

VISUAL REASONING IN SOLVING MATHEMATICAL
PROBLEMS ON FUNCTIONS AND THEIR DERIVATIVES
AMONG MALAYSIAN PRE-UNIVERSITY STUDENTS

HALIZA ABD HAMID

THESIS SUBMITTED IN FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

INSTITUTE OF GRADUATE STUDIES
UNIVERSITY OF MALAYA
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ABSTRACT

The main objective of this study was to develop a framework and assess the visual reasoning process adopted by pre-university students when integrating Cartesian graphs to solve mathematical problems on functions and derivatives. The study identified the usage levels of graphs, method of preference and graph-reasoning ability, and subsequently, the correlation among them. The study also investigated the misconceptions and difficulties faced by the students. The study employed a 3-phase descriptive quantitative method. The development of framework in Phase 1 involved a three-stage process: the document analysis on theories and models on visual reasoning and Cartesian graphs, focus group discussion among experts in the domain content and visual reasoning, and a 3-round Delphi method to confirm the framework. In Phase 2, three instruments were prepared; the Visual Representation Usage Level on four categories of using graphs, the Mathematical Visuality Test to measure the students' preference method and the Graph-Reasoning Test measuring their graph-reasoning ability. Phase 3 involved the collection and analysis of data on 194 pre-university students. The developed framework consisted of seven categories of encoding and five categories of decoding processes. Results indicated between 41.75% to 84.02% of the students were very positive towards the use of graphs and diagrams in the teaching and learning of mathematics although between 56.70% and 78.87% said they faced difficulties in constructing and interpreting them. Students exhibited fluctuating patterns of visual reasoning ability. In the encoding process, the students were categorised into three types of mathematical visuality; 26.8% were visual, 16.5% were partially-visual and 56.7% were non-visual. This exhibits their reluctance to sketch graphs although they managed to obtain the correct solutions which indicates that they had a predominant preference for algebraic method as compared to visuals. Responses in the decoding process were based on the three levels of graph reasoning. At least 68% of the

students managed to get correct solutions for all the *read graph* items. More than 70% of the students managed to obtain 75% and 43% of the items correct for the *read between graph* and *read beyond graph* respectively. These indicate that as the tasks get harder visually, more cognitive load is needed for the students to read and interpret graphs. Strong positive correlation values of at least 0.91 were obtained among the three decoding levels. They perform fundamental, operational and subjective types of errors and encountered the non-use of graph, generic and idiosyncratic difficulties when relating the algebraic forms of functions and derivatives to their visual representations on graphs and vice versa. The results of the study were significant in providing reliable and important ideas depicting the development of visual reasoning that is useful in the students' thinking and understanding, to guide the development of instructional materials to improve students' understanding and reasoning, for educators and curriculum developers to enhance the learning outcomes and teaching strategies to challenge students' thinking and reasoning skills, and to reduce gap in the literature and knowledge on visual reasoning in the Malaysian educational system.

ABSTRAK

Tujuan utama kajian adalah untuk membangun rangkakerja dan mentaksir proses penaakulan visual yang digunapakai oleh pelajar pra-universiti dengan mengintegrasikan graf Cartesian dalam menyelesaikan masalah fungsi dan terbitan. Kajian mengenalpasti tahap penggunaan graf, kaedah pilihan dan keupayaan penaakulan graf, korelasi di antara ketiga-tiganya dan menganalisa kesukaran dan kesilapan yang dihadapi oleh pelajar. Rekabentuk kajian adalah berdasarkan kaedah 3-fasa deskriptif kuantitatif. Fasa 1 melibatkan proses 3-peringkat pembangunan rangkakerja: analisa dokumen ke atas teori-teori dan model-model berkaitan penaakulan visual dan graf Cartesian, perbincangan kumpulan fokus melibatkan pakar-pakar dalam domain kandungan dan penaakulan visual, dan kaedah Delphi 3-pusingan bagi proses pengesahan rangkakerja. Tiga instrumen utama telah disediakan dalam Fasa 2: *Visual Representation Usage Levels* bagi mengukur empat kategori pelajar dalam penggunaan graf, *Mathematical Visuality Test* bagi mengukur kaedah pilihan dan *Graph-Reasoning Test* bagi mengukur kebolehan penaakulan visual pelajar. Fasa 3 melibatkan proses pengumpulan dan penganalisan data ke atas 194 orang pelajar pra-universiti. Rangkakerja yang dibangunkan mengandungi tujuh kelas pengekodan dan lima kelas penyahkodan. Seramai 41.57% hingga 84.02% pelajar adalah sangat positif terhadap penggunaan graf dan gambarajah dalam pembelajaran dan pengajaran matematik walaupun seramai 56.70% hingga 78.87% pelajar menghadapi kesukaran dalam membina dan mengtafsir graf. Pelajar mempamerkan corak kebolehan penaakulan visual yang turun naik. Dalam proses pengekodan, pelajar dapat dikelaskan dalam tiga kumpulan: 26.8% visual, 16.5% separa-visual dan 56.7% bukan-visual. Ini mempamerkan keengganan pelajar untuk melakar graf tetapi mereka mampu untuk menyelesaikan masalah tersebut. Ini menunjukkan bahawa kaedah algebra adalah menjadi pilihan pelajar berbanding kaedah visual. Analisis proses penyahkodan adalah berdasarkan tiga tahap penaakulan graf.

Sekurang-kurangnya 68% pelajar berjaya menyelesaikan masalah bagi semua item *baca graf*. Lebih dari 70% pelajar berjaya menyelesaikan 75% dan 43% masing-masing daripada item *baca antara graf* dan *baca luar graf*. Ini menunjukkan bahawa beban kognitif yang lebih diperlukan apabila tahap visual yang dipaparkan semakin sukar untuk dibaca dan ditaksir. Ketiga-tiga tahap decoding mempunyai korelasi positif yang kukuh dengan nilai sekurang-kurangnya 0.91. Pelajar telah mempamerkan jenis kesilapan asas, operasi dan subjektif, dan menghadapi kesukaran dari segi tidak menggunakan graf, generik dan pelbagai bila mengaitkan hubungan di antara bentuk algebra dan graf serta sebaliknya. Hasil kajian adalah sangat signifikan dalam menyediakan ide ketara dan yang boleh dipercayai bagi menggambarkan pembangunan penaakulan visual yang berguna dalam pemikiran dan pemahaman pelajar, membimbing untuk pembangunan bahan pengajaran dan pembelajaran bagi membaikpulih proses pemahaman dan penaakulan pelajar, para pendidik dan penyedia kurikulum boleh memperkuatkan hasil pembelajaran dan strategi pengajaran yang boleh mencabar proses pemikiran dan penaakulan pelajar, dan membantu merapatkan jurang dalam tinjauan literatur dan pengetahuan tentang penaakulan visual setiap peringkat dalam sistem pembelajaran Malaysia.

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LIST OF ABBREVIATIONS

CDC	Curriculum Development Centre
CGCS	Correct graph with correct solution
CGIS	Correct graph with incorrect solution
CSIR	Correct solution with invalid reason
CSNR	Correct solution with no reason
CSVR	Correct solution with valid reason
CMI	Communication of Mathematical Information
CR	Cartesian Graph
GRT	Graph Reasoning Test
IELTS	International English Language Testing System
IGCS	Incorrect graph with correct solution
IGIS	Incorrect graph with incorrect solution
ISINR	Incorrect solution with invalid reason or no reason
Int.	International
JPA	Jabatan Perkhidmatan Awam
Loc.	Local
KBSM	Kurikulum Bersepadu Sekolah Menengah
MARA	Majlis Amanah Rakyat
MC	Mathematical Content
ME	Mathematic Education
MKSA	Mathematical Knowledge and Skills and Their Application
MMP	Mathematical Modeling and Problem Solving
MPI	Mathematical Processing Instrument
MVT	Mathematical Visuality Test
NA	No answer / Not attempted

NCTM	National Council of Teachers of Mathematics
NGCS	No graph with correct solution
NGIS	No graph with incorrect solution
PETRONAS	Petroleum Nasional
SACE	South Australian Certificate of Education
SAM	South Australian Matriculation
SPM	Sijil Pelajran Malaysia
SPSS	Statistical Package for the Social Science
VR	Visual Reasoning
VR-G	Visual Reasoning over Graph
VRUL	Visual Representation Usage Levels
WISC-R	Wechsler Intelligence Scale for Children-Revised
YTN	Yayasan Tenaga Nasional

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CHAPTER 1: INTRODUCTION

1.1 Background of the Study

The role of visual reasoning in solving mathematical problems, such as functions and derivatives, at pre-university level has been a main area of interest in the mathematics educational research. For more than four decades, visual tools such as Cartesian graphs were regarded as essential in the work of mathematicians (Arcavi, 2003; Bayazit & Aksov, 2010; Esmeralda, 2011; Herbert, 2008; Huang, 2015; Ismail & Yusof, 2010; Kultur, Ozdemir & Konyalioglu, 2011; Oehrtman, Carlson & Thompson, 2008; Stylianou & Silver, 2004). The reform in the calculus teaching in 1980s acknowledged visual ability as an important cognitive tool to support the understanding of concepts in functions and derivatives and hence to help explore and solve related mathematical problems (Cheng, 2004; Goerdts, 2007; Hollebrands, 2007; Leng, 2011; Leung & Chan, 2004; Mariotti, Laabourdes & Façade, 2003; Orhun, 2012). Therefore, visual reasoning is regarded as a major strategy that underlies the teaching and learning of functions and derivatives through the manipulation of graphs at pre-university level (Boesen, Lithner & Palm, 2010; Calder, 2008; Herbert, 2008; Pjanic, Lidan & Kurtanovic, 2015; Rosken & Rolka, 2006).

Polya (1945), Presmeg (1986, 1989) and Zimmermann and Cunningham (1991), among the earliest group of researchers who had promoted the visual approach, emphasized the importance of visual thinking and the use of various visual representations that usually parallels success in solving mathematical problems. The Malaysian curriculum is, in general, very traditional. Visual displays play very limited roles and are regarded as illustrative graphics or guiding tools to help in solving the problems (Freitas & Sinclair, 2012; Natsheh & Karsenty, 2013). Visual reasoning should not be imposed as to explain, establish or provide new information or to be used

in the proving of mathematical concepts and properties. Realizing these states of affairs and that the methods of delivery are equally vital to equip and prepare students with understanding and readiness for higher levels of university learning (Bosse, Adu-Gyamfi & Cheetham, 2011; Font, Bolite & Acevedo, 2010; Noraini, 2006; Presmeg, 2006), ministries of education and curriculum developers globally (Japan: Ministry of Education, Culture, Sports, Science and Technology, 2010; NCTM, 2000; SACE, 2014; Singapore: Soh, 2008; UK, 2010) including Malaysia (CDC, 2010) have called for a search on different and upgraded perspectives on the teaching and learning of functions and derivatives (Guler & Ciltas, 2011; Yavuz, 2010) that focus on graphs as visual representations of relationships and connections between reasoning and conceptual knowledge (Ainsworth & Loizou, 2003; Calder, 2008; Dubinsky & Wilson, 2013)

1.1.1 Learning of Functions and Derivatives

The concept of function is fundamental to the understanding of derivatives (Brijlall & Maharaj, 2010; Gagatsis, Elia, Panoura, Gravvani & Spyrou, 2006; Mahir, 2010; Sofronas, DeFranco, Vinsonhaler, Gorgievski, Schroeder, & Hamelin, 2011). In the Malaysian classroom scenario, students are introduced to the notion of functions through the algebraic expression of $y = f(x)$ while NCTM (2000) emphasizes for the introduction of functions to be in the forms of words, tables and graphs and to be done as early as grade 3 through grade 5. Teachers, on the other hand, need to highlight on the importance of interpreting various visual representations such as Cartesian graphs as guides for the students to be able to relate and manipulate various types of functions and their derivatives (Bayazit & Aksov, 2010; Elia & Spyrou, 2006; Esmeralda, 2011; Judson & Nishimori, 2005).

Most calculus curriculums introduce derivative as the ratio of the change in the independent variable which is usually referred to as the x -value, with respect to a second

related variable which is usually referred to as the y -value, and is written as $\frac{dy}{dx}$. The matriculation curriculum of the South Australian Certificate of Education (SACE, 2015) proposed for functions and derivatives to be taught in three different perspectives: numerical, algebraic and graphical, and further be expressed through various representation systems such as symbols, expressions and graphs. The common learning outcome worldwide is for the students to be able to present and communicate the concepts of functions and derivatives in a variety of ways. The representations that students use indicate the meaning they attribute to the concepts of functions and derivatives (Abbey, 2008; Berry & Nyman, 2003; Herbert, 2008; Ismail & Yusof, 2010; Kultur, Ozdemir & Konyalioglu, 2011; Tokgoz, 2012). The numerical representation of derivative refers to the slope of the function at a particular point while the symbolic representations are used in determining the differentiation techniques. Solving derivatives to search for the properties of any function such as the stationary points and intervals of concavity demonstrate the need for graphical representations (Brijlall & Ndlovu, 2013; Hahkioemi, 2005; Huang, 2003; Orhun, 2012).

Concepts in differentiation are intertwined as how rings are connected in a chain. One concept is linked to the others encountered before or after it. Students who face a problem in understanding one concept may have difficulties in understanding the other related concepts (Dikici & Isleyen, 2004; Habre & Abboud, 2005; Kultur et al, 2011; Maharaj, 2013; Siyepu, 2012; Tarmizi, 2010; Ubuz, 2007). Among the important concepts in differentiation are limit, slope of tangent, properties of functions and their applications in various mathematical contexts and other disciplines such as engineering, chemistry, biology, and economics. In order for the students to understand and make sense of these concepts and their applications, they need to be well-equipped with strong conceptual knowledge in functions and Cartesian graphs (Alkharusi, Kazem, & Al-Musawai, 2011; Kultur et al, 2011; Lim & Hwa, 2007). Lack of understanding in

any of these concepts may lead to students facing difficulties in handling related problems and subsequently perform various types of misconceptions and errors.

Over the last 30 years, mathematics researchers and educators have identified that most of the students' misconceptions and difficulties in understanding and applying the concepts of derivatives were rooted in their weak understanding on the concepts of functions (Engelke, Oehrtman, & Carleson 2005; Ferguson, 2012; Makonye, 2011; Oehrtman, 2004, 2008a, 2008b ; Smith, 2003) and their inability to use functions to represent, relate, and reason on the relationships between any two related quantities and how they change with respect to one and the other (Carlson, Oehrtman & Engelke, 2010; Herbert, 2008). These lead to the students' lack of competency in understanding the main ideas on derivative such as limit and the first principles, tangent and normal, properties of functions and their applications into the real life situations (Brijlall & Maharaj, 2014; Cetin, 2009; Oehrtman, Carlson & Thompson, 2008; Smith, 2003).

1.1.2 Graphs as Visual Tools

Graphs in general, are indispensable visual tools used to encode and decode abstract ideas, to organize and analyse data (Batanero, Arteaga & Ruiz, 2009, Wall & Benson, 2009), present and communicate mathematical concepts and information (Heiser & Tversky, 2006; Tversky Lozano, Heiser, Lee & Daniel, 2005), and to stimulate creative and innovative reasoning or thinking (Booth & Koedinger, 2012; Boyce & DiPrima, 2009). Cartesian graphs, in specific, are efficacious in their use in different mathematical areas such as calculus and trigonometry, and other disciplines of study such as physics and economics (Kultur, Ozdemir & Konyalioglu, 2011; Lambertus, 2007; Syed Mustapha, 2007). Among the uses of Cartesian graphs are: 1) to present complicated and complex data in a concise and precise manners for readers to be able to make meaning of (Alacaci, Lewis, O'Brien & Jiang, 2011, Bowen & Roth, 2005; Elia &

Philippou, 2004; Monteiro & Ainley, 2004), 2) to determine the co-variation and correlation between the data and variables for the readers to be able to relate them easily (Bowen & Roth, 2005; Connery, 2007; Glazer, 2011; Leung & Chan, 2004; Uesaka & Manalo, 2007, 2011), 3) to clarify the meaning of the data for the reader to make decision and deduction (Belenky & Schalk, 2014). They are also utilized in the textbooks, examinations and other education contexts (Edens & Potter, 2008) to reduce number of solution steps and procedures, and to coordinate comparisons of variables which usually require heavier cognitive load.

The calculus contents, be it functions or derivatives, consist of many visual components especially graphs. The approach of 'draw a graph' is strongly encouraged as a tool for solving problems on functions and derivatives (NCTM, 2000). A graph is particularly an effective visual representation because it utilises the spatial arrangement of the related variables to depict a clear and logical relationship which holistically represent complex structures and processes (Stern, Aprea & Ebner, 2003; Terwel, van Oers, van Dijk & van den Eden, 2009; Yavuz, 2010). The process of generating graphs is able to assist students to conceptualize the problem structure that later leads to successful problem solution. On the other hand, improper graphical representation of a problem, which may result from the students' perception on its use and efficacy or inadequate knowledge about the structure and properties of graph, may limit students' capabilities to arrive to the solutions (Lassak, 2009; Roth & Jin Lee, 2004; Sheehan & Nillas, 2010; Stavridou & Kakana, 2008; Uesaka, Manalo & Emmanuel, 2011).

Graphs as visual representations of information play key roles in determining students' understanding on the ideas of functions and derivatives, and in the reasoning or making sense of their concepts. These make drawing and interpreting graphs as essential mathematical skills in the calculus courses (Ubus, 2007). The ability to retrieve and reason about information embedded in graphs is a skill which require the

complex interaction of three primary elements: the cognitive ability of the students (Bowen & Roth, 2002, 2003; Glazer, 2011; Grueber, 2011; Presmeg, 1986, 1989; Sharma, 2006, 2013), the graphical characteristics and properties of the graph (Friel, Curcio & Bright, 2001; Lee, Khng, Ng & Ng, 2013), and the requirements of the tasks and subject content (Munez, Orrantia & Rosales, 2013; Shah & Hoeffner, 2002; Uesaka & Manalo, 2011). Therefore, it is important to highlight the call for the development of visual intuition in students when dealing with graphs in solving mathematical problems.

Despite the many advantages and positive aspects of graphs in the learning and understanding of functions and derivatives, some researchers had also identified numerous setbacks related to their uses. In order to efficiently use graphs for solving mathematical problems, students must be well-equipped with the knowledge about the graphs and their related content domain (Eraslan, Aspinwall, Knot & Evitts, 2007; Gravemeyer & Cox, 2008). Students were also found to be reluctant to use graphs as tools to help them solving mathematical problems (Uesaka, Manalo & Ichikawa, 2007, 2010). Although students have been exposed to some ideas of basic graphs since their secondary schooling, they were still not highly efficacious in answering questions that requires visualization or compel for higher order thinking skills (Ferrini-Mundy & Gucler, 2009). They tend to revert quickly to algebraic manipulations or appear to read or interpret graphs in such a way that portray their lack of understanding on the underlying principles (Li, 2006; O'Connor & Robertson, 2005; Ryken, 2009; Sharma, 2013; Stylianou, 2010).

1.1.3 Visual Reasoning in Learning Differentiation

In general, visual reasoning provides an effortless way of acquiring new information and is able to reduce complexity in dealing with handful information (Giaquinto, 2007; Kadunz & Straber, 2004; Mudaly, 2007; Naidoo, 2007; Pulido, 2006; Singh, 2007). The

last decade has seen the rise of research and studies on pre-university students' difficulties in acquiring and understanding the concepts of functions and derivatives (Calder, 2008; Herbert, 2008; Rasmussen, 2003; Rasmussen and Blumenfeld, 2007; Rowland, 2006). Problems involving functions and derivatives that employ graphs as reference is the most proposed approach to the teaching and learning of differential calculus (Font, Bolite & Acevedo, 2010; Hahkioniemi, 2004; Kendal & Stacey, 2003; Roorda, Vos & Goedhart, 2006) since this will develop their visual reasoning skills (Costa, 2011; Habre & Abboud, 2006, Liu, 2010; Lowrie, Diezmann & Kay, 2011; Presmeg, 1986, 1989, 2006) and help to cultivate students' cognitive ability. To be competent in graphing, students need to be equipped with graph constructional skills (Gerofsky, 2010; Monteiro & Ainley, 2003; Temiz & Tan, 2009) and graphs interpretational skills (Amer & Ravindran, 2010; Aoyama, 2007; Glazer, 2011; Lowrie, et al., 2011; Sharma, 2013).

At the lower educational level, the Ministry of Education, through its Curriculum Development Centre (CDC, 2010) and *Kurikulum Bersepadu Sekolah Menengah* (KBSM) (2006), encourages the use of technological tools such as graphic calculators and computer software (Nik Rafidah, Zarita & Safian, 2008; Noraini, 2006; Pumadevi, 2004; Rosihan, 2004; Sharifah Zarina, 2008) in the learning of Additional Mathematics so as to provide students with massive visual interface and as preparation for the pre-university level. The SACE (2014) curriculum developer implemented the use of graphic calculators in both of the mathematical subjects, Mathematical Studies and Specialist Mathematics, as a way to help students strategize their methods of solving problems and to enhance their understanding of mathematical concepts visually through the manipulation of graphs. Graphics calculators and other technological software are able to emphasize graphical representations of any objects, concepts and processes. The functions 'zoom in' and 'zoom out' allow students to explore in details,

for example, on what happens to the chord as the horizontal difference is getting smaller. Graphic calculator also assists students to experience the relationship between the numerical values of the derivatives and their 'situation' of the turning points (Kissane & Kemp, 2006).

Visual representations of mathematical objects, concepts and processes (Rivera, 2011) such as Cartesian graphs are considered efficient representational approaches in differential calculus. The importance of using graphs in derivatives can be explained through the contributions they make to the development of conceptual understanding (delos Santos & Thomas, 2005; Lowrie & Diezmann, 2007), intuitional (Hattikudor, Prather, Asquith, Alibali & Knuth, 2012; Leung & Chan, 2004) and perceptual (Haciomeroglu, Aspinwall & Presmeg, 2010; Hahkioniemi, 2004; Moore, 2012, 2014) perspectives. A visual understanding of derivative should include appreciation of the main ideas underlying the concept of derivative, namely the rate of change, the limit, the slope of chord and tangent and the relationships among them (Bingolbali & Monaghan, 2008). Therefore, in order to develop students' conceptual understanding of functions and derivatives, Malaysia, in line with the rest of the world, has proposed the emphasis on reasoning, representing, and describing relationships and information visually through the use of graphs.

Allowing students to experience and practice visual reasoning as a tool for solving problems is of great advantage because; first, visual reasoning is an important and a powerful strategy in mathematics (Mahir, 2010; Peeble & Cheng, 2003; Tappenden, 2005), and second, for the students to adjust their views on mathematics, which had always been on the negative or more 'difficult' perspective, and what it means to do mathematics (Carter, 2010; Lappan and Evan, 1990; Mancosu, Jorgensen, & Pedersen, 2005; Miller & Cohen, 2001). Visual understanding of a given situation is 'stronger' and is more likely to be remembered by the students in the longer term than a

purely algebraic manipulation, thus allowing them to build knowledge through their metacognitive mind (Balacheff & Gaudin, 2010; Kakihana, Fukuda and Shimizu, 2000; Starikova, 2012).

The study on functions and derivative using graphs as visual tools to reason is essential for several reasons. Functions and derivative are the central concepts of differential calculus and calculus in general, which provide the foundation for various subjects and fields at higher levels of education. At the same time, graphs provide a rich source of visuals which is important in understanding the concepts of functions and derivatives. They also help to provide students with greater power in ‘seeing’ the relationship between two related quantities (Huetinck & Munshin, 2004; Maharaj, 2010; Roorda, Vos & Goedhart, 2009; Stewart, 2009; Tall, 2010; Uygur & Ozdas, 2005, 2007) which in turn is the foundation for understanding and solving mathematical problems (Ministry of Education Malaysia, 2013; NCTM, 2000). A loose base of conceptual understanding of functions and derivatives at pre-university level might become critical at university levels where students need to encounter more complicated and advanced concepts and applications of derivatives (Sofronas, 2011).

In Malaysia, the emphasis on developing visual reasoning is fairly new and little is known about the use and types of visual reasoning adopted by the students. Considering limited roles that visual reasoning plays in the pre-university curriculum and judging the potential aspects of how visual reasoning contributes to the conceptual understanding of functions and derivatives, this study aimed at assessing the types of visual reasoning employed by the pre-university students when using graphs to solve problems on functions and derivatives.

1.2 Statements of the Problem

The notion of functions and derivative and the development of students' ability in solving problems are vital in differential calculus (KBSM, 2006; NCTM, 2000; SACE, 2015) but many students struggle to comprehend it and to nurture this potential respectively. Although they had informally dealt with derivatives in the form of rate of change in their daily context, unfortunately many are unable to associate this casual knowledge to a more mathematical way in the classroom environment. A commonly cited reason for a high non-performing rate in understanding calculus especially the functions and derivatives at pre-university level is on how the materials are delivered to the students and consequently on how the students understand the concepts taught in order to apply them to solve related mathematical problems. Most researchers focussed on the functions of graphs and the effects of using graphs (e.g., Cheng, 2004; Hipkins, 2011; Gray, Loud & Sokolowski, 2009; Yerushalmy & Swidan, 2012). They had also revealed that pre-university students do not have a sound understanding of the concepts of functions (Mousoulides & Gagatsis, 2015) which later affects their understanding on derivatives. Students' difficulties with derivatives emerged from their struggles when learning about functions, graphs and other related concepts in algebra (Judson & Nishimori, 2005). Their immature and weaknesses in understanding the notions of functions led to many misconceptions, which start from the basic slope of chord to the applications of the concepts of derivatives (Muzangwa & Chifamba, 2012; Pillay, 2008).

National educational organizations, curriculum developers and policy makers at the pre-university level have repeatedly calls for the calculus curriculum to greatly emphasize on understanding the notion of functions and derivatives for students to be able to continue smoothly to their applications at higher levels of educations. The design of the educational process, and instructional methods and materials are of utmost

important in conceptualizing functions and derivatives. The aim is for students to adopt mathematical thinking by using mathematical language, perform analysis and solve related problems. Students in general regard functions and derivatives as a pool of formulae and rules which are mere procedural knowledge and in abstract form that are not understood by the students. Graphs of functions and their derivatives contain all the details and the required information regarding the properties of functions or the behaviour of the related quantities represented by the functions. Pre-university students should be able to overcome a lot of misconceptions and difficulties by using graphs to solve related problems. Unfortunately they hardly use them. Thus, their operational or procedural knowledge dominate as compared to their conceptual knowledge and it is unlikely to coordinate the concepts of functions and derivatives to their graphs.

