f is a constant function on M. Thus, M becomes simply a Riemannian product manifold of invariant and pointwise slant submanifolds. The proof is completed.

**Theorem 5.3.9.** Let  $\phi : M = M_T \times_f M_\theta \to \widetilde{M}(c)$  be an isometric immersion of a compact orientable proper warped product pointwise semi-slant submanifold in a cosymplectic

space form  $\widetilde{M}(c)$ . Then M is simply a Riemannian product if and only if satisfies

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = \frac{c}{4} n_2(n_1 - 1), \qquad (5.3.30)$$

where  $h_{\mu}$  is a component of h in  $\Gamma(\mu)$ . Further,  $n_1$  and  $n_2$  are dimensions of  $M_T$  and  $M_{\theta}$ , respectively.

*Proof.* Let us consider the equality case holds in (5.3.19). Then, we estimate

$$||h(\mathscr{D},\mathscr{D})||^{2} + ||h(\mathscr{D}^{\theta},\mathscr{D}^{\theta})||^{2} + 2||h(\mathscr{D},\mathscr{D}^{\theta})||^{2} = \frac{c}{2}n_{2}(n_{1}-1) + 2n_{2}(||\nabla \ln f||^{2} - \Delta(\ln f)). \quad (5.3.31)$$

As by hypothesis of Theorem 5.3.6,  $M_T$  is totally geodesic in  $\widetilde{M}(c)$ , i.e.,  $h(\mathscr{D}, \mathscr{D}) = 0$ and  $M_{\theta}$  is totally umbilical in  $\widetilde{M}(c)$ , then h(Z,W) = g(Z,W)H. But, M is minimal submanifold in  $\widetilde{M}$ . Hence, H = 0 and immediately follows the Theorem 5.3.7, it implies that h(Z,W) = 0. This means that  $||h(\mathscr{D}^{\theta}, \mathscr{D}^{\theta})||^2 = 0$ , thus (5.3.31), becomes

$$2||h(\mathscr{D},\mathscr{D}^{\theta})||^{2} + 2n_{2}\Delta(\ln f) = \frac{c}{2}n_{2}(n_{1}-1) + 2n_{2}||\nabla \ln f||^{2}.$$
 (5.3.32)

Let  $X = e_i$  and  $Z = e_j$  for  $1 \le i \le n_1$  and  $1 \le j \le n_2$ , respectively and using the definition from (2.3.9). Then,

$$h(e_i, e_j) = \sum_{r=1}^{n_2} g(h(e_i, e_j), Fe_r^*) Fe_r^* + \sum_{l=1}^{2m+1} g(h(e_i, e_j), \tau e_l) \tau e_l.$$

for  $\tau \in \Gamma(\mu)$ . Taking summing up over the vector fields on  $M_T$  and  $M_{\theta}$ , then using adapted frame 5.3.1 for pointwise semi-slant submanifolds, we derive

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(h(e_i, e_j), h(e_i, e_j)) = \csc^2 \theta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(e_i, e_j^*), Fe_k^*)^2 + \csc^2 \theta \sec^2 \theta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(e_i, Pe_j^*), Fe_k^*)^2 + \csc^2 \theta \sec^2 \theta \sum_{i=1}^{d_1} \sum_{j,k=1}^{d_2} g(h(\varphi e_i, e_j^*), FPe_k^*)^2$$

$$+ \csc^{2} \theta \sec^{2} \theta \sum_{i=1}^{d_{1}} \sum_{j,k=1}^{d_{2}} g(h(\varphi e_{i}, e_{j}^{*}), FPe_{k}^{*})^{2}$$

$$+ \csc^{2} \theta \sec^{4} \theta \sum_{i=1}^{d_{1}} \sum_{j,k=1}^{d_{2}} g(h(\varphi e_{i}, Pe_{j}^{*}), FPe_{k}^{*})^{2}.$$

$$+ \csc^{2} \theta \sec^{2} \theta \sum_{i=1}^{d_{1}} \sum_{j,k=1}^{d_{2}} g(h(\varphi e_{i}, Pe_{j}^{*}), Fe_{k}^{*})^{2}$$

$$+ \csc^{2} \theta \sec^{4} \theta \sum_{i=1}^{d_{2}} g(h(\varphi e_{i}, e_{j}^{*}), Fe_{k}^{*})^{2}$$

$$+ \csc^{2} \theta \sec^{4} \theta \sum_{i=1}^{d_{1}} \sum_{j,k=1}^{d_{2}} g(h(e_{i}, Pe_{j}^{*}), FPe_{r}^{*})^{2}$$

$$+ \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} \sum_{r=n+n_{2}+1}^{2m+1} g(h(e_{i}, e_{j}), e_{r})^{2}.$$

In view of Lemma 5.3.1, it can be easily seen that

$$||h(\mathscr{D}, \mathscr{D}^{\theta})||^{2} = n_{2}(\csc^{2}\theta + \cot^{2}\theta)||\nabla \ln f||^{2} + \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} ||h_{\mu}(e_{i}, e_{j})||^{2}.$$
 (5.3.33)

Then (5.3.33) and (5.3.32) implies that

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 - \frac{c}{4} n_2(n_1 - 1) = n_2 \Delta(\ln f) + n_2 \cot^2 \theta ||\nabla \ln f||^2.$$
(5.3.34)

Now taking the integration over the volume element dV of compact orientate warped product pointwise semi-slant *M*. Then (2.3.16) gives

$$\int_{M} \left( \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} ||h_{\mu}(e_{i}, e_{j})||^{2} \right) dV = \int_{M} \left( \frac{c}{4} n_{2}(n_{1} - 1) \right) dV + 2n_{2} \int_{M} \left( \cot^{2} \theta ||\nabla \ln f||^{2} \right) dV.$$
(5.3.35)

for the first case, if (5.3.30) is satisfied, then from (5.3.35) implies that f is constant function on proper warped product pointwise semi-slant submanifold M. It means that M is a Riemannian product of invariant and pointwise slant submanifolds  $M_T$  and  $M_{\theta}$ , respectively.

*Conversely*, suppose that *M* is simply a Riemannian product, then warping function *f* must be constant, i.e.,  $\nabla \ln f = 0$ . Thus from (5.3.35) implies the equality (5.3.30). This completes proof of the theorem.

**Remark 5.3.2.** If we consider  $\theta = \frac{\pi}{2}$  in Theorem 5.3.9 with globally constant function  $\theta$ . Then, Theorem 5.3.9 generalizes the result for contact CR-warped products in cosymplectic space forms.

**Corollary 5.3.3.** Assume that  $M = M_T \times_f M_{\theta}$  be a warped product pointwise semi-slant submanifold in a cosymplectic space form  $\widetilde{M}(c)$ . Let  $M_T$  be a compact invariant submanifold and  $\lambda$  be non-zero eigenvalue of the Laplacian on  $M_T$ . Then

$$\int_{M_T} ||h||^2 \mathrm{d}V_T \ge \int_{M_T} \left(\frac{n_2 c}{2} n_2 (n_1 - 1)\right) \mathrm{d}V_T + 2n_2 \lambda \int_{M_T} (\ln f)^2 \mathrm{d}V_T.$$
(5.3.36)

(ii) In the warped product pointwise semi-slant submanifold both  $M_T$  and  $M_{\theta}$  are totally geodesic.

Proof. Thus, using the minimum principle property, we obtain

(*i*)  $\Delta \ln f = \lambda \ln f$ .

$$\int_{M_T} ||\nabla \ln f||^2 dV_T \ge \lambda \int_{M_T} (\ln f)^2 dV_T.$$
 (5.3.37)

From (5.3.19) and (5.3.37) we get the required result (5.3.36). This completes proof of the corollary.

**Corollary 5.3.4.** Let  $\ln f$  be a harmonic function on  $M_T$ . Then there does not exist any warped product pointwise semi-slant  $M_T \times_f M_{\theta}$  into cosymplectic space form  $\widetilde{M}(c)$  with  $c \leq 0$ .

*Proof.* Let us consider that there exists a warped product pointwise semi-slant submanifold  $M_T \times_f M_{\theta}$  in cosymplectic space form  $\widetilde{M}(c)$  with  $\ln f$  is a harmonic function  $M_T$ . Thus, Theorem 5.3.6 gives c > 0. This completes the proof of the corollary.

Another straightforward characterization of the Theorem 5.3.5.

**Corollary 5.3.5.** There does not exist a warped product pointwise semi-slant submanifold  $M_T \times_f M_{\theta}$  into cosymplectic space form  $\widetilde{M}(c)$  with  $c \leq 0$  such that  $\ln f$  be a positive eigenfunction of the Laplacian on  $M_T$  corresponding to an eigenvalue  $\lambda \geq 0$ .

#### 5.3.6 Consequences to Hessian of warping functions

Throughout study of this subsection, we construct some applications in terms of Hessian of positive differentiable warping functions. We derive some necessary and sufficient conditions under which a warped product pointwise semi-slant submanifold into a cosymplectic space form becomes a Riemannian product manifold with non-compactness.

**Theorem 5.3.10.** Let  $\phi : M = M_T \times_f M_{\theta}$  be an isometric immersion of a warped product pointwise semi-slant into a cosymplectic space form  $\widetilde{M}(c)$ . If the following inequality holds, i.e.,

$$||h||^{2} \ge 2n_{2} \left\{ \frac{c}{4}(n_{1}-1) + \sum_{i=1}^{d_{1}} \left( (H^{\ln f}(e_{i},e_{i}) + H^{\ln f}(\varphi e_{i},\varphi e_{i})) \right\},$$
(5.3.38)

where  $H^{\ln f}$  is Hessian of warping function  $\ln f$ . Then M is trivial warped product pointwise semi-slant submanifold of  $M_T$  and  $M_{\theta}$ .

*Proof.* Thus from (2.3.14), we have

$$\begin{split} \Delta(\ln f) &= -\sum_{i=1}^{n} g(\nabla_{e_i} grad \ln f, e_i) \\ &= -\sum_{i=1}^{2d_1} g(\nabla_{e_i} grad \ln f, e_i) + \sum_{j=1}^{2d_2} g(\nabla_{e_j} grad \ln f, e_j) + g(\nabla_{\xi} grad \ln f, \xi). \end{split}$$

The warped product submanifold of cosymplectic manifold is defines such that  $g(\nabla_{\xi} grad \ln f, \xi) = 0$ , by follows the fact  $(\xi \ln f) = 0$ . Thus, we arrive at

$$\begin{split} \Delta(\ln f) &= -\sum_{i=1}^{d_1} g(\nabla_{e_i} grad \ln f, e_i) - \sum_{i=1}^{d_1} g(\nabla_{\varphi e_i} grad \ln f, \varphi e_i) \\ &- \sum_{j=1}^{d_2} g(\nabla_{e_j} grad \ln f, e_j) \\ &- \sec^2 \theta \sum_{i=1}^{d_2} g(\nabla_{Pe_j} grad \ln f, Pe_j). \end{split}$$

Due to the fact that  $\nabla$  is Levi-Civitas connection on *M* and taking account (2.3.13), we

derive

$$\begin{split} \Delta(\ln f) &= -\sum_{i=1}^{d_1} \left[ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \right] \\ &- \sum_{j=1}^{d_2} \left[ e_j g(\operatorname{grad} \ln f, e_j) - g(\nabla_{e_j} e_j, \operatorname{grad} \ln f) \right]. \\ &- \operatorname{sec}^2 \theta \sum_{j=1}^{d_2} \left[ Pe_j g(\operatorname{grad} \ln f, Pe_j) - g(\nabla_{Pe_j} Pe_j, \operatorname{graid} \ln f) \right]. \end{split}$$

We arrive at with the help of gradient property warped function (2.3.12), thus

$$\begin{split} \Delta(\ln f) &= -\sum_{i=1}^{d_1} \left[ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \right] \\ &- \sum_{j=1}^{d_2} \left[ e_j(e_j \ln f) - (\nabla_{e_j} e_j \ln f) \right] \\ &- \sec^2 \theta \sum_{j=1}^{d_2} \left[ Pe_j(Pe_j \ln f) - (\nabla_{Pe_j} Pe_j \ln f) \right] \\ &= -\sum_{i=1}^{d_1} \left[ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \right] \\ &- \sum_{j=1}^{d_2} \left[ e_j \left( \frac{g(gradf, e_j)}{f} \right) - \frac{1}{f} g(\nabla_{e_j} e_j, gradf) \right] \\ &- \sec^2 \theta \sum_{j=1}^{d_2} \left[ Pe_j \left( \frac{g(gradf, Pe_j)}{f} \right) - \frac{1}{f} g(gradf, \nabla_{Pe_j} Pe_j) \right]. \end{split}$$

By hypothesis of warped product pointwise semi-slant submanifold,  $M_T$  defines as totally geodesic in M. It implies that  $grad f \in \Gamma(TM_T)$ , and Lemma 2.4.1(ii), we obtain

$$\begin{aligned} \Delta(\ln f) &= -\sum_{i=1}^{d_1} \left[ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \right] \\ &- \sum_{j=1}^{d_2} \left[ g(e_j, e_j) ||\nabla \ln f||^2 + \sec^2 \theta g(Pe_j, Pe_j) ||\nabla \ln f||^2 \right]. \end{aligned}$$

Using (2.3.58) in the last term of right hand side, finally, we obtain

$$\Delta(\ln f) = -\sum_{i=1}^{d_1} \left[ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \right] - 2n_2 ||\nabla \ln f||^2.$$
(5.3.39)

It follows from (5.3.19) and (5.3.39) that

$$||h||^{2} \ge 2n_{2} \left( \frac{c}{4} (n_{1} - 1) + (n_{2} + 2) ||\nabla \ln f||^{2} + \sum_{i=1}^{d_{1}} \left[ H^{\ln f}(e_{i}, e_{i}) + H^{\ln f}(\varphi e_{i}, \varphi e_{i}) \right] \right).$$
(5.3.40)

If the inequality (5.3.38) holds, then,(5.3.40) implies that  $||\nabla \ln f||^2 \leq 0$ , which is impossible by definition of norm of a positive function. Thus it can be concluded that  $grad \ln f = 0$ , then it means that f is a constant function on M. Hence M becomes a trivial warped product pointwise semi-slant submanifold. This completes the proof of the theorem.