The calculus content of most pre-university curriculums, together with examinations and assignments tasks, composed of concepts that require students to present and analyse their work using graphs (Gundersen & Steihaug, 2010; Hausknecht & Kowalczyk, 2008). At the same time, there is an apparent increase in the problems on pre-university students learning and understanding the concepts of derivative such as tangents and slope functions (Salleh, 2006) and facing difficulties or performing various mistakes when solving problems related to the applications of functions and derivatives that rooted from their unable to conceptualize graphs. Students faced confusion when reading and interpreting even the constant rate of change or slopes of straight lines due to their difficulty in visualizing rate of change of two different quantities. To most students, derivatives are collections of differentiation rules, with neither visuals nor reasons (Siyepu, 2013a,b). Students may be very competent to solve algebraic differentiation tasks but most cannot explain the meaning of derivative when relating it to the basic ideas of rate of change, limit and slope of chord and tangent. Students' weak understanding of derivative may due to: 1) their misconception on particular parts or

topics in differential calculus, 2) teaching that focuses on the procedural knowledge than the conceptual knowledge, 3) concentration on algebraic and symbolic representations to emphasize concepts, 4) superficial understanding on co-variational reasoning and 5) lack of visual teaching techniques used in solving problems.

In the Malaysian classroom practices, function- and derivative-related definitions and theorems are presented using formulae, and later drilled through algebraic manipulations. Being an abstract subject, most students fail to grab the concepts taught, perceive the learning of differential calculus being very difficult and consequently dislike it. Students experienced little opportunity to discover and conceptualize the mathematical concepts using graphs and consequently barred them from optimal learning. Therefore, students must be well-equipped and should be allowed to personally examine and explore graphs to understand concepts in depth, their relative representations and applications to other fields of mathematics as well as real life situations.

Visualization is very important in the process of understanding the concepts of function and derivative. Students may be able to efficiently construct the graphs of the function $y = f(x)$ and its derivative $y = f'(x)$, but most would struggle to interpret them effectively. Generally, for the students, graphs do not bring so much meaning but as an object to display functions. Using graphs as visual tools to solve problems require the ability to read, understand and interpret them effectively. The Matriculation Division of Malaysia (2006) reported that students were unable to convert problem statement(s) into graph(s) and performed various errors in visualizing mathematical concepts. The complicated cognitive processes involved and the convention used prohibit students from opting to graphs as aid for solutions. Lacking of this skill together with their perception about the efficacy and difficulties of graph usage may contribute to the reluctance to use graphs. Taking graphs as illusory and at the same time being oblivious

of their efficiency, students fail to read or extract relevant data or information, consequently cause them to revert to verbal or algebraic explanations for clarifications, assuming text alone provide all the required information.

Assessing the learning and understanding of derivative using graphs is not easy. Some students were still weak in understanding the concepts of derivative even after been given the opportunity to work with graphs. Even those with good mathematical mind, may not possess the skills to visualize or are not visualizers. Students do not necessarily perceive what teachers appreciate in graphs. Those who are lack in understanding of the main concepts see 'irrelevancies' which are dismissed by the teacher's vision. Teachers need to be aware of this and make effort to understand what students perceive in visual representations and consequently provide guidance in constructing and interpreting them.

In many classroom practices, teachers unconsciously convey the idea that visual approaches in mathematics are inferior to analytics or algebraic approaches. In applying certain concepts, using graphs are neither a correct nor a valid method. For example, in solving or proving for the interval of increasing or decreasing or calculating the stationary points, students are not allowed to solve or show them using the graph of the particular function. Although educators and mathematicians utilize visual methods in their works, when it comes to teaching, they tend to employ analytic or algebraic methods of processing information, relying on sequential or procedural steps. Graphs are complex and concentrated with information, and therefore are more effective since they explicitly show important conceptual links among parts of information. Unfortunately their complexities need extra cognitive processing to make sense of. Students usually do not address graphs spontaneously to start solving any problem even if teachers practiced them in the classroom. The mode of how graphs are displayed is

critical in students' sense-making that may cause potential conflict between the conceptual and perceptual features of the graphs.

As literature confirms students' difficulties with functions and derivatives, it does not come to surprise that as a mathematics lecturer, I have encountered many pre-university students' struggle with both concepts and methods of solving problems. For the last fifteen years, I have taught and worked with specifically pre-university students from various mathematical background and ability, ranging from those with very little understanding of mathematical concepts to those with excellent and high thinking skills. Despite their different abilities, prevailing issues that are common to all pre-university students are their reluctance to sketch graphs when there is no explicit instruction for using them, and having difficulty in reading and interpreting graphs. Lastly, there has been almost no Malaysian-based research that examined pre-university (and other levels of education) students' visual reasoning through the use of graphs to solve problem on functions and their derivatives. Therefore, it seemed that a constructive way of discovering the scenario would be to assess how the Malaysian pre-university students' employ graphs as tools when solving tasks demanding various types of visual reasoning skills.

1.3 Objectives of the study

The main purpose of this study is to assess the pre-university students' visual reasoning when they solve mathematical problems involving functions and their derivatives.

Specifically, the study is aimed to:

1. develop an effective framework for assessing levels of pre-university students' visual reasoning when using graphs in solving mathematical problems on functions and their derivatives.
2. examine what are the pre-university students' :

- i. usage level of graphs when solving mathematical problems on functions and their derivatives
 - ii. preference when solving mathematical problems on functions and their derivatives.
 - iii. graph reasoning ability when using graphs to solve mathematical problems on functions and their derivatives.
3. investigate the correlation between the pre-university students' :
 - i. usage levels of graphs and their preference in using graphs when solving mathematical problems on functions and their derivatives.
 - ii. usage levels of graphs and graph reasoning ability when solving mathematical problems on functions and their derivatives.
 - iii. preference and their graph reasoning ability when solving mathematical problems on functions and their derivatives.
 4. investigate the misconceptions and difficulties faced by pre-university students when using graphs in solving mathematical problems on functions and their derivatives.

The second, third and fourth objectives are achieved following the completion of the first objective.

1.4 Research Questions

This study focuses on answering the following research questions:

1. What is an effective framework for assessing levels of pre-university students' visual reasoning when using graphs in solving mathematical problems on functions and their derivatives?
2. What are the pre-university students'

- i. usage levels of graphs when solving mathematical problems on functions and graphs?
 - ii. preference when solving mathematical problems on functions and derivatives?
 - iii. graph reasoning ability when solving mathematical problems on functions and derivatives.
3. What is the correlation between the pre-university students' :
- i. usage levels of graphs and their preference in using graphs when solving mathematical problems on functions and their derivatives
 - ii. usage levels of graphs and graph reasoning ability when solving mathematical problems on functions and their derivatives.
 - iii. preference in using graph and their graph reasoning ability when solving mathematical problems on functions and their derivatives.
4. What are the misconceptions and difficulties encountered by pre-university students when using graphs in solving mathematical problems on functions and their derivatives.

The second, third and fourth research questions are answered following the completion of the first research question.

1.5 Definition of terms

The important terms as used in this study are operationally defined as follows :

Reasoning. This was defined as the process of thought students adopted to reach solutions or conclusions in solving problems on functions and derivatives that appears in the students' written sequence of worked solutions. The act of reasoning does not

necessarily base on rules or formal logic as long as students are able to support them rationally (Bergqvist & Lithner, 2012).

Visual Reasoning. This was defined as the act of understanding and applying the objects, concepts and processes of functions and derivatives through reasoning activities on visual elements, such as graphs in this study. Students undergo the processes of encoding and decoding graphs as visual tools. According to Lowrie et al. (2011), the encoding process allows the students to compose meaningful visual communication from the text such that they are expected to sketch graphs to explain solutions. The decoding process requires the students to interpret and make meaning from the visual messages where they are to use the information embedded in the accompanied graphs in order to search for solutions.

Graph reasoning. This was referred to as the decoding process and defined as the act of understanding, interpreting and making meaning of the Cartesian graphs where they need to use the data or information embedded in the graphs in order to solve the problems

Visualization. This was referred to the ability for students to process and produce, through the interpretation and reflection upon graphs, on paper with the purpose of communicating information and enhancing understanding (Pulido, 2006). Mathematical visualization refers to the process of encoding functional and logical properties and relationships of mathematical objects, concepts and process (functions and derivatives in this study) in visual form which is the Cartesian graphs.

Preference. This was referred to the act of spontaneity with which students, under their own volition, solve algebraically or draw or use graphs (through the use of graphic calculators) when dealing with derivative problems. In the tasks provided, no graphs were supplied, no hints were given for their use and no instructions were provided to encourage the use of graph in solving the derivative problems (Uesaka & Manalo, 2010).

Usage level. This was referred to the feedback, responses, behaviour and choices provided by the students when confronted with a task. The usage levels were determined by their self-concepts, self-efficacies and personal ideas or theories with reference to their knowledge and the ability to use graphs and diagrams in specific domains (Panaoura & Michael, 2010).

Mathematical visibility. This was referred to the encoding process and defined as the degree to which students preferred to use graphs when attempting the tasks on functions and derivatives which can be solved using either the graphical method or non-graphical/algebraic method.

Conceptual knowledge. This was referred to a skilful process of thinking on concepts, rules or problems presented in various forms. Conceptual knowledge can be differentiated from the procedural knowledge by the students' consciousness on they used the knowledge. While procedural knowledge indicates the students' use of visual representation, the conceptual knowledge, on the other hand, signifies the establishment of connections between the algebraic representation and the visual representations, the graphs.

1.6 Significance of the Study

While there is literature on students' understanding of functions and derivatives, there is no study in the Malaysian educational context that connects the concepts with a focus on the use of visual reasoning skills. There is also no study involving these ideas that uses Cartesian graphs as a method of data collection. Therefore, this study sought to provide a reliable and significant idea depicting the types of visual reasoning employed by pre-university students in order to understand their reasoning and thinking and in improving the instructional methods and materials when dealing with functions and derivatives.

In collaboration with several models and theories, the study developed a framework to assess the visual reasoning of pre-university students when they are solving tasks on functions and derivatives. The analysis on their usage levels, preference method and types of visual reasoning will contribute to a better understanding of how students comprehend the concepts of functions and derivatives and the use of Cartesian graphs. This knowledge can be used to revise the course curriculum to include and emphasize on the applications of functions and derivatives with respect to the use of graphs and at the same time to identify the difficulties and misconceptions struggled by the students. Providing the students with a strong foundation in the use of graphs at pre-university level will help them to be in a better position to apply the concepts to other mathematical areas and disciplines at higher educational levels.

The Malaysian Mathematics Education yearns to ensure that all pupils and students are engaged in visual reasoning in solving mathematical problems (CDC, 2006; KBSM, 2006; Shahrul, 2011). One significant method is to challenge their thinking through tasks that guide their exploration of concepts and understanding (Rivera, 2011; Saifulnizam, 2011). Thus the study is hope to lead and help adjusting the learning environment from the 'product' or 'cognitive' of learning to the 'process' or 'meta-

cognitive' learning (Gilbert, 2005). Performance assessment can be to assess 'product' such as simplifying algebraic expressions or solving algebraic equations. On the other hand, assessing 'process' allows teachers to learn about students' thinking and reasoning in completing tasks in functions and derivatives through the use of graphs as communication tools. Students are able to understand which are the key ideas and proceed to grasp the heuristic values of the ideas. They then employ the ideas strategically to solve non-routine problems, avoiding common misunderstanding and acquire inflexible knowledge and skills. Metacognitive ability is higher level of cognitive skills that allow students to use their prior knowledge, on functions and derivatives, strategize plan to use graphs to produce information through non-linear approach. These reflect the quality of the students' thinking and reasoning.

Curriculum for all subjects could be designed to focus on how pre-university students draw, interpret, and understand graphs effectively. Ng and Lee (2009) proposed 'meta-visualization' as visual reasoning skills and graph literacy to be included in the 'thinking curriculum' (McCulloch, 2011; Novick, 2006). The introduction of educational software and technological tools in the classroom environment to facilitate visual reasoning and thinking processes is of greatest help. The visual effects of graphs, could assist students to draw, interpret and understand concepts of functions and derivatives through their intuitions and experiences (Presmeg, 2006; Rivera, 2010). Thus, teachers who have learned and became skilful in the use of visualization technique to reason would be able to reinforce concepts of derivatives to improve the learning process in the classroom (Rahim & Siddo, 2010)

Teaching and learning methods should essentially be equipped with tools that promote visual reasoning and visual thinking. Educators are able to redefine the classroom objectives and redesign the classroom activities to improve and upgrade approaches to teaching that gear towards the use of visual tools such as graphs.

Indirectly, the study is able to help characterize features of the appropriate classroom tasks that may reveal various forms or types of visual reasoning. Breen and O'Shea (2011) advised on linking formal mathematical representations of functions and derivatives to students' informal understanding of real life problems to enhance understanding of classroom mathematical concepts, independent of the use of formulae or algebra. Students' mathematical concepts can be enhanced by viewing, analysing and adjusting graphs (Clark & Lovric, 2008; SACE, 2015). Teachers with strong visualizing power are able to train students to connect mathematics with other thinking abilities and aspects relevant to the real world.

The study may be of benefits to the education ministry, curriculum developers and assessment designers into enhancing the goals and learning outcomes at all levels of education. Since graphs, not to forget other visuals such as diagrams and geometry, encompass the understanding of visual phenomena (Arzarello & Robutti, 2010; Freitas & Sinclair, 2012), they are encouraged to be included massively into the curriculum, not only in mathematics but across all subjects (Noraini, 2008; Pierce, Stacey, Wander & Ball, 2011). The curriculum developers and assessment designers are able to analyze the requirement of graphical literacy, the types of visuals appropriate for particular teaching goals and learning outcomes (Ruthven, Deaney & Hennessy, 2009; Sheehan & Nillas, 2010), their assessments that should emphasize on relevant constructs instead of emphasizing algebra and procedural knowledge, and the levels to introduce to them starting from the primary through pre-university levels and pre-service teachers as response to the Malaysian Ministry of Education's recommendation.

Visuals are likely to enliven dull materials such as words or difficult concepts. Authors and publishers of textbooks could profits by adding more visuals such as diagrams and graphs for pupils and students to easily grasp concepts besides capturing their interests and motivations. In addition, the study would help to identify and

recommend on when, how and to what extent should visuals be used in the textbook or classroom presentation. The understanding of cognitive coordination and the knowledge of visual representations could help educational target to promote the use of visuals in mathematical problem solving setting.

The study helps to tackle and reduce gap in the literature and knowledge on visualization and visual reasoning in functions and derivatives, differential calculus and other mathematical areas or other educational disciplines. It will also open opportunities for researchers to proceed with related or unanswered phenomenon (Presmeg, 2006;) or effective instructional methods or strategies to adopt (Huntley & Davies, 2008; Moore, Teuscher & Carlson, 2011; Shepherd, Selden & Selden, 2012).

1.7 Conclusion

As implicitly suggested, the intended result of the study was an effective framework to assess students' use of Cartesian graphs as visual tools to reason in solving problems on functions and derivative. Subsequently, the study will investigate how the pre-university students make use of Cartesian graphs to relate and understand the concepts of functions and derivatives to solve problems together with the identification of the errors that lead to some difficulties and misconceptions, both in the constructing and reading or interpreting graphs. The study does not intend to make general claims about the way that all pre-university students use visual reasoning. Instead, these pre-university students' use of visual reasoning serves as an illustrative example on how it is possible to use graphs to help the thinking and reasoning process as preparation to encounter more challenges concepts and applications at the university levels.

CHAPTER 2: LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

2.1 Introduction

The primary purpose of this study was to develop a framework and hence to assess the types of visual reasoning adopted by pre-university students in solving problems on functions and derivatives through the use of Cartesian graphs. The study investigated the students' usage level of graphs as visual tools during their learning of mathematics, their preference method adopted and their graph-based reasoning ability. Subsequently, the study proceeds to identify the misconceptions and difficulties encountered when dealing with problems on functions and derivatives using graphs.

The first part of this chapter presents the review of related studies in the literature regarding the visual reasoning process in mathematics education focusing on pre-university students and on the teaching and learning of functions and derivatives. This includes literature on Cartesian graphs as visual tools to solve mathematical problems and students' conceptual understanding on functions and derivatives. This part also reviewed studies on student's conceptual understanding of functions and derivatives together with the difficulties and misconceptions that they encountered when constructing, reading and interpreting graphs. The second part of the chapter presents the selected theories and models that contribute to the development of the framework for this study.

2.2 Review of Literature

The review of literature is presented in the following sections: (a) defining visual reasoning, (b) visual reasoning in mathematics education, (c) conceptual understanding of functions and derivative, (d) defining graphs, (e) making sense of graphs and (f) visual reasoning models.

2.2.1 Defining Visual Reasoning

Mathematics is a branch of knowledge domain with vast number of entities to be visualised. Many educators, mathematicians and researchers have emphasized the importance of visual learning, visual communication and visual reasoning (Arcavi, 2003; Booth & Koedinger, 2012; Diezmann & Lowrie, 2009; Friendly, 2009; Holvikivi, 2007; Lee, Khng, Ng & Ng; 2013; Orhun, 2012; Presmeg, 2006; Sinclair & Whiteley, 2004; Tallman & Carlson, 2012; Tarmizi, 2010; Trigueros & Martiinez-Planell, 2010; Wall & Benson; 2009) in the learning of mathematics and observed that a lot more research in mathematics education are to be carried out on these topics (Ahmad Tarmizi, Mohd Ayub & Abu Bakar, 2010; Huang, 2015; Pjanic, Lidan & Kurtanovic, 2015; Presmeg, 2006; Rivera, 2011). A number of visualization and visual reasoning approaches are conceptualised, designed and developed by various educators, mathematicians and researchers in the literature. These visual reasoning approaches include the use of many forms, variations and aspects of visual representations (Lam, Bertini, Isenberg, Palisant & Carpendale, 2012). While they have difference importance in the ways students adopt their reasoning, a common thread among these reasoning techniques is the focus on how students use visual to relate concepts and solve mathematical problems. Concerning the terms 'visual reasoning' and 'visualization', disagreement and even confusion, are common among educators, mathematicians and researchers (Van Garderen, 2006). In most situations visualization always parallel visual reasoning, therefore the terms have often been used interchangeably to describe the learning and thinking processes that involve visuals such as diagrams, pictures, graphs, tables and other non-written representations. Mathai (2004) refers, in the most basic gist, 'visual' as elements seen simultaneously, continuously and directly from the surrounding. The brain will then sort the perceived information into various paths

accordingly, such as the properties of the object or their locations with respect to other related objects (Ball & Ball, 2007; Naidoo, 2007; Roorda, Vos & Goedhart, 2007).

Visual reasoning concerns with the understanding and comprehending problems, concepts, objects or processes in terms of visuals. Park and Kim (2007) defined visual reasoning as, preceded by the process of observation and interpretation of the visual information, a two-way process that goes beyond the visual provided: the first way is to transform visuals based on the rules or models and the second is to make judgement and generalization from the visuals. Earlier, Zimmermann and Cunningham (1991) emphasized that the influence of mental and physical attributes on the visualization process to consists of constructing images mentally, with pencil and paper or with the aid of technology, and subsequently using such images for effective mathematical discovery and understanding. From the mathematical teaching and learning points of view, employing visualization and visual reasoning as tools and methods seems to enhance students understanding and comprehending of various concepts, not only in mathematics but in other disciplines such as physics, biology, chemistry, applied statistics and other areas such as architecture, designs and engineering. Literature also reveals that the ability to 'see' can be learned and induced instead of an individual natural practice (Goerd, 2007; Goldin, 2004). Rodriguez, Espinosa and Uriza (2007) classified four different visualization approaches from the mathematics educational viewpoint: 1) visualization as a link between reasoning and intuition (Clark, Nguyen & Sweller, 2006; Woleck, 2001), 2) visualization as a way to form mental images, (Hitt, 2002; Presmeg, 2006; Zimmerman & Cunningham, 1991), 3) visualization is the connection of different representations of mathematical object (Goldin, 2004), and 4) visualization as mental process to represent, transform, generate, describe, maintain and reflect visual information (Aparicio, Rodriguez-Vasquez & Cantoral, 2003; Cantoral & Montiel, 2001; Rodriguez-Vasquez, 2003).

A similar definition of visualization is a skill, a product and a way of creativity and interpretation, a reflection of the diagrams in the minds and is significant in understanding and steering steps to solve problems (Owens & Clements, 2004). Sinclair and Whiteley (2007) specified visual reasoning to mathematics and mathematics education as to understand and to apply mathematical concepts, objects and processes using visually based information or representations. Using Zimmermann and Cunningham' definition, Rodrigues, Esinosa and Uriz (2007) proposed a three-step activity of action-formulation-validation for students to visually reasoned mathematical concepts through recognizing the conceptual characteristics and establishing relationships to their graphical forms.

Liu and Stasko (2010) presented a four-level cognitive processing that describes visual reasoning as the interplay between internal graphical representations or the mental models, and external graphical representations. In the first level of 'internalization', the process of encoding involved the information being extracted from the stage of perception in the long-term memory. In the 'processing' level, the internal representations make sense of the new external representations using different structural properties preserved in the long-term memory while in the 'augmentation' level, the internal representations are developed and referred to for the sense-making and reasoning. The last level of 'creation' involved the cognitive process of creativity and innovation that give rise to new concepts in visual forms.

In summary, visual presented to the students by teachers or used by teachers in the teaching process, although the students' perception on the benefits of these visuals may not necessarily consistent to that of the teachers, tend to influence the students' understanding and application skills of the mathematical concepts, objects and processes onto their solving of mathematical problems.

2.2.1.1 Visual Reasoning in Mathematics Education

Research examining the impact of visual reasoning and visualization on mathematics teaching and learning, and academic achievement has mostly indicated positive results in various subject matters and for most levels of educations. Earlier in the 1960s, psychologist Rudolf Arnheim argued that educators had failed to notice visual thinking as one of the most compelling and powerful human cognition. Gyorges and his colleagues, followed by Ferguson, Miller, Gooding and more others strove to value visual reasoning and visualization as essential and fundamental parts in the problem solving process across multiple domains (Jacobson & Turner-Rahman, 2007). Bishop's work on visualization in mathematics education, between 1970 and 1990s, ended up with three main findings: 1) students takes more time and cognitive load in developing mental images as compared to analytical method and process, 2) various systems and schemes with different effectiveness were identified by encouraging the use of visual in learning and understanding mathematical objects, concepts and processes and 3) students' reluctance to visualize in learning mathematics at all levels, must be taken seriously and could not be analyzed in simple terms.

Earlier in 1991, Presmeg conducted a study on how 13 high school teachers employed visualization in their daily classroom practices. Her interesting findings include three levels of visual skills: strong, average and weak. Those teachers with strong visualization skills will try to connect mathematics with other thinking skills instead of applying visualization ability solely. They allowed students to associate mathematics to real world situations through their creativity, playfulness, self-awareness and openness to their own experience. Teachers with average visualizations skill tend to emphasize on the values of visualization skills and approach while those teachers with weak or no visualization skills opted for the symbolic or algebraic manipulations which led to rote memorization as a mean of solving mathematical problems. Later studies

also seem to lend support to the positive impact of visual reasoning on mathematics understanding and achievement.

In Canada, Lam, Bertini, Isenberg, Palisant & Carpendale, (2012) reviewed 800 publications on visual reasoning and visualization and categorised them into seven themes that were able to guide researchers and educators to adopt the most effective evaluation approach for their students. The themes were formed based on the goals of the articles or research, areas of focus of the research objectives and research questions. The themes focused on evaluating: 1) the environment and work practices (Plaisant, 2004), 2) visual data analysis and reasoning (Isenberg, Tang & Carpendale, 2008; Saraiya, North, Lam & Duca, 2006), 3) communicating through visualization (Hinrichs, Schmidt & Carpendale, 2008; Pousman, Stasko & Mateas, 2007), 4) collaborative data analysis (Pirolli & Card, 2005), 5) user performance (Greenberg, 2008), 6) user experience (Eccles, Kapler, Harper & Wright, 2008), and 7) automated evaluation of visualizations (Haroz & Ma, 2006). For each theme, Lam et al. (2012) outlined the popular types of goals and outputs, the typical research questions and the applicable methodology adopted. The categorization of themes captured the current practices of visual reasoning and visualization activities and was able to guide or monitor researchers and educators on the various approaches and for them to decide on the most applicable approach to adapt.

Visualization skills had been empirically proven to correlate to the success of mathematical problem solving. Van Garderen (2006) investigated visual and spatial ability of 66 grade six students when solving mathematical word problems. They were categorised as students with learning disabilities, average-achiever students and full-scale scorers on the Wechsler Intelligence Scale for Children-Revised (Wechsler, 1976). They adopted the 13 items in the Mathematical Processing Instrument (MPI) developed by Hegarty and Kozhevnikov (1999). Four categories of measurements were based on:

the number of correct solutions, the use of visual and the use of pictorial or schematics types of visual. The results confirmed the positive correlation between the visualization skills and mathematical word problems achievement on the MPI. Additional outcome suggested that the full-scale scorers performed better than those with learning disabilities and average achiever students.

Bremigan (2005) investigated how high school students used prepared diagrams to help them in solving applied calculus problems. The results managed to alert educators on the various methods on how students made use of diagrams, modified them and sketched new ones. The study further examined the relationships between the number of diagrams produced by both groups of high- and low-achievers and their accomplishment of the problem. Results indicate that the males produced less diagrams than the female students although they were more successful in solving the problems (Lowrie, 2005). The diagrams produced by the male students were also found to be more direct and simple.

On the other hand, some studies identified negative results on visual reasoning. Despite the positive views by researchers and educators on the importance of visualization and visual reasoning, there are some tendencies for visualization and visual reasoning to be under appreciated in mathematics teaching and learning and consequently students, although were able to visualize mathematically, swapped for non-visual or algebraic approaches when solving problems.

In Cyprus, Pantziara, Gagatsis and Pitta-Pantazi (2004) explored the use of diagrams as visuals to solve non-routine problems. They administered two tests for the students, one that allowed students to use any method of their preference while the other guided the students to use the diagrams that accompanied the tasks. As suggested by some studies in the literature, the visuals provided in the tasks did not seem to help the students to handle the non-routine mathematical problems (Woolner, 2004). Although

some of the visuals were repetitive of some that had been used in the classroom context, they still failed to read and interpret them efficiently. This led to the fact that the experience that they have in handling visuals to solve problems together with their ability in reading and interpreting visuals do not really determine the success or failure of the students when using visuals as tools to solve mathematical tasks.

In her study, Bardelle (2010) tried to prove the Pythagoras and Convergence theorems using only geometrical figures. She discovered that students were very weak and faced difficulties in employing visual to reason or to justify ideas. The majority of them preferred to prove the theorems algebraically. They either ignored or did not notice the details provided in the figures, but had considered them as basic tools that need only be used to help them in the proving processes. She proposed that these could be due to the students' lack of concepts of geometrical knowledge which then led them to be unable to analyse the figures in detailed. The lack of quality in methods of proving was indicator of a weak understanding in the related mathematical concepts. Similar study carried out by Uesaka and Manalo (2011) on students' spontaneous use of diagrams to solve problems confirmed the results. They observed that promoting students to the use of diagram is affected by their perceptions on the 'efficacy of diagram use' and their 'diagram construction skills'.

In Malaysia, Rohani (2010) conducted a study on 20 undergraduate students' performance together with the difficulties they faced in solving problems on calculus. The students were also interviewed after they had solved the problems to probe into their thought processes. The analysis was coded based on Polya's four-problem-solving-steps. Results of the study concluded that students did perceive calculus as difficult and consequently misunderstood the idea of functions and applied procedural methods as an alternative. Cheah (2007) also identified some constraints that hindered the implementation of a more constructive and progressive approach, such as the use of

visual, to promote mathematical reasoning and thinking. Among the constraints were: 1) teachers tended to instruct and inform students on what and how to do mathematics instead of letting them work and construct their own mathematical ideas, 2) the culture on exam-oriented tended to lead to teachers emphasizing procedural competency in order to arrive to the correct answers, and 3) the belief on practice-makes-perfect and hard-working are the main elements to success in mathematical learning.

From this review, it seems that a variety of visual reasoning formats and aspects were of interest in order to enhance understanding and achievement in various mathematical areas. The effectiveness of visual representations in many forms of mathematical understanding and achievement appears generally positive especially for the pre-university students. In terms of the research design, most studies adopted case studies with intact classes. The assessment of mathematical understanding and achievement also varied, with both standardized and researchers' constructed tests or items being used.

2.2.1.2 Visual consideration in problem solving

In some aspects of pre-university and university mathematical teachings, visual considerations are naturally prominent. Be it on the board or the use of electronic technology, educators would put some thought on the layout of the presentation such as the fonts and sizes of the words and the quantity and quality of information on each page, so that students are able to see everything that are supposed to be delivered. Specifically in mathematics, visual works usually involve graphs or other types of mathematical diagrams to enhance students' ability to generate reasoning on mathematical concepts and relationships rather than manipulating symbols and expressions.

In 2014, Anderson-Pence, Moyer-Packenham, Westenskow, Shumway and Jordan took some efforts to restructure the relationships between the usage of visual tools and the students' written worked solutions. The two open-ended tasks were distributed to 371 students, one with diagram for them to refer to and another one, a word problem for them to sketch diagrams or graphs to help the solution process, to trace the patterns of solutions and errors performed. Students were found to lack flexibility in the reading and interpreting graphs either from those provided in the tasks or the ones that they had to construct by themselves. It was also detected that their exposure to various types of mathematical representations influenced their choice of solution methods and hence their understanding of related concepts.

The use of diagrams which are visual in nature is regarded as one of the most effective ways to encourage students to strategize their method of solving mathematical problems (Ainsworth & Loizou, 2003; Cheng, 2004; Mayer, 2003; Stern, Aprea & Ebner, 2003). Nevertheless Uesaka, Manalo and Ichikawa (2007, 2010) had identified that students were reluctant to use diagrams when solving mathematical word problems. They were unaware of the diagrams' efficacy when dealing with word problems on real life situations. In their series of studies on the area, Uesaka and Manalo (2011) identified factors related to the students' lack of urge to use diagrams. In their first experiment on 125 Japanese students, they identified that students were prone to use diagrams on problems that require more mental efforts as compared to problems involving length or distance measurements. The National Curriculum of New Zealand (2007) stressed on the importance of both teaching students to understand diagrams and the use of diagrams as communication tools. Therefore, in their second experiment, they made a comparison between the same Japanese students and 323 New Zealand students. The tasks were translated to English language for the New Zealand student. As

expected, a significantly higher proportion of the New Zealand students exhibited their preference to the use of diagrams when solving mathematical word problems.

Many researchers and educators highlighted the importance of visual reasoning in the teaching and learning of differential calculus and had proposed that there were a lot to cultivate in the topics (Presmeg, 2006). Kannemeyer (2005) noted that teachers emphasized on the completion of the syllabus through the typical process on recurring problems instead of stressing on the handling of application or non-routine problems. Visuals such as diagrams, graphs or other representations serve both as tools for solving problems and communication purposes. Therefore designing suitable tasks or real life problems so as to promote the use of visual in solving the problems is vital (Doerr & English, 2006). Francisco & Maher (2005) carried out a study on the nature and types of visual tasks that should be used for classroom purposes and determined that the more complex the task was, the more cognitive efforts and reasoning skills that were required from the students.

In 2004, Leung and Chan's students, Kevin, experienced a visual process of understanding the global features of graphs of functions through the manipulation of local parts using the zooming capabilities of graphing software, the graphic calculator. He was able to view the whole continuous and separated portions of graphs together in one screen. The zooming function allows him to scrutinize visually the situation of *separated curves* that led him to his own idea of *law of continuity*. This allowed him to combine all his prior knowledge on the visual information to explain his understanding on functions through graphing.

Teacher's knowledge on the subject content has a large effect on how students learn and grasp concepts. In Singapore, Toh (2009) gathered information on 27 new (less than five years experienced) in-service mathematics teachers from various secondary schools. He adopted Amit and Vinner's (1990) model using a questionnaire

to stimulate the teachers' knowledge in calculus. Two, out of the seven tasks, were graphed-based. The tasks were mostly dealing with the definition and images of various concepts in derivative and calculus essential for the secondary levels. Among the mistakes that they had performed were: 1) failure in grasping the essential principles to solve problems, 2) did not recognize the discontinuity of the graphs of functions, 3) did not manage to identify the correct values of limits, and 4) unable to link the concepts to the tasks. He identified that most of the pre-service teachers did not possess strong or convincing concepts images related to the derivative concepts. They would generally favour the procedural understanding in handling the tasks.

2.2.2 Conceptual Understanding of Functions and Derivatives

The notions of functions are among the most important concepts in mathematics. While the ideas of functions had been introduced to the students since the early stages of schooling, students at pre-university level are still facing difficulty in understanding what it actually means (Abdullah, 2010; Akkoc & Tall, 2005; Carlson, 1998; Clement, 2001; Cooney & Wilson, 1996; Dubinsky & Wilson, 2013; Oehrtman, Carlson, & Thompson, 2008; Sajka, 2003). Among the common difficulty for students is the transition between the algebraic form and the graphical forms (Eraslan, 2005, 2008; Kotsopoulos, 2007; Metcalf, 2007). Some researchers proposed the use of graphics calculator, or other technological software, to help students overcome this particular complication (McCulloch, 2011; Mesa, 2007). However, the problem proceeds to exist in the calculus classroom at the pre-university level where the students struggled on the concepts of derivatives graphically (Borgen & Manu, 2002; Ellis & Grinstead, 2008). In their study, Ellis and Grinstead (2008) discovered that students had the misconception that changing the coefficient of the leading coefficient of quadratic function will not

affect the location of the vertex, or stationary point. The use of graphics calculator will help them to view the changes in the graphs while manipulating the expressions.

Within the topics in differential calculus, there are a number of important concepts that students need to be well-equipped with before they pursue to higher mathematical explorations at undergraduate level. For the last thirty years, educators, mathematicians and researchers had documented difficulties students experienced when learning concepts of functions, rate of change or derivatives that reflect their inability to comprehend and to reason about the problems. Studies also indicated that the success and failure in derivative and calculus are likely to be caused by the firm understanding on the concepts of functions (Carlson, Oehrtman & Engelke, 2010). The idea of movement concerning the teaching and learning of derivatives has sparked debate among researchers and educators. In the past, the teaching of derivative concepts had focused on drills and memorization, manipulations of signs and symbols, or not linking algebraic mathematical concepts to other representations such as graphs (Ainsworth & Loizou, 2003; Aspinwall, Shaw & Presmeg, 1997; Borgen & Manu, 2002; Breen & O'Shea, 2011; Cates, 2002; del Mas, Garfield & Ooms, 2005; Mesa, 2007; Orton, 1983; Stroup, 2002; Ubuz, 2007; Viholainen, 2005; Zandieh, 2000) with the hope that students will master the procedures and be able to answer them in the examinations (Chazan & Yerusalmi, 2003; Parmjit & White, 2006). There were massive concerns in the failure to develop a conceptual understanding of calculus topics that rooted from the rote and manipulative learning that took place at the introductory course (Bingolbali & Monaghan, 2008; Biza & Zachariades, 2010; Eraslan, 2005; Lam, 2009; Şahin, Yenmez-Aydoğan, & Erbaş, 2015). This has led to the encouragement and increased on research to investigate students' ability to solve derivative graphically (Abbey, 2008; Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 2004; Aspinwall & Shaw, 2002; Baker, Chapell & Kilpatrick, 2003; Cooley, Trigueros & Baker, 2007;

Haciomeroglu, Aspinwall & Presmeg, 2010; Maharaj, Brijlall & Govender, 2008; Nemirovsky & Tierney, 2001; Roddick, 2001; Tiwari, 2007; Ubuz & Kirkpinar, 2000)

Topics in mathematics such as algebra, trigonometry and geometry contain essential concepts required in the building of mathematical skills before embarking into any calculus or derivative courses. However, Habre (2006) discovered that 87% of his pre-university students performed errors when solving inequality questions or when sketching graphs as the solution sets. About 71% of the students produced incorrect answers when they were asked to find the equation of line having slope of -2 and passes through the point (-1,2). This basic equation of line is important and largely used in finding the equations of tangents and their normals. When they were asked questions on the definition of some basic trigonometric ratios, as preamble to understanding derivatives, only 24% of the students provided complete answers. Given that these are simple skills required in all concepts and applications of calculus, the figures are alarming.

In her doctoral study, Biza (2008) focused on 182 Greek's first year university students' understanding on the properties of tangent lines that they had learned at pre-university level a few months back. Her study aimed for the students with graphs given to them, to be able to detect directly the properties, that may not generally valid and to create new (again, may not valid) properties out of the information provided graphically. The students were assigned tasks where they had to describe the tangent line in their own words. The correctness of students' solutions were analysed quantitatively. Results show that the solution methods were largely influence by the properties of circle tangent of Deductive Geometry.

Engineering and advanced level mathematics students require higher levels of concepts in derivatives for a smooth learning path throughout their undergraduate study. In 2012, Tokgoz observed that the undergraduate and graduate mathematics and

engineering students had better understanding in conceptual knowledge of derivatives. He interviewed 17 students after they had completed 15 tasks on the first and second derivatives of composite and quotient functions. The students' works were evaluated based on Piaget and Garcia's (1989) scheme development model. Results revealed that students tend to form wrong concept images in their mind and consequently caused misconceptions in the understanding of the concepts in derivative.

One of the difficult concepts in derivative that students faced was the idea of rate of change. This could be due to their pre-existing concepts of derivative captured previously. Herbert and Pierce (2008) conducted a phenomena-graphic analysis on pre-calculus students in Australia. Eight conceptual understandings of rate of change emerged from the students' responses: 1) rate as quantity, 2) rate as speed, 3) rate related to numbers as single quantity, 4) rate related to numbers as two quantities, 5) rate related to formula as single quantity, 6) rate related to formula as two quantities, 7) rate related to quantity as single quantity and 8) rate related to quantity as two quantities. These conceptions were able to explicitly exhibit the difficulties students encountered. They were also discovered to be attached to their prior basic knowledge which indirectly barred them from accepting new concepts and ideas, especially those related to the real life situations.

Recently in Turkey, Sahin, Aydogan Yenmez and Erbas (2015) employed Skemp's definition of relational and instrumental understanding to investigate students' understanding and awareness of the relationships among the concepts of derivative. The modeling tasks were in the form of model-exploration activities to inspire students to construct and reflect ideas on derivatives. Results of the study indicated that students were inclined to instrumental understanding where students knew the rules or formulae without justification or not making sense of the meaning of the concepts and the relationships

Doerr and O' Neill (2012) examined 33 pre-university students' development in creating and interpreting models of rate of changes. Using a motion detector of their own bodily motion, students were instructed to create graphs and to compare and analyze them based on the changing of speeds and directions. The Rate of Change Concept Inventory used consisted of items representing four categories: algebraic expression, symbolic and graphical interpretation, and was purely contextual. They worked individually on the model exploration tasks but were then encouraged to discuss in groups of three or four students. After six weeks, the results of the post-tests showed a significant improvement in the students' understanding of the concept of rate of change as compared to the results of the pre-test. Some of the students were able to reason when comparing the data and provided meaningful interpretation of the graphs.

In 2006, Habre and Abboud conducted two experiments carried out during two consecutive semesters to identify how students viewed their understanding on the concepts of functions and their derivatives. Diagnostic tests were distributed to 89 students at the beginning of the first semester. It was found that the students were very lack of pre-calculus conceptual knowledge. A total of 56 students remain to continue to the second semester and unfortunately 12 of them failed the course. Students' progresses and performances were followed through very closely: the students' personal notes on their thinking after class sessions and copies of exam papers were collected, and two sets of interviews were conducted. Results of the experiment indicated that the approach of the traditional teaching method was not suitable for the majority of the weak students but rewarding for those with better mathematical skills. At the same time, it was also determined that procedural method and algebraic representations were still dominating the students' mind and thinking.

Among the greatest invention in mathematics, the history of derivative began in the 1600s and continued to be the base of analysis. Grabnier (1983), a mathematician,

examined the process of derivative historically as: ‘the derivative was first used; it was then discovered; and it was finally defined’. Consequently, in the mid-17th century, Fermat and others treated the derivative as a tangent followed by the discovery by Newton and Leibniz towards the end of the century. The notion was rapidly developed and explored during the 18th century and finally defined in the 19th century. In 1790, Lagrange defined derivative algebraically, followed by Cauchy (1820) who defined derivative from the notions of limits and infinitesimals and later, in 1870, Weierstrass introduced the concepts of epsilon and delta.

The concepts of derivatives are complex and multi-faceted. It is complicated to determine the degree to which a student understands the concepts. Researchers and mathematicians use the terms such as scheme, structure, connections and relations to describe the understanding of the concepts of derivative while Hiebert and Carpenter (1992) described the understanding of concepts as depending on the way the information is represented. The degree of understanding depends on the link and connections between facts, procedures or ideas and their visual representations. Tall and Vinner (1981) described *concept image* as the ‘total cognitive structure that is associated with a concept’. Therefore we have to investigate how students relate concepts to their visual representations graphically in order to describe their understanding of derivatives.

There are broad and important concepts within derivative and calculus that equip students to higher levels of understanding and explorations in mathematics and other disciplines such as engineering, sciences and social sciences. The central concepts of derivative are limits, tangent, properties of functions and graphs (Haese & Haese, 2010). Similarly, the SACE (2015) curriculum identifies the central topics for the Mathematical Studies to be functions and graphs, limit, tangent and normal, the properties of functions and their derivatives, and the applications of differentiation.

Understanding the concepts of derivative demands the knowing and making sense of their relationships together with being able to ‘put’ them on graphs. For example, a student who understands that slopes of chord lines approximate the slope of tangent, must be able to distinguish how the two lines are situated when drawn on the graphs of functions as shown in Figure 2.1, or in another example, the locations of the coordinates of stationary points and inflection points when relating them to the graphs of functions or the graphs of their first or second derivatives.

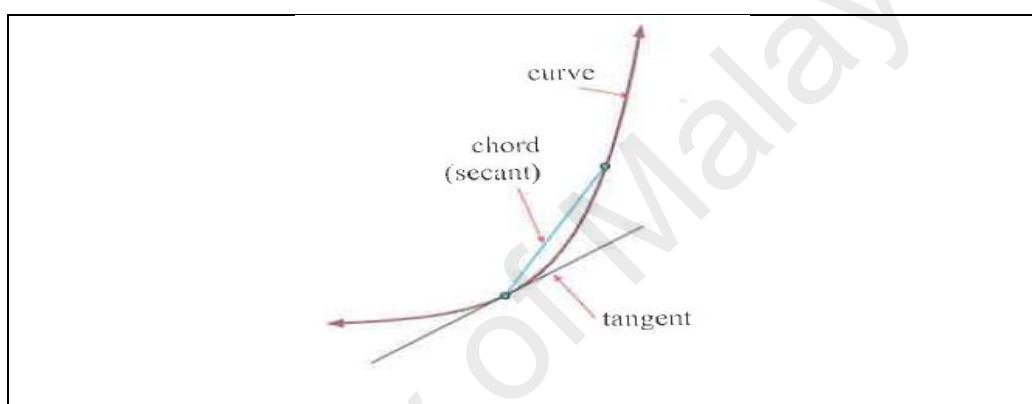


Figure 2.1: The concepts of chord and tangent. Adopted from Haese & Haese (2010). *Mathematical Studies for Year 12*. Adelaide, Australia: Haese & Haese Publications

A derivative is an assessment of how a function changes as its variables change. Lightly speaking, a derivative can be reasoned as how much one dependent quantity is changing with regards to the changes in the independent quantity. For example, the derivative of the position (s) of a moving object with respect to time (t), given algebraically as $\frac{ds}{dt}$ is the object's instantaneous velocity (v). The derivative can be geometrically interpreted as the slope of a curve of a function or the slope of the tangent at a particular point and physically as a rate of change of the vertical distance with respect to the horizontal distance. Understanding variability and change is essential to

the concepts of derivatives that help students making sense of mathematics to the real world. The teaching of the pre-university calculus needs to shift from the focus on the procedural by simplifying algebraic expressions and solving equations towards emphasizing the applications of concepts. Through observing graphs that are representing the real life situations, students are able to learn how to read, describe, interpret, extend and predict patterns depicted by the graphs of the functions.

Textbooks for matriculation or pre-university levels introduce slope as the ratio of vertical change to horizontal change or ‘rise over run’ as shown in Figure 2.2. Students practice calculating this ratio by taking the difference of the two ordered pairs of the y -value and x -value, i.e. using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Although most students are able to calculate the ratio correctly, through memorizing and applying formula, they may not grasp the concepts of slope computed in order to represent the rate of change and hence derivative without seeing how they are related graphically.

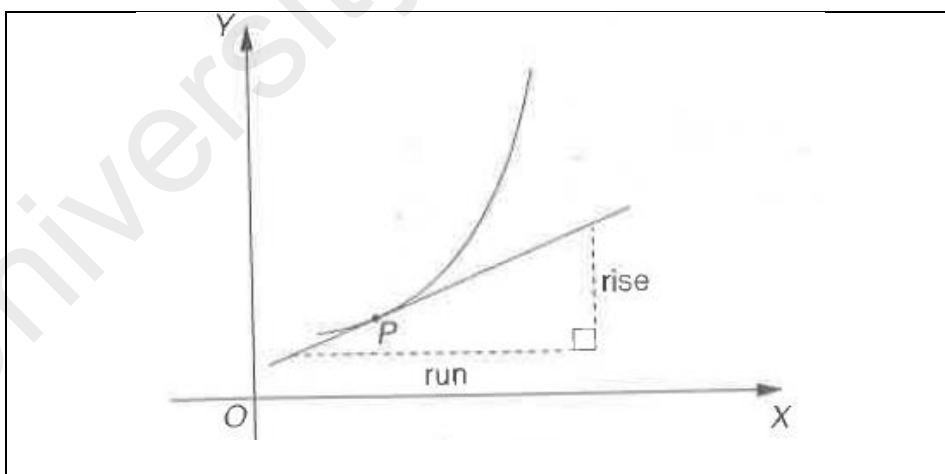


Figure 2.2: The concept of slope represented visually. Adopted from Haese & Haese (2010). *Mathematical Studies for Year 12*. Adelaide, Australia: Haese & Haese Publications

2.2.2.1 The derivative function

The derivative function, denoted algebraically by $f'(x)$ or $\frac{dy}{dx}$ is defined as the slope of the tangent line to the graph or the slope of the function at any point on the graph. It is also a function itself. Demonstrating and estimating the derivative at any point on the graph can be done by placing a straight line to represent the tangent line as shown in Figure 2.3. It is important to notice that every point on the graph of the function will have its own derivative value or the slope of the tangent at that particular point.

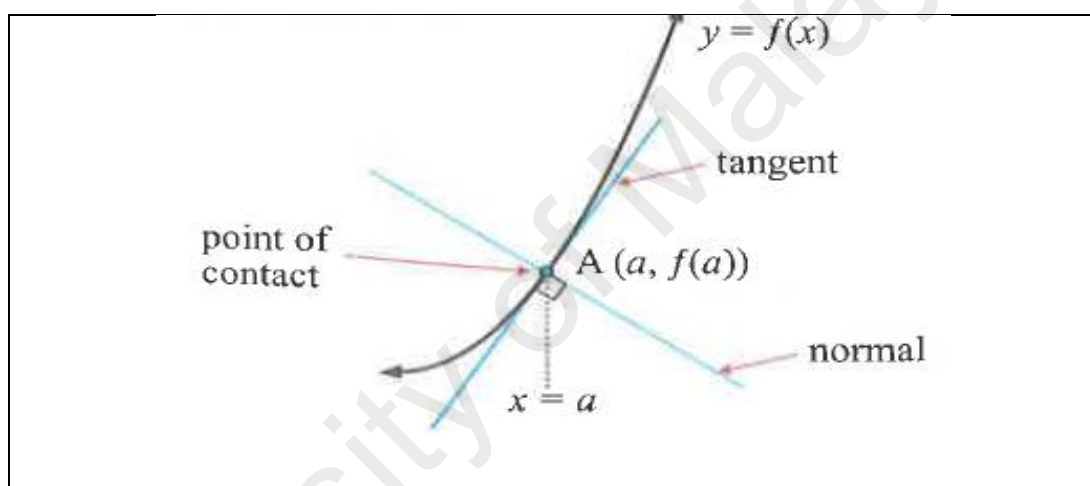


Figure 2.3: Illustration of the idea of tangent. Adopted from Haese & Haese (2010). *Mathematical Studies for Year 12*. Adelaide, Australia: Haese & Haese Publications

The derivative functions, both the first ($f'(x)$ or $\frac{dy}{dx}$) and the second derivatives ($f''(x)$ or $\frac{d^2y}{dx^2}$), are further used to identify the properties of graphs. The signs of the derivative function tell when the graph of the function is increasing, decreasing or stationary. The magnitude of the value of the derivative of the function indicates the steepness of the tangent line to the graph of the function. Where the function is

increasing, the tangent line is sloping up and therefore the value of the derivative function is positive. Similarly, where the function is decreasing, the tangent line is sloping down, and therefore the value of the derivative function is negative. The zero value of the derivative function represents the stationary or turning point of the graph of the function (Figure 2.4) or specifically the maximum, minimum or the inflection point of the graph of the function.

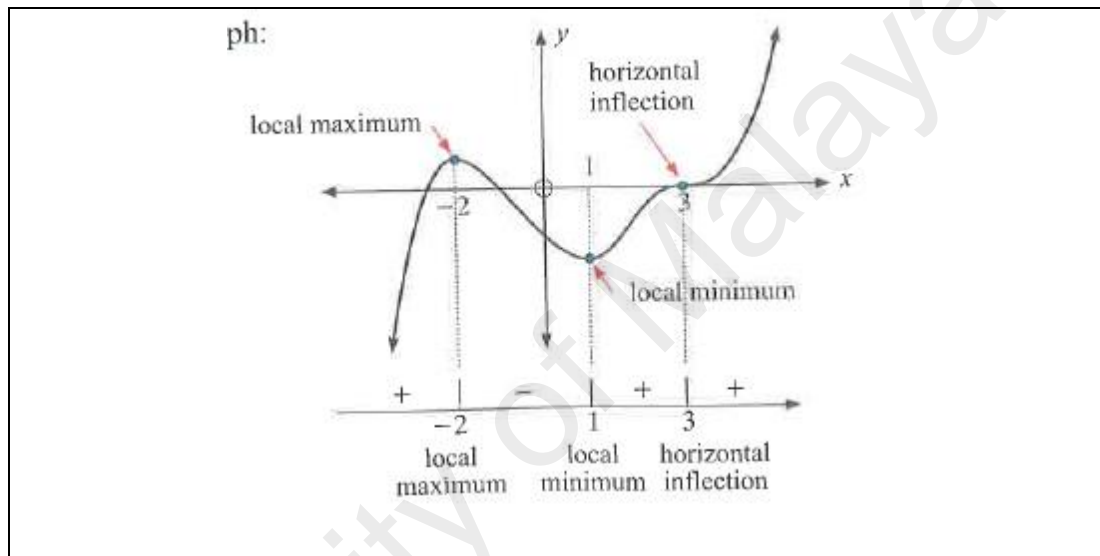


Figure 2.4: Illustration on the properties of graphs. Adopted from Haese & Haese (2010). *Mathematical Studies for Year 12*. Adelaide, Australia: Haese & Haese Publications

The second derivative is derived algebraically from the differentiation process of the first derivative or graphically, the slope of the tangent line of the graphs of the first derivatives. Since the first derivative of a function indicates whether the function is increasing or decreasing of the function, the second derivative in turn, will indicate the increasing or decreasing of the first derivative or the slope of the tangent to the graph of the first derivative and results in the concavity (Figure 2.5) and the inflection points of the of the graph of the functions.

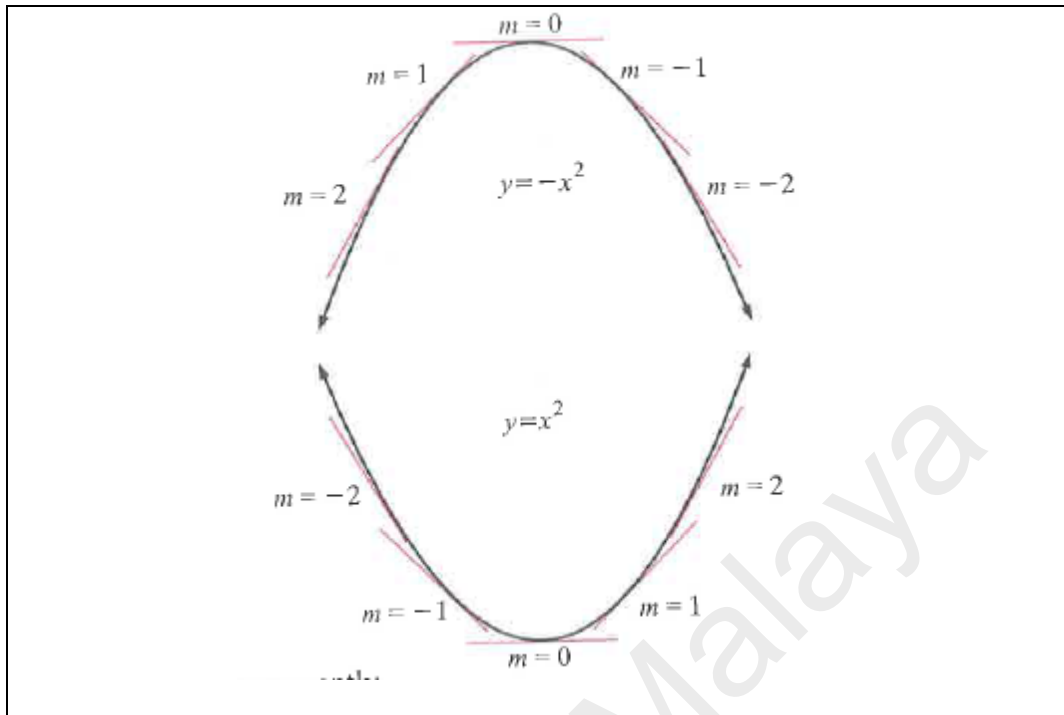


Figure 2.5: Illustration on the changes in the values of the first derivative.