**Theorem 5.3.11.** Let  $\phi : M = M_T \times_f M_{\theta}$  be an isometric immersion of a warped product pointwise semi-slant into cosymplectic space form  $\widetilde{M}(c)$  such that a slant function  $\theta \neq \operatorname{arc} \operatorname{cot} \sqrt{(n_2)}$ . Then M is a trivial warped product if and only if provided the condition

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = n_2 \frac{c}{4} (n_1 - 1) + n_2 \sum_{i=1}^{d_1} \left[ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \right].$$
(5.3.41)

*Proof.* From the relations (5.3.34) and (5.3.39) with considered equality of inequality 5.3.19, we derive

$$\frac{1}{n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = \sum_{i=1}^{d_1} \left[ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \right] \\ + 2 \left( n_2 - \cot^2 \theta \right) ||\nabla \ln f||^2 + \frac{c}{4} (n_1 - 1).$$
(5.3.42)

By the hypothesis of the theorem, if (5.3.41) holds, then (5.3.42) indicates that  $||\nabla \ln f||^2 = 0$ , which implies that f is a constant function on M, i.e., M is simply a Riemannian product of  $M_T$  and  $M_{\theta}$  (M is trivial). Similarly, the converse part can be proved by using (5.3.42) and (5.3.41). This completes the proof of theorem.

**Theorem 5.3.12.** There does not exist a warped product pointwise semi-slant  $M_T \times_f M_{\theta}$ into a cosymplectic space form  $\widetilde{M}(c)$  with  $c \leq 0$  such that  $M_T$  is a compact invariant submanifold tangent to  $\xi$  and  $M_{\theta}$  is a pointwise slant submanifold of  $\widetilde{M}(c)$ .

*Proof.* Assume there exists a warped product pointwise semi-slant  $M_T \times_f M_{\theta}$  in a cosymplectic space form e  $\widetilde{M}(c)$  with  $c \leq 0$  such that  $M_T$  is compact. Then the function  $\ln f$  has an absolute maximum at some point  $p \in M_T$ . At this critical point, the Hessian  $H^{\ln f}$  is non-positive definite. Thus (5.3.38) leads to a contradiction.

## 5.3.7 Applications to kinetic energy functions, Hamiltonian and Euler-Lagrangian equations

In view of connected, compact Riemannian manifolds with nonempty boundary, i.e.,  $\partial M \neq \emptyset$ , and applications of Green lemma, we apply these to warping function for deriving geometric necessary and sufficient conditions of warped product pointwise semi-slant submanifolds of cosymplectic space form to be Riemannian products in terms of kinetic energy, Hamiltonian and Euler Lagrange equation of warping function.

**Theorem 5.3.13.** Assume that  $\phi : M = M_T \times_f M_{\theta}$  is an isometric immersion of a warped product pointwise semi-slant into cosymplectic space form  $\widetilde{M}(c)$ . A connected, compact warped product  $M_T \times_f M_{\theta}$  is simply Riemannian product of  $M_T$  and  $M_{\theta}$  if and only if the kinetic energy satisfies

$$E(\ln f) = \frac{1}{4n_2} \tan^2 \theta \left\{ \int_M \left( \frac{n_2 c}{4} (n_1 - 1) - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_\mu(e_i, e_j)||^2 \right) dV - 8n_2 \int_M \left( \csc^2 \theta \cot \theta \left( \frac{d\theta}{dV} \right) E(\ln f) \right) dV \right\},$$
(5.3.43)

where  $E(\ln f)$  represents the kinetic energy of the warping function  $\ln f$  and dV is volume element on M.

*Proof.* The above theorem can be easily prove by using (5.3.34), (2.3.17), and the same method we have used in the Theorem 5.2.17.

Similarly, we derive some characterization in terms of Hamiltonian.

**Theorem 5.3.14.** An isometric immersion  $\phi : M = M_T \times_f M_\theta$  of a non-trivial connected, compact warped product pointwise semi-slant submanifold  $M_T \times_f M_\theta$  into a cosymplectic space form  $\widetilde{M}(c)$  is a trivial warped product if and only if it satisfies

$$H(d(\ln f), p) = \frac{1}{4n_2} \tan^2 \theta \left( \frac{c}{4} n_2(n_1 - 1) - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 \right),$$
(5.3.44)

where  $H(d(\ln f), p)$  is the Hamiltonian of warping function  $\ln f$  at point  $p \in M$ .

*Proof.* The proof is follows to the Theorem 5.2.18.

Another interesting application of Theorem 5.3.6 is the following.

**Theorem 5.3.15.** Assume that  $\phi: M = M_T \times_f M_{\theta}$  be an isometric immersion of a compact warped product pointwise semi-slant into a cosymplectic space form  $\widetilde{M}(c)$ . Suppose that warping function satisfies the Euler-Lagrange equation. Then, the necessary condition of M to be a trivial warped product is

$$||h||^2 \ge \frac{c}{2}n_2(n_1-1).$$
 (5.3.45)

*Proof.* The above theorem can prove easily with taking the help of the Theorem 5.2.19 and Eqs (5.3.19).

**Theorem 5.3.16.** Assume that  $\phi : M = M_T \times_f M_{\theta}$  be an isometric immersion of a compact warped product pointwise semi-slant into a cosymplectic space form  $\widetilde{M}(c)$ . Suppose that the warping function satisfies the Euler-Lagrange equation. Then the necessary and sufficient condition of the warped product  $M_T \times_f M_{\theta}$  to be trivial is

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = \frac{c}{4} n_2(n_1 - 1).$$
(5.3.46)

*Proof.* Similarly, the proof is similar as the proof Theorem 5.2.20. This completes the proof of the theorem.

### 5.4 GEOMETRY OF WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS OF SASAKIAN MANIFOLDS

Following the previous sections of this Chapter, we prove the existence of warped product pointwise semi-slant submanifolds by it characterizations in terms of Weingarten operator and then, we obtain a geometric inequality for the second fundamental form in terms of intrinsic invariants by using Gauss equation. Moreover, we give some applications of these inequality for Sasakian space forms and compact Riemannian submanifolds as well. As results, we provide some conditions which are necessary and sufficient to prove the non-existence of compact warped product pointwise semi-slant submanifolds in Sasakian space forms in terms of Hamiltonian, Hessian of warped functions. The same results, we generalizes for contact CR-warped products into Sasakian space forms.

#### 5.4.1 Warped product pointwise semi-slant submanifold in Sasakian manifold

If the two factors of warped product submanifold are invariant and pointwise slant submanifolds, then it is called warped product pointwise semi-slant submanifold. Therefore, there are two type warped product pointwise semi-slant submanifolds in Sasakian manifold

(a) 
$$M_{\theta} \times_f M_T$$
 (b)  $M_T \times_f M_{\theta}$ .

In such cases the following results proven by Park (2014) are;

**Theorem 5.4.1.** There do not exist proper warped product pointwise semi-slant submanifolds  $M = M_{\theta} \times_f M_T$  in a Sasakian manifold M such that  $M_{\theta}$  is a proper pointwise slant submanifold tangent to  $\xi$  and  $M_T$  is an invariant submanifold of M.

*Lemma* 5.4.1. Let  $M = M_T \times_f M_\theta$  be a warped product pointwise semi-slant submanifold of a Sasakian manifold  $\widetilde{M}$  such that  $\xi$  is tangent to  $M_T$ . Then

$$g(h(X,Z),FPW) = -(\varphi X \ln f)g(Z,PW) - \cos^2 \theta(X \ln f)g(Z,W)$$
$$+ \eta(X)g(W,PZ),$$
(5.4.1)

$$g(h(Z, \varphi X), FW) = (X \ln f)g(Z, W) - (\varphi X \ln f)g(Z, PW),$$
 (5.4.2)

$$g(h(Z,X), FW = -\sin^2 \theta \eta(X)g(Z,W) - (\varphi X \ln f)g(Z,W)$$
$$+ (X \ln f)g(Z,PW), \qquad (5.4.3)$$

for any  $X \in \Gamma(TM_T)$  and  $Z, W \in \Gamma X(TM_{\theta})$ .

Now we give the following main characterization theorem of warped product pointwise semi-slant submanifolds:

Theorem 5.4.2. Let M be a proper pointwise semi-slant submanifold of a Sasakian man-

ifold  $\widetilde{M}$ . Then M is locally a warped product submanifold if and only if

$$A_{FZ}\varphi X - A_{FPZ}X = (1 + \cos^2\theta)(X\lambda)Z + \eta(X)PZ, \qquad (5.4.4)$$

for each  $X \in \Gamma(TM_T)$  and a  $C^{\infty}$ -function  $\lambda$  on M with  $Z\lambda = 0$  for each  $Z \in \Gamma(\mathcal{D}^{\theta})$ .

*Proof.* First, we consider that M be a warped product pointwise semi-slant submanifold of a Sasakian manifold  $\widetilde{M}$ . Then the first part is directly proved by (5.4.1) and (5.4.3) of Lemma 5.4.1.

The converse part can be prove as similar of the Theorem 5.3.4 by using structure equation of Sasakian manifold (2.2.10)(a)-(b). This complete proof of the theorem.

# 5.4.2 Chen type inequality for warped product pointwise semi-slant in Sasakian manifolds

In this section, we derive some geometric properties of the mean curvature for warped product pointwise semi-slant submanifolds, and applying these result to construct a general inequality by means of Gauss equation. For this, we shall use the same orthonormal Frame 5.3.1 with replacing  $d_1 = \alpha$  and  $d_2 = \beta$ , and give some preparatory propositions.

**Proposition 5.4.1.** On a non-trivial warped product pointwise semi-slant submanifold M in a Sasakian manifold  $\widetilde{M}$ . Then

$$g(h(X,X),FZ) = g(h(X,X),FPZ) = 0,$$
(5.4.5)

$$g(h(\varphi X, \varphi X), FZ) = g(h(\varphi X, \varphi X), FPZ) = 0, \qquad (5.4.6)$$

$$g(h(X,X),\tau) = -g(h(\varphi X,\varphi X),\tau), \qquad (5.4.7)$$

for any  $X \in \Gamma(TM_T)$ ,  $Z \in \Gamma(TM_{\theta})$  and  $\tau \in \Gamma(\mu)$ .

*Proof.* We left the proof of the above proposition for verification to reader due to similarity of Lemma 5.3.2 as base manifold is a Sasakian manifold instead of cosymplectic manifold. This completes proof of proposition.

**Proposition 5.4.2.** Let  $\phi : M = M_T \times_f M_\theta \to \widetilde{M}$  be an isometrically immersion of a warped product pointwise semi-slant  $M_T \times_f M_\theta$  into a Sasakian manifold  $\widetilde{M}$  such that  $M_T$  is invariant submanifold tangent to  $\xi$  of  $\widetilde{M}$  and  $M_\theta$  is a pointwise slant submanifold

of M. Then the squared norm of mean curvature of M is given by

$$||H||^{2} = \frac{1}{n^{2}} \sum_{r=n+1}^{2m+1} (h_{n_{1}+1n_{1}+1}^{r} + \dots + h_{nn}^{r})^{2},$$

where *H* is the mean curvature vector and  $n_1$ ,  $n_2$ , *n* and 2m + 1 are dimensions of  $M_T$ ,  $M_{\theta}$ ,  $M_T \times_f M_{\theta}$  and  $\widetilde{M}$  respectively.

*Proof.* The above proposition can be proved in similar as of Lemma 5.2.5.

**Theorem 5.4.3.** Let  $\phi : M = M_T \times_f M_\theta \to \widetilde{M}$  be an isometrically immersion of an ndimensional warped product pointwise semi-slant  $M_T \times_f M_\theta$  into an 2m + 1-dimensional Sasakian manifold  $\widetilde{M}$ . Then

(i) The squared norm of the second fundamental form of M is given by

$$||h||^{2} \ge 2\left(n_{2}||\nabla \ln f||^{2} + \widetilde{\tau}(TM) - \widetilde{\tau}(TM_{T}) - \widetilde{\tau}(TM_{\theta}) - n_{2}\Delta(\ln f)\right), \quad (5.4.8)$$

where  $n_2$  is the dimension of pointwise slant submanifold  $M_{\theta}$ .

(ii) The equality sign holds in (5.4.8) if and only if  $M_T$  is totally geodesic and  $M_{\theta}$  is totally umbilical submanifolds in  $\widetilde{M}$ . M is minimal submanifold of  $\widetilde{M}$ .

#### 5.4.3 Some interesting applications of the Theorem 5.4.3 to Sasakian space forms

**Theorem 5.4.4.** Assume that  $\phi: M = M_T \times_f M_\theta \to \widetilde{M}(c)$  be an isometric immersion of a warped product pointwise semi-slant  $M_T \times_f M_\theta$  into a Sasakian space form  $\widetilde{M}(c)$ . Then

(i) The squared norm of the second fundamental form of M is defined as

$$||h||^{2} \ge 2n_{2} \left( ||\nabla \ln f||^{2} + \frac{c+3}{4}p - \frac{c-1}{4} - \Delta(\ln f) \right),$$
 (5.4.9)

where  $n_1$  and  $n_2$  are the dimensions of invariant  $M_T$  and pointwise slant submanifold  $M_{\theta}$ , respectively.

(ii) The equality sign holds in (5.4.9) if and only if  $M_T$  is totally geodesic and  $M_{\theta}$  is totally umbilical submanifolds of  $\widetilde{M}(c)$ , respectively. Moreover, M is minimal submanifold in  $\widetilde{M}(c)$ .