Adopted from Haese & Haese (2010). *Mathematical Studies for Year 12*. Adelaide, Australia: Haese & Haese Publications

Graphically, the first derivative is the gradient of the function while the second derivative is the gradient of the first derivative. Consequently, the second derivative indicates how the gradient of the function changes along the x -axis. For a function with non-constant gradients, the second derivative indicates the shape or curvature of the graph. Applying the same concepts as the relationship between a function and its derivative, the positive values of the second derivative indicate the increasing of the first derivative. In other words, the gradient of the tangent line of the function is increasing as x increases. Graphically, the curve of the graph is said to be concave up or open upwards. Likewise, the negative values of the second derivative indicate the decreasing of the first derivative. In other words, the gradient of the tangent line of the function is decreasing as x decreases. Graphically, the curve of the graph is said to be concave down

or open downwards. When the value of the second derivative is zero, the graph can be either concave up or concave down, or it may be at the changing situation from concave up to concave down or vice versa.

The second derivative is also used to determine the nature of any stationary point, a local maximum or local minimum. As mentioned earlier, the first derivative is zero for any stationary point of a graph. The positive values of the second derivative indicate that the first derivative is increasing and the graph is concave up. Therefore the stationary point is a local minimum. Likewise, the negative values of the second derivative indicate that the first derivative is decreasing and the graph is concave down. Therefore the stationary point is a local maximum. On another note, the zero value of the second derivative indicates that the graph has an inflection point and the graph is changing from concave up to concave down and vice versa.

2.2.2.2 The Concepts of Limits

Understanding the idea of limit is the fundamental and critical in understanding what is going on in differential calculus. In the study of calculus, it is important to know what happens to the function or the dependent variable at the vicinity of a particular point or as the independent variable get closer and very close to a particular value. Students faced difficulty in understanding the idea of limits. They used the formulae

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

to calculate the derivative function

or the derivative of the function at a particular point. Figure 2.6 demonstrates the situation. The derivative or the slope of the tangent at the point A is found by moving the point B along the graph of the function towards the point A . The chord or secant line AB will later become the tangent at A and subsequently, the slope of the chord or secant

line AB , $\frac{f(x+h)-f(x)}{h}$ or $\frac{f(x)-f(a)}{x-a}$, will become the slope of the tangent at A . The

idea of limit can be clearly seen and understood through the use of graph.

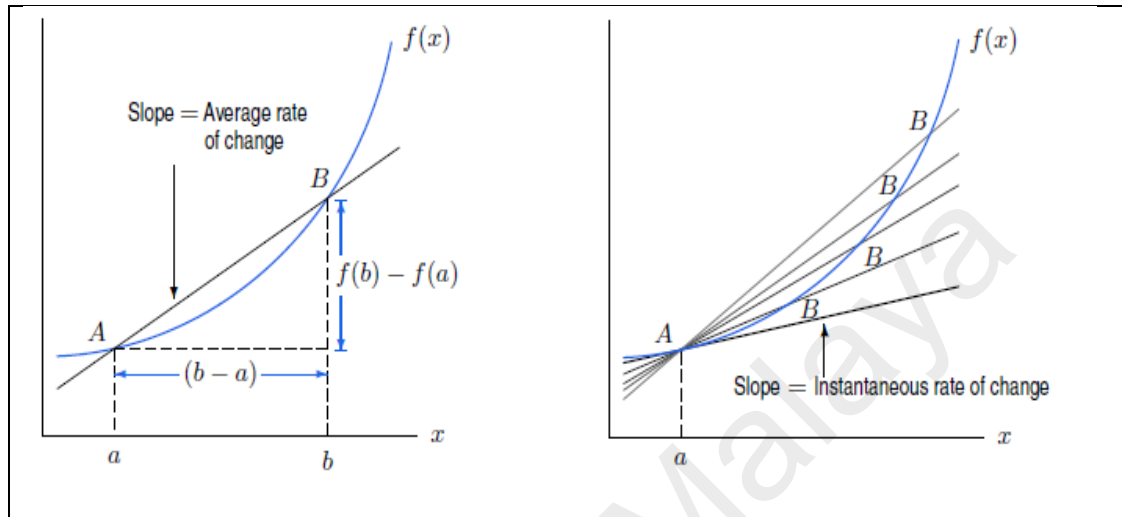


Figure 2.6: Illustration of the idea of limits. Adopted from Haese & Haese (2010). *Mathematical Studies for Year 12*. Adelaide, Australia: Haese & Haese Publications

2.2.2.3 The Application of Derivative - Rate of Change

The three ideas of rate of change are the constant rate of change, the instantaneous rate of change and the average rate of change. The concepts of rate of change are best explained using the real life situation and by the use of graphs. A common distance-time graph illustrating a car traveling at 60 km/hour as shown in Figure 2.7 helps to emphasize understanding the idea of constant rate of change. Students are able to see that from the start of the journey, the distance is increasing at a constant rate of 60 km every hour and therefore the car will travel a total of 300 km after 5 hours. The slopes of the line are always $\frac{300}{5} = 60$ for the whole graph and represent the constant rate of change of the distance with respect to the time.

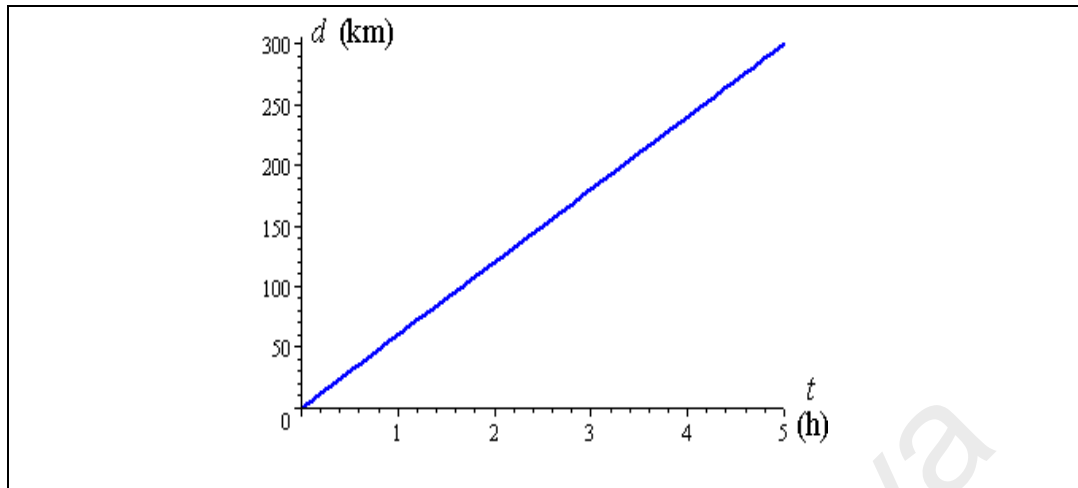


Figure 2.7: Graph of distance-time car traveling at constant velocity. Adopted from Haese & Haese (2010). *Mathematical Studies for Year 12*. Adelaide, Australia:

Haese & Haese Publications

Other real life examples to illustrate the constant rate of change include :

- A candle with its length decreasing at a constant rate of 4 cm per hour
- Water is flowing out from a tap at a constant volume of 3 litres per minute
- A block of ice left to melt at a rate of 5 cm^2 per hour

The average rate of change and the instantaneous rate of change are best illustrated using graph of functions to represent specific quantity. For example, in Figure 2.8, the graph of the function depicts the volume of soil V cubic metre dug by a team of labourers. If the total amount of soil dug was 500 cubic metre in 100 minutes, the average rate of the soil dug is $500/100 = 5$ cubic metre per minute. Of course, this does not mean that the labourers dug exactly 5 cubic metre per minute. It can also be seen from the diagram, the instantaneous rate of change at which the labourers were digging at the count of every 20 minutes were different, shown by the slope of the tangents to the graph of function at points B through D .

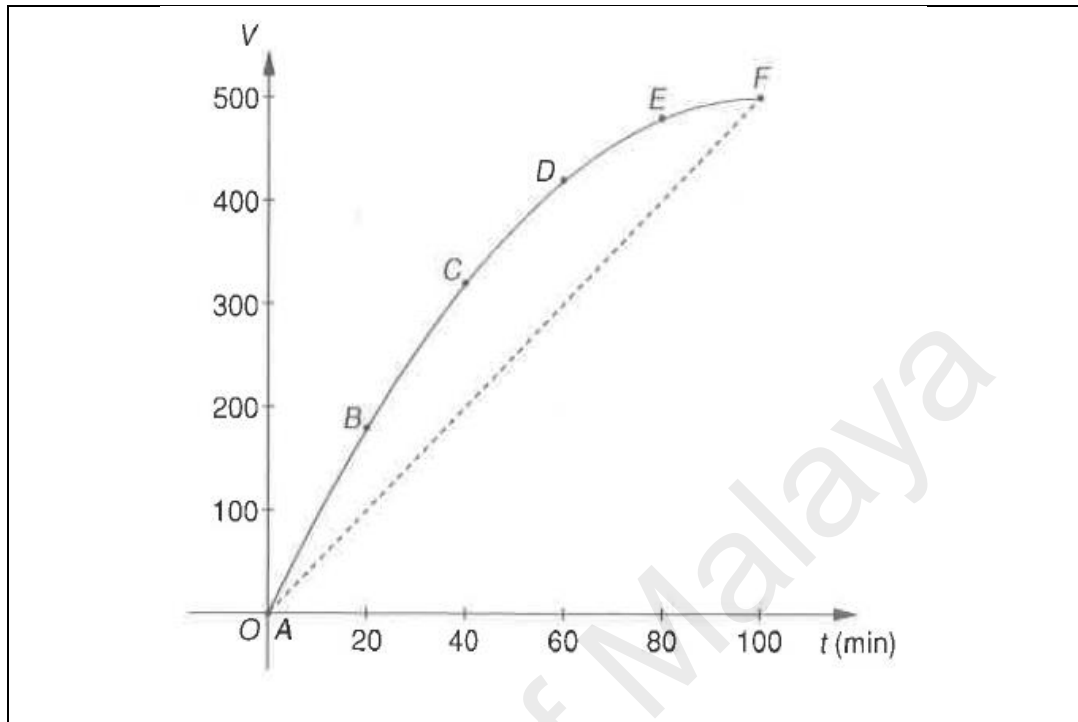


Figure 2.8: Illustration on the average and instantaneous rate of change

2.2.3 Defining Graphs

Data analysis refers to the visual displays of quantitative data through the use of graphical representations such as Cartesian graphs. Graphs are typically used to portray mathematical functions and display data from the aspects of science or social. It is among the essential part in the elementary mathematics curriculum and therefore students should be able to read, understand and utilize the information to solve mathematical problems. Earlier, Fry (1984) provided a more generic definition of graph as information transferred by the location of the point, line or curve, or area between lines or curves, on a two-dimensional set of axes.

Chein, Mugnier and Croitoru (2010) defined the graph based approach on the idea of graph theory. They elaborated it as a structure made up of a set of points and connections among the points. In their proposed graph-based approach, they shared

some benefits upon using graphs to represent knowledge: graphs are basic mathematical objects consist of points, line or curves and relations that can be visualized, graphs are rich assemble of well-organized algorithms, and graphs are equipped with logical semantics. Operations with graphs, either as a sequence of operations or as an overall operation, can be easily demonstrated to students due to their visual in nature.

In classroom context, graphs are used in most subjects such as Mathematics, Sciences, Economics and even English. Reading and constructing graphs are regarded as inter-disciplinary skills (Dhakulkar & Nagarjuna, 2006). Graphs serve two purposes in the science subjects: to present data in meaningful and comprehensive manners and to show relationships between two quantities (Baker, 2012; Ferrini-Mundy, & Gucler, 2009). In economics or other social science subjects, graphs are used to mainly display statistical data (Belenky & Schalk, 2014; Booth & Koedinger, 2012). In mathematics, Cartesian graphs are mostly dedicated to functions and non-functions such as circles. Students used the combined set of functions and graphs as symbolic system to understand algebraic expression of the function and patterns of data (Van de Walle, 2007; Wall & Benson, 2009).

All Cartesian graphs share similar structural components (Friel et al, 2001). The *framework* of a graph consists of the axes and scales to provide information on the data to be measured and the types of measurements being used. The framework is of an L-shape, with one leg as the horizontal or the x -axis for the independent data while the other leg of the vertical or the y -axis provides information about the measurements being used. The *specifiers* are used to represent data values. These are the lines or curves that denote the relations among the data represented within the framework. The *labels* for each leg of the framework named the type of measurements being made. This includes the title of the graph itself. The *background* of a graph may include colours, the

grid, or images superimposed on the graph, but may not be so distinguished or important in the Cartesian graph system.

2.2.3.1 Making Sense of Graphs

The National Council of Teachers of Mathematics (NCTM, 1989) has exclaimed for more emphasis on the students to reason from graphs, and to describe and represent relationships among graphs and functions. NCTM had also strongly encouraged for the use of computer-based graphing utilities to enhance on understanding and reasoning from graphs instead of using the traditional paper and pencil to learn the technicality of plotting graphs. The National Mathematics Advisory Panel (2008) had also made recommendations for research to investigate on particular use of technologies, specifically graphing calculator, and their effects on students' conceptual understanding and computational skills, and solving mathematical problems.

As literacy is the ability to read texts, graphicacy, on the other hand, is the ability to read, understand and present representations such as graphs, diagrams, sketches, charts etc. Dhakulkar and Nagarjuna (2006) analysed 28 school textbooks from grade 5 to grade 10 which were approved by the Indian National Curriculum Framework 2005 (NCERT, 2005) to get a trend on the graphs used. The textbooks catered for all subjects except languages, and being grouped into major areas: mathematics, sciences and social sciences. They detailed the analysis on the different types of graphs used and their frequencies of occurrence and the preference of one subject as compared to the other in using graphs. The results of the analysis showed that the presence of graphs in the science textbooks was the least although the reading, understanding and sketching of graphs are of utmost important in chemistry, physics and biology.

In graph comprehension, the processes of perception and conception are compulsory for decoding information from graphs. In 2008, Ratwani, Trafton and Boehm-Davis put forward a new framework to incorporate the visual and cognitive in the process of extracting and integrating information from graphs. Ten undergraduate students studying psychology were assigned with four groups of three to ten graphs. The first experiment sought to find the pattern of the processes through verbal protocol on the extraction of information. This was followed by the integration part to uncover the multiple processing cycles (Carpenter & Shah, 1998) and the forming and interpreting of the visual clusters formed by the students. In the second experiment, they additionally gathered the eye movements as the students answered the questions in order to understand the visual and cognitive integration as they tried to decode information from graphs. Experiment 3 concentrated on the integration part solely to examine for stronger evidence on the cognitive integration of the process. Outcomes of the study revealed that the visual clusters created by the students during the visual integration helped to reason about the graphs in the cognitive integration. It was also noticed that as the complexity in reading the graph increases, the number of visual clusters formed also increased and be used to compare and make meaning of information in the graphs (Uesaka & Manalo, 2007).

In the previous study, Paoletti (2004) had found that students frequently ignored graphics when reading texts even when they were warned to analyse or summarize the materials. Paoletti (2006) then conducted a study on 100 undergraduate students studying psychology at the Trieste University in Italy, to ascertain the degree to which students make use of the information embedded in the graphs incorporated into texts. The study aimed to acquire in detailed the students' inclinations to read the required materials and at the same time to focus on the quantitative information within the text-graph relationships. They were provided with a three-page-long text accompanied by

two graphs. They were then asked to go through the text thoroughly and examine the graphs for any related information. Several text-graph inconsistencies were also inserted to enhance indicators. The students were tested individually and their behaviours video-taped. Results discovered that most of the students need at least two trials in examining the text-graph tasks. Most of them did look at the graph during their first reading, but tend to ignore or gave insufficient responds to it. Only when they attended to the graph later and understood it, were they able to perform the integration.

Due to the fact that appropriate use of representations will facilitate the process of learning (Vekiri, 2002), research were carried out to examine the effective use of graphs to present quantitative data in various subjects and disciplines (Lowrie & Diezmann, 2009). In responding to the claims, Ozcelik and Tekman (2010) conducted a study to examine the guidelines needed for research in using graphs for educational purposes. They explored how graph comprehension was affected by the types of graphs used to display information, the conceptual understanding of the subject domain and the perception on the information organizational system. Forty-two undergraduate students in Turkey were given eight different types of graphs to describe. Results of the analysis revealed that students reasoning using graphs were influenced mostly by how they perceived information in clusters as compared to the types of graphs. They faced difficulties when the unfamiliar settings of the graphs were presented to them. The results indicated that the students worked in separated or smaller parts rather than the whole and they tend to memorize the format of the typical graphs.

Kalchman and Koedinger (2005) introduced the term *ungrounded competence* to describe students who were able to carry out the procedural knowledge and processes, and quantitative skills efficiently in certain areas or contexts but unfortunately performed errors or faced difficulty in other areas or contexts. This indicates the lack of conceptual or qualitative understanding. In their study to overcome the problems, they

proposed three teaching approaches with the aim to develop meaningful understanding on graphs of functions: first, teaching should start with contexts that are familiar to the students to allow them to recall on their prior knowledge. Second, teaching should start with simple contents and concepts to allow students to grab the essence of the big ideas and at the same time avoiding the pre-perceptions on difficulties in understanding and applying concepts. Third, teaching should allow students to express their thinking and understanding of concepts using their own invented terminologies and natural languages.

Friel, Bright and Curcio (2001) analysed students' understanding of graphs and recognized six category of behaviours that are related to making sense of graphs and at the same time suited with one of the Curcio's three levels of data :1) recognizing the parts of graphs and speaking the language of graphs as reading the data, 2) understanding the relationships among the parts of graph and analysing while maintaining the objective stance as reading between the data, and 3) interpreting the information in the graph to predict and extrapolate its context as reading beyond the data.

Monterio and Ainley (2004) described that the interpretation of graphs as a complicated process that involved specifically three main interrelated elements: cognitive, affective and contextual. The cognitive aspects incorporated informal knowledge related to intuitions and can be connected to beliefs and some of the affective elements. They explored 118 school student teachers' critical sense in graphing. They responded to two tasks based on media context and conducted interview session prior to enrolling into a data handling curriculum methods course in primary school mathematics.

2.2.4 Empirical support for graphs as visual tools in functions and derivatives

Taking into account the extensive utilization of graphs in various contexts, curriculum developer around the world incorporated graphing as one of the main topic in all levels of mathematics education. These inclusions are for students to be able to use graphs as visual tools to understand objects, concepts and processes in mathematics (Riveria, 2011).

To make sense and understand the concepts of derivative involve the conceptual and subject knowledge of various mathematical areas and their relationships, among others: geometry, functions, limit, tangent, slope and rate of change (Bingolbali, 2008). Kulfur et al, 2011) identified that rich conceptual visual representations such as graphs are able to overcome learning difficulties of pre-service teachers in understanding graphs of functions and derivatives. It was discovered that they were lack of fundamental and geometric interpretation of derivatives and consequently attempted to memorize formulae.

Graphs incorporated in model eliciting activities are able to challenge students to explain, describe and predict meaningful situations. Doer and O'Neill (2012) let students created and described the rate of change based on their exploration on motion detector of their own body motion. The students were able to: 1) correctly justify about the velocity or the average rate of change for the whole motion based on the features of the graph, and 2) correctly reasoned the position of the two cars based on the graphs representing their velocities.

Visualizers can be defined as individuals who are prone to use visual method when solving tasks which may be attempted by both, the visual and the non-visual ways (Presmeg, 2006). In response to calls for changes in instructional methods, Habre and Abboud (2006) studied the understanding of function and its derivative with university students in Lebanon. They compared the students' understanding and reasoning

between the traditional approach using the algebraic and symbolic representation to after the experimental session. Due to its visualization capabilities, technology was introduced to assist in reflecting, analysing and modifying their thinking. Students, especially for those with high mathematical ability exhibited a higher level of understanding and were positive to the new method of teaching.

Monteiro & Ainley (2010) discussed some features of the socio-historical graphs and the different context in the use of the graphs. In school context, the interpretation of graphs developed specific characteristics that were different from the enquiry context or the reading context. The use of graphs in classroom situations is generally related to the intentional purpose of the subjects being taught.

Zazkis (2013, 2014) was interested in problem solving strategy and conducted in-class activities using the Geometer Sketchpad as visual tools to examine students' difficulties in relating the graphs of functions and their derivatives. The students exhibited various strategies in sketching the graph of a function from the graph of its derivative that was not accompanied by any formula. The students integrated both the algebraic reasoning and the graphical reasoning in their thinking and were able to switch from one representation to another.

Ratwani, Trafton and Boehm-Davies (2008) proposed a framework that employed the combination of visual integration and cognitive integration. Visual integration involved the process of pattern recognition to form visual groups of information which were then being utilized in the cognitive integration part to reason on the information embedded in the given graphs. The process incorporated the verbal protocol and eye movement data. Results showed that as the complexity of extracting information from the graph increases, the integrative processes also scaled up.

Biza (2011a, 2011b, 2010 & 2008) conducted a numerous studies on students' conceptions about derivative specifically on the notion of tangent line. She identified

five factors influencing students' perception and thinking about the tangent line with regards to how they were perceived graphically. Students were presented with various situations of lines and curves and asked to justify for tangents. Additionally, they were requested to construct tangent lines for some different graphs and reasoned on their drawings. Students' supports on their thinking can be mainly categorized as based on their prior knowledge on the circle properties. Other studies on students' understanding of the functions and derivatives had drove many researchers to check into students' ability to read and sketch, and to read and interpret derivative functions (Aspinwall, Shaw, Edwards, & Graham, 2002; Chappell & Kilpatrick, 2003; Hallett, 2001; Roddick, 2001; Ubuz et al., 2000).

2.2.5 Visual Reasoning Models

Visual Reasoning Model by Park and Kim (2007) was originally designed as a mean to assess the types of visual reasoning and the related cognitive activities among architectural students when sketching their designs. Tversky (1999) proposed two ways to go beyond the visual information; one is to transform to visual information according to the predetermined rules and second, to make deduction and conclusion on the visual information. When students solve the given tasks, their visual reasoning can be summarized in terms of three broad categories consisting of eight interrelated types: perception, analysis, and interpretation in the seeing, generation, transformation, and maintenance in the imagining, and internal representation and external representation in the drawing as illustrated in Figure 2.9.

In the seeing process, the activities of visual perception, analysis, and interpretation occur. During the perception activity, basic properties of the visual information and their combinations are recognized and identified. The image of the object is attained as and when it is observed. This is a very selective process and this

selectivity is accountable for the qualitative value in the subsequent visual images to be produced. The difference contexts and purposes in which the visual are perceived and generated play important role in creating the final visual images. During the analysis activity, the observation on the relationships among the properties and the exploration about the characteristics of the visual information occur. During the interpretation activity, the naming, categorization, and giving new meaning to the perceived visual information occur. These activities in the seeing process bring about the extraction of characteristics as required for new visual generation and transformation.

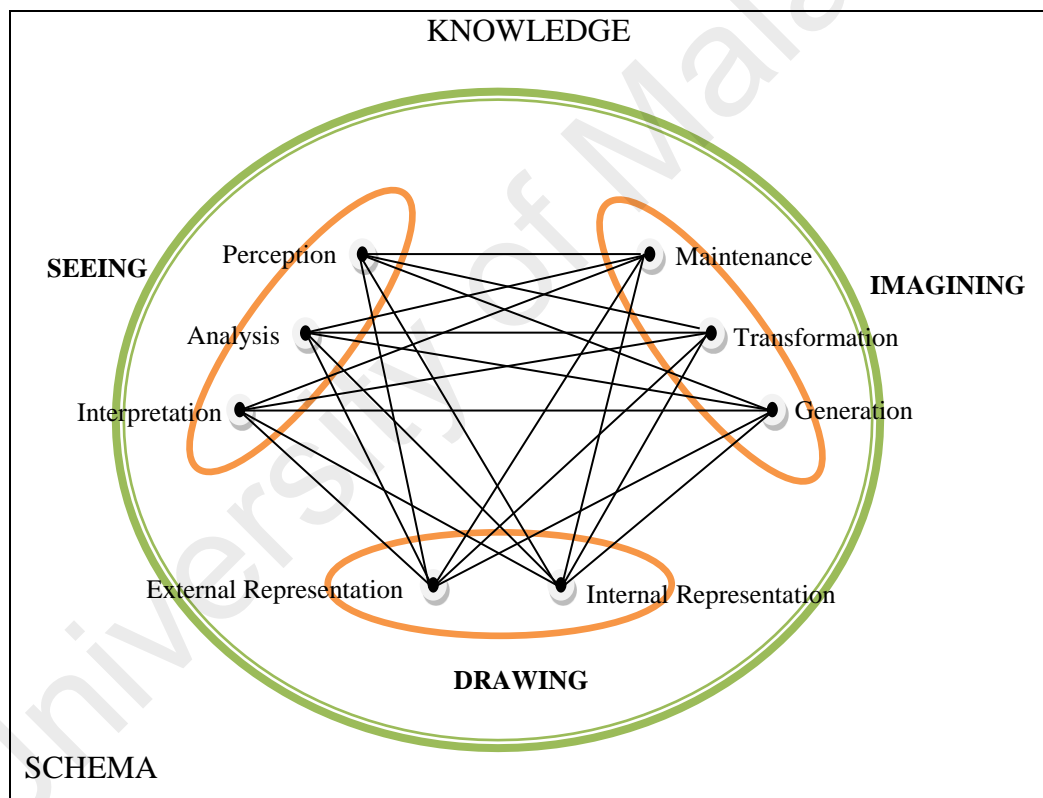


Figure 2.9: The Visual Reasoning Model (Park & Kim, 2007)

The imagining process enables the synthesizing of conceptual information for the new visual representation. Imagining process can be classified into the generation, transformation, and maintenance activities. In an earlier study conducted by Kavakli & Gero (2002), they proved that the generation and transformation activities were very critical in creating visual information. Visual generation occurs in two ways: the first is

from the perceptual input during the seeing process while the other one emerged from the activated knowledge and schema that were stored in the long-term memory (Kosslyn, 1994). Visual transformation can be differentiated into two types: congruent transformation and pattern change transformation (Park & Kim, 2007). Kosslyn (1994) defined congruent transformation as equivalent to the actual perception such as the mental rotation or the resize of visual objects. On the other hand, Oxman (2002) suggested for the pattern change transformation to involve the developing or evolving of visual objects. Following the visual transformation, the maintenance activity takes place to store the internal representations.

The drawing process enables visual objects to be represented through both the internalization and externalization. In internal representation, the transformed visuals are to be confirmed. This drawing process occurs through interactions with imagining and seeing processes. In addition, the external representation serves as external memory, in which ideas are settled as visual tokens, and to be revisited later for inspection, if necessary (Suwa, Purcell, & Gero, 1998). The process of generating the imagined objects might also occur during the process of converting from internal representation to external representation. As a result, the drawing process is important in visual reasoning. In addition, the drawing process also make possible for the visual information to be manipulated and transformed.

Knowledge and schema are engaged in the interaction within the visual reasoning activities. A schema is a collection of objects, processes and actions and other previously constructed schemas that are coordinated and synthesized by the individual to form structures utilized in problems situations (Sabella & Redish, 2005). The retrieval of visual knowledge from long term memory becomes a cue to match between visual input and visual memory for visual perception in seeing process. The visual schema retrieved from the long term memory becomes a rule for the extraction of the

characteristics of the visual information. The iterative process of seeing and imagining make it possible to reorganize, transform and modify the existing visual input in imagining process. Oxman (2002) highlighted the importance on how to transform and to access schema of basic structure in reformulating visuals, since the order and pattern of visuals can cause different types of reasoning. The schema, therefore, plays a critical role to link between the conceptual and perceptual processes in drawing process. As a result, diverse manipulation or interpretation of images can be generated. In the visual reasoning process, seeing, imagining, and drawing processes do not occur independently but interactively with knowledge and schema, together with interaction between perceptual and conceptual knowledge.