*Proof.* A Sasakian manifold is called Sasakian space form with constant  $\varphi$ -sectional curvature *c* if and only if the Riemannian curvature tensor  $\widetilde{R}$  is given by

$$\widetilde{R}(X,Y,Z,W) = \frac{c+3}{4} \left( g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \right) + \frac{c-1}{4} \left( \eta(X)\eta(Z)g(Y,W) + \eta(W)\eta(Y)g(X,Z) - \eta(Y)\eta(Z)g(X,W) - \eta(X)g(Y,Z)\eta(W) + g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) + 2g(X,\varphi Y)g(\varphi Z,W) \right).$$
(5.4.10)

Substituting  $X = W = e_i$ ,  $Y = Z = e_j$  in the above equation, we derive

$$\begin{split} \widetilde{R}(e_i, e_j, e_j, e_i) &= \frac{c-3}{4} \Big( g(e_i, e_i) g(e_j, e_j) - g(e_i, e_j) g(e_j, e_i) \Big) \\ &+ \frac{c-1}{4} \Bigg( \eta(e_i) \eta(e_j) g(e_i, e_j) - \eta(e_j) \eta(e_j) g(e_i, e_i) \\ &+ \eta(e_i) \eta(e_j) g(e_i, e_j) - \eta(e_i) \eta(e_i) g(e_j, e_j) \\ &+ g(\varphi e_j, e_j) g(\varphi e_i, e_i) - g(\varphi e_i, e_j) g(\varphi e_j, e_i) \\ &+ 2g^2(e_i, \varphi e_j) \Bigg). \end{split}$$

Taking summing up over the vector fields on  $TM_T$ , one can show that

$$\widetilde{\tau}(TM_T) = \frac{c+3}{4} \left( n_1(n_1-1) \right) - \frac{c-1}{4} \left\{ 2(n_1-1) - 3 ||P||^2 \right\}.$$

Since  $\xi(p)$  is tangent to  $TM_T$  for  $n_1$ -dimensional invariant submanifold, we have  $||P||^2 = n_1 - 1$ , then we get

$$2\widetilde{\tau}(TM_T) = \frac{c+3}{4} \{n_1(n_1-1)\} + \frac{c-1}{4}(n_1-1).$$

Similarly, for pointwise slant submanifold  $TM_{\theta}$ , we put  $||P||^2 = n_2 \cos^2 \theta$ , then

$$\widetilde{\rho}(TM_{\theta}) = \frac{c+3}{4} \{n_2(n_2-1)\} + \frac{c-1}{4} (3n_2\cos^2\theta),$$

Taking summing up over basis vectors of *TM* such that  $1 \le i \ne j \le n$ , it is easy to obtain

that

$$2\widetilde{\tau}(TM) = \frac{c+3}{4}n(n-1) + \frac{c-1}{4} \Big( 3\sum_{1 \le i \ne j \le n} g^2(Pe_i, e_j) - 2(n-1) \Big).$$
(5.4.11)

As *M* be a proper pointwise semi-slant submanifold of Sasakian space form  $\widetilde{M}(c)$ . Thus, the following relation comes from (5.3.24) by changing indices

$$\sum_{i,j=1}^{n} g^2(Pe_i, e_j) = 2(\alpha + \beta \cos^2 \theta).$$
 (5.4.12)

From (5.4.11) and (5.4.12), it follows that

$$2\tilde{\tau}(TM) = \frac{c+3}{4}n(n-1) + \frac{c-1}{4} \left[ 6(\alpha + \beta \cos^2 \theta) - 2(n-1) \right].$$
(5.4.13)

Therefore, using the above relations in (5.4.8), we derive the required result (5.4.9). The equalities cases directly follow the Theorem 5.4.3(ii). This completes the proof of the Theorem.

Some immediately consequences of the Theorem 5.4.4 the following,

**Corollary 5.4.1.** Let  $\widetilde{M}(c)$  be Sasakian space forms with  $c \leq -3$ . Then there does not exist a warped product pointwise semi-slant  $M_T \times_f M_{\theta}$  into  $\widetilde{M}(c)$  such that f is eigenfunction of Laplacian on  $M_T$  with respect to eigenvalue  $\lambda > 0$ .

**Corollary 5.4.2.** Let  $\widetilde{M}(c)$  be Sasakian space forms for thwith  $c \leq -3$ . Then there does not exist a warped product pointwise semi-slant  $M_T \times_f M_{\theta}$  into  $\widetilde{M}(c)$  such that f is harmonic function on invariant submanifold  $M_T$ .

**Corollary 5.4.3.** Assume that  $\phi : M = M_T \times_f M_\perp \to \widetilde{M}(c)$  be an isometrically immersion of an n-dimensional contact CR-warped product  $M_T \times_f M_\perp$  into 2m + 1-dimensional Sasakian space form  $\widetilde{M}(c)$  such that  $M_\perp$  is anti-invariant submanifold and  $M_T$  is an invariant submanifold tangent to  $\xi$  of  $\widetilde{M}(c)$ . Then

(i) The squared norm of the second fundamental form of M is given by

$$||h||^{2} \ge 2n_{2} \left( ||\nabla \ln f||^{2} + \frac{c+3}{4}n_{1} - \frac{c-1}{4} - \Delta(\ln f) \right),$$
 (5.4.14)

where  $n_2$  is the dimension of anti-invariant submanifold  $M_{\perp}$ .

(ii) The equality sign holds in (5.4.14) if and only if  $M_T$  is totally geodesic and  $M_{\perp}$  is totally umbilical submanifolds in  $\widetilde{M}(c)$ . Moreover, M is minimal in  $\widetilde{M}(c)$ .

#### 5.4.4 Applications to compact warped product pointwise semi-slant

This subsection give the brief discussion of compactness to warped product pointwise semi-slant submanifold, and using integration theory on manifold, we give some characterizations.

**Theorem 5.4.5.** Let  $M = M_T \times_f M_{\theta}$  be a compact orientable non-trivial warped product pointwise semi-slant submanifold of Sasakian space form  $\widetilde{M}(c)$ . Then M is trivial warped product if and only if

$$||h||^2 \ge \frac{c+3}{2}n_1n_2 - \frac{c-1}{2}n_2,$$

where  $n_1$  and  $n_2$  are dimensions of  $M_T$  and  $M_{\theta}$ , respectively.

*Proof.* From the integration theory on manifolds, i.e., compact orientable Riemannian manifold without boundary on M, then (5.4.9) and (2.3.16) implies that

$$\int_{M} \left( \frac{c+3}{2} n_1 n_2 - \frac{c-1}{2} n_2 + ||\nabla \ln f||^2 - \frac{1}{2n_2} ||h||^2 \right) dV \le \int_{M} \Delta(\ln f) dV = 0.$$

Let us consider the following inequality holds

$$||h||^2 \ge \frac{c+3}{2}n_1n_2 - \frac{c-1}{2}n_2.$$

Then

$$\int_M (||\nabla \ln f||^2) dV \le 0.$$

Since, integration always be positive for positive functions. Hence, we derive  $||\nabla \ln f||^2 \le 0$ , but  $||\nabla \ln f||^2 \ge 0$ , which implies that  $\nabla \ln f = 0$ , i.e., f is a constant function on M. Thus M becomes Riemannian product manifold of  $M_T$  and  $M_\theta$  respectively. Converse part is straightforward. This completes proof of theorem.

**Theorem 5.4.6.** Assume that  $\phi: M = M_T \times_f M_\theta \to \widetilde{M}(c)$  be an isometric immersion of a compact orientable proper warped product pointwise semi-slant  $M_T \times_f M_\theta$  into Sasakian

space form  $\widetilde{M}(c)$ . Then M is simply a Riemannian product if and only if

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = n_2 \Big(\frac{c+3}{4}n_1 - \frac{c-1}{4} - \sin^2\theta\Big), \quad (5.4.15)$$

where  $\theta$  is a real value function define on  $T^*M$  and its called a slant function. Moreover,  $n_1$  and  $n_2$  are the dimensions of invariant  $M_T$  and pointwise slant submanifold  $M_{\theta}$ , respectively.

*Proof.* We assume the equality holds in the inequality (5.4.9) of the Theorem 5.3.10, simplification gives

$$||h(\mathscr{D},\mathscr{D})||^{2} + ||h(\mathscr{D}^{\theta},\mathscr{D}^{\theta})||^{2} + 2||h(\mathscr{D},\mathscr{D}^{\theta})||^{2} = \frac{(c+3)}{2}n_{1}n_{2} - \frac{(c-1)}{2}n_{2} + 2n_{2}\{||\nabla \ln f||^{2} - \Delta(\ln f)\}.$$
 (5.4.16)

Similarly, the following equality can obtained by using same arguments of equation (5.3.32) and fact that  $\phi$  be a  $M_{\theta}$ -minimal isometric immersion of a warped product pointwise semi-slant submanifold  $M_T \times_f M_{\theta}$  into Sasakian manifold  $\widetilde{M}$ , if  $M_{\theta}$  is totally umbilical in  $\widetilde{M}$ , then  $\phi$  is  $M_{\theta}$ -totally geodesic. That is from (5.4.16), one derives

$$\frac{p(c+3)}{4} - \frac{c-1}{4} - \frac{1}{q} ||h(\mathscr{D}, \mathscr{D}^{\theta})||^2 + ||\nabla \ln f||^2 = \Delta(\ln f).$$
(5.4.17)

As we assumed that M is compact orientable submanifold, then M taking integration over the volume element dV of M, using (2.3.16), we arrive at

$$\int_{M} \left( \frac{c+3}{4} n_1 n_2 - \frac{c-1}{4} n_2 \right) dV = \int_{M} \left( ||h(\mathscr{D}, \mathscr{D}^{\theta})||^2 - n_2 ||\nabla \ln f||^2 \right) dV.$$
(5.4.18)

Now let us define  $X = e_i$  and  $Z = e_j$  for  $1 \le i \le p$  and  $1 \le j \le q$ , respectively. Taking summation over the vector fields  $M_T$  and  $M_{\theta}$  and using adapted frame for pointwise semi-

slant submanifolds, we can expressed to second fundamental form as:

$$\begin{split} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(h(e_i, e_j), h(e_i, e_j)) &= \csc^2 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(e_i, e_j^*), Fe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(e_i, Pe_j^*), Fe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, e_j^*), FPe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, Pe_j^*), FPe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, Pe_j^*), FPe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, Pe_j^*), FPe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, Pe_j^*), FPe_k^*)^2 \\ &+ \csc^2 \theta \sec^2 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, Pe_j^*), FPe_k^*)^2 \\ &+ \csc^2 \theta \sec^4 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, Pe_j^*), FPe_k^*)^2 \\ &+ \csc^2 \theta \sec^4 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, Pe_j^*), FPe_k^*)^2 \\ &+ \csc^2 \theta \sec^4 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, Pe_j^*), FPe_k^*)^2 \\ &+ \csc^2 \theta \sec^4 \theta \sum_{i=1}^{\alpha} \sum_{j,k=1}^{\beta} g(h(\varphi_{e_i}, Pe_j^*), FPe_k^*)^2 \end{split}$$

Applying the Lemma 5.4.1 in the above equation, it easily obtain

$$||h(\mathscr{D}, \mathscr{D}^{\theta})||^{2} = n_{2}(\csc^{2}\theta + \cot^{2}\theta)||\nabla \ln f||^{2} + n_{2}\sin^{2}\theta + \sum_{i=1}^{n_{1}}\sum_{j=1}^{n_{2}}||h_{\mu}(e_{i}, e_{j})||^{2}.$$
(5.4.19)

Now putting the above value in (5.4.18), ones get

$$\int_{M} \left( \frac{c+3}{4} n_{1} n_{2} - \frac{c-1}{4} n_{2} - n_{2} \sin^{2} \theta \right) dV = \int_{M} \left( \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} ||h_{\mu}(e_{i}, e_{j})||^{2} \right) dV + 2 \int_{M} \left( \cot^{2} \theta ||\nabla \ln f||^{2} \right) dV.$$
(5.4.20)

If the condition (5.4.15) is satisfied, then from (5.4.20) implies that f is constant function on proper pointwise semi-slant submanifold M. Thus M is a Riemannian product of invariant and pointwise slant submanifolds  $M_T$  and  $M_{\theta}$ , respectively.

*Conversely*, assume that *M* is simply a Riemannian product then warping function *f* must be constant, i.e.,  $\nabla \ln f = 0$ . Thus from relation (5.4.20) implies the equality (5.4.15).

This completes the proof of theorem.

**Note:-** Similarly, we can generalize some results for contact CR-warped product submanifolds in Sasakian space forms. If slant function  $\theta$  becomes globally constant and then we setting  $\theta = \frac{\pi}{2}$  in Theorems 5.4.5, 5.4.6. Thus, we give the corollaries.

**Corollary 5.4.4.** Let  $M = M_T \times_f M_\perp$  be a compact orientate CR-warped product submanifold in Sasakian space form  $\widetilde{M}(c)$ . Then, M is trivial CR-warped product if and only if

$$||h||^2 \ge \frac{c+3}{2}n_1n_2 - \frac{c-1}{2}n_2$$

where  $n_1$  and  $n_2$  are dimensions of  $M_T$  and  $M_{\perp}$ , respectively.

**Corollary 5.4.5.** Assume that  $M = M_T \times_f M_\perp$  be a compact CR-warped product submanifold in Sasakian space form  $\widetilde{M}(c)$  such that  $M_T$  is invariant submanifold tangent to  $\xi$ and  $M_\perp$  is anti-invariant submanifold in  $\widetilde{M}(c)$ . Then M is simply a Riemannian product if and only if

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = n_2 \left(\frac{c+3}{4}n_1 - \frac{c-1}{4} - 1\right).$$

where  $n_1$  and  $n_2$  are dimensions of  $M_T$  and  $M_{\perp}$ , respectively.

**Theorem 5.4.7.** Assume that  $M = M_T \times_f M_{\theta}$  be an isometric immersion of a warped product pointwise semi-slant into Sasakian space form  $\widetilde{M}(c)$ . Let  $\lambda_T$  be non-zero eigenvalue of the Laplacian on compact invariant submanifold  $M_T$ . Then

$$\int_{M} ||h||^{2} dV_{T} \geq \int_{M} \left( \frac{c+3}{2} n_{1} n_{2} - \frac{c-1}{2} n_{2} \right) dV_{T} + 2n_{2} \lambda_{T} \int_{M} (\ln f)^{2} dV_{T},$$
(5.4.21)

where  $dV_T$  is volume element on  $M_T$ .

- (*i*)  $\Delta \ln f = \lambda \ln f$ .
- (ii) In the warped product pointwise semi-slant submanifold both  $M_T$  and  $M_{\theta}$  are totally geodesic.

Proof. Thus using the minimum principle property, we have

$$\int_{M} ||\nabla \ln f||^2 dV_T \ge \lambda_T \int_{M} (\ln f)^2 dV_T.$$
(5.4.22)

From (5.4.22) and (5.4.9), we required the result (5.4.21). It complete proof of corollary.

**Theorem 5.4.8.** Let  $M = M_T \times_f M_{\theta}$  be a warped product pointwise semi-slant into Sasakian space form  $\widetilde{M}(c)$  such that  $M_T$  is compact submanifold and  $\lambda_T$  be non-zero eigenvalue of the Laplacian on  $M_T$ . Then

$$\int_{M} \left( \sum_{i=1}^{n_{1}} \sum_{j=1}^{n_{2}} ||h_{\mu}(e_{i}, e_{j})||^{2} \right) dV_{T} \ge \int_{M} \left( \frac{c+3}{4} n_{1}n_{2} - \frac{c-1}{4} n_{2} - n_{2} \sin^{2} \theta \right) dV_{T} + 2n_{2} \cot^{2} \theta \lambda_{T} \int_{M} (\ln f)^{2} dV_{T}.$$
(5.4.23)

- (i)  $\Delta \ln f = \lambda \ln f$ .
- (ii) In the warped product pointwise semi-slant submanifold both  $M_T$  and  $M_{\theta}$  are totally geodesic.

*Proof.* The proof follows from (5.4.20) and (5.4.22).

#### 5.4.5 Applications to Hessian of warping functions

Throughout study of this subsection, we shall construct some applications in terms of Hessian of warped function. We derive conditions under which a warped product pointwise semi-slant isometrically immersed into Sasakian space form to be a Riemannian product manifold.

**Theorem 5.4.9.** Let  $M = M_T \times_f M_{\theta}$  is a warped product pointwise semi-slant submanifold into a Sasakian space form  $\widetilde{M}(c)$ . If the following inequality holds

$$||h||^{2} \ge 2n_{2} \left\{ \frac{c+3}{4}n_{2} - \frac{c-1}{4} + \sum_{i=1}^{\alpha} \left( (H^{\ln f}(e_{i}, e_{i}) + H^{\ln f}(\varphi e_{i}, \varphi e_{i})) \right) \right\},$$
(5.4.24)

where  $H^{\ln f}$  is Hessian of warping function  $\ln f$ . Then M is a simply Riemannian product.

**Theorem 5.4.10.** Let  $\phi : M = M_T \times_f M_\theta \longrightarrow \widetilde{M}(c)$  be an isometric immersion from a non-trivial warped product pointwise semi-slant submanifold into Sasakian space form  $\widetilde{M}(c)$  such that a slant function  $\theta \neq \operatorname{arc} \operatorname{cot} \sqrt{n_2}$ . Then necessary and sufficient condition for M to a trivial warped product is given by

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = \frac{c+3}{4} n_1 n_2 - \frac{c-1}{4} n_2 - n_2 \sin^2 \theta + n_2 \sum_{i=1}^{\alpha} \left[ H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \right].$$
(5.4.25)

*Note.* The proof of the Theorem 5.4.9 and Theorem 5.4.10 can be derived by following the Theorem 5.3.10 and Theorem 5.3.11, respectively.

#### 5.4.6 Applications to kinetic energy and Hamiltonian

In this section, we considered warped product pointwise semi-slant submanifold as a connected, compact warped product pointwise semi-slant submanifold with a nonempty boundary, that is,  $\partial M \neq \emptyset$ . Thus, we constructed some necessary and sufficient conditions in terms of kinetic energy and Hamiltonian whose positive differentiable function is warping function, and classify non-trivial warped products turning into trivial warped product, which isometrically immersed into a Sasakian space form.

**Theorem 5.4.11.** Assume that  $\phi : M = M_T \times_f M_\theta \longrightarrow \widetilde{M}(c)$  be an isometric immersion from a connected, compact warped product pointwise semi-slant submanifold in a Sasakian space form  $\widetilde{M}(c)$ . Then M is a Riemannian product of  $M_T$  and  $M_\theta$  if and only if the kinetic energy satisfies following equality

$$E(\ln f) = \frac{1}{4} \tan^2 \theta \left\{ \int_M \left( \frac{c+3}{4} n_1 - \frac{c-1}{4} - \sin^2 \theta - \frac{1}{n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_\mu(e_i, e_j)||^2 \right) dV - 8 \int_{M^n} \left( \csc^3 \theta \cos \theta \left( \frac{d\theta}{dV} \right) E(\ln f) \right) dV \right\},$$
(5.4.26)

where  $E(\ln f)$  represent the kinetic energy of the warping function  $\ln f$  and dV is volume element on M.

Similarly, we derive some characterization in terms of Hamiltonian.

**Theorem 5.4.12.** Let  $\phi : M = M_T \times_f M_\theta \longrightarrow \widetilde{M}(c)$  be an isometric immersion from a warped product pointwise semi-slant into Sasakian space form  $\widetilde{M}(c)$ . If M is connected, compact submanifold, then M is a trivial warped product pointwise semi-slant if and only if the Hamiltonian of warping function satisfies the following equality

$$H(d(\ln f), x) = \frac{1}{4} \tan^2 \theta \left\{ \frac{c+3}{4} n_1 - \frac{c-1}{4} - \sin^2 \theta - \frac{1}{n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 \right\},$$
(5.4.27)

where  $H(d(\ln f), x)$  is Hamiltonian of warped function  $\ln f$ .

The proof of above theorems can be derived by following the Theorem 5.2.17 and Theorem 5.2.18, respectively.

#### **CHAPTER 6**

# SOME INEQUALITIES OF WARPED PRODUCT SUBMANIFOLDS FOR DIFFERENT AMBIENT SPACE FORMS

#### 6.1 INTRODUCTION

The following well-known result of Chen (2002), a sharp relationship between the squared norm of mean curvature and the warping function f of the warped product  $M \times_f M_2$  isometrically immersed into real space form.

**Theorem 6.1.1.** Let  $\phi : M_1 \times_f M_2 \longrightarrow \mathbb{R}^m(c)$  be an isometrically immersion of an ndimensional warped product into a Riemannain m-manifolds of constant sectional curvature c. Then

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} ||H||^2 + n_1 c$$

where  $n_i = \dim M_i$ , i = 1, 2 and  $\Delta$  is the Lapalcian operator of  $M_1$ . Moreover, the equality holds in the above inequality if and only if  $\phi$  is mixed totally geodesic and  $n_1H_1 = n_2H_2$ such that  $H_1$  and  $H_2$  are partial mean curvature.

Motivated from Chen (2002) result several inequalities were obtained some other geometers for warped products and doubly warped products in different setting of the ambient manifolds such as Chen (2004); Murathan et al. (2006); Olteanu (2010, 2014); Yoon et al. (2004); Yoon (2004).

In this Chapter, we establish some inequalities for warped product semi-slant isometrically immersed into Kenmotsu space form in terms of the squared norm of mean curvature vector which relate to the squared norm of warping function. Similary, for *C*-totally real doubly warped product isometrically immersed into locally conformal almost cosymplectic manifold with pointwise  $\varphi$ -sectional curvature *c* following by case study of Chen (2002). The equality cases in the statement of inequalities are also considered. Moreover, some applications from obtained results are derived.

## 6.2 INEQUALITIES FOR WARPED PRODUCT SEMI-SLANT SUBMANIFOLDS OF KENMOTSU SPACE FORMS

#### 6.2.1 Existence of warped product semi-slant submanifold in a Kenmotsu manifold

It is well known that, M. Atçeken (2010) studied the non existence of warped product semi-slant submanifolds of Kenmotsu manifold such that structure vector  $\xi$  is tangent to fiber. While, Uddin et al. (2012) and Srivastava (2012) were proved that the warped product semi-slant submanifold of Kenmotsu manifold exists of the forms  $M = M_T \times_f M_{\theta}$ and  $M = M_{\theta} \times_f M_T$ , in case when the structure vector field  $\xi$  is tangent to  $M_T$  and  $M_{\theta}$ , respectively. Moreover, we found that Cioroboiu (2003) and Aktan et al. (2008) obained some inqualities for semi-slant submanifolds by constructing its orthonormal frame. But overlooks the suitable conditions for inequalities of warped product semi-slant submanifold, there a need to derive the inequalities for the mean curvature and warping functions with slant angles of warped product semi-slant in Kemotsu space forms. In the current study, we are extending the case studies of Cioroboiu (2003) to warped product semi-slant submanifolds in a Kenmotsu space form. We also generalize some other inequalities for CR-warped product submanifolds by special cases.

#### 6.2.2 Main Inequalities

We obtain an inequality for warped product semi-slant isometrically immersed into Kenmotsu space form such that  $\xi$  tangent to first factor of warped product. First, we recall the following lemma which is useful to derive inequalities and its given by Chen (1993).

*Lemma* 6.2.1. Let  $a_1, a_2, \dots, a_n, a_{n+1}$  be n+1  $(n \ge 2)$  be real numbers such that

$$(\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + a_{n+1}).$$
(6.2.1)

Then  $2a_1a_2 \ge a_3$  with the equality holds if and only if  $a_1 + a_2 = a_3 = \cdots, a_n$ .

Now we construct the following result next page.

**Theorem 6.2.1.** Assume that  $\phi : M = M_T \times_f M_\theta \to \widetilde{M}(c)$  be an isometric immersion from a warped product semi-slant  $M_T \times_f M_\theta$  into Kenmotsu space form  $\widetilde{M}(c)$  such that c is  $\varphi$ -sectional constant curvature and  $\xi$  is tangent to  $M_T$ . Then

*(i) The relation between warping function and the squared norm of mean curvature is obtained* 

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} ||H||^2 + \frac{c-3}{4}n_1 - \frac{c+1}{4n_2} \Big\{ 3d_1 + d_2(2+3\cos^2\theta) \Big\}, \tag{6.2.2}$$

where  $n_i = dim M_j$ , i = 1, 2 and  $j = T, \theta$  and  $\Delta$  is the Laplacian operator on  $M_T$ .

(ii) The equality case holds in (6.2.2) if and only if  $n_1H_T = n_2H_{\theta}$ , where  $H_T$  and  $H_{\theta}$  are partially mean curvature vectors of  $M_T$  and  $M_{\theta}$ , respectively. Moreover,  $\phi$  is mixed totally geodesic immersion.

*Proof.* Let  $M_T \times_f M_{\theta}$  be a warped product semi-slant submanifold in a Kenmotsu space form  $\widetilde{M}(c)$  and its curvature tensor  $\widetilde{R}$  is defined as;

$$\widetilde{R}(X,Y,Z,W) = \frac{c-3}{4} \left( g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \right) + \frac{c+1}{4} \left( g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W) - 2g(X,\varphi Y)g(Z,\varphi W) - g(X,W)\eta(Y)\eta(Z) + g(X,Z)\eta(X)\eta(W) - g(Y,Z)\eta(X)\eta(W) + g(Y,W)\eta(X)\eta(Z) \right),$$
(6.2.3)

where *c* is the function of constant  $\varphi$ -sectional curvature of  $\widetilde{M}(c)$  (see Murathan et al. (2006). By substitute  $X = Z = e_A$ , and  $Y = W = e_B$  for any  $1 \le A, B \le n$  in (6.2.3). Hence, based on the Gauss formula (2.3.2) and (6.2.3), we derive

$$2\rho = \frac{c-3}{4}n(n-1) + \frac{(c+1)}{4} \left( 3\sum_{1 \le A \ne B \le n} g^2(Pe_A, e_B) - 2(n-1) \right) + n^2 ||H||^2 - ||h||^2.$$
(6.2.4)

From (5.3.25) and (6.2.4), it follows that

$$2\rho = \frac{c-3}{4}n(n-1) + \frac{(c+1)}{4} \left( d_1 + d_2(3\cos^2\theta - 2) \right) + n^2 ||H||^2 - ||h||^2.$$
(6.2.5)

Now we consider that,

$$\delta = 2\rho - \frac{c-3}{4}n(n-1) - \frac{(c+1)}{2} \left( d_1 + d_2(3\cos^2\theta - 2) \right) - \frac{n}{2} ||H||^2.$$
(6.2.6)

Then from (6.2.5) and (6.2.6), it implies that

$$n^{2}||H||^{2} = 2(\delta + ||h||^{2}).$$
(6.2.7)

Now consider a locall orthonormal frame  $\{e_1, e_2, \dots e_n\}$  such that the equation (6.2.7) takes the form

$$\left(\sum_{r=n+1}^{2m+1}\sum_{A=1}^{n}h_{AA}^{r}\right)^{2} = 2\left(\delta + \sum_{r=n+1}^{2m+1}\sum_{A=1}^{n}(h_{AA}^{r})^{2} + \sum_{r=n+1}^{2m+1}\sum_{A$$

which more simplfying as:

$$\frac{1}{2} \left( h_{11}^{n+1} + \sum_{A=2}^{n_1} h_{AA}^{n+1} + \sum_{B=n_1+1}^{n} h_{BB}^{n+1} \right)^2 = \delta + (h_{11}^{n+1})^2 + \sum_{A=2}^{n_1} (h_{AA}^{n+1})^2 + \sum_{A=2}^{n_1} (h_{AA}^{n+1})^2 + \sum_{B=n_1+1}^{n} (h_{BB}^{n+1})^2 - \sum_{2 \le j \ne l \le n_1} h_{jj}^{n+1} h_{ll}^{n+1} - \sum_{n_1+1 \le B \ne S \le n} h_{BB}^{n+1} h_{SS}^{n+1} + \sum_{A < j=1}^{n} (h_{Aj}^{n+1})^2 + \sum_{r=n+1A, j=1}^{n} (h_{Aj}^r)^2.$$
(6.2.9)