Costa (2010) finalized a model to understand four different modes of the visual-spatial thinking: from perception, from mental manipulation of images, from the mental construction of relationships among images and from the exteriorization of thinking. The visual-spatial thinking that resulted from perception used visual information that are represented based on movement. It involved different individual perceptions referred to as concrete images and memory images when images of experiences were recalled. The thinking processes engaged in the process were intuitive inference, visual recognition, construction of visual, recalled visual representation, evaluation of images, identifying of objects and images, recognition of abstraction and concepts generation. Among the mental processes that took place in the visual-spatial thinking that resulted from mental manipulation of images were the secondary and anticipatory stages of intuitions which involved a stable cognitive attitude on understanding reasoning on more common situations. Other processes were mental transformation, constructive and synthesizing, coordinating spatial structure and visual construction.

The visual-spatial thinking that resulted from the mental construction of relationships among images involved the mental construction of how visuals were

related and comparing the models, ideas and concepts. The thinking processes involved include the searching for relationships among images, facts and properties, and continuous evaluation along the process of solving a problem. Lastly, the visual-spatial thinking that resulted from the exteriorization of thinking involved the mental processes of translation, describing the mental dynamics through verbalization and gestures and using the analogies.

The abstract mathematical objects, concepts and processes can best be experienced by students through the use of visual representations. Therefore, there is a need of clear meaning on how visual processing can help to solve mathematical problems. Gorgorio and Jones (1996) described three distinctive components of visualization process that resulted from the ability to mentally manipulate, influence and transform visual images and visual representations. Starting with crude visualization where students are able to draw diagrams with either pencil or pen, or with the help of technological software, visuals were used to represent mathematical objects, concepts, of processes and subsequently to use them to understand and help in the solving of mathematical problems by interpreting the technical rules or mathematical formula. This was then followed by the visualization be regarded as the activity to read the visual information where the interpretation of the relationships among the properties of the visual representations. The final part of visual processing involved the ability to manipulate and transform the visual images and visual representations mentally and conceptually.

Due to the increase in the number of tools that are able to help users interact with mathematical visualization, Sedig (2009) presented three frameworks describing the interaction design of mathematical visualization; the micro-level interaction framework, the micro-level interactivity framework and the macro interaction framework. The micro-level interaction framework characterizes the interaction in the

context of exhibiting low-level cognitive tasks and epistemic behaviours. The interaction framework organized the user activities into basic (conversing, manipulating, and navigating) and task-based (animating, cutting, filtering, rearranging and scoping). The second level of micro-level interactivity framework organized the user activities into factors such as cognitive offloading, constraints, flexibility, focus, scaffolding and transition. Lastly, the third macro-level interaction framework listed the design space into four categories ; access-, annotation- , construction- and combination-based.

2.3 Conceptual Framework

According to Lowrie et al. (2011), visual representations and hence visual reasoning fell under two non-separated processes, the encoding and decoding processes. Their relationships involved the ability to transform mathematical information into graphics during the encoding process and to extract mathematical information from graphics during the decoding process. Students develop visual sense gradually as a result of creating graphics and using already designed graphics in a variety of contexts that require them to make sense of the data embedded in the graphics (Friel, Curcio & Bright, 2001).

2.3.1 The Visual Reasoning Constructs

The study conducted by Lowrie et al. (2011) was the researcher's attempt elaborate on the processes of visual reasoning that support the learning outcomes of mathematics education. The two learning outcomes of the SACE curriculum (2015) were for the students to be competent on the literacy and numeracy skills. The literacy skills in mathematics education deliberate the ability to *shift between verbal, graphical, numerical and symbolic forms of representations* in order to understand concepts, solve

mathematical problems and communicate information. On the other hand, the numeracy skills expect the students to be able to *understand, analyse, reason and use mathematical knowledge and skills* to apply in ranges of contexts such as : 1) gathering, representing, interpreting and analysing data, 2) using spatial sense and geometric reasoning, and 3) working with graphical and algebraic representations and other mathematical models. In particular, the SACE curriculum for topic Working with Function and Graphs using Calculus described teaching and learning strategies that covers specific key areas of learning. Some samples of the key ideas and the teaching and learning strategies are as listed in Table 2.1.

Table 2.1: Samples of Key Ideas and Teaching and Learning Strategies extracted from the SACE Mathematical Studies

Key Questions and Key Ideas	Considerations for Developing Teaching and Learning Strategies
<p>How will functions arise?</p> <p>What makes one model more appropriate than another?</p> <ul style="list-style-type: none"> • Students need to be able to discuss the appropriateness of the model on the basis of its features and the structure of the problem or context <p>Maxima, minima, limiting behaviour (horizontal asymptotes), points of inflection, points of discontinuity (vertical asymptotes)</p>	<p>Students can be reminded of the work that they have done on modelling, and can re-examine models and their construction in contexts that require numerical, algebraic, and graphical approaches. This could be done in a number of ways:</p> <ul style="list-style-type: none"> • numerical data \rightarrow graphical representation \rightarrow algebraic model • algebraic model \rightarrow numerical data \rightarrow graphical representation
<p>What is a rate of change?</p> <p>How can a constant rate of change be identified?</p>	<ul style="list-style-type: none"> • Numerically, in a table with a constant adder • Algebraically, as a property of a linear function • graphically (and geometrically) by considering gradients of chords across graphs of curves (graphics calculators, interactive geometry, and graphing software provide invaluable visual support, immediacy, and relevance for this concept).

The visual reasoning construct is built upon two processes, namely the encoding process and decoding process, that are used to describe the students' ability to construct and interpret respectively, graphs in terms of conceptual knowledge and performance standard.

2.3.2 The Encoding Process

The encoding process helps students to use graphs to explain the verbal or written information. Studies that investigated students encoding skills make use of graphs to communicate their understanding of concepts and their solution tasks. Earlier, Simon (1986a) conducted interviews with the pre-university students who needed to undergo remedial classes due to their lack of foundation understanding on functions. His findings proposed a set of skills displayed by the students when drawing diagrams to solve mathematical problems, as listed in Table 2.2. The skills described the levels of ability to sketch or draw diagrams effectively. Later, Diezmann (1999), employed the levels of sub-skills in his experimental study, as external control measures, to guide the students' works. He discovered that those students who were in the treatment group appreciated the suggestions on the criteria for effective drawing and had actually sketched a higher quality and complete diagrams as compared to those in the controlled group.

Table 2.2: Simon's (1986) Diagram Drawing Sub-skills

Sub-skill	Description
Sub-skill 1	representing all relevant information
Sub-skill 2	creating an integrated diagram that are critical to the conceptualisation of the problem
Sub-skill 3	labelling completely
Sub-skill 4	checking the accuracy of the diagram
Sub-skill 5	drawing multiple representations are not critical
Sub-skill 6	verbalising what is represented and what needs to be represented

In 1998, Carlson undertook a study on undergraduate students to solve co-variational tasks that required them to interpret and represent functional situations. She identified five mental actions that categorized the activities that students performed when sketching diagrams as shown in Table 2.3. The mental actions provided a way to classify each student based on the overall images that he/she produced to support various types of thinking when in context with the tasks.

Table 2.3: Carlson's (1998) Mental Actions of the Co-variation Framework

Mental Action	Description	Actions
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	<ul style="list-style-type: none"> labelling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	<ul style="list-style-type: none"> constructing an increasing straight line verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	<ul style="list-style-type: none"> plotting points/constructing secant lines verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of- change of the function with uniform increments of change in the input variable.	<ul style="list-style-type: none"> constructing contiguous secant lines for the domain verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	<ul style="list-style-type: none"> constructing a smooth curve with clear indications of concavity changes verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

The advantage of constructing a diagram relates to how problems are conceptualised (Rivera, 2011; Uesaka & Manalo, 2007, 2011; Van de Walle, 2007). Therefore, the content of an instructional material should emphasize on the sketching of diagrams to enhance understanding. Hergaty and Kozhevnikov (1999) introduced the term mathematical visuality to describe an individual's preference in using diagrams when solving mathematical problems. Prior to that, Krutetskii (1976) classified the students' visual ability into three broad categories as shown in Table 2.4.

Table 2.4: Krutetskii's (1976) categories of visual ability

Category of preference	Description
Analytical type	Individuals who prefer verbal-logical rather than imagery modes when attempting to solve problems
Geometric type	Individuals who prefer to use diagrams or images rather than the verbal modes when attempting to solve problems
Harmonic type	Individuals who have no tendency to any one of the type in particular.

2.3.3 The Decoding Process

The decoding process allows the students to read, interpret and make sense of the visual information embedded in diagrams. Studies that investigated students' decoding process and skills took into account the extent to which students interpreted and made sense of data and information in various graphics of different structures and purposes. The ability to read and reason on how one quantity varies depending on another related quantity is of utmost importance.

The study conducted by Friel, Curcio and Bright (2001) was the researchers' attempt to identify students' levels of interpreting graphs in school context. Their study was an enhancement of the study conducted by Curcio (1987) earlier on how fourth and seventh grades students strategized to understand graphs. Curcio also detected that students' prior knowledge on the properties and structures of graph played the main

factor in influencing their ability to understand the mathematical relationships embedded in graphs. In 1998, Friel and Bright detailed the study on data exploration, data comparison and data prediction. Their finding indicated that besides the fact that students had encountered or had been exposed to many types of graphs in and/or out of school context, they were still lack of competency in tackling tasks that require higher order thinking skills. Later in 2001, building on their series of previous works, Friel, Curcio and Bright listed six behaviours that they presumed to be essential in understanding graphs. The lists of both findings are as tabulated in Table 2.5.

Table 2.5: Levels of graph comprehension by Friel, Curcio & Bright (2001)

Level	Behaviour	Description
Read	Recognising components of graphs	'lifting' information to answer explicit questions for which the obvious answer is right there in the graph
Describe	Speaking the language of graphs	
Interpret	Understanding relationships among tables, graphs and data	interpolating and finding relationships in the data presented in a graph
Analyze	Making sense of graph but avoiding personalization and maintaining an objective stance while reading the graphs	
Predict	Interpreting information in a graph and answering questions about it	extrapolating, predicting, or inferring from the representation to answer implicit questions
Extrapolate	Recognising appropriate graphs for a given data set and its context	

In the same year, based on his research on pre-service teachers reasoning when solving mathematical problems, Yumus (2001) established four levels of reasoning as shown in Table 2.6.

Table 2.6: Yumus's (2001) levels of reasoning

Level	Description
Level 1	Unable to produce any reasoning
Level 2	Aware of models, known facts, properties and relationships used as basis of reasoning, but cannot produce any arguments
Level 3	Able to provide reasons although arguments are weak
Level 4	Able to provide strong arguments to support reasoning

Students should pay more attention and try to understand the data displayed in graphical form. Sharma (2013) conducted a meta-analysis on studies that investigated students' thinking and interpreting graphs (and tables). She discovered a wide range of responses, from those who exhibited no or very little characteristics of visual thinking related to mathematical concepts, to overly considering the mathematical concepts and visual thinking. She described a five-stage framework to establish students' ability in reading and interpreting graphs (and tables) as shown in Table 2.7.

Table 2.7: Sharma's (2013) framework for interpreting graph.

Stage	Description	Behaviours
Stage 0	Informal/idiosyncratic	Students at this stage are exhibiting characteristics of pre-structural thinking
Stage 1	Consistent non-critical	Students at this stage are exhibiting characteristics of pre-structural thinking or at most uni-structural thinking
Stage 2	Consistent non-critical	Students at this stage are exhibiting characteristics of uni-structural and multi-structural thinking
Stage 3	Early critical	Students at this stage are beginning to exhibit characteristics of relational thinking. Students at this stage can attend to more than one relevant aspects of the data and are beginning to integrate the aspects
Stage 4	Advanced critical	Students at this stage are integrating statistical and contextual knowledge that exhibits extended abstract thinking

2.3.4 Knowledge and Scheme

2.3.4.1 Making sense of graphs

Alacaci, Lewis, O'Brien and Jiang (2011) established for graphs to compose of four structural components; 1) the framework of a graph that refers to the elements related to the measurements such as axes and scales, 2) the specifiers which represent the data such as the point, line or curve, 3) the labels to indicate the variables representing the quantities and the relationships between them, and 4) the background that add to the aesthetical value of the structure in order to enhance the visual presentation of the system of axes such as the colours or shading.

A Cartesian graph is used to transmit information through its spatial characteristics such as the location of a point that is represented by paired values or the lines or curves to represent the related quantities. Graphs sketched by students during the problem solving process play important and multiple roles for both the students and the solution process (Friel et al, 2001; Hodges & Conner, 2011; Mesa, 2007; Tang, 2004). They serve as external memory to complement the limitation of human cognitive abilities (Goldin & Kaput, 1996). Graphs also act as a medium that students used to communicate and enable them to reason on the problem. The kinds of perception and their functions (Carney, 2002; Elia & Philippou, 2004; Dorler, 1991) in the solving process and the relationship between perceptions and the appreciative system are related to the functional references attached to the graphs themselves. The inappropriate dichotomy between analytic and synthetic thinking process emphasizes that mathematical visual reasoning was not equivalent to vision but a production of thought via visual imagery (Canham & Hegarty, 2010; Enns, 2004; Wang, 2012). Problem solving was therefore considered essential as the interplay between two types of knowledge - conceptual and perceptual, which are linked by the cognitive process

known as visual reasoning (Long, 2005; Siyepu, 2013a; Stylianides & Stylianides, 2007; Wittmann, 2006).

The elementary phase of investigating graphs, reading the graph, focuses on the extracting data directly as how they are seen on the graphs (Persmeg, 1986, 1991, 2006). Students are to find, locate and translate information based on the specific rules or conditions (Moore, Paoletti & Musgrave, 2013; Ubuz, 2007). Translation requires a change in the form of a communication. In order to translate between words and graphs, students need to describe the specific structures of the graphs (Adu-Gyamfi, Bose & Stiff, 2012). In reading between the graph, the intermediate phase of interpreting graphs focuses on interpolating and finding connection in the data shown on the graphs. Students are to integrate or pull together two or more pieces of information (Chapman, 2013), make comparisons and to observe relationships among the specifiers or between the specifiers and the labelled axes. To interpret, students need to rearrange and prioritize information in the order of their importance (Stayridou & Kakana, 2008). The phase of reading beyond the graph or the advance phase of applying graphs focuses on extrapolating information and analysing the relationships implicitly out of the data shown in the graphs. Students are to generate, predict and make inferences. To extrapolate, students need to extend the interpreting phase by stating not only the essence of the communication but to identify some of the consequences through noting the trend perceived by the data or specifying implications based on personal background knowledge (Tiwari, 2007). The hierarchical levels of decoding skills can be viewed and regarded as being built from the previous level and as the progression of levels of understanding the functions and derivatives.

2.3.4.2 Performance standard

Based on SACE Curriculum Statement (2014), mathematics is not only a collection of concepts and skills but a technique to evolve into new tasks and challenges by exploring, displaying, *reasoning*, *visualising*, and solving, with the goal to communicate the relationships exhibited and the solved problems. The Mathematical Studies Performance Standard (Appendix A) of the SACE Curriculum Statement (2014), outlines three main areas of measure or criteria as guide on how the students are progressing in their learning: 1) Mathematical Knowledge and Skills and Their Application (MKSA), 2) Mathematical Modeling and Problem Solving (MMP) and 3) Communication of Mathematical Information (CMI). For the MKSA, students are expected to demonstrate their knowledge of content and understanding of mathematical concepts and relationships. They are expected to use mathematical algorithms and techniques to find solutions to routine and complex problems, application of knowledge and skills to solve problems in different contexts, selection and the use of technology. MMP requires the development of mathematical models that lead to mathematical results, development of mathematical results for problems set in familiar and unfamiliar contexts, interpretation of mathematical results in the context of the problem, understanding of the reasonableness and possible limitations of the interpreted results, and recognition of assumptions made and possible new mathematical questions to be investigated. CMI focuses on the communications of mathematical ideas and reasoning to develop logical arguments, use of appropriate mathematical notation, representation and terminology.

When setting tasks to test students on their skills and applications proficiency, teachers should prepare the questions or information in both the written and diagrammatic form including graphs. These are to allow the students to demonstrate their understanding on relationships among ideas and concepts. Students are also able to

make choices on the appropriate techniques or methods based on the nature of the tasks (Hegarty & Kozhevnikov, 1999), be it a routine, interpretive or analytical, that maybe set in personal or global context. Students are also strongly encouraged to support their solution and steps taken by arguments and explanation through the use of appropriate or correct terminologies, notations and representations. They are also advised to make use of the technology, graphics calculators in this study, to aid and enhance their understanding and supports to solution process.

2.3.4.3 Conceptual knowledge

Haapasalo and Kadijevich (2000) explained conceptual knowledge as an action of ‘using the knowledge’ instead of a mere ‘knowledge of’ which is best to describe the procedural knowledge. Therefore, students’ conceptual knowledge can be determined through their use of this knowledge to solve any mathematical problems. While procedural knowledge requires only the use of visual representations, conceptual knowledge, on the other hand, demands the making of connections among the visual representations and texts or other types of representations. For example, in relating the properties of functions and their derivatives graphically, the reading of the value of the function or making comparison of the values of functions at several points through the vertical height from the x -axis of each position means ‘knowledge of’. The conceptual knowledge refers to the students making connections between the sign of the slopes of the chord lines and the increasing and decreasing of the functions or the existence of the stationary points and their natures, being the specific local maxima, local minima or stationary inflection.

2.3.4.4 Perceptual knowledge

Hoffer (1977, p.85) defines visual perception as ‘the ability to see and interpret’ which include the perceiving ability such as to perceive figure-ground and spatial relationships, and the processing of visual information such as to distinguish and memorize visuals. Individuals are very dependent on visuals as compared to other forms of information, in which Sinnett, Spence and Soto-Faraco (2007) referred to as ‘visual dominance effect’. Initially, concentration on graphs tends to be objects rather than processes. The focus interest is placed largely on the figural properties, i.e. the figure and shape of graphs as perceived through the senses and interpreted by mental reflection (Goldin, 2001; Liu, 2010). Visual perception processing involves mental representation in the mind which varies significantly among individuals in term of their visual images and their use in solving mathematical problems (Presmeg, 2006). Visual processing, on the other hand, involves ‘visualization and the translation of abstract relationships and non-figural information into visual terms that includes the manipulation and transformation of visual representations and visual imagery’ (Bishop, 1983, p.184). Unfortunately, difficulties and errors may result in making mathematical generalization due to inappropriate thoughts and/or knowledge that students had pinned to graphs.

Graph perception refers to ‘the part played by visual perception in analyzing graphs’ (Legge, Gu, & Luebker, 1989, p. 365). Similarly they outlined that to understand graph perceptual processes, one must identify mental processes that: (a) affect early vision and establish a mental representation, (b) operate on the graph to enable one to identify or to make inferences about non-obvious properties, and (c) integrate one’s understanding of context with the mental representation to generate a task-appropriate response. They firstly addressed the syntax of graph perception (i.e., visual decoding) and secondly acknowledged the importance of operations that involve

the use of the syntactic properties of graphs (i.e., judgment tasks). Lastly, they took into account the semantic content of a graph (i.e., context).

Knowledge about graphs contains a variety of attributes and their relationships, and attributes of the situations in which they are used (Diezmann, Lowrie & Kozak, 2007; Roth, & Jin Lee, 2004). Consider a derivative for example. Knowledge of this derivative includes information about the properties of graphs, non-visual features such as the rate of change and situational information such as real life situations associated with the use of derivatives. It is the student's knowledge on graph comprehension that is hoped to influence in some way with visualization and visual reasoning. For example, making certain processes visually explicit may facilitate students' ability to make new links between concepts in their knowledge, understand new uses and application of the graphs.

2.3.5 Framework for assessing visual reasoning in this study

In this study, the structure of responses that refers to the worked solutions and reasoning of the students can be traced out from the encoding and decoding processes. The processes are ordered in terms of various ways to extract information embedded in the graph from as simple as spotting to exploring and to extracting and interpreting the related concepts.

The pre-university students are assumed to be able to exhibit the encoding and decoding processes to making sense of the graphs. The conceptual framework, as shown in Figure 2.10, had been developed along with the expected students' visual reasoning ability across the processes for the content domains of functions and derivatives.

This study suggested that the visual reasoning ability can be assessed based on these activities. Five derivatives problems were to be assigned to the students letting them to demonstrate such activities. The nature of the problems represents five content domains

of derivatives: slope, tangent, properties of graphs, graphs of functions and their derivatives and applications of derivatives, which are based on the SACE syllabus. Performing the standards set by the curriculum based on the three main areas; Mathematical Knowledge and Skills and Their Application, Mathematical Modelling and Problem Solving, and Communication of Mathematical Information through correct responses of the parts of the problems are indicator of them to use graphs as visual reasoning tools.

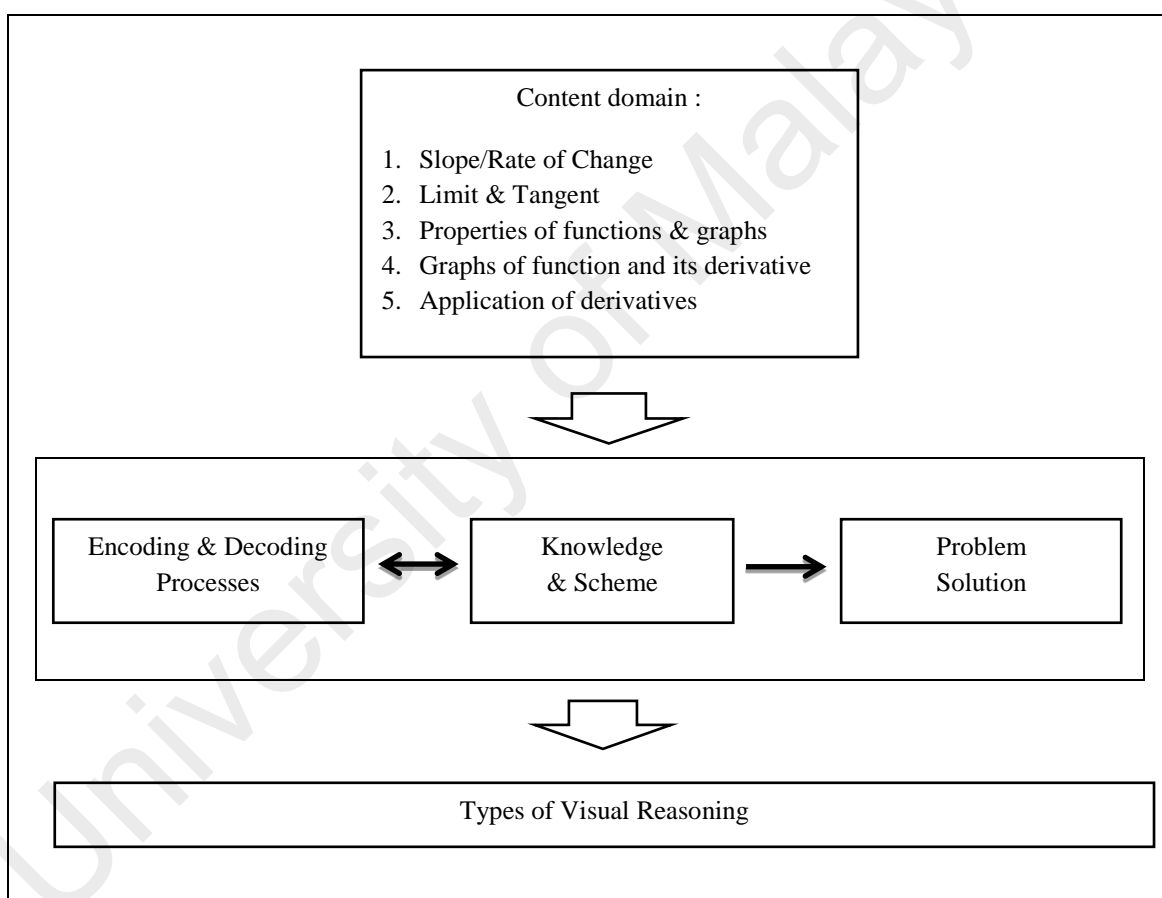


Figure 2.10: Conceptual framework of the study

2.4 Summary

The main reason for using the visual reasoning framework for the present study rested on the premise the use of visual reasoning strategy to promote the abilities to read and

interpret could improve the metacognition (Kultur, Ozdemir & Konyalioglu, 2011; Lim & Noraini, 2007) and achievement among pre-university students. The focus of the study required a practical model that could facilitate between instructor and learning outcomes. To this end, the making sense of graphs provided a pedagogical recommendation of reasoning instructional to help students making meaningful connections among learning, understanding and reasoning, and achievements (Leung & Cheng, 2004).

In the context of the present study, this method of reasoning meant assisting learning strategies that would foster understanding of relations among the concepts of derivative via the active utilization of the encoding and decoding processes. The scope of the study was thus confined to designing tasks to enhance understanding of functions and derivatives and visual reasoning, predicted from the encoding and decoding processes. There was no intention to examine the physical actions and development aspects such as those involving body gestures such as eye or hand movements.

CHAPTER 3: METHODOLOGY

3.1 Introduction

The main purpose of this study is to develop a framework that can be used to assess the visual reasoning ability of the pre-university students when they are using graphs to solve problems on functions and derivatives. The foci of the study are to examine the students' usage level of graphs (and diagrams) during their daily learning of mathematics inside and outside of the classroom contexts, their preference method and their graph reasoning ability that they adopted when using Cartesian graphs to solve tasks involving functions and derivatives. This study also proceeded to investigate the correlations among the students' usage levels in using graphs (or diagrams), their preference method and their graph reasoning ability when solving problems on functions and derivatives. The study then investigated the difficulties and misconceptions that the students faced when constructing and reading or interpreting graphs. Data were gathered using document analysis on theories and models on related visual reasoning for the development of the framework. The data on the visual reasoning ability of the students were collected using a questionnaire and two sets of the mathematical tasks on functions and derivatives. Quantitative data analysis was conducted using Excel and Statistical Package for the Social Science (SPSS).

This chapter presents the methodology of this study in five main sections comprising of : a) the research design, b) participants, c) instrumentation, d) data collection, and e) data analysis.

3.2 Research Design

A research design is the complete layout or plan to answer the research questions of the study in order to support and strengthen the prospect of representing the real situation (Noraini, 2010). It strategizes to handle and overcome complexities encountered in conducting the research processes (Polit & Beck, 2009) that could impede the validity and reliability of the research findings (Burn & Grove, 2003). This study employed the quantitative approach in order to explain the precise measurement and quantification of the single variable, namely visual reasoning, that the pre-university students exhibited when solving graph-related problems on functions and derivatives.