Assuming that  $a_1 = h_{11}^{n+1}$ ,  $a_2 = \sum_{A=2}^{n_1} h_{AA}^{n+1}$  and  $a_3 = \sum_{B=n_1+1}^{n} h_{BB}^{n+1}$ . Taking account of Lemma 6.2.1 in (6.2.9), we derive

$$\frac{\delta}{2} + \sum_{A < B = 1}^{n} (h_{AB}^{n+1})^2 + \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{A,B=1}^{n} (h_{AB}^r)^2 \le \sum_{2 \le A \ne l \le n_1} h_{AA}^{n+1} h_{ll}^{n+1} + \sum_{n_1+1 \le B \ne s \le n} h_{BB}^{n+1} h_{ss}^{n+1},$$
(6.2.10)

with equality holds in (6.2.10) if and only if satisfies

$$\sum_{A=1}^{n_1} h_{AA}^{n+1} = \sum_{B=n_1+1}^{n} h_{BB}^{n+1}.$$
(6.2.11)

Furthermore, (2.4.4) and (2.3.10) implies that

$$\frac{n_2\Delta f}{f} = \rho - \sum_{1 \le A \ne k \le n_1} K(e_A \wedge e_k) - \sum_{n_1 + 1 \le B \ne s \le n} K(e_B \wedge e_s).$$
(6.2.12)

Taking help from (6.2.3), we obtain

$$\frac{n_{2}\Delta f}{f} = \rho - \frac{c-3}{8}n_{1}(n_{1}-1) + \frac{(c+1)}{4}(n_{1}-1) 
- \frac{3(c+1)}{4}\sum_{1\leq A\neq k\leq n_{1}}g^{2}(Pe_{A},e_{k}) - \sum_{r=1}^{2m+1}\sum_{2\leq A\neq k\leq n_{1}}(h_{AA}^{r}h_{kk}^{r} - (h_{Ak}^{r})^{2}) 
- \frac{3(c+1)}{4}\sum_{n_{1}+1\leq B\neq s\leq n}g^{2}(Pe_{B},e_{s}) - \frac{c-3}{8}n_{2}(n_{2}-1) 
- \sum_{r=1}^{2m+1}\sum_{n_{1}+1\leq t\neq s\leq n}(h_{BB}^{r}h_{ss}^{r} - (h_{Bs}^{r})^{2}).$$
(6.2.13)

Taking into account of (6.2.10) and (6.2.13), it is easily observed that

$$\frac{n_2\Delta f}{f} \le \rho - \frac{c-3}{8}n(n-1) + \frac{c-3}{4}n_1n_2 + \frac{c+1}{4}(n_1-1) - \frac{3(c+1)}{4}(n_1-1) - \frac{3(c+1)}{4}n_2\cos^2\theta - \frac{\delta}{2}.$$
(6.2.14)

Therefore using (6.2.7), then the inequality (6.2.14) reduce to

$$\frac{n_2\Delta f}{f} \le \frac{n^2}{4} ||H||^2 + \frac{c-3}{4}n_1n_2 + \frac{c+1}{4}\left(-3d_1 - 3d_2\cos^2\theta - 2d_2\right).$$
(6.2.15)

This is implies the inequality (6.2.2). The equality sign holds in (6.2.2) if and only if the leaving terms in (6.2.10) and (6.2.11) satisfies that

$$\sum_{r=n+2}^{2m+1} \sum_{A=1}^{n_1} h_{AA}^r = \sum_{r=n+2}^{2m+1} \sum_{B=n_1+1}^{n_1} h_{BB}^r = 0, \qquad (6.2.16)$$

and  $n_1H_T = n_2H_{\theta}$ , where  $H_T$  and  $H_{\theta}$  are partially mean curvature vectors on  $M_T$  and  $M_{\theta}$ , respectively. Moreover, from (6.2.10), we find that

$$h_{AB}^r = 0$$
, for each  $1 \le A \le n_1$ ,  $n_1 + 1 \le B \le n$   $n+1 \le r \le 2m+1$ . (6.2.17)

This means that  $\phi$  is mixed totally geodesic immersion. But converse of (6.2.17) may not be true in warped product semi-slant in Kenmotsu space form. This completes the proof
of theorem.

If we reverse the factors of warped product semi-slant subamnifold, we get the following result.

**Theorem 6.2.2.** Let  $\phi : M = M_{\theta} \times_f M_T \to \widetilde{M}(c)$  be an isometric immersion of a warped product semi-slant  $M_{\theta} \times_f M_T$  into Kenmotsu space form  $\widetilde{M}(c)$  such that  $\xi$  is tangent to  $M_{\theta}$ . Then

*(i) The relation between warping function and the norm of squared mean curvature is given by* 

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} ||H||^2 + \frac{c-3}{4}n_1 - \frac{c+1}{4n_2} \Big( 3d_2 + d_1 \{2 + 3\cos^2\theta\} \Big), \tag{6.2.18}$$

where  $n_i = dim M_j$ , i = 1, 2 and  $j = T, \theta$  and  $\Delta$  is the Laplacian operator on  $M_{\theta}$ .

(ii) The equality case holds in (6.2.18) if and only if  $n_1H_T = n_2H_{\theta}$ , where  $H_T$  and  $H_{\theta}$  are partially mean curvature vector fields on  $M_T$  and  $M_{\theta}$  respectively and  $\phi$  is mixed totally geodesic immersion.

*Proof.* The proof of Theorem 6.2.2 is similar as the Theorem 6.2.1 by considering the structure vector field  $\xi$  is normal to fiber.

The generalization of CR-submanifolds in the sense of Papaghiuc (1994), we directly obtain the following corollaries by using Theorem 6.2.1, and Theorem 6.2.2 with  $\theta = \frac{\pi}{2}$ , such that,

**Corollary 6.2.1.** Assume that  $\phi : M = M_T \times_f M_\perp \to \widetilde{M}(c)$  be an isometric immersion of a CR-warped product  $M_T \times_f M_\perp$  into Kenmotsu space form  $\widetilde{M}(c)$  with c is  $\varphi$ -sectional constant curvature such that  $\xi$  is tangent to  $M_T$ . Then

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} ||H||^2 + \frac{c-3}{4}n_1 - \frac{c+1}{4n_2} \left(3d_1 + 2d_2\right), \tag{6.2.19}$$

where  $n_i = dim M_j$ , i = 1, 2 and  $j = T, \perp$ , rsepectively, and  $\Delta$  is the Laplacian operator on  $M_T$ .

**Corollary 6.2.2.** Let  $\phi : M = M_{\perp} \times_f M_T \to \widetilde{M}(c)$  be an isometric immersion of a CRwarped product submanifold  $M_{\perp} \times_f M_T$  into Kenmotsu space form  $\widetilde{M}(c)$  such that  $\xi$  is tangent to  $M_{\perp}$ . Then

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} ||H||^2 + \frac{c-3}{4}n_1 - \frac{c+1}{4n_2} \Big( 3d_2 + 2d_1 \Big), \tag{6.2.20}$$

where  $n_i = dimM_j$ , i = 1, 2 and  $j = T, \perp$ , respectively, and  $\Delta$  is the Laplacian operator on  $M_{\perp}$ .

<sup>1</sup> Follows the case study of Chapter 5, some of the following results for the second fundamental form h can be proved easily of warped product semi-slant submanifold in a Kenmotsu space forms. That'is,

**Theorem 6.2.3.** Assume that  $\phi: M = M_T \times_f M_\theta \to \widetilde{M}(c)$  is an isometric immersion of a warped product semi-slant  $M_T \times_f M_\theta$  into a Kenmotsu space form  $\widetilde{M}(c)$ . Then

(i) The squared norm of the second fundamental form of M is defined as

$$|h||^{2} \ge 2n_{2} \left( ||\nabla \ln f||^{2} + \frac{c-3}{4}n_{1} - \frac{c+1}{4} - \Delta(\ln f) \right), \tag{6.2.21}$$

where  $n_1$  and  $n_2$  are the dimensions of invariant  $M_T$  and slant submanifold  $M_{\theta}$ , respectively.

(ii) The equality sign holds in (6.2.21) if and only if  $M_T$  is totally geodesic and  $M_{\theta}$  is a totally umbilical submanifold in  $\widetilde{M}(c)$ . Moreover, M is a minimal submanifold in  $\widetilde{M}(c)$ .

Moreover, we are giving some applications of above theorem based on the minimum principal properties.,

**Theorem 6.2.4.** Assume that  $\phi : M = M_T \times_f M_\theta \to \widetilde{M}(c)$  is an isometric immersion of a warped product semi-slant submanifold into a Kenmotsu space form  $\widetilde{M}$ . Let  $\lambda_T$  be a

<sup>&</sup>lt;sup>1</sup>We skip the of the theorem 6.2.3 due to similar methods which we used in chapter 5. Moreover, others results which are applications of this theorem easily seen that, just replacing semi-slant submanifold instead of pointwise semi-slant submanifold and change the ambient space forms.

non-zero eigenvalue of the Laplacian on the compact invariant submanifold  $M_T$ . Then

$$\int_{M_T} ||h||^2 \mathrm{d}\mathbf{V}_T \ge \int_{M_T} \left( \frac{c+3}{2} n_1 n_2 - \frac{c-1}{2} n_2 \right) \mathrm{d}\mathbf{V}_T + 2n_2 \lambda_T \int_{M_T} (\ln f)^2 \mathrm{d}\mathbf{V}_T,$$
(6.2.22)

where  $dV_T$  is the volume element of  $M_T$ .

**Theorem 6.2.5.** Assume that  $\phi: M = M_T \times_f M_\theta \to \widetilde{M}(c)$  is an isometric immersion of a compact orientable proper warped product semi-slant submanifold  $M = M_T \times_f M_\theta$  into a Kenmotsu space form  $\widetilde{M}(c)$ . Then M is simply a Riemannian product if and only if

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = n_2 \left( \frac{c-3}{4} n_1 - \frac{c+1}{4} + 2\cot^2 \theta + 1 \right), \tag{6.2.23}$$

where  $\theta$  is a slant angle defined on TM and  $h_{\mu}$  is the component of the second fundamental form h in  $\Gamma(\mu)$ .

**Theorem 6.2.6.** Let  $\phi: M = M_T \times_f M_\theta \to \widetilde{M}(c)$  is an isometric immersion of a compact orientable proper warped product semi-slant submanifold  $M = M_T \times_f M_\theta$  into a Kenmotsu space form  $\widetilde{M}(c)$ . Then M is a trivial warped product if

$$||h||^2 \ge \frac{c-3}{2}n_1n_2 - \frac{c+1}{2}n_2,$$

where  $n_1$  and  $n_2$  are dimensions of  $M_T$  and  $M_{\theta}$ , respectively.

Similarly, we generalizes some results from the above theorem for contact CR-warped product submanifolds into Kenmotsu space forms.

**Corollary 6.2.3.** Let  $\phi: M = M = M_T \times_f M_\perp \to \widetilde{M}(c)$  be an isometric immersion of a compact orientable CR-warped product submanifold into a Kenmotsu space form  $\widetilde{M}(c)$ . Then, M is trivial CR-warped product if and only if

$$||h||^2 \ge \frac{c-3}{2}n_1n_2 - \frac{c+1}{2}n_2,$$

where  $M_T$  and  $M_{\perp}$  are invariant and anti-invariant submanifolds, respectively.

**Corollary 6.2.4.** Assume that  $\chi : M = M_T \times_f M_\perp \to \widetilde{M}(c)$  is an isometric immersion of a compact orientable CR-warped product submanifold into a Kenmotsu space form  $\widetilde{M}$  such

that  $M_T$  is an invariant submanifold tangent to  $\xi$  and  $M_{\perp}$  is an anti-invariant submanifold in  $\widetilde{M}$ , then M is simply a Riemannian product if and only if

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = n_2 \Big(\frac{c+3}{4}n_1 - \frac{c-1}{4} + 1\Big),$$

where  $h_{\mu}$  is the second fundamental form of component in  $\Gamma(\mu)$ .

The following results are direct consequences of the Theorem 6.2.3,

**Corollary 6.2.5.** Let  $\widetilde{M}(c)$  be a Kenmotsu space form with  $c \leq 3$ . Then there does not exists a warped product semi-slant  $M_T \times_f M_{\theta}$  into  $\widetilde{M}(c)$  such that  $\ln f$  is the eigenfunction of the Laplacian on  $M_T$  with respect to the eigenvalue  $\lambda > 0$ .

**Corollary 6.2.6.** Assume that  $\widetilde{M}(c)$  is a Kenmotsu space form with  $c \leq 3$ . Then there does not exist a warped product semi-slant submanifold  $M_T \times_f M_{\theta}$  into  $\widetilde{M}(c)$  such that  $\ln f$  is a harmonic function on the invariant submanifold  $M_T$ .

**Theorem 6.2.7.** Let  $M = M_T \times_f M_\theta \to \widetilde{M}(c)$  be a warped product semi-slant submanifold into a Kenmotsu space form  $\widetilde{M}(c)$ . If the inequality

$$||h||^{2} \ge 2n_{2} \left\{ \frac{c-3}{4}n_{1} - \frac{c+1}{4} + 1 + \sum_{i=1}^{\alpha} \left( H^{\ln f}(e_{i}, e_{i}) + H^{\ln f}(\varphi e_{i}, \varphi e_{i}) \right) \right\},$$
(6.2.24)

holds, where  $H^{\ln f}$  is Hessian of warping function  $\ln f$ , then M is simply a Riemannian product of  $M_T$  and  $M_{\theta}$ , respectively.

**Theorem 6.2.8.** Let  $M = M_T \times_f M_{\theta}$  be a non-trivial warped product semi-slant submanifold into a Kenmotsu space form  $\widetilde{M}(c)$  such that a slant angle  $\theta \neq \operatorname{arc} \operatorname{cot} \sqrt{n_2}$ . Then necessary and sufficient condition of M to be a trivial warped product is given by

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 = \frac{c-3}{4} n_1 n_2 - \frac{c+1}{4} n_2 + 1 + n_2 \left(1 + 2\cot^2\theta\right) + n_2 \sum_{i=1}^{\alpha} \left(H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i)\right).$$
(6.2.25)

**Theorem 6.2.9.** Assume that  $M = M_T \times_f M_{\theta}$  is a connected, compact warped product semi-slant submanifold in a Kenmotsu space form  $\widetilde{M}(c)$ , then M is a Riemannian product

of  $M_T$  and  $M_{\theta}$  if and only if the kinetic energy satisfies the following equality

$$E(\ln f) = \frac{1}{4} \tan^2 \theta \left\{ \int_M \left( \frac{c+3}{4} n_1 - \frac{c-1}{4} + 2\cot^2 \theta + 1 - \frac{1}{n_2} \sum_{i=1}^{n_2} \sum_{j=1}^{n_2} ||h_\mu(e_i, e_j)||^2 \right) d\mathbf{V} \right\},$$
(6.2.26)

where  $E(\ln f)$  represents the kinetic energy of the warping function  $\ln f$  and dV is ithe volume element on  $M^n$ 

In similar way, we derive a characterization in terms of Hamiltonian operator.

**Theorem 6.2.10.** Let  $M = M_T \times_f M_{\theta}$  be a warped product semi-slant into a Kenmotsu space form  $\widetilde{M}(c)$ . If M is connected compact, then M is a trivial warped product semi-slant if and only if the Hamiltonian of warping function satisfies the following equality

$$H(d(\ln f), x) = \frac{1}{4} \tan^2 \theta \left\{ \frac{c+3}{4} n_1 - \frac{c-1}{4} + 2\cot^2 \theta + 1 - \frac{1}{n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} ||h_{\mu}(e_i, e_j)||^2 \right\}.$$
 (6.2.27)

# 6.3 CURVATURE INEQUALITIES FOR *C*- TOTALY REAL DOUBLY WARPED PRODUCT OF LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLDS

## 6.3.1 Motivations

We study *C*-totally real doubly warped product isometrically immersed into a locally conformal almost cosymplectic manifold. The inequalities which we obtain are fascinating and provide upper bound and lower bound for warping functions in terms of squared mean curvature, scalar curvature and pointwise constant  $\varphi$ -sectional curvature *c*. The obtained results generalize some other inequalities as special cases.

The idea of doubly warped product manifolds studied by B. Ünal (2001). He defined these manifolds as follows:

**Definition 6.3.1.** Let  $M_1$  and  $M_2$  be two Riemannian manifolds of dimensions  $n_1$  and  $n_2$  endowed with their Riemannian matrics  $g_1$  and  $g_2$  such that  $f_1 : M_1 \to (0, \infty)$  and  $f_2 : M_2 \to (0, \infty)$  be positive differentiable functions on  $M_1$  and  $M_2$ , respectively. Then doubly

warped product  $M =_{f_2} M_1 \times_{f_1} M_2$  is defined to be the product manifold  $M_1 \times M_2$  with equipped metric  $g = f_2^2 g_1 + f_1^2 g_2$ . Moreover, If we consider  $\gamma_1 : M_1 \times M_2 \to M_1$  and  $\gamma_2 : M_1 \times M_2 \to M_2$  are natural projections on  $M_1$  and  $M_2$ , respectively. Then the metric gon doubly warped product is defined as:

$$g(X,Y) = (f_2 \circ \gamma_2)^2 g_1(\gamma_1 \star X, \gamma_1 \star Y) + (f_1 \circ \gamma_1)^2 g_2(\gamma_2 \star X, \gamma_2 \star Y),$$
(6.3.1)

for any vector fields X, Y tangent to M, where  $\star$  is the symbol of the tangent map. Thus, the functions  $f_1$  and  $f_2$  are called *warping functions* on  $M_1$  and  $M_2$ , respectively. If both  $f_1 = 1$  and  $f_2 = 1$ , then M is called a simply *Riemannian product manifold*. If either  $f_1 = 1$  or  $f_2 = 1$ , then M is called a (single) warped product manifold. If  $f_1 \neq 1$  and  $f_2 \neq 1$ , then M is said to be a non-trivial doubly warped product manifold. Further, let  $M =_{f_2} M_1 \times_{f_1} M_2$  be a non-trivial doubly warped product manifold of an arbitrary Riemannian manifold  $\widetilde{M}$ . Then

$$\widetilde{M}$$
. Then  
 $\nabla_X Z = \nabla_Z X = (Z \ln f_2) X + (X \ln f_1) Z,$ 
(6.3.2)

$$\nabla_X Y = \nabla_X^1 Y - \frac{f_2^2}{f_1^2} g_1(X, Y)(\ln f_2), \qquad (6.3.3)$$

for any vector fields  $X, Y \in \Gamma(TM_1)$  and  $Z \in \Gamma(TM_2)$ . Further,  $\nabla^1$  and  $\nabla^2$  Levi-Civitas connections of induced Riemannian metrics on Riemannian manifolds  $M_1$  and  $M_2$ , respectively.

Assuming that  $\phi: M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}$  be isometric immersion of a doubly warped product  $_{f_2}M_1 \times_{f_1} M_2$  into a Riemannian manifold  $\widetilde{M}$  with constant sectional curvature *c*. Moreover, let  $n_1$ ,  $n_2$  and *n* be the dimensions of  $M_1$ ,  $M_2$  and  $M_1 \times_f M_2$ , respectively. Then for unit vector fields *X* and *Z* tangent to  $M_1$  and  $M_2$  respectively, we have

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z)$$
  
=  $\frac{1}{f_1} \{ (\nabla_X^1 X) f_1 - X^2 f_1 \} + \frac{1}{f_2} \{ (\nabla_Z^2 Z) f_2 - Z^2 f_2 \}.$  (6.3.4)

If we consider the local orthonormal frame  $\{e_1, e_2, \dots, e_{n_1}, e_{n_1+1}, \dots, e_n\}$  such that  $e_1, e_2, \dots, e_{n_1}$  and  $e_{n_1+1}, \dots, e_n$  are tangent to  $M_1$  and  $M_2$ , respectively. Then the sectional

curvature in terms of general doubly warped product is defined by

$$\sum_{1 \le i \le n_1} \sum_{n_1 + 1 \le j \le n} K(e_i \land e_j) = \frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2}, \tag{6.3.5}$$

for each  $j = n_1 + 1, \dots, n$ .

#### 6.3.2 Existence and non-existence of doubly warped product submanifold

A (2m+1)-dimensional smooth manifold  $\widetilde{M}$  is called *locally conformal almost cosymplectic* manifold with almost contact structure  $(\varphi, \xi, \eta, g)$  which satisfies the following:

$$(\widetilde{\nabla}_U \varphi) V = \vartheta \left( g(\varphi U, V) - \eta(V) \varphi U \right), \tag{6.3.6}$$

$$\widetilde{\nabla}_U \xi = \vartheta \left( U - \eta(U) \xi \right), \tag{6.3.7}$$

for any U, V tangent to  $\widetilde{M}$  and  $\omega = \vartheta \eta$ , where  $\vartheta : M \to \mathbb{R}$  is locally conformal function (see Yoon et al. (2004). Let the conformal function  $\vartheta = 0$  and  $\vartheta = 1$ , then  $\widetilde{M}$  becomes cosympelctic manifold and Kenmotsu manifold, respectively (see Yoon (2004); Murathan et al. (2006)). For an almost contact metric manifold  $\widetilde{M}$ , a plane section  $\sigma$  in  $T_p\widetilde{M}$  of  $\widetilde{M}$ is said to be a  $\varphi$  – section if  $\sigma \perp \xi$  and  $\varphi(\sigma) = \sigma$ . The sectional curvature  $\widetilde{K}(\sigma)$  does not depend on the choice of the  $\varphi$ -scetion  $\sigma$  of  $T_p\widetilde{M}$  at each point  $p \in \widetilde{M}$ , then  $\widetilde{M}$  is called a manifold with pointwise constant  $\varphi$  – sectional curvature. In this case for any  $p \in \widetilde{M}$  and for  $\varphi$ -scetion  $\sigma$  of  $T_p\widetilde{M}$ , the function c defined by  $c(p) = \widetilde{K}(p)$  is said to be  $\varphi$  – sectional curvature of  $\widetilde{M}$ . Thus for a locally conformal almost cosymplectic manifold  $\widetilde{M}$  of dimension  $\geq 5$  with pointwise  $\varphi$ -sectional curvature c, its curvature tensor  $\widetilde{R}$  is defined as:

$$\widetilde{R}(X,Y,Z,W) = \frac{c-3\vartheta^2}{4} \left( g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \right) + \frac{c+\vartheta^2}{4} \left( g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W) - 2g(X,\varphi Y)g(Z,\varphi W) \right) - \left(\frac{c+\vartheta^2}{4} + \vartheta'\right) \left( g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(X)\eta(W) + g(Y,Z)\eta(X)\eta(W)g(Y,W)(X)\eta(Z) \right),$$
(6.3.8)

for any X, Y, Z, W tangent to  $\widetilde{M}(c)$ , where  $\vartheta$  is the conformal function such that  $\omega = \vartheta \eta$ and  $\vartheta' = \xi \vartheta$ . Moreover, *c* is the function of pointwise constant  $\varphi$ -sectional curvature of  $\widetilde{M}$ . If we consider the function  $\vartheta = 0$  and  $\vartheta = 1$ , then  $\widetilde{M}(c)$  generalizes to cosymplectic space form and Kenmotsu space form, respectively (see Yoon et al. (2004); Murathan et al. (2006)).

On the other hand, first we analyze general doubly warped products in locally conformal almost cosymplectic manifold such that  $M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}$  be an isometric immersion from a doubly warped product  $_{f_2}M_1 \times_{f_1} M_2$  into a locally conformal almost cosymplectic manifold  $\widetilde{M}$ . Assume that  $\xi \in \Gamma(TM_1)$  and  $X \in \Gamma(TM_2)$  and from (6.3.7), we obtain

$$\nabla_X \xi = \vartheta \{ X - \eta(X) \xi \}, \tag{6.3.9}$$

which implies by using (2.3.2) and  $\eta(X) = 0$ , that

$$\nabla_X \xi = \vartheta X, \quad h(X,\xi) = 0. \tag{6.3.10}$$

Using (6.3.2) in the first relation of above equation, we find

$$(X\ln f_2)\xi + (\xi\ln f_1)X = \vartheta X. \tag{6.3.11}$$

Thus taking the inner product with  $\xi$  in (6.3.11), we obtain  $X \ln f_2 = 0$ , i.e.,  $f_2$  is constant on  $M_2$ ., it means that, there is no doubly warped product in a locally conformal almost cosymplectic manifold with  $\xi$  tangent to  $M_1$ . Moreover, if  $\xi \in \Gamma(TM_2)$  and  $Z \in \Gamma(TM_1)$ , then again from (6.3.7), we have

$$\overline{\nabla_Z}\xi = \vartheta Z, \tag{6.3.12}$$

From (2.3.2), we get

$$\nabla_Z \xi = \vartheta Z, \quad h(Z,\xi) = 0. \tag{6.3.13}$$

Again using (6.3.2) in (6.3.13) and then taking inner product with  $\xi$ , it is easy to see that  $f_1$  is also constant function on  $M_1$ . Therefore, in both cases, we find that any one of the warping function is constant. Thus, we conclude that there does not exist doubly warped product submanifold in a locally conformal almost cosymplectic manifold with  $\xi$  tangent to submanifold. If we choose  $\xi$  normal to submanifold M, then there is a non-trivial doubly warped product in a locally conformal almost cosymplectic manifold which is called C-totally real doubly warped product. In the next section, we obtain some geometric inequalities for such type doubly warped product immersions.

## 6.3.3 Main inequalities of *C*-totally real doubly warped products

**Theorem 6.3.1.** Let  $\widetilde{M}(c)$  be a (2m+1)-dimensional locally conformal almost cosymplectic manifold and  $\phi :_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$  be an isometric immersion of an n-dimensional *C*-totally real doubly warped product into  $\widetilde{M}(c)$  such that *c* is pointwise constant  $\varphi$ -sectional curvature. Then

*(i) The relation between warping functions and the squared norm of mean curvature is given by* 

$$\frac{n_2\Delta_1 f_1}{f_1} + \frac{n_1\Delta_2 f_2}{f_2} \le \frac{n^2}{4} ||H||^2 + \frac{c - 3\vartheta^2}{4} n_1 n_2, \tag{6.3.14}$$

where  $n_i = \dim M_i$ , i = 1, 2 and  $\Delta^i$  is the Laplacian operator on  $M_i$ , i = 1, 2.

(ii) The equality sign holds in the above inequality if and only if  $\phi$  is mixed totally geodesic immersion and  $n_1H_1 = n_2H_2$ , where  $H_1$  and  $H_2$  are partial mean curvature vectors on  $M_1$  and  $M_2$ , respectively.