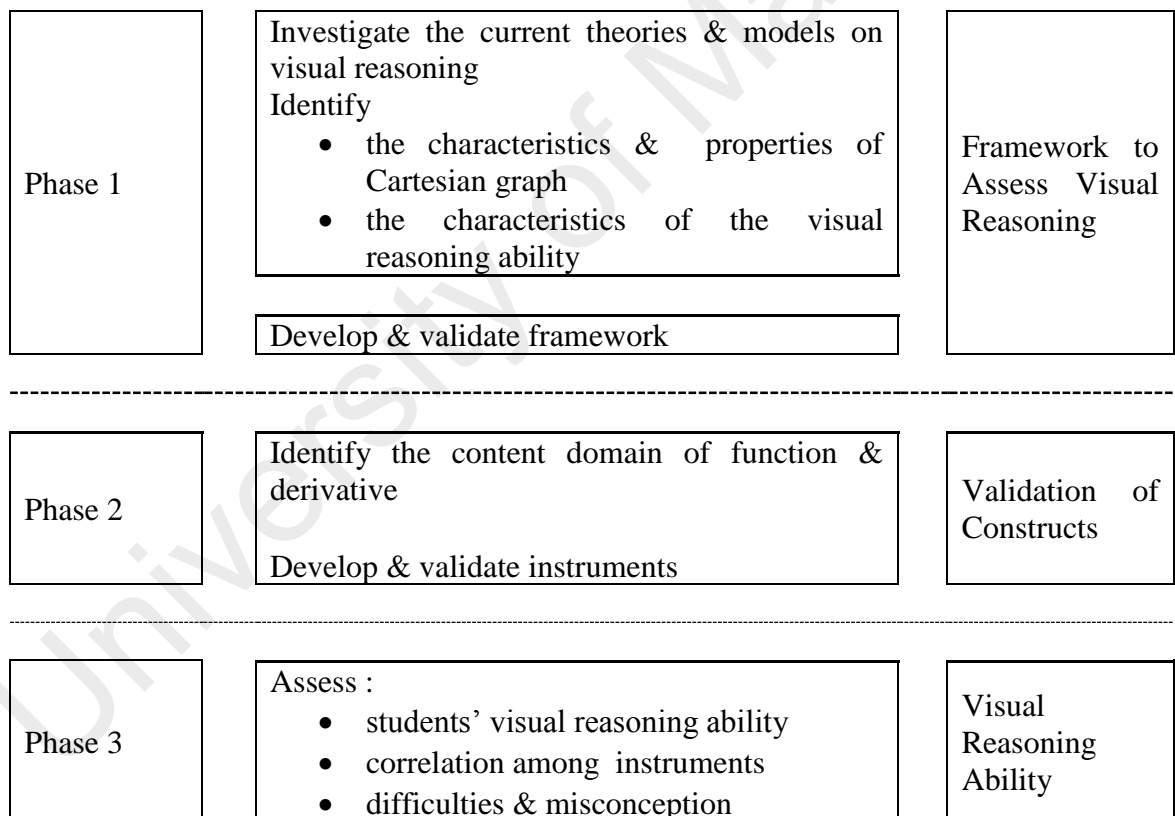


Figure 3.1: Flowchart of the phases in the research design

The overall structure of the research design is demonstrated by the flowchart in Figure 3.1. In general, this study can be divided into three phases. Phase 1 of the study was the investigation on the related theories, models and frameworks in order to prepare

the required frameworks to assess the visual reasoning ability. It engages a three-stage process. Stage 1 involved the document analysis on the literature describing the current trends and practices on using graphs or other visual tools in the teaching and learning of mathematics, and the literature to identify the characteristics of visual reasoning ability and visual reasoning process. The study employed Turner's method (1990, 1991, 1998) of theory synthesis which involves drawing together existing theories, models and frameworks to extract and integrate key ideas to generate a meaningful framework that has relevance to practicality and methodological of the visual reasoning ability. The search of the theories, models and frameworks started with both hand-search and electronic search on mathematics educational journals, articles and books. The study excluded the large body of research into lay experiments and perceptions of visualizations and visual reasoning, although empirical papers covering related theories were included. Instead of using an official definition of theory, the study adhered to proposal by Sutton and Staw (1995) that theory is about answering the *why* questions and about the relationships among trends and phenomena. Turner's method includes three steps: 1) planning of synthesis where the existing theories, models and frameworks were clarified and relevant, plausible and useful related information were extracted, 2) synthesis where theories, models and frameworks were itemized and classified to compare points of convergence and 3) refining the synthesis where the products from Stage 2 were further analysed including examination on fundamental processes in order to generate further theoretical agreements and a more robust framework.

Stage 2 involved the refinement of the initial framework identified in Stage 1 through the individual social psychology perspective type of focus group (Belzile & Oberg, 2012; Farnsworth & Boon, 2010; Kamberelis & Dimitriadis, 2013). The ideas offered by the participants through the individualistic social psychological perspective

are characterized as stable individual reasoning and thinking expressed (Eagly & Chaikan, 2007; Fazio, 2007; Markova, Linell, Grossen & Orvig, 2007) and elaborated promptly in the focus group setting (Belzie & Oberg, 2012). The setting of the focus group was designed to facilitate thought-provoking and interaction among each other. The interactions among the participants were organized to encourage verbal exchanges of ideas (Farnsworth & Boon, 2010; Lezaun, 2007). The initial framework was then refined. Stage 3 proceeded to a further refinement and finalizing the framework using 3-round Delphi method. The refined framework from Stage 2 was sent to 50 local and international experts in the areas of visual reasoning, mathematical contents (functions and derivatives), Cartesian graphs, and mathematics education. The final framework was then sent to an international expert for the final validation.

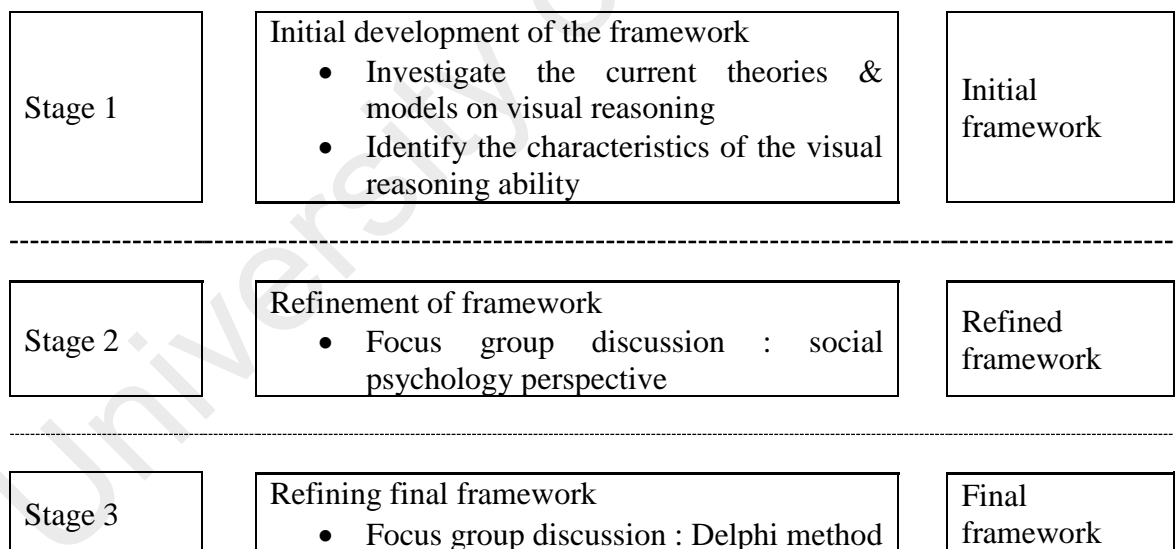


Figure 3.2: Flowchart of the development of framework for assessing visual reasoning in Phase 1

Phase 2 involves the setting of the content domain of functions and derivatives and the development of the three instruments and their validations. The detail description of Phase 2 will be explained in the section 3.4.

Phase 3 involves the use of the instruments to assess the students' usage levels of graphs, their preference method and graph reasoning ability together with the correlations among the results and finally the difficulties and misconceptions faced by the students. These involved the process of data collection and data analysis. Cross-sectional survey design with direct administration of paper-and-pencil task items was used to describe the variable in the study due to each set of the data being collected at one point in time and involved a large group of students. The term survey is designated as any research activity in which the investigator gathers quantitative data from participants for the purpose of examining the characteristics, opinion or intentions of those participants (Noraini, 2010; Polit & Beck, 2009). A quantitative research rest upon numbers aggregated into statistics, to enable researchers to interpret the obtained data and reach conclusions (Moru, 2006; Spinato, 2011) and therefore be fairly structured to enhance objectivity. Quantitative data can be transposed into numbers or coding in a formal, systematic process to obtain information and to describe variables and their relationships (Moru, 2006; Noraini, 2010; Spinato, 2011). The study utilised structured questionnaires which enabled me to quantify the responses and to conduct statistical analysis and maintained objectivity through structured data collection. The study described: 1) the usage levels of the students with regards to graphs or diagrams when learning mathematics, 2) their preferred method when solving mathematical tasks, 3) the types of visual reasoning that they adopted when using graphs to extract information, 4) the correlation among the results of the three instruments, and 5) the errors they performed, in order to solve problems on functions and derivatives. The research setting refers to the place where the data collection is taking place (Noraini,

2010). In this study, data were collected in a classroom environment where the students were having their daily learning so as to provide a natural and familiar setting for them to perform the tasks. Within the context of this study, the visual reasoning adopted by Malaysian students, at any levels, has yet to be documented.

3.3 Participants

Noraini (2010) defines participants as the entire aggregation of the cases that meet the criteria to participate in a study and about which the researcher is interested to make description of. The participants of the focus group discussion were seven experts in the areas of functions and derivatives and are attached to both the public and private higher institutions in Selangor. They have been in the teaching profession for more than six years. All of them had been teaching calculus at pre-university level and are very well-versed in using Cartesian graphs to solve mathematical problems. The study assumed that the experts participated in the focus group had their own ideas about the methods of solving mathematical problems and the use of graphs in solving problems on functions and derivatives. This focus group was set at one of the private colleges in Selangor and within easy reach by all experts. All of the participants gave their consent that their participations will be strictly confidential and their opinions and views will be used only for the academic and research purposes. They were also reminded that the purpose of the discussion was to seek their ideas on the matters discussed and there was no right or wrong answers in the discussion.

The participants to respond to the instruments were 194 pre-university students enrolled to study the South Australian Matriculation (SAM) Programme at one of the colleges in Selangor. Most of the students were selected excellent academic performers in the national examination, Sijil Pelajran Malaysia (SPM) nationwide including Sabah and Sarawak. They were mostly sponsored by the *Jabatan Perkhidmatan Awam (JPA)*

and other semi-government authorities such as *Majlis Amanah Rakyat (MARA)*, *PETRONAS*, *Yayasan Tenaga Nasional (YTN)* and some state governments. The South Australian Matriculation (SAM) programme adhered to the syllabus of the South Australian Certificate of Education (SACE), which is based in Adelaide, South Australia. Upon completing the 18-month pre-university or matriculation programme with satisfactory university entry points together with their sponsors' cut-off point requirements, they will pursue to the tertiary level of education majoring in various disciplines such as Engineering, Sciences or Commerce, at top-ranked universities in Australia and New Zealand. The distribution of students by gender, majors and races is as in Table 3.1. For the purposes of the analysis, the participants were not distinguished among their majors, classes and academic achievements. Their ages ranged from 18 to 19 years old.

Table 3.1: Distribution of students' demographic details

Characteristics		Number of students
Gender	Male	101
	Female	93
Race	Malay	158
	Chinese	21
	Indian	9
	Others	6
Major	Engineering	72
	Science	58
	Social Science	64
SPM Mathematics	A+	187
	A	7
SPM Additional Mathematics	A+	102
	A	37
	A-	30
	B+	19
	Not taken	6

All students studying at the SAM programme are required to take Mathematical Studies as one of the subjects for the South Australian Certificate of Education examination, regardless their intended major at the university. The syllabus is based on the curriculum statement and the subject outline set by the SACE board (SACE, 2014). Despite being offered a ticket to study abroad on the basis of very competitive SPM results, and the fact that the students had been exposed to the notions of function and derivative since the upper secondary levels, they showed various capabilities with regards to their mathematical ability and in understanding in the related concepts. At the time of the study, the participants had already completed the syllabus of the SACE and were in the revision months for the final external examination. They, therefore, had been working with tasks on conceptual and application of functions and derivative and were fully readied with the knowledge and understanding of the topics. They were expected to draw upon those experiences and knowledge to complete the questionnaire and tasks. The reasons for including the entire batch (for the pilot study and actual study) were that it was a manageable size and the data were collected at a localised setting. The response rate for the students was 100% and very satisfactory. This resulted from the fact that the researcher monitored the setting herself and oversaw the processes of completing the questionnaires and tasks.

The students were from the same background and eligibility criteria. Eligibility criteria defined as the set of measures that specify the characteristics of the subjects in the population must possess in order to take part in the study. The eligibility criteria of the students to be included in this study are that they : 1) must have finished their SPM examination, 2) must be a student enrolled to the SAM programme and is familiar with the SACE curriculum and syllabus content, and 3) must have been exposed to the concepts of function and derivative.

3.4 Instrumentation

Three main instruments: (a) the Visual Representation Usage Levels (VRUL), (b) the Mathematical Visuality Test (MVT) and (c) the Graph Reasoning Test (GRT) were employed to collect the data of the study.

3.4.1 Visual Representation Usage Level (VRUL)

3.4.1.1 Description of VRUL

The VRUL (Appendix B) consisted of two sections. The first, Section A: The General Information about Respondent, required students to complete their demographic data: gender, race, major, and the grades that they had obtained for the Mathematics and Additional Mathematics at the SPM level. The distribution of students and their demographic data are as displayed in Table 3.1.

In the second section, Section B: The Visual Representation Usage Level, the students were asked for their views on the use of graphs and diagrams in their daily learning of mathematics. This section is used to address research question 2(i). The instrument consists of 17-Likert scale items that fall under four different constructs :

- 1) five items on the students' usage levels of graphs or diagrams in their daily learning behaviour,
- 2) three items on the usefulness of graphs or diagrams in solving mathematical problems,
- 3) four items on the students' difficulty in using graphs or diagrams in solving mathematical problems,
- 4) five items on the teacher's behaviours in using graphs or diagrams in teaching mathematics.

In the first category of the students' usage of graphs or diagrams in their daily learning behaviour, students were asked about their usage of graphs or diagrams and those graphs or diagrams used by their teachers and the textbooks in helping them to solve mathematical problems. An example of the items in the first category is *Do you usually use graphs or diagrams in solving mathematical problems?*. The second set of items in the second category on the usefulness of graphs or diagrams in solving mathematical problems sought on the students' efficiency in using graphs or diagrams to assist them in solving mathematical word problems. An example of the items in the second category is *Do you think the use of graphs or diagrams is helpful in efficiently solving mathematical word problems?*.

The third category of the students' difficulty of the use of graphs or diagrams in solving mathematical problems looked on the ease of the students to construct graphs or diagrams to help them solving mathematical word problems. An example of the items in the third category is *In general, do you know how to construct graphs or diagrams for solving mathematical word problems?*. Lastly, the fourth set of items of students' view on teacher's behaviours in using graphs or diagrams in teaching mathematics searched for the students' views on their teachers' usage and encouragement on the use of graphs or diagrams in solving mathematical word problems. An example of the items in the fourth category is *Do your mathematics teachers use graphs or diagrams to explain how to solve mathematical word problems?*.

3.4.1.2 Validity and reliability of the VRUL

For the purposes of this study, the content validity of the VRUL was defined as the extent to which the measures or scales, 'Not at all', 'Slightly', 'Moderately', 'Very much' and 'Definitely', accurately reflected students view on the usage level of graphs and diagrams in the teaching and learning of mathematics. In this study, five

mathematics experts, with at least six years of teaching experiences were consulted on the contexts, language and terminologies that were used in the questionnaire so that students are able to understand and respond to them correctly.

Based on their comments and feedbacks, adjustments were made to suit the students' understanding on the terminologies. Three feedbacks and actions were taken:

- 1) The original instrument (permission was requested as in Appendix C) required for the students to rate each item on a five-point Likert scale, with only end points scales with labelled 1 for 'Not at all' and 5 for 'Definitely'. This caused confusion to the students in estimating their answers. Therefore, specific terms of levels were assigned to the instruments. The levels are; 1 for 'Not at all', 2 for 'Slightly', 3 for 'Moderately', 4 for 'Very much' and 5 for 'Definitely'. These were more familiar and 'clearer' terms to the students in ensuring that their responses are more accurate.
- 2) The word 'image' brings too vague or too broad meaning. It may take, for example, the form of maps that are very unlikely to be used in the teaching and learning of derivatives. Therefore, since the study focused on the use of graphs, the word 'image' in the original instruments was replaced by the word 'graph'.
- 3) The two words, 'difficulty' and 'troublesome' in two of the items triggered the same meaning to the students, and be regarded as repetitive. The items are 'How difficult is it for you to make diagrams by yourself for solving mathematical word problems?' and 'How troublesome is it for you to use diagrams / graphs in solving mathematical word problems?'. As a result, one of the items, 'How troublesome is it for you to use diagrams / graphs in solving mathematical word problems?' was removed due to the fact that students may have the idea that it is seeking for the same concept as the other item. In addition, the word 'difficulty',

in the item was replaced with the word 'easy' so as to bring consistency on the positivity of the views in the items.

The VRUL was reported to be a reliable instrument to assess the students' usage levels of diagrams in solving mathematical problems (Uesaka, Manalo, & Ichikawa, 2007). The adapted version of the VRUL was pilot-tested on 50 students to determine the reliability of the VRUL for this study. The students were studying the same programme and with the same education backgrounds, in terms of their academic performance, but not included as participants in the actual study. Table 3.2 shows the reliability estimates as measured by Cronbach Alpha for the overall VRUL and its four main categories. The coefficients ranged from 0.64 to 0.89 for the categories and 0.87 for the overall reliability.

Table 3.2: Reliability coefficients of the VRUL and its categories

Category	Alpha Cronbach coefficients
Overall VRUL	0.87
The usage of graphs or diagrams in their daily learning behaviour.	0.74
The usefulness of graphs or diagrams in solving mathematical problems	0.81
The students' difficulty on the use of graphs or diagrams in solving mathematical problems	0.64
The teacher's behaviours in using graphs or diagrams in solving mathematical problems.	0.89

3.4.2 Mathematical Visuality Test (MVT)

3.4.2.1 Description of MVT

This instrument (Appendix D) was designed to investigate encoding process of the students specifically on the method that they prefer or spontaneously adopted when solving word problems on functions and derivatives. The instrument elaborates on a construct called mathematical visuality that is useful in addressing research question 2(ii). The mathematical visuality clarifies productive ways of different methods of preference students employed to assist them when solving mathematical word problems. Three levels of mathematical visuality: visual, partially visual and non-visual, clarify important difference among the students' choice of method of solving mathematical word problems. In detailing the three categories, the study drew categories of mathematical visuality from prior studies over the past four decades (Kang, 2012; Krutetskii, 1976; Presmeg 1986, 1993, 2006).

The MVT consists of word problems that can be solved either graphically or algebraically, which indirectly acquire for the students' understanding on the concepts of functions and derivatives. It was a paper-and-pencil test with an open-ended format. The items were constructed based on the SACE curriculum with some being adapted from the main textbook for Mathematical Studies (Haese & Haese, 2010). SACE final examination questions are designed to inter-relate other areas in the syllabus such as integration, trigonometry and statistics. Therefore, it was not appropriate to adopt the questions for it offers little content validity to the domain of functions and derivatives that are covered in the study and testing objectives (Roweton, 2003). At present, there is no standardised test available, specifically in Malaysia, to test on students' method of preference when answering word problems on functions and derivatives. Other studies globally concentrate on general mathematical word problems focusing on other aspects such as basic understanding of the concepts of functions (Dubinsky & Wilson, 2013),

functions to categorize students' ways of thinking (Moore & Thompson, 2015) and students' co-variational and quantitative reasoning (Moore, 2014; Weber, 2012).

The final instrument comprised of 5 items with each item to have a few follow-up parts. All items were word problems and were set such that students should be able to solve them either by using the algebraic method or by sketching graphs to represent and explain solutions. Each item represented one content-domain : rate of change, slope and limits, properties of function and its derivatives and applications of derivatives. The descriptions of the items are as follows:

Item 1 : Rate of change

The task in item 1 assessed students' understanding on the concepts of different types of rates of change; the constant rate of change, the average rate of change and the instantaneous rate of change. The problem required students to explain each of the types in relation to the graphs of functions. Students were supplied with the word 'graph' as a hint for them to use or sketch/draw graphs as a method of explaining or describing the rates of change. An example of the item is as shown in Figure 3.3.

Explain how are the following related to graphs of function?

(a) Constant rate of change

Figure 3.3: An example of item on rate of change in the MVT

The study anticipated that students would illustrate the explanation using graphs of straight lines with, either sloping up or sloping down, to represent the constant rate of change. For both of the other two, the average rate of change and the instantaneous rate of change, it was expected that students sketch curved graphs of functions and consequently use the concepts of chord between two points and tangent at a particular

point to illustrate and described the average and instantaneous rates of change respectively. They may also enhance their explanation through the use of right-angled triangle to show the increment in both the horizontal and vertical directions.

Item 2 : Limit and tangent

The task in item 2 assessed students' understanding on the concepts of limits and tangent. Students were presented with the formula of slope between two points and the formula for limit. The problem required the students to justify their understanding on the relationships between the ideas of limit and the concepts of tangent. An example of the item is as shown in Figure 3.4.

Explain what you understand of the formula

(a)
$$\frac{f(x) - f(a)}{x - a}$$

Figure 3.4: An example of item on slope and limits in the MVT

The study anticipated that students would illustrate the explanation using the chord between any two points and consequently illustrate the situation when the horizontal distance between the two points is getting smaller and eventually approaching zero, relate it to the graph of curved to represent the idea of limits and hence the tangent of the function at a particular point.

Item 3 : Properties of functions and graphs

The task in item 3 assessed students' understanding on the concepts of functions and graphs and their properties. Students were presented with an algebraic expression of logistic function and follow-up parts that requested them to look for the basic properties

of functions such as the domain and range, the x -intercept and y -intercept, the vertical and horizontal asymptotes and the behaviour of the function as the variable x get very small ($-\infty$) and very big ($+\infty$) values. The item continued with tasks for the students to analyse some properties of the first and second derivatives. An example of the items is as shown in Figure 3.5.

As mentioned earlier, the students have indeed been using the graphic calculator during most of their daily learning of mathematics. They in fact should be very well-versed with its use. Therefore, the study anticipated that students would make use of their graphic calculator to sketch the graph of the function and hence to read out most of the information from it. They should also be able to extract information required on the first derivative and need to further draw the graph of the first derivative or the second derivative in order to extract information related to the second derivative of the function. They should be able to use the ‘zoom-in’ or ‘zoom-out’ function keys to adjust the screen in order to view those patterns or trends of the graphs as required by the questions such as the domain and range.

Consider the function $f(x) = \frac{50}{2 + 3e^{-x}}$

- (a) State the domain of the function
- (c) Find the x -intercept(s).
- (f) Discuss $f(x)$ as $x \rightarrow +\infty$

Figure 3.5: An example of item on properties of graph in the MVT

If the students were to sketch the graph of $y = f'(x)$, they should be able to reason the sign of the derivative function based on the location of the graphs either above or below the x -axis. Subsequently, to interpret the function $y = f(x)$, based on the

sign of $y = f''(x)$, would require the basic knowledge on the relationship between both functions. In looking for the point of inflection, students may use the graphic calculator to sketch the graph of $y = f''(x)$, and read-off its zero(s).

Item 4 : Graphs of functions and its derivative

The task in item 4 assessed students' understanding on the concepts of function and its derivative. The item required students to describe the original function based on the given conditions of its first and second derivatives. Students were presented with information on the initial value of unemployed people, u , to be 800,000. The first derivative and the second derivatives were set to be negative and positive respectively. An example of the item is as shown in Figure 3.6.

The number of unemployed people u at time t was studied over a period of time. At the start of this period, the number of unemployed was 800 000.

Throughout the period, it is observed that $\frac{du}{dt} < 0$ and $\frac{d^2u}{d^2t} > 0$.

Describe the number of unemployed people over time.

Figure 3.6: An example of item on graph of function and its derivative in the MVT

The study hope for the students to use the conditions set for the first and second derivative to sketch the graph for the unemployed people $y = u(t)$. The students should know that the negativity of the first derivative implies that the number of unemployed people is decreasing while the positivity of the second derivative indicate the shape of the graph of unemployed people to curve up (or convex as referred to the mathematical terminology). The study also expect the students to set the y -intercept of the graph at 800,000 indicating the initial number of unemployed people and not to sketch their

graph to the left of the y-axis, or towards the negative values of the horizontal variable, the time, t . They should also realize that the number of people will always be positive and therefore they cannot extend the graph to below the horizontal axis.

Item 5 : Applications of derivatives

The task in item 5 assessed students' understanding on the concepts of function and derivative when applied to the real life situation. Students were given an algebraic expression representing the number of students logged into an educational website over a five-hour period and were asked to find information such as the interval of time when the students and the rate of change of the students logged onto the website is increasing. An example of the item is as shown in Figure 3.7.

The number $A(t)$ of students logged onto an educational website at any time t , over a five-hour period is approximated by the formula $A(t)=175+18t^2-t^4$, $0 \leq t \leq 5$.
Find :
(a) the rate of change of the number of students logged onto the website after 2 hours

Figure 3.7: An example of item on applications of derivatives in the MVT

The study anticipated the students to draw the graph of the function using their graphic calculator and read off the data from it in order to answer the follow-up questions. Students must be able to recall the relationships among the properties of functions and derivatives that represent the real life situations. Students should also be alert on the limits imposed for the situation, which is over the first five hours only.

3.4.2.2 Validity and reliability of the MVT

The items were written in English as the medium of instruction in the classroom was English. I have been teaching the pre-university programme for at least 25 years and

therefore am proficient in the language to be able to construct the items. The initial instruments comprised of five questions. For its content validity, the instrument was sent to three experts in the area : 2 locals and 1 international (Appendix E) for their relevance and concordance with the content-domain and syllabus. Adjustments were made based on the feedbacks from the experts :

- 1) The title of the survey was proposed to be more specific instead of ‘Cross Sectional Survey 1’. Therefore it was change to Mathematical Visuality Test since the purpose of the instrument is to look for the students’ mathematical visuality when answering the mathematical word problems.
- 2) The words and terminologies used in the instruction section are to be as simple as possible so as to make sure that students were able to understand and abide to.
- 3) Item 2 and 3 of the original instrument were advised to be replaced since students were still able to solve for the answers using the information given and without to sketch any graph.
- 4) In item 5 of the original instrument, the instruction to find $\frac{dA}{dt}$ was proposed to be excluded since it would indirectly lead students to solve it using algebraic manipulation.

The instrument was pilot-tested with the same 50 students for the VRUL test. The worked solutions by the students were marked based on the final rubric to increase consistency of scoring (Moskal & Leydens, 2000). The rubric was based on the categories listed under the encoding process of the framework for assessing the visual reasoning.