*Proof.* Assume that  $_{f_2}M_1 \times_{f_1} M_2$  be a *C*-totally real doubly warped product submanifold in a locally conformal almost cosymplectic manifold  $\widetilde{M}(c)$  with pointwise constant

 $\varphi$ -sectional curvature c. Then from Gauss equation (2.3.6) and (6.3.8), we derive

$$2\rho = \frac{c - 3\vartheta^2}{4}n(n-1) + n^2||H||^2 - ||h||^2.$$
(6.3.15)

By assuming

$$\delta = 2\rho - \frac{c - 3\vartheta^2}{4}n(n-1) - \frac{n^2}{2}||H||^2.$$
(6.3.16)

Then from (6.3.15) and (6.3.16), it follows that

$$n^{2}||H||^{2} = 2(\delta + ||h||^{2}).$$
(6.3.17)

Thus from the orthonormal frame  $\{e_1, e_2, \cdots, e_n\}$ , the above equation takes the form

$$\left(\sum_{r=n+1}^{2m+1}\sum_{i=1}^{n}h_{ii}^{r}\right)^{2} = 2\left\{\delta + \sum_{r=n+1}^{2m+1}\sum_{i=1}^{n}(h_{ii}^{r})^{2} + \sum_{r=n+1}^{2m+1}\sum_{i
(6.3.18)$$

which simplifies as:

$$\frac{1}{2} \left( h_{11}^{n+1} + \sum_{i=2}^{n} h_{ii}^{n+1} + \sum_{t=n_1+1}^{n} h_{tt}^{n+1} \right)^2 = \delta + (h_{11}^{n+1})^2 + \sum_{i=2}^{n_1} (h_{ii}^{n+1})^2 + \sum_{t=n_1+1}^{n} (\delta_{tt}^{n+1})^2 - \sum_{2 \le j \ne l \le n_1} h_{jj}^{n+1} h_{ll}^{n+1} - \sum_{N_1+1 \le t \ne s \le n} h_{tt}^{n+1} h_{ss}^{n+1} + \sum_{i< j=1}^{n} (h_{ij}^{n+1})^2 + \sum_{r=n+1}^{n} (h_{ij}^{r})^2.$$
(6.3.19)

Assume that  $a_1 = h_{11}^{n+1}$ ,  $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$  and  $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$ . Then applying the Lemma 6.2.1 in (6.3.19), it is easily seen that

$$\frac{\delta}{2} + \sum_{i< j=1}^{n} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \le \sum_{2 \le j \ne l \le n_1} h_{jj}^{n+1} h_{ll}^{n+1} + \sum_{n_1+1 \le t \ne s \le n} h_{tt}^{n+1} h_{ss}^{n+1}.$$
(6.3.20)

The equality holds in (6.3.20) if and only if satifies

$$\sum_{i=1}^{n} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}.$$
(6.3.21)

On the other side, from (6.3.8), we find that

$$\frac{n_2\Delta_1 f_1}{f_1} + \frac{n_1\Delta_2 f_2}{f_2} = \rho - \sum_{1 \le j \ne k \le n_1} K(e_i \land e_k) - \sum_{n_1 + 1 \le t \ne s \le n} K(e_t \land e_s).$$
(6.3.22)

The equations (6.3.5) and (6.3.22) imply that

$$\frac{n_2\Delta_1 f_1}{f_1} + \frac{n_1\Delta_2 f_2}{f_2} = \rho - \frac{c - 3\vartheta^2}{8} n_1(n_1 - 1) - \sum_{r=1}^{2m+1} \sum_{2 \le j \ne k \le n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) - \frac{c - 3\vartheta^2}{8} n_2(n_2 - 1) - \sum_{r=1}^{2m+1} \sum_{n_1 + 1 \le t \ne s \le n} (h_{tt}^r h_{ss}^r - (h_{ts}^r)^2). \quad (6.3.23)$$

After combining (6.3.20) and (6.3.22), it can be easily seen that

$$\frac{n_2\Delta_1 f_1}{f_1} + \frac{n_1\Delta_2 f_2}{f_2} \le \rho - \frac{c - 3\vartheta^2}{8}n(n-1) + \frac{c - 3\vartheta^2}{4}n_1n_2 - \frac{\delta}{2}.$$
(6.3.24)

Hence, from (6.3.16), the inequality (6.3.24) reduce to

$$\frac{n_2\Delta_1 f_1}{f_1} + \frac{n_1\Delta_2 f_2}{f_2} \le \frac{n^2}{4} ||H||^2 + \frac{c - 3\vartheta^2}{4} n_1 n_2, \tag{6.3.25}$$

which is the inequality (6.3.14). On the other hand, the equality sign holds in (6.3.14) if and only if from (6.3.21), we get  $n_1H_1 = n_2H_2$ . Moreover, from (6.3.20), we find that  $h_{ij}^r = 0$ , for each  $1 \le i \le n_1$ ,  $n_1 + 1 \le j \le n$  and  $n + 1 \le r \le 2m + 1$ , which means that  $\phi$  is a mixed totally geodesic immersion. The converse part is straightforward. This completes the proof of the theorem.

Now, we have the following applications of Theorem 6.3.1

**Remark 6.3.1.** If we substitute either  $f_1 = 1$  or  $f_2 = 1$  in Theorem 6.3.1, then Theorem 6.3.1 turn into *C*-totally real warped product, i.e.,

**Corollary 6.3.1.** Let  $\widetilde{M}(c)$  be a (2m+1)-dimensional locally conformal almost cosymplectic manifold and  $\phi: M_1 \times_f M_2 \to \widetilde{M}(c)$  be an isometric immersion of an n-dimensional *C*-totally real warped product into  $\widetilde{M}(c)$  such that *c* is pointwise constant  $\varphi$ -sectional curvature. Then

*(i) The relation between warping function and the squared norm of mean curvature is given by* 

$$\frac{n_2 \Delta f}{f} \le \frac{n^2}{4} ||H||^2 + \frac{c - 3\vartheta^2}{4} n_1 n_2, \tag{6.3.26}$$

where  $n_i = \dim M_i$ , i = 1, 2 and  $\Delta$  is the Laplacian operator on  $M_1$ .

(ii) The equality sign holds in the above inequality if and only if  $\phi$  is mixed totally geodesic immersion and  $n_1H_1 = n_2H_2$ , where  $H_1$  and  $H_2$  are partial mean curvature vectors of  $M_1$  and  $M_2$ , respectively.

**Remark 6.3.2.** If we put either  $f_1 = 1$  or  $f_2 = 1$  and  $\vartheta = 0$  in Theorem 6.3.1, then it is the same inequality Theorem 3.2 which obtained by Yoon et al. (2004).

**Remark 6.3.3.** If we consider either  $f_1 = 1$  or  $f_2 = 1$  and  $\vartheta = 1$  in Theorem 6.3.1, then the Theorem 6.3.1 is exactly same as Lemma 2.1 which also derived by Murathan et al. (2006).

**Corollary 6.3.2.** Let  $\phi: M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$  be an isometric immersion of an *n*-dimensional *C*-totally real doubly warped product into a locally conformal almost cosympelctic manifold  $\widetilde{M}(c)$  with *c* a pointwise constant  $\varphi$ -sectional curvature such that the warping functions are harmonic. Then, *M* is not a minimal submanifold of  $\widetilde{M}$  with inequality

$$\vartheta > \sqrt{\frac{c}{3}}.$$

**Corollary 6.3.3.** Let  $\phi: M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$  be an isometric immersion of an *n*-dimensional *C*-totally real doubly warped product into a locally conformal almost cosympelctic manifold  $\widetilde{M}(c)$  with *c* a pointwise constant  $\phi$ -sectional curvature. Suppose that the warping functions  $f_1$  and  $f_2$  of *M* are eigenfunctions of Laplacian on  $M_1$  and  $M_2$  with corresponding eigenvalues  $\lambda_1 > 0$  and  $\lambda_2 > 0$ , respectively. Then *M* is not a minimal submanifold of  $\widetilde{M}$  with inequality

$$\vartheta \ge \sqrt{\frac{c}{3}}.$$

**Corollary 6.3.4.** Let  $\phi: M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$  be an isometric immersion of an *n*-dimensional *C*-totally real doubly warped product into a locally conformal almost cosympelctic manifold  $\widetilde{M}(c)$  with *c* a pointwise constant  $\varphi$ -sectional curvature. Suppose that one of the warping function is harmonic and other one is eigenfunction of Laplacian with corresponding eigenvalue  $\lambda > 0$ . Then *M* is not minimal in  $\widetilde{M}$  with inequality

$$\vartheta \ge \sqrt{\frac{c}{3}}$$

Now motivated from Chen (2004) study, we establish the following sharp relationship for the squared norm of the mean curvature vector in terms of intrinsic invariants and the scalar curvature.

**Theorem 6.3.2.** Let  $\widetilde{M}(c)$  be a (2m+1)-dimensional locally conformal almost cosymplectic manifold and  $\phi : M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$  be an isometric immersion of an n-dimensional C-totally real doubly warped product into  $\widetilde{M}(c)$  such that c is pointwise constant  $\varphi$ -sectional curvature. Then

(i)

$$\left(\frac{\Delta_1 f_1}{n_1 f_1}\right) + \left(\frac{\Delta_2 f_2}{n_2 f_2}\right) \ge \rho - \frac{n^2 (n-2)}{2(n-1)} ||H||^2 - \left(\frac{c-3\vartheta^2}{4}\right)(n+1)(n-2), \quad (6.3.27)$$

where  $n_i = \dim M_i$ , i = 1, 2 and  $\Delta^i$  is the Laplacian operator on  $M_i$ , i = 1, 2.

- *(ii) If the equality sign holds in* (6.3.27)*, then equalities conditions* (6.3.40) *holds auto-matically.*
- (iii) If n = 2, then equality sign in (6.3.27) holds identically.

*Proof.* Assume that  $_{f_2}M_1 \times_{f_1} M_2$  be a *C*-totally real doubly warped product in a locally conformal almost cosymplectic manifold  $\widetilde{M}(c)$  with pointwise constant  $\varphi$ -scetional curvature *c*. Then from Gauss equation, we find

$$2\rho = \left(\frac{c-3\vartheta^2}{4}\right)n(n-1) + n^2||H||^2 - ||h||^2.$$
(6.3.28)

Now we consider that

$$\delta = 2\rho - \left(\frac{c-3\vartheta^2}{4}\right)(n+1)(n-2) - \frac{n^2(n-2)}{n-1}||H||^2.$$
(6.3.29)

Then from (6.3.28) and (6.3.29), it follows that

$$n^{2}||H||^{2} = (n-1)\left[||h||^{2} + \delta - \left(\frac{c-3\vartheta^{2}}{2}\right)\right].$$
(6.3.30)

Let  $\{e_1, e_2, \dots, e_n\}$  be an orthonormal frame for *M*, the above equation can be written at

the following form

$$\left(\sum_{r=n+1}^{2m+1}\sum_{i=1}^{n}h_{ii}^{r}\right)^{2} = (n-1)\left\{\delta + \sum_{r=n+1}^{2m+1}\sum_{i=1}^{n}(h_{ii}^{r})^{2} + \sum_{r=n+1}^{2m+1}\sum_{i< j=1}^{n}(h_{ij}^{r})^{2} + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^{2} - \left(\frac{c-3\vartheta^{2}}{2}\right)\right\},$$

which implies that

$$\left(h_{11}^{n+1} + \sum_{i=2}^{n_1} h_{ii}^{n+1} + \sum_{t=n_1+1}^{n} h_{tt}^{n+1}\right)^2 = \delta + (h_{11}^{n+1})^2 + \sum_{i=2}^{n_1} (h_{ii}^{n+1})^2 + \sum_{t=n_1+1}^{n} (h_{tt}^{n+1})^2 - \sum_{2 \le j \ne l \le n_1} h_{jj}^{n+1} h_{ll}^{n+1} - \sum_{2 \le j \ne l \le n_1} h_{tt}^{n+1} h_{ss}^{n+1} + \sum_{i< j=1}^{n} (h_{ij}^{n+1})^2 + \sum_{r=n+1}^{n} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \left(\frac{c-3\vartheta^2}{2}\right).$$
(6.3.31)

Now we setting  $a_1 = h_{11}^{n+1}$ ,  $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$  and  $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$ . Then from Lemma 6.2.1 and equation (6.3.31), we get

$$\frac{\delta}{2} - \left(\frac{c-3\vartheta^2}{2}\right) + \sum_{i< j=1}^n (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n (h_{ij}^r)^2 \le \sum_{2\le j \ne l \le n_1} h_{jj}^{n+1} h_{ll}^{n+1} + \sum_{n_1+1 \le t \ne s \le n} h_{tt}^{n+1} h_{ss}^{n+1}.$$
(6.3.32)

with equality holds in (6.3.32) if and only if

$$\sum_{i=1}^{n} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}.$$
(6.3.33)

On the other hand, from (6.3.32) and (2.3.10), we have

$$K(e_{1} \wedge e_{n_{1}+1}) \geq \sum_{r=n+1}^{2m+1} \sum_{j \in P_{1n_{1}+1}} (h_{1j}^{r})^{2} + \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{j \in P_{1n_{1}+1}} (h_{ij}^{r})^{2} + \sum_{r=n+1}^{2m+1} \sum_{j \in P_{1n_{1}+1}} (h_{n_{1}+1j}^{r})^{2} + \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n_{1}+1} (h_{ij}^{r})^{2} + \frac{\delta}{2}.$$
(6.3.34)

where  $P_{1n_1+1} = \{1, \dots, n\} - \{1, n_1+1\}$ . Then it implies that

$$K(e_1 \wedge e_{n_1+1}) \ge \frac{\delta}{2}.\tag{6.3.35}$$

As  $M =_{f_2} M_1 \times_{f_1} M_2$  is a C-totally real doubly warped product submanifold, we have

 $\nabla_X Z = \nabla_Z X = (X \ln f_1)Z + (Z \ln f_2)X$  for any unit vector fields X and Z tangent to  $M_1$  and  $M_2$ , respectively. Then from (6.3.4), (6.3.29) and (6.3.35), the scalar curvature derive as;

$$\rho \leq \frac{1}{f_1} \{ (\nabla_{e_1} e_1) f_1 - e_1^2 f_1 \} + \frac{1}{f_2} \{ (\nabla_{e_2} e_2) f_2 - e_2^2 f_2 \} + \frac{n^2 (n-2)}{2(n-1)} ||H||^2 \\
+ \left( \frac{c - 3\vartheta^2}{4} \right) (n+1)(n-2).$$
(6.3.36)

Let the equality holds in (6.3.36), then including all leaving terms in (6.3.32) and (6.3.35), we obtain the following conditions, i.e.,

$$h_{1j}^{r} = 0, \ h_{jn_{1}+1}^{r} = 0, \ h_{ij}^{r} = 0, \ where \ i \neq j, \ and \ r \in \{n+1, \cdots, 2m+1\};$$
  
$$h_{1j}^{r} = h_{jn_{1}+1}^{r} = h_{ij}^{r} = 0, \ and \ h_{11}^{r} + h_{n_{1}+1n_{1}+1}^{r} = 0, \ i, j \in P_{1n_{1}+1}, \ r = n+2, \cdots, 2m+1.$$
  
(6.3.37)