The scoring procedure began with the allocation of points to each of the category in the encoding process as listed in the framework for assessing visual

reasoning, as shown in Table 3.3. The students' works were then checked and points were assigned based on the respective category. Frequencies, percentage, mean and standard deviation were then calculated for all parts of the items. As a measure of precaution to refine the rubric, 5 students' works were selected as 'anchor papers', sets of scored solutions which reflect a variety of different solutions to the items and different aspects of the rubric. Minor adjustments were made based on the 'anchor papers'.

Table 3.3: The rubric for the MVT

Point	Code	Description
6	CGCS	Correct graph with correct solution - Produces correct graph to explain and represent the solutions and managed to arrive to the correct solution
5	CGIS	Correct graph with incorrect solution - Produces correct graph to explain and represent the solutions but did not manage to arrive to the correct solution
4	IGCS	Incorrect graph with correct solution - Produces incorrect graph to explain and represent the solutions and managed to arrive to the correct solution based on the wrong graphs. Solutions may differ from the original solutions set.
3	IGIS	Incorrect graph with incorrect solution - Produces incorrect graph to explain and represent the solutions and did not manage to arrive to the correct solution
2	NGCS	No graph with correct solution - Produces no graph to explain and represent the solutions and managed to arrive to the correct solution
1	NGIS	No graph with incorrect solution - Produces no graph at all to explain and represent the solutions and did not manage to arrive to the correct solution
0	NA	No answer / Not attempted - Left the item un-attempted – no graphs or any algebraic solutions.

Reliability is achieved when any other one student of similar characteristics is able to obtain the same score regardless of when the student completed the test and

when it is being marked and who marked the test (Noraini, 2010). Two local experts in the area and subject were assigned to mark the worked solutions by the students and subject to inter-rater reliability analysis. Inter-rater reliability is defined by Noraini (2010) as when scores by two independent experts or raters are consistent due to a well-constructed rubric and scoring criteria for each level or criteria. The overall reliability of 0.94 measured with Cohen's Kappa of the MVT was based on the inter-rater reliability score of the two experts and indicates that the MVT was reasonably reliable for the study. Although scoring rubrics may not eliminate variations that occur among the raters completely, they do reduce the occurrence of discrepancies. The main objective is for the raters to come to the same score for the same student.

An item analysis was performed on the results of the pilot test. Those items that were outside the ranges of 0.2 and 0.8 (Singh, 2012) for both the difficulty index and discriminant index, respectively, were modified. Difficulty index indicates the total number of students who were able to correctly solve each item. These values would be able to identify the vagueness or complexity of each item for the majority of the students (Kaplan & Saccuzzo, 2005). Discriminant index determines if one student had done well in one part or item will also performed well in the whole set of item. These values would be able to differentiate students with varying ability in terms of the subject content. Items that caused confusion to the students were re-worded or reviewed for clarity (Ghadi, Abu Bakar & Alwi, 2013). The final instrument had appropriate levels of difficulty ranged from 0.6 to 0.97 and levels of discriminant ranged from 0.66 to 0.89.

3.4.3 Graph Reasoning Test (GRT)

3.4.3.1 Description of GRT

This instrument (Appendix F) was designed to investigate the decoding process of the students and how students use graphs to solve problems on functions and derivatives

and to measure aspects of acquiring the understanding on the concepts of functions and derivatives. The instrument elaborates on a construct called graph reasoning that is useful in addressing research question 2(iii). The graph reasoning clarifies how students make use of the visual information depicted on graphs when solving problems on functions and derivatives. Three levels of decoding or graph reasoning ability: reading the graph, reading between the graph and reading beyond the graph clarify important difference among students' ability to read, extract and interpret data or information embedded in graphs. In determining the three levels of decoding process, the study drew categories of visual reasoning from prior studies over the past three decades (Friel et al, 2001).

The GRT consists of problems that are accompanied by Cartesian graphs which require the students to look for answers through reading and interpreting them. The tasks indirectly acquire for the students' understanding on the concepts of functions and derivatives visually. As with the MVT, it was a paper-and-pencil test with an open-ended format. The items were also constructed based on the SACE curriculum with some being adapted from the main textbook for Mathematical Studies (Haese & Haese, 2006). SACE final examination questions are designed to inter-relate other areas in the syllabus such as integration, trigonometry and statistics. Therefore, it was not appropriate to adopt the questions for it offers little content validity to the domain of functions and derivative that are covered in the study and testing objectives (Roweton, 2003). The framework for the GRT, based on the content domain and the decoding level is as illustrated in Table 3.4.

Table 3.4: Framework for the Graph Reasoning Ability

	Reading the graph	Reading between the graph	Reading beyond the graph
Slope	<ul style="list-style-type: none"> Identify the y-coordinate of a given point Identify as the increment in one variable with respect to another related variable 	<ul style="list-style-type: none"> Making comparison of the slopes 	<ul style="list-style-type: none"> Make relationships between instantaneous and average slope
Tangent	<ul style="list-style-type: none"> Identify the coordinate of a point on the graph and line as the point of contact The location / position of graphs of functions (one above the other) 	<ul style="list-style-type: none"> Calculating the slope of the tangent at the point of contact Relationship of the two functions in terms of the distance between them 	<ul style="list-style-type: none"> Notice the relationship of the slope of the tangents at particular points or particular conditions
Properties of graphs	<ul style="list-style-type: none"> Identify the coordinates of zeros Read off the vertical asymptote and the horizontal asymptote 	<ul style="list-style-type: none"> Identify the equation of the vertical asymptote Making connection of the zeros of $y=f'(x)$ as the stationary points of $y=f(x)$ 	<ul style="list-style-type: none"> Make decision on the nature of stationary points through the signs of $y=f'(x)$ Make connection of the visible shape of the graph with the signs of $y=f''(x)$ and $y=f'(x)$
Graphs of functions and their derivatives	<ul style="list-style-type: none"> Identify the increasing and decreasing parts of the graph 	<ul style="list-style-type: none"> Identify the shapes of the graph Evaluating $y=f(x)$ as x positive negative infinity 	<ul style="list-style-type: none"> Make connections of the graphs and their derivatives
Application of derivatives	<ul style="list-style-type: none"> Comparing the real life situations verbally and the graphs of the situation 	<ul style="list-style-type: none"> Describing the slope of the graphs as rate of change 	<ul style="list-style-type: none"> Identify the properties of the second derivative from the graph of $y=f(x)$ Relating important points and properties of the graphs to the real life situation

At present, there is no standardised test available, specifically in Malaysia, to test on students' ability to read and interpret Cartesian graphs when solving problems on functions and derivatives. Other studies globally that had concentrated on general mathematical problems focusing on other mathematical areas such as geometry (Bardelle, 2010; Clements & Sarama, 2012) which is very visual in nature, statistics (Lee, Khng, Ng & Ng, 2013) which make use of more types of graphs to represent the data and other areas that used other types of visual information such as diagrams (Booth & Koedinger, 2012; Fathulla & Hameed, 2009) and representations (Koedinger, Alibali & Nathan, 2008).

The GRT comprised of three scales that make up the constructs of the decoding processes in graph reasoning (Friel et al, 2001; Lowrie et al., 2011, Sharma, 2013) : 1) reading the graph, 2) reading between the graph and 3) reading beyond the graph. Reading the graph measured students' ability to extract information directly from the graph. In reading between the graph, students need to be able to understand the relationships among the information shown in the graph and reading beyond the graph required students to interpret the information displayed in the graph. Table 3.5 list the distribution of each part of the items to the scales describing the decoding ability.

Table 3.5: Scales of the decoding process for the items in GRT

Decoding scale	Item
Reading the graph	1(a)(i), 1(b)(i), 2(a)(i) 3(b)(i)(1), 3(b)(i)(2), 3(b)(i)(3), 3(b)(ii)(1) 4(a)(i), 4(a)(ii), 5(b)(ii)
Reading between the graph	1(a)(ii), 1(b)(ii), 1(b)(iii), 1(c)(i), 1(d) 2(a)(ii), 2(b)(i), 3(b)(ii)(2) 4(b), 5(a), 5(b)(i)
Reading beyond the graph	1(c)(ii), 1(e), 2(b)(ii) 3(a), 4(c), 5(b)(iii), 5(b)(iv)

The final instrument comprised of 5 items with each item to have follow-up parts. All items were graph-accompanied problems on functions and derivatives and were set such that students need to refer to graphs for solutions, or in other words, the information are in the graphs. Each of the item may contain one or a mixture of content-domains : slope, tangent, properties of functions, graphs of functions and its derivatives and applications of derivatives.

The descriptions of the items based on the content domain and the decoding scales are as follows:

a) Descriptions on items based on content domain and their graphical representations

Item 1 : Properties of functions and graphs

The tasks in item 1 assessed students understanding on the basic properties of functions and graphs. No specific expression of the function is given in order to avoid from the students manipulating it algebraically. The problem was accompanied by a Cartesian graph and it required students to relate the algebraic expression of the properties of the function to their visual representation on graph. Grids were added in the background of the axes system in order to help students with the spatial relationship among the properties. An example of the item is as shown in Figure 3.8.

Conceptually, it is anticipated that students are able to relate the algebraic expression $f(4)$ and $\frac{f(3)-f(1)}{3-1}$ onto their visual representations as point and gradient respectively, on graph. Part (b) of the item tested the students reasoning ability to decide the larger values of the pairs of expressions. Students need to compare visually the vertical position of the points, the steepness or slanting of the chords and the tangent lines. Part (c) of the item required students to associate the algebraic expression to the positivity or negativity or the directions of the slopes of the tangent and chord and

subsequently to illustrate them on the graph and to recall the relationship between them. Part (d) of the item tested the students on the visual representations of both the chord and the tangent while part (e) of the item required the students to visualize the relationship between them.

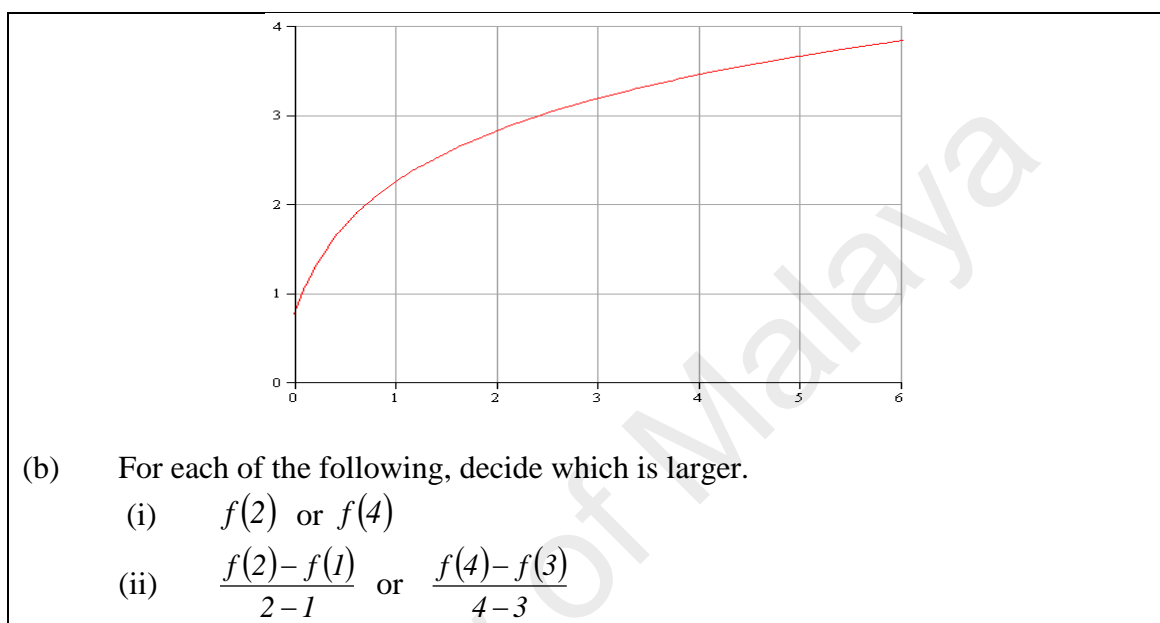


Figure 3.8: An example of item on slope in the GRT

Visually, the students were provided with basic graph of a function and gridded background, therefore, they were expected to be able to visualize the information by reading the data directly from the shape of the graph. The positions of $f(2)$ and $f(4)$ are easily read-off from the grid while some thinking might need to be imposed in locating the gradient of a few pairs of points. Students should also be able to reason the decision for the gradients of chords and gradients of tangents in part (b)s by the shape of the curve which is an increasing function but decreasing in the rate of change. Part (c) of the item required the students to reason their answer by referring to the direction or the increasing pattern of the curve. Part (d) and part (e) needed for the students relate some concepts of the derivatives and to go beyond what were illustrated in the graphs.

Item 2 : Limit & tangent

The tasks in item 2(a) assessed students understanding on the concepts of limit and tangent. No specific expression was given for neither the function nor the tangent line in order to avoid from the students manipulating them algebraically. The problem was accompanied by part of a Cartesian graph and it required for the students to relate the coordinates of the point of contact on the graph to another point on the tangent line. An example of the item is as shown in Figure 3.9.

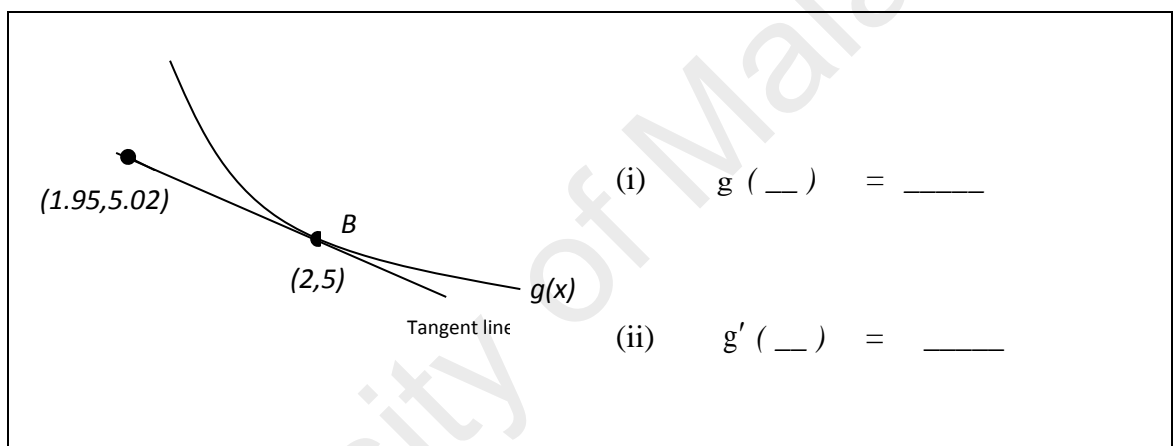


Figure 3.9: An example of item on tangent in the GRT

Conceptually, the study anticipated that students are able to relate the gradient of the tangent line to the gradient of the curve at point B , which is also represented algebraically by $g'(x)$. By checking the coordinates of both points, students should be able to realize the small difference in both the horizontal and vertical directions which is the basic idea of limit. Part (b) of the item required the students to understand the relationship between the two functions in terms of their vertical location or vertical distances.

Visually, the students were provided with part of the graph showing a graph of a function and its tangent. Two set of coordinates for two specific points on the tangent

line were given, one was the point of contact and the other one was any point that is very near to the point of contact, with a horizontal difference of 0.05 and a vertical difference of 0.02. Part (i) of the item just needed the students to read directly from the graph while part (ii) required them to notice that the gradient of the curve at point B is also the gradient of the tangent line. Part (b) of the item needed the students to express the higher function minus the lower function in order to get the correct positive value of any distance. Part (ii) required them to go beyond what were illustrated and evaluate the relationship of the derivative of both functions at specific point, $x = c$.

Item 3 : Properties of function and its derivative

The task in item 3(a) assessed the students understanding on the properties of the function through the information given on the graph of its derivative. Students were given the graph of the derivative and students needed to look for the minimum value of the function. Conceptually, the students were expected to know the relationship between function and its derivative. Students are to understand that the positive and negative values of the derivative would indicate the decreasing and increasing of the function and hence the minimum or maximum of the functions. Visually, it was expected for the students to relate that the position of the derivative graph to be above or below the x -axis would indicate the increasing or decreasing respectively, of the function and hence the position of the maximum or minimum of the function. Students may alternatively draw the sign diagram of the derivative function and determine the nature of the stationary point out of it.

The tasks in item 3(b) assessed students' knowledge on the properties of the function specifically on the asymptotes and second derivative of the function. Conceptually, the study expected the students to realize that the graph represented a rational function and therefore they would break and that asymptotes are among the

main features. The students should also understand that the properties of the function for various values of the first and second derivatives of the function. Visually, the students were expected to know that the vertical or horizontal dotted lines represent vertical and horizontal asymptotes. On the same note, students should also realize that both axes may also be the asymptotes but students may ignore them due to the missing dotted line. These can actually be analysed by inspecting the shapes of the graphs when discontinuity happens and as the horizontal variable gets bigger in both directions, which is usually indicated by an arrow at the end of the graph. Students should also recognize that the values of the first and second derivative, as requested by the tasks, can be gathered from the direction of the graph of the function, either increasing or decreasing, and the shape of the graph of the function, either convex or concave. An example of the item is as shown in Figure 3.10.

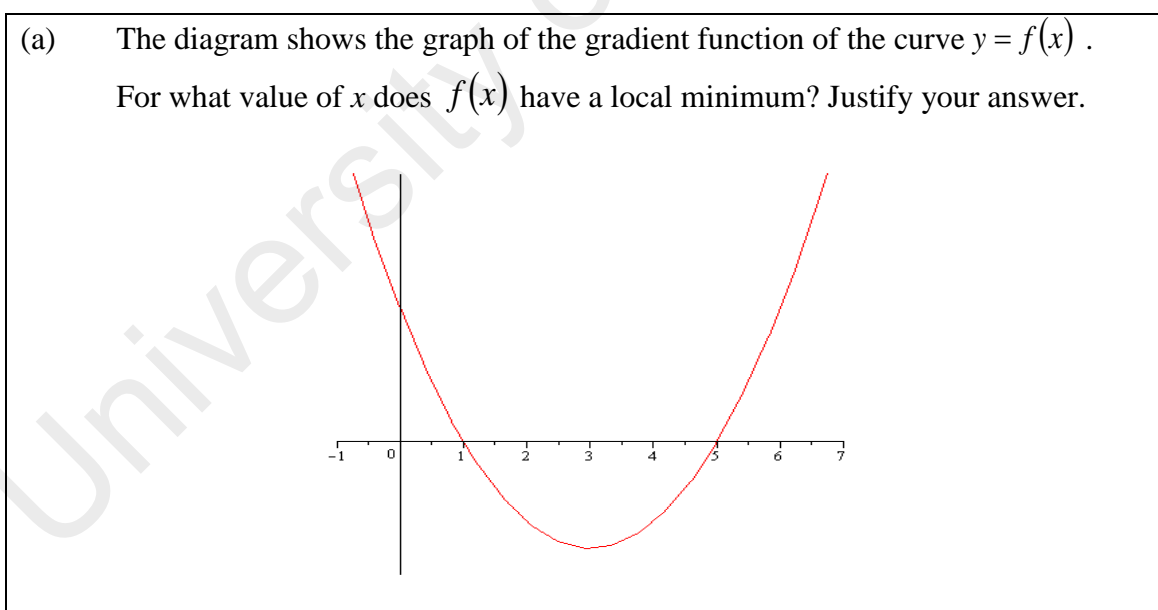


Figure 3.10: An example of item on properties of functions in the GRT

Item 4 : Properties of functions and its derivative

Similar to the tasks in item 3, item 4 assessed the relationship between the function and its derivative with an additional that students need to draw the graph of the derivative function of the given graph of the function. Conceptually, students were again tested on some properties of the function and its derivative such as the increasing, concavity and the extreme ends of the graph of the function. Visually, for the last part of the item, it was expected for the students to be able to relate the increasing and decreasing part of the graph to the positivity and negativity of the derivative function and hence its location to be above or below the horizontal axis. Students should also understand that the maximum point on the graph of the function is where the graph of the derivative should cut the horizontal axis. Lastly, students must be able to recognize that the part of the graph that tend to go flat towards the right of the horizontal axis indicates that the derivative is approaching the zero value and therefore its graph should be approaching the horizontal axis. An example of the item is as shown in Figure 3.11.

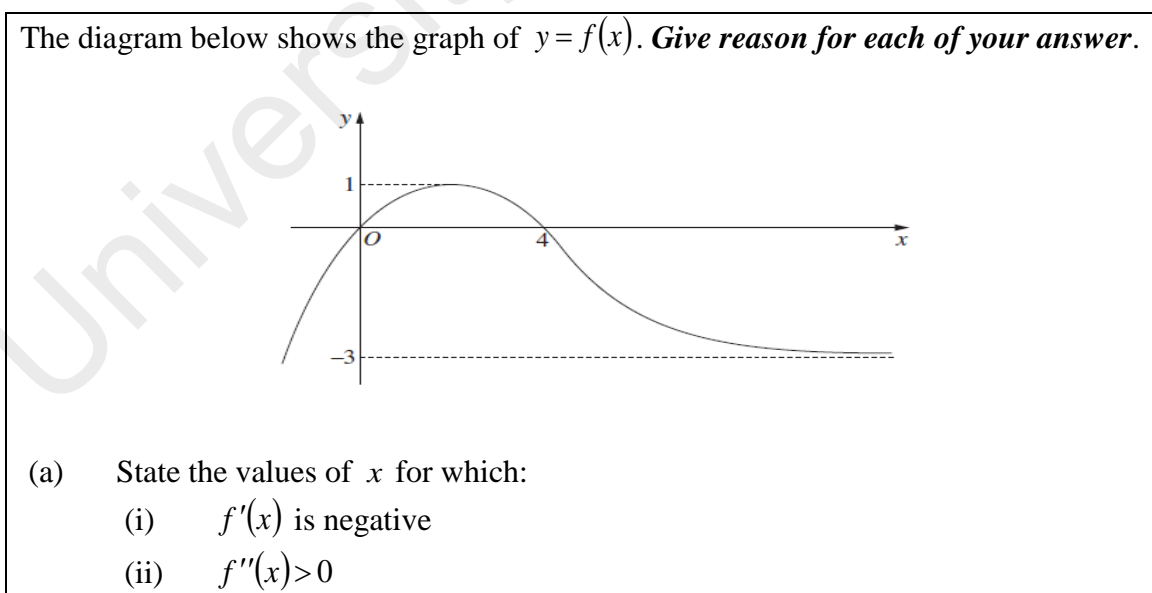


Figure 3.11: An example of item on graphs of functions and their derivatives in the

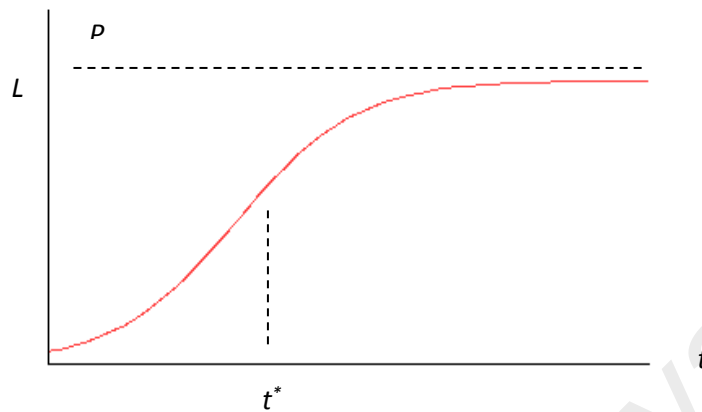
GRT

Item 5 : Rate of change & application of derivative

The tasks in item 5(a) assessed the students' visual reasoning on the application of rate of change. Students were given three graphs and three descriptions of motion to match. Conceptually, the study expects the students to have the knowledge on the types of rate of change, constant rate of change in this case. Visually, the study anticipated that the students were able to relate the graphs of various straight lines and combinations of straight lines to the idea of motion and speed undertaken by the vehicles. Students needed to look beyond the graphs provided in order to match with the real-life descriptions.

The tasks in item 5(b) portray another problem on real-life situation. It required the students to explore the growing trend of a population in a confined area, in terms of its rate of change, and thus subjected to some restrictions. Students were given a logistic curve to represent a growing population. Conceptually, the students were expected to relate the idea of rate of change to how the population was increasing and the relationship between the function and its second derivative. Visually, the students were expected to describe the rate of change of how the population is increasing by examining the changes in the gradient of the graph of the function. They should also recognize that the inflection point is to represent the maximum rate of change due to the steepness of the gradient and horizontal asymptote is to indicate the maximum capacity of the confined area. Finally, it was expected that the students were able to relate the shape or concavity of the graph of the function to the characteristics of the second derivative of the function. An example of the item is as shown in Figure 3.12.

(b) A population, P , growing in a confined environment often follows a logistic growth curve, as shown in the diagram below. *Give reason for your answers.*



- (i) Describe how the *rate* at which the population is increasing changes over time.

Figure 3.12: An example of item on applications of gradient of derivative in the GRT

b) Description on items based on decoding scales

The decoding process demonstrates three scales on how students may extract information embedded in graphs. All scales are incorporated in all items in the GRT, as listed in Table 3.4 and Table 3.5.

Reading the graph

The decoding process of reading the graph required the students to extract the information on the properties of functions and their derivatives directly from what they can see directly or as shown on the graph. In item 1(a)(i) and 2(a)(i), the students were expected to be able to locate the position of particular points directly based on the grid provided. Items 1(b)(i) required the students to decide on the larger between the two values based on their vertical positions. Items 3(b)(i)(1), 3(b)(i)(2), 3(b)(i)(3), and 3(b)(ii)(1) needed the students to be able to read off the properties of the functions, such as the asymptotes, the concavity or the shape of the graph and the stationary points,

directly from the graph. Items 4(a)(i) sought the students' knowledge on the decreasing part of the graph to relate to the negativity of the derivative function while item 4(a)(ii) needed the students to associate the positivity of the second derivative to the concavity of the graph of the function. Item 5(b)(ii) required the students to describe the rate of the population increasing by looking at the gradient of the graph of the function.

Reading between the graph

The decoding process of reading between graph required the students to interpolate and look for relationships among the data presented visually on a graph with regards to the properties of functions and their derivatives. Item 1(a)(ii) and 1(b)(ii) required the students to interpret the expression for the slope of chords and check their steepness in order to determine which is the larger in value. Similarly, item 1(b)(iii) required the students to make comparisons on the steepness of the tangent lines in order to determine the larger of the given two expressions representing the gradients of the tangent lines. On the other hand, item 1(c)(i) needed for the students' knowledge to interpret the expression $f'(l)$ and relate to the tangent line being slanting to the right or to the left in order to determine the sign of the value. Item 1(d) needed for the students to illustrate the difference between the chord and the tangent line.

Item 2(a)(ii) needed for the students to realize that the gradient of the line between two points represents the gradient of the tangent line and further to transfer the interpretation into the symbolic form. In item 2(b)(i), students should be able to realize that the expression for the vertical distance between the two graphs is obtained by subtracting the lower function from the higher function. Item 3(b)(ii)(2) needed the students to identify the intervals for the function to be above the horizontal axis and at the same time to relate to the portion of the graph of the function being decreasing. Item 4(b) needed for the students to interpret the part of the graph to the right of the

horizontal axis which is flattened to horizontal. Students are expected to relate this to the first derivative of the function and explain it in terms of the gradients instead of the function itself.