Similarly, we extend the relation (6.3.36) as follows

$$\rho \leq \frac{1}{f_1} \{ (\nabla_{e_{\alpha}} e_{\alpha}) f_1 - e_{\alpha}^2 f_1 \} + \frac{1}{f_2} \{ (\nabla_{e_{\beta}} e_{\beta}) f_2 - e_{\beta}^2 f_2 \} + \frac{n^2 (n-2)}{2(n-1)} ||H||^2 \\
+ \left( \frac{c - 3\vartheta^2}{4} \right) (n+1)(n-2), \tag{6.3.38}$$

for any  $\alpha = 1, \dots, n_1$  and  $\beta = n_1 + 1, \dots n_n$ . Taking the summing up  $\alpha$  from 1 to  $n_1$  and up  $\beta$  from  $n_1 + 1$  to  $n_2$  respectively, we arrive at

$$n_{1}.n_{2}.\rho \leq \frac{n_{2}.\Delta_{1}f_{1}}{f_{1}} + \frac{n_{1}.\Delta_{2}f_{2}}{f_{2}} + \frac{n^{2}.n_{1}.n_{2}(n-2)}{2(n-1)}||H||^{2} + \left(\frac{c-3\vartheta^{2}}{4}\right)n_{1}n_{2}(n+1)(n-2).$$
(6.3.39)

Similarly, the equality sign holds in (6.3.39) identically. Thus the equality sign in (6.3.39) holds for each  $\alpha \in \{1, \dots, n_1\}$  and  $\beta \in \{n_1 + 1, \dots, n\}$ . Then we get the following;

$$h_{\alpha j}^{r} = 0, \ h_{ij}^{r} = 0, \ h_{ij}^{r} = 0, \ where \ i \neq j, \ and \ r \in \{n+1, \cdots, 2m+1\};$$
  
 $h_{\alpha j}^{r} = h_{ij}^{r} = h_{ij}^{r} = 0, \ and \ h_{\alpha \alpha}^{r} + h_{\beta \beta}^{r} = 0, \ i, j \in P_{1n_{1}+1}, \ r = n+2, \cdots, 2m+1.$   
(6.3.40)

Moreover, If n = 2, then  $n_1 = n_2 = 1$ . Thus from (6.3.4), we get  $\rho = \Delta_1 f_1 + \Delta_2 f_2$ . Hence,

the equality in (6.3.27) holds. This proves the theorem completely.

We also have the following applications of Theorem 6.3.2.

*Remark* 6.3.4. If either  $f_1 = 1$  or  $f_2 = 1$  in Theorem 6.3.2, then we get following corollary for a *C*-totally real warped product.

**Corollary 6.3.5.** Let  $\widetilde{M}(c)$  be a (2m+1)-dimensional locally conformal almost cosymplectic manifold and  $\phi : M = M_1 \times_f M_2 \to \widetilde{M}(c)$  be an isometric immersion of an n-dimensional C-totally real warped product into  $\widetilde{M}(c)$  such that c is pointwise constant  $\varphi$ -sectional curvature. Then

(i)

$$\left(\frac{\Delta f}{n_1 f}\right) \ge \rho - \frac{n^2(n-2)}{2(n-1)} ||H||^2 - \left(\frac{c-3\vartheta^2}{4}\right)(n+1)(n-2), \tag{6.3.41}$$

where  $n_i = \dim M_i$ , i = 1, 2 and  $\Delta$  is the Laplacian operator on  $M_1$ .

- *(ii)* If the equality sign holds in (6.3.41), then the condition (6.3.40) holds automatically.
- (iii) If n = 2, then equality sign in (6.3.41) holds identically.

We also have the following special cases of Theorem 6.3.2.

**Corollary 6.3.6.** Let  $\phi: M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$  be an n-dimensional *C*-totally real doubly warped product into a locally conformal almost cosympelctic manifold  $\widetilde{M}(c)$  with *c* a pointwise constant  $\varphi$ -sectional curvature such that the warping functions are harmonic. Then *M* is not minimal in  $\widetilde{M}$  with inequality

$$c > \frac{4\rho}{(n+1)(n-2)} + 3\vartheta^2.$$

**Corollary 6.3.7.** Let  $\phi: M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$  be an n-dimensional *C*-totally real doubly warped product into a locally conformal almost cosympelctic manifold  $\widetilde{M}(c)$  with c a pointwise constant  $\varphi$ -sectional curvature. Suppose that the warping functions  $f_1$  and  $f_2$  of *M* are eigenfunctions of Laplacians on  $M_1$  and  $M_2$  with corresponding eigenvalues  $\lambda_1 > 0$  and  $\lambda_2 > 0$ , respectively. Then *M* is not minimal in  $\widetilde{M}$  with inequality

$$c \geq \frac{4\rho}{(n+1)(n-2)} + 3\vartheta^2$$

**Corollary 6.3.8.** Let  $\phi: M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$  be an n-dimensional C-totally real doubly warped product into a locally conformal almost cosympelctic manifold  $\widetilde{M}(c)$  with c a pointwise constant  $\varphi$ -sectional curvature. Suppose that one of the warping function is harmonic and other one is eigenfunction of the Laplacian with corresponding eigenvalue  $\lambda > 0$ . Then M is not minimal in  $\widetilde{M}$  with inequality

$$c \geq \frac{4\rho}{(n+1)(n-2)} + 3\vartheta^2.$$

If we combine both Theorem 6.3.1 and Theorem 6.3.2, then we get the following interesting result.

**Theorem 6.3.3.** Assume that  $\phi: M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$  be an isometric immersion of an n-dimensional C-totally real doubly warped product into a (2m+1)-dimensional locally conformal almost cosymplectic manifold  $\widetilde{M}(c)$  such that c is pointwise constant  $\varphi$ -sectional curvature. Then

$$\begin{split} \rho - \frac{n^2(n-2)}{2(n-1)} ||H||^2 - \left(\frac{c-3\vartheta^2}{4}\right)(n+1)(n-2) &\leq \left(\frac{\Delta_1 f_1}{n_1 f_1}\right) + \left(\frac{\Delta_2 f_2}{n_2 f_2}\right) \\ &\leq \frac{n^2}{4n_1 n_2} ||H||^2 + \left(\frac{c-3\vartheta^2}{4}\right). \end{split}$$

*Remark* 6.3.5. Theorem 6.3.3 present an upper and lower bounds for warping functions of a C-totally real doubly warped product submanifold.

*Remark* 6.3.6. If either  $f_1 = 1$  or  $f_2 = 1$  in Theorem 6.3.3, then we get following corollary for a *C*-totally real warped product.

**Corollary 6.3.9.** Assume that  $\phi: M = M_1 \times_f M_2 \to \widetilde{M}(c)$  be an isometric immersion of an n-dimensional C-totally real doubly warped product into a (2m+1)-dimensional locally conformal almost cosymplectic manifold  $\widetilde{M}(c)$  such that c is pointwise constant  $\phi$ -sectional curvature. Then

$$\begin{split} \rho - \frac{n^2(n-2)}{2(n-1)} ||H||^2 - \left(\frac{c-3\vartheta^2}{4}\right)(n+1)(n-2) &\leq \left(\frac{\Delta f}{n_1 f}\right) \leq \frac{n^2}{4n_1 n_2} ||H||^2 \\ &+ \left(\frac{c-3\vartheta^2}{4}\right). \end{split}$$

## **CHAPTER 7**

# **CONCLUSION AND FUTURE WORK**

## 7.1 INTRODUCTION

The present chapter mainly focus to summarize the work done in this thesis and to propose some open problems related to all previous chapters. We divide this chapter into two different sections. The first section summarizes some research findings in this thesis. In the second section, we provide some future work of this research in geometric of Riemannian submanifolds and their warped products.

## 7.2 SUMMARY OF FINDINGS

We consider some significantly important findings related to CR-warped product submanifolds and some characterizations of CR-warped products.

# **7.2.1** CR-warped product submanifolds of the form $M_T \times_f M_{\perp}$

It should be noted that CR-structures are so important from a mathematical and physical point of view, particularly general relativity theory. Thus CR-structure form to CRproducts. A CR-submanifold is called a CR-product if it is a Riemannian product of a holomorphic (invariant) submanifold and a totally real (anti-invariant) submanifold. Similarly, a CR-warped product is the product of holomorphic and totally real submanifolds with non-constant warping function. Numerous characterizations of CR-warped products can be found in Chen (2013) for different ambient manifolds in terms of shape operator. In the present thesis, we use the properties of tensor fields P and F to characterise CR-submanifold to be a CR-warped product in nearly Sasakian manifolds. Moreover, we define a new structure as T-manifold which generalizes to Kaehler manifold and cosymplectic manifold under some special cases. In this sequel, we also define T-space form with constant sectional curvature c which also generalize to complex space and cosymplectic space from. It can be concluded that the results which we obtain for CR-warped product in T-manifold and T-space forms are generalized to Kaehler and cosymplectic manifolds, and complex space form and cosymplectic space form, respectively, (see Chapter 3).

**Result 1:** We notice that in Chapter 4, it is difficult to obtain the first inequality of Chen for the squared norm *h* of warped product pseudo-slant submanifolds of any kind of structures which relate to warping function and slant immersions. Alternatively, the mixed totally geodesic warped product submanifold idea comes out in the Riemannian geometry. First, Sahin (2009b) gave the characterization of warped product pseudo-slant submanifold in a Kaehler manifold and obtain an inequality for extrinsic invariant in case of mixed totally geodesic. Following Sahin (2009b), we develope some geometric inequalities for mixed totally geodesic warped product pseudo-slant submanifolds in locally Riemannian product manifolds and nearly Kenmotsu manifold which are the classes of almost Hermitian and almost contact metric manifolds, respectively. Also, we give a characterization theorem and establish some examples on warped product pseudo-slant of locally Riemannian product manifolds. Hence, our results achieved the objective of finding some inequalities in the Riemannian geometry.

**Result 2:** It is well known that the semi-slant submanifolds were introduced as a generalization of proper slant and proper CR-submanifolds. It is more interesting to see that the warped product semi-slant submanifold does not succeed to generalize CR-warped product submanifolds in most the structures such as Kaehler, cosymplectic and Sasakian manifolds. Based on our study in Chapter 5, we find a contradiction toward warped product semi-slant submanifold. That is the notion of pointwise slant submanifold introduced as the generalization of slant, invariant and anti-invariant submanifolds. It follows that the warped product pointwise semi-slant submanifold with pointwise slant fiber was studied by many mathematicians which successfully generalize to CR-warped products in Kaehler, cosymplectic and Sasakian manifolds such as Sahin (2013); Park (2014, 2015). By the isometrically embedding theorem of Nölker (1996), we extend the notion of

warped product pointwise semi-slant isometrically immersed into Kaehler, cosymplectic and Sasakian manifolds, then obtain geometric inequalities Theorem 5.2.9, 5.3.5, 5.4.2 for the second fundamental form in terms of scalar curvature and Laplacian of submanifold by using Gauss equation. Applying these Theorems to complex space form, cosymplectic space form and Sasakian space form, we establish the relation between the second fundamental form, Laplacian of warping function and constant sectional curvatures. The chapter is an attempt to study the geometry of the second fundamental form of a compact orientable Riemannian submanifold isometrically immersed into complex, cosymplectic and Sasakian space forms, when a submanifold M is product manifold and the first fundamental form of the immersion  $i: M \to \widetilde{M}(c)$  is a warped product. The proof of the results are based on the so called Bochner technique (see Yano & Kon (2012)). We use Green theorem on a compact manifold M and the given a smooth function  $f: M \to \mathbb{R}$ , one has  $\int_M \Delta f dV = 0$ . We attribute the result of Yano & Kon (1985), immediately follows:  $\Delta f = -div(\nabla f)$  and from Green lemma  $\int_M div(X) dV = 0$ , where X is any arbitrary vector field. We derive geometric necessary and sufficient for warped product pointwise semi-slant submanifolds to be Riemannian products by using the definitions of Hessian, Hamiltonian, Kinetic energy function and Euler-Lagrangian equation. Therefore, this thesis is providing some fascinating applications in physical sciences and finding magnificent applications in applied geometry as well.

**Result 3:** The doubly warped product submanifold is a new concept in differential geometry. This concept arises from the comparison between differents kind of warped products. By considering the C- totally real submanifold case, we discover that some of doubly warped product exist as the name of C- totally real doubly warped product submanifolds in locally conformal almost cosymplectic manifolds. Taking this concept in Chapter 6, we establish very interesting inequalities in terms of the warping function, the squared mean curvature, and scalar curvature which we can see both the upper bound and the lower bound of the functions  $\frac{\Delta f}{f}$ .

#### 7.3 FUTURE WORK

Although our research has achieved its objectives, we find that there are many open problems that we hope to solve in our future work. Taking a quick look to the existence of CR-warped products in nearly Sasakian and T-manifold in terms of endomorphisms, one can conclude that some characterization for CR-warped product submanifolds are left to be proved. Referring to this situation, some of the open problems that may arise are:

**Problem 7.1.** What are the necessary and sufficient conditions for the existence of CRwarped product submanifold of type  $M = M_T \times M_{\perp}$  in nearly Kenmotsu and nearly Trans-Sasakian manifolds in terms of endomorphisms?.

**Problem 7.2.** Do the characterization of proper warped product semi-slant submanifold of types  $M_T \times_f M_{\theta}$  in a nearly cosymplectic and a nearly Trans-Sasakian manifolds?.

For the next problems, we believe that warped product pseudo-slant submanifolds of the types  $M_{\theta} \times_f M_{\perp}$  exist in locally conformal Kaehler and nearly Trans-Sasakian manifolds. If they eist, it is not proven yet.

**Problem 7.3.** To obtain the geometric inequalities for mixed totally geodesic warped product pseudo-slant submanifold  $M_{\theta} \times_f M_{\perp}$  in locally conformal Kaehler and nearly Trans-Sasakian manifolds in the form the second fundamental form and slant immersions?

Taking into account of Chapter 5, we may find the following problem.

**Problem 7.4.** To obtain the inequality for second fundamental form with constant curvature *c* for warped product semi-slant submanifolds of locally conformal almost complex space forms by using Gauss method?.

These open problems shall remain as an on-going research in this study of Riemannian submanifold and their warped products.

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