Students were provided with three descriptions to match with the three sets of graphs in item 5(a). The graphs of time against distance displayed three different situations or motions of a vehicle between two towns. Students were expected to be able to interpret the motion in terms of the speed which is described through the changes of gradients throughout the journey. Item 5(b)(i) tested the students' ability to relate the pattern on how the graph is increasing to the rate of change of the population in the problem.

Reading beyond the graph

The decoding process of reading beyond graph required the students to extrapolate, predict, or infer based on what were shown on the graphs to answer implicit questions. Items 1(c)(ii) required students to make a decision on various positions of new points in relation to the fixed point $x=1$ and to decide the sign of the algebraic value of the gradient based on the patterns of the slant directions of the lines. Item 1(e) continued to let the students display the situations (together with item 1(c)(i)) graphically. Students were expected to sketch a few chords to describe the formation of a tangent line from the chord in terms of the concepts of limits. Item 2(b)(ii) is the extension to item 2(b)(i). Graphs, although drawn to scale as in many cases, exhibited that the point in question (*point* $x=c$) is where the distance is the largest and therefore students should be able to relate the idea of optimization or maxima/minima points. Item 3(a) required the students to relate the location of the graph of the derivative function to the sign of the derivative function and subsequently identify the minimum point of the function through the changing from decreasing to increasing portion of the graph of the function.

Students may also add a sign diagram of the derivative function to help with the solution. Item 4(c) required the students to sketch the graph of the derivative function of the graph provided that does have a specific algebraic function. Students were expected to base their judgement on their knowledge of the increasing and decreasing portion and the concavity of the shape of the graph. Items 5(b)(iii) and 5(b)(iv) involved into the further interpretation by requiring the students to relate some characteristics of the graph to represent some real life situations.

3.4.3.2 Validity and reliability of the GRT

As with the instrument for MVT, the items were also written in English. The initial instrument comprised of eight questions and was sent to the same three experts assigned for the instrument MVT to check for their relevance and concordance with the content-domain and syllabus. Adjustments were made based on the feedbacks from the experts.

- 1) The title of the survey was proposed to be more specific instead of ‘Cross Sectional Survey 1’. Therefore it was change to Graph Reasoning Test since the purpose of the instrument is to look for the students’ reasoning when employing graphs to answer the mathematical problems on functions and derivatives.
- 2) The words and terminologies used in the instruction are to be as simple as possible so as to make sure that students were able to understand and abide to.
- 3) Item 8 of the original instrument were advised to be eliminated or placed in the MVT since it was a word problem and suited the nature of the MVT.
- 4) Six items (items 1, 3,5 6, and 9) of the original instrument were proposed to be removed or amended.

New items were replaced and submitted to the experts for approval. A pilot test was conducted on the instrument using the same 50 students for the VRUL and the MVT test. The worked solutions of the students in the pilot test were marked based on the final rubric to increase consistency of scoring (Moskal & Leydens, 2000). The rubric was based on the categories listed under the decoding process of the framework for assessing the students' visual reasoning.

The scoring procedure began with the allocation of points to each of the category in the decoding process as listed in the framework for assessing visual reasoning, as shown in Table 3.6. The students' works were checked and points were assigned based on the respective category. Frequencies, percentage, mean and standard deviation were calculated for all parts of the items. As a measure of precaution to refine the rubric, 5 students' works were selected as 'anchor papers', sets of scored solutions which reflect a variety of different solutions to the items and different aspects of the rubric. Minor adjustments were made based on the 'anchor papers'.

Table 3.6: The descriptions of the final rubric for the GRT

Point	Category	Description
4	Correct solution with valid reason	Produces correct solution based on the graph and managed to provide valid reason(s) to arrive to the correct solution
3	Correct solution with invalid reason	Produces correct solution based on the graph but did not manage to provide valid reason(s) to arrive to the correct solution
2	Correct solution with no reason	Produces correct solution based on the graph but did not manage to provide any valid reason(s) to arrive to the correct solution
1	Incorrect solution with invalid reason / no reason	Produces incorrect solution based on the graph and did not manage to provide valid reason(s) to arrive to the correct solution
0	No answer / Not attempted	Left the item un-attempted.

Two local experts in the area and in the subject were assigned to mark the worked solutions by the students and subject to inter-rater reliability analysis. The overall reliability of 0.91 measured with Cohen's Kappa of the GRT was based on the inter-rater reliability score of the two experts and indicates that the GRT was reasonably reliable for the study. The average variation between their scores was between 3.5% and 5.5%.

An item analysis was performed on the results of the pilot test. Those items that were outside the ranges of 0.2 and 0.8 (Singh, 2012) for both the difficulty index and discriminant index, respectively, were modified. Items that caused confusion to the students were re-worded or reviewed for clarity (Ghadi, Abu Bakar & Alwi, 2013). The final instrument had appropriate levels of difficulty ranged from 0.69 to 0.95 and levels of discriminant ranged from 0.71 to 0.93. On the overall, the summary of the instruments is as shown in Table 3.7.

Table 3.7: Divisions and nature of the instruments

Instrument	Number of items	Level of measurement	Aspect covered
Visual Representation Usage Levels (VRUL) Section A	5	Nominal	Demographic data
Visual Representation Usage Levels (VRUL) Section B	17	Ordinal	Students usage level on the use of visual (graphs or diagrams) in their learning of mathematics
Mathematical Visuality Test (MVT)	5	Ordinal	Students preference on the visual (graph) or algebraic method in solving mathematical problems (functions and derivatives)
Graph Reasoning Test (GRT)	5	Ordinal	Students ability to use visual (graph) reasoning to solve mathematical problems (functions and derivatives)

3.5 Data Collection

This study employed document analysis and scoping focus group in the data collection process to facilitate the development of the criteria in the framework to assess students' visual reasoning ability when solving problems on functions and derivatives. The purpose of the scoping focus group was to generate potential responses regarding how students would employ graphs when solving problems on functions and derivatives, both, when given only word problems and when graphs are supplied in the contexts of problems. The criteria selected for the first part of the framework were to explore the preference methods that students would prefer to use when they were given word problems which can be solved both by the algebraic method or by the used of graphs. The second part of the framework outlined the criteria for students to have to make use of graphs to find data or information in order to solve the tasks. Besides the important findings of the students' usage of graphs, both parts of the framework took into consideration the conceptual knowledge on functions and derivatives.

The study used standardized questions and probes to strengthen replicability. The discussion began by explanation on the purpose of the study and explanations on some of the theories, models and frameworks on visual reasoning and the use of graphs and other visual representations in mathematics and in solving mathematical problems. The proposed framework and the proposed instruments to be used in the study were distributed and 20 minutes were allowed for the experts to go through them individually. The initial instructions were standard questions to encourage interactions among the experts and not prompt by me, as the moderator. When necessary, the researcher interrupted the conversations to ensure that all the experts' views were heard and offered probes to stimulate additional perspectives or to explore potential and promising consistencies. The questions and probes are as listed in Appendix G.

This study used three instruments, a questionnaire and two sets of paper-and-pencil task items. All three instruments were distributed to the students in the pilot test as well as the actual study. The series of instruments took over six-month period before they sat for their trial examination. This was to ensure that the students had covered all the required topics and syllabi and that they were well-equipped with the conceptual knowledge on functions and derivatives. Prior to carrying out the study, consents were granted from the top management of the college and the students were sought permission on their participations. They were informed about the study and its purposes, and that their participations were voluntary and will not affect the performance or results of their internal assessment. The students were also asked to write their names on the questionnaire and the tasks sets to make sure that data were not mixed-up among the three instruments. Therefore, the students were granted confidentiality and permissions were obtained from them to use the data collected for academic purposes only. The researcher handled the monitoring of the questionnaire and both instruments herself. The sessions were set outside the class lecture hours so as not to interrupt with the students' time and be carried out at their normal classroom setting so as to provide a familiar and relaxing environment.

The first instrument, VRUL, was distributed about six months before their trial examinations. The students were given flexible time to complete the questionnaire. All of them managed to complete them between 30 – 60 minutes. The researcher was there for the whole duration to prepare for any queries or misunderstood of the questions or terminologies.

The second instrument, MVT, and the third instruments, GRT, were distributed to the students about three and two months, respectively, before they sat for their trial examination. This is to ensure that the students had completed their learning on functions and derivatives. Students were allowed to use their graphic calculators when

answering both instruments in order to help them answering the questions. Since the purpose of the study was to gather information on how the students make use of graphs to solve problem on functions and derivatives, they were not subjected to any time-constrained. Each of the student worked individually and they were given ample time to complete both of the instruments. Most of the students completed the MVT within 1 hour and the GRT within 1 – 2 hours.

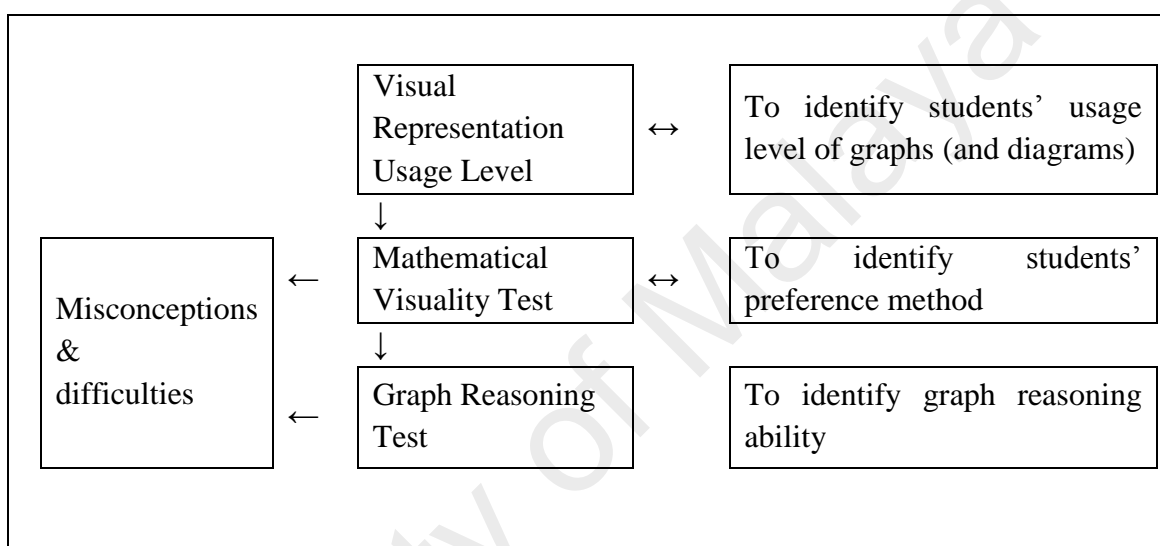


Figure 3.13: The framework of the instrumentation

The instrument MVT was used to assess students' method of solving either algebraically or graphically. They were provided with mathematical word problems on functions and derivatives and students had the choice whether to use any of the methods to answer them. On the other hand, the instrument GRT was used to assess students' use of graphs to answer problems on functions and derivatives. They were provided with graph-accompanied questions on functions and derivatives where they had to use the information embedded in the graphs in order to solve for the tasks. The students were encouraged to show in detail all steps of solutions required to arrive to the answer. They were also requested to elaborate them in order to capture their thinking and reasoning. The students' misconceptions and difficulties in sketching graphs and in reading or

interpreting the given graphs were also extracted from their worked solutions for both the instruments, MVT and GRT respectively. The framework of the instruments is as shown in Figure 3.12.

3.6 Data Analysis

The study employed a quantitative approach to analyse the data collected. Descriptive statistics were calculated to summarize the students' responses to all items. Descriptive statistics enable researchers to reduce, summarize and describe quantitative data obtained from empirical evidence (Noraini, 2010). For the first questionnaire, frequencies and percentages were used to describe the demographics data of the participants while the students' responses to each item in the VRUL were described through their frequencies, percentages, mean and standard deviations.

The findings of the focus group discussion are not generalizable, the researcher followed a standardised protocol with structured questions in order to maintain objective stance. The data analysis focused predominantly on verbal content. No attention was paid into analysing the interactions among the experts or how the information or criteria were socially expressed or constructed (Belzile & Oberg, 2012). Although they mostly agreed on the categories and criteria set in the framework, there were still minor disagreements in the expert's opinions on the arrangement of the categories. However, the differences of ideas appeared to reflect the experts' teaching experience more than their personal perspectives. Similar patterns of disagreement were identified across the categories in the framework which assess the preference methods and the use of graphs in solving mathematical problems involving functions and derivatives. The final framework was then sent to an international expert in visual reasoning for its content validity.

Rubrics, based on the framework designed to evaluate the students' visual reasoning ability, were prepared to assess the mathematical visuality and visual reasoning ability of the students. For instrument MVT, the students' worked solutions were assigned to one of the seven categories listed as the students' preference to use graph or not when responding to the word problems in functions and derivatives, and at the same time checking for their conceptual understanding of the concepts of functions and derivatives. On the other hand, for the instrument GRT, the students' worked solutions were assigned to one of five listed categories of the students' visual reasoning ability in terms of the reading or interpreting the three levels of information embedded in the graph, and at the same time checking their conceptual understanding on the concepts of functions and derivatives. The mean scores for individual students were also recorded so as to find the correlations among their responses in the questionnaire and instruments. The errors that the students performed in their worked solutions for the MVT were calculated based on the incorrect graph drawn and incorrect solution performed while for the GRT, the errors were calculated based on the incorrect solutions and invalid reason provided when solving the mathematical problems on functions and derivatives.

CHAPTER 4: ANALYSIS OF RESULTS

4.1 Introduction

This study purports to develop a framework and subsequently to assess the visual reasoning adopted by Malaysian pre-university students in solving problems on functions and derivative graphically. Data were collected quantitatively using a questionnaire, the Visual Representation Usage Level, and two sets of task-based instruments, the Mathematical Visuality Test and the Graph Reasoning Test. The participants of the study were students from one of the higher institutions in the state of Selangor. Specifically, the study aim to answer the following research questions:

1. What is an effective framework for assessing levels of pre-university students' visual reasoning when using graphs in solving mathematical problems on functions and their derivatives?
2. What are the pre-university students'
 - i. usage levels of graphs when solving mathematical problems on functions and graphs?
 - ii. preference when solving mathematical problems on functions and derivatives?
 - iii. graph reasoning ability when solving mathematical problems on functions and derivatives.
3. What is the correlation between the pre-university students' :
 - i. usage levels of graphs and their preference in using graphs when solving mathematical problems on functions and their derivatives
 - ii. usage levels of graphs and graph reasoning ability when solving mathematical problems on functions and their derivatives

- iii. preference in using graph and their graph reasoning ability when solving mathematical problems on functions and their derivatives.
4. What are the misconceptions and difficulties encountered by pre-university students when using graphs in solving mathematical problems on functions and their derivatives.

This chapter presents the results of analysis and is organized into five main sections. Firstly, the framework for assessing students' visual reasoning is developed. The second part reports the descriptive analysis of the students' responses on their usage levels of graphs and diagrams, their method of preference and their graph reasoning ability in using graphs to solve mathematical problems on functions and derivatives. These are followed by the analysis on the correlation among the three characteristics and the profile of errors performed by the students which subsequently led to the identification of the misconceptions and difficulties encountered by pre-university students when sketching and using graphs to solve mathematical problems on functions and their derivatives. A summary of the chapter is provided in the last section.

4.2 Phase 1 : Development of the framework to assess visual reasoning

The framework for assessing the pre-university students' visual reasoning in this study was developed in three stages as outlined in chapter three.

4.2.1 Stage 1: Initial development of the framework

Stage 1 of the initial development of the framework consisted of three steps of locating and synthesizing related theories, models and frameworks.

4.2.1.1 Step 1: Planning of synthesis

The document analysis on the articles and books through hand-search and electronic-search resulted in one hundred and twenty one articles and books were ascertained for full examination. The details of the search strategy are as shown in Figure 4.1. Out of these, twenty nine were considered relevant. The references from the one hundred and twenty one articles and books were further scrutinized and produced a further twenty four promising articles and books. In addition, another two articles and a book were detected by chance and which gave a total of fifty six relevant articles and books that are associated to twenty one different theories, models and frameworks, as shown in Table 4.1. Eight of these theories, models and frameworks (or for some, only parts of them) were related to visual reasoning and visualization to some aspects on the use of graphs and diagrams and therefore were selected as the materials for the synthesis.

The theories, models and frameworks span over forty decades with various epistemological backgrounds and were analysing visual reasoning and visualization in distinctively different techniques. All the theories, models and frameworks were developed by mathematicians, mathematics educators and mathematics researchers. The theories, models and frameworks are relevant to all ages of pupils and students although most were developed on the basis of working with secondary and pre-university students. The types of visual reasoning activities considered by the eight theories, models and frameworks are varied and inclusive of cognitive domains.

Figure 4.1

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Table 4.1: Theories identified for synthesis

Theory/Models/framework approach	References (abbreviated)
1 Theories on representations	<i>Lowrie et al. (2011)</i> , Goldin (1992), Gravemeijer, Bowers, & Stephan (2003), Ainsworth (2006)
2 Visual reasoning model	<i>Park & Kim (2007)</i> , Gattis & Holyoak (1996), Metros (2008)
3 Diagram drawing sub-skills	<i>Simon (1986b)</i>
4 Mental actions of the co-variation	<i>Carlson (1998)</i> , Liu (2010), Goldin (2001)
5 Characteristics of visualisers	<i>Alcock & Simpson (2004)</i> , <i>Krutetskii (1976)</i>
6 Understanding of tables and graph	<i>Sharma (2013)</i> , Peebles & Cheng (2003)
7 Levels of reasoning	<i>Yumus (2001)</i>
8 Levels of graph comprehension	Friel, Curcio & Bright (2001), Curcio (1987), Friel & Bright (1998), Shaughnessy (2007), Watson & Callingham (2003)
9 Visual-spatial thinking	Costa, Matos & e Silva (2010), Hulmann e al (2011), Gutierrez (1996), Ball & Ball (2007)
10 Mathematical inscriptions	<i>Gagatsis & Elia (2003)</i>
11 Functions of picture	Carney (2002), Elia & Philippou (2004), Diezmann & Lowrie (2008)
12 Types of image schemata	Dorfler (1991), Hegarty & Kozhevnikov (1999), Blackwell & Engelhardt (2002)
13 Types of visual imagery	<i>Presmeg (1986, 1992, 2006)</i>
14 Elements of visualization process	Gorgorio & Jones (1996), Gray & Tall (2001), Jones (1998)
15 Interactive visualization	Sedig (2009), Meyer (1998), Pike (2007), Kabaca (2013), Stern, Aprea, & Ebner (2003)
16 Scalable visual reasoning	Pike et al (2007)
17 Graph based reasoning	Peebles & Cheng (1999), Lohse (1997)
18 Connections between representations	Zandieh (2000), Roorda (2007), Cox & Grawemeyer (2003), Huang (2013)
19 Concept image	Gagatsis et al. (2006), Likwambe & Christiansen (2008), Ubuz (2001), Lambertus (2007)
20 Spatial visualization	Van Garderen (2004) ,
21 Visualization - semiotic	<i>Kazunz & Strasser (2004)</i>
Total theories = 21	Total articles = 56

Note : Synthesized theories, models and frameworks are those in bold.

4.2.1.2 Step 2: Synthesis

Synthesis involves the process of extracting, clarifying and summarising those ideas and aspects of the theories, models and frameworks to suit the nature of this study. Following are the theories, models and frameworks and their main thoughts that had been taken into consideration for the synthesis.

(a) Theory on representations

Lowrie et al. (2011), partly on the basis that the use of visual and graphic are increasingly taking placed to influence how students make sense of their mathematical concepts, developed a theory to explain how the process of thinking that shapes the students' mind when dealing with mathematics and mathematical ideas. He alerted on the drastic shift of how mathematical ideas and concepts are being presented and communicated in the last decade although the curriculum had not change much. His theory described the encoding and decoding processes to explain how students composed their own representations based on the textual descriptions and the technique used to employ diagrams provided in order to make sense of situations respectively. He argued that the encoding process was a support system to help students apprehend the reality of problem solving. He decomposed the decoding process into three levels of elementary, intermediate and advanced levels to describe how information is extracted or interpreted from the data in the graphics. He argued against the current practice of providing graphics for the students as compared to letting the students to construct them which will enhanced their thinking skills and understanding. Word problems incline to provide platform for students to practice the encoding techniques in order to understand the mathematical concepts and ideas. On the other hand, various skills of decoding are also required due to different graphics are composed of different elements and structures.

(b) **Visual reasoning model**

Park and Kim (2007) defined visual reasoning as to progress further than the visual information displayed in two different paths: one is to transform the information based on their conceptual rules or formulae and the other one is to make deductions or implications. The overall process of visual reasoning involved the visual analysis through seeing, the synthesis through imagining and the modelling process through the drawing process. Three activities of visual perception, analysis and interpretation occurred during the seeing process, while another three activities of generation, transformation and maintenance took place during the imagining process. The drawing process involved the evaluation of the internal and external representations. These physical actions take place in the interaction with the conceptual knowledge and perceptual activities. They identified that the visual reasoning activities engaged the visual knowledge to complement the perceived visual and the memory system to produce the visual information. The visual schema from the memory system guides the transformation and reorganization of the visual perception. The arrangement and relationships in the visual displayed may cause different types of activities during the visual reasoning process based on the complexity of the structure of the visuals. They concluded that visual reasoning process is an essential cognitive activity that has specific relation to any visual process and therefore, students ought to be trained on the reasoning activities through well-constructed and well-structured visual systems.

(c) **Characteristics of visualisers**

By the use of grounded theory methods, Alcock and Simpson (2004) developed a theory to assess students mathematical performance that resulted from their tendency to visualize or not to visualize, and their self-belief about themselves and their roles as mathematics learners. The results exhibited three major indicators to describe the

students' pattern to visualize: 1) introduced and made use of diagrams when solving problems, 2) used gesture when explaining solutions or arguing on concepts, 3) prefer to think in diagrams rather than algebraic expressions. Subsequently, those students who were categorized as visualisers were more focused on the mathematical concepts as objects, quick-thinking for drawing initial conclusions and were more confident in their own solutions and decisions. The study led to the awareness on how students' understandings on mathematical concepts were influenced by their learning environment. Only patterns (1) and (3) were considered in the present study.

(d) Diagram drawing sub-skills

Due to the importance of diagram drawing as heuristic strategy in solving mathematical problems, Simon (1986b) identified a set of six sub-skills that described how pre-calculus students attempted to use diagrams to solve mathematical problems: 1) represent all relevant information, 2) creating an integrated diagram that are critical to the conceptualisation of the problem, 3) labelling completely, 4) checking the accuracy of the diagram, 5) drawing multiple representations that are not critical, and 6) verbalising what is represented and what needs to be represented. He had also discovered five factors that contributed to whether students may or may not opt for diagrams to help them search for solutions: 1) their understanding on the mathematical concepts and arithmetic related to the problems, 2) their previous knowledge and skills to drawing diagrams, 3) their understanding of mathematical concepts, 4) their self-concept in mathematics, and 5) their motivation to correctly solve the mathematical problems. Feedback given to students resulted in them providing higher quality diagrams which indicate that it is a necessity for them to gain some metacognitive skills to successfully draw diagrams in their mathematical learning.

(e) Mental actions of the co-variation

Carlson's (1998) co-variation framework incorporated five groups of mental actions that were observed when students reasoned in representing and interpreting graphical model of live operating event on concepts of rate of change. The mental actions include; 1) visualizing two variables that change simultaneously, 2) visualizing weak relationship of two variables that changes with respect to each other such as the increasing and decreasing functions, 3) visualizing specific change in one variable with respect to a specific change in the other variable, 4) visualizing continuous changes of the function over the domain, and 5) visualizing changes of rates over the domain of the function. The framework was based on multiple refinements and analysis of the co-variational reasoning can be detected to a finer degree. It can also assist to guide the structuring of teaching and learning activities.

(f) Understanding of tables and graph

Sharma (2013), on the basis of meta-analysis on various research investigating the students' thinking claimed that students ought to start probing *worry questions* and able to justify their opinions on any graphical representations or relationships to data values in tables and algebraic expressions. Her study identified a broad range of ability, from *no* to *over* considerations on the contexts of mathematical education. One of her findings was that teaching students to extract information from graphs and tables was much easier as compared to assist them to mature in their questioning with *how* and *why* the need to gather and compare within and between categories and to further thinking about the data in the specific contexts. She finally provided a conceptual framework that can be used to assess information that is displayed in data representations and guide teachers and curriculum developers with firm pedagogical teaching and learning of mathematical concepts. Her framework outlined five stages of behaviours when

students dealt with graphs to solve statistical problems. The five stages were informal or idiosyncratic, consistent non-critical, consistent non-critical, consistent, early critical and advanced critical. For each of the behaviour, students were revealing the characteristics of their thinking starting from pre-structural thinking, uni-structural thinking, multi-structural thinking, relational thinking to extended abstract thinking.

(g) Level of reasoning

Yumus's (2001) level of reasoning emphasized on the importance of transforming students' instrumental understanding of the basics mathematical rules and concepts without referring to reasons, to more relational understanding that involved the detailed of how rules and concepts worked. The first part of the levels deals with the *what* and *how* while the latter involved the *why* for the *what* and *how*. The levels of reasoning include: 1) unable to produce any reasoning, 2) aware of models, known facts, properties and relationships used as basis of reasoning, but cannot produce any arguments, 3) able to provide reasons although arguments are weak, and 4) able to provide strong arguments to support reasoning.

(h) Levels of graph comprehension

The six behaviours of reading, describing, interpreting, analysing, predicting and extrapolating data stated by Friel, Curcio and Bright (2001) were based on the follow-up of two main findings. One of the findings was the results of the research carried out by Curcio (1987) on fourth and seventh grades student where she identified three levels of graph comprehension and the other one was determined by Friel and Bright (1998) on how students make sense of information on graphs. Curcio argued students' prior knowledge on structural components of graphs do affect their ability to read and understand the mathematical information and relationships shown on graphs. It was also

identified that they struggled in responding to tasks that need higher order thinking skills, for example, when the information is not displayed on the graphs. The researchers noted that the students tended to manipulate or to interpret that proved their inconsistent understanding on the concepts. They concluded that the process of dealing with a massive data and the structural components of the graphs contribute to the ability to read and interpret graphs.

4.2.1.3 Step 3: Refinement of synthesis

The theories, models and frameworks were evaluated, compared and linked with each other for the points of convergence as illustrated in Table 4.2 for the visual reasoning, Table 4.3 for the encoding process and Table 4.4 for the decoding process. To enable these comparisons, the theories, models and frameworks were described and broken down into respective aspects of each category.

(a) Comparison of theories/models/frameworks on visual reasoning

The three theories, models and frameworks considered for the overall visual reasoning were developed to understand the phenomena and to improve understanding on how students use representations in their daily learning through open-based tasks. Lowrie et al.'s theory on encoding and decoding was based on his empirical findings (Diezmann, Lowrie & Kozak, 2007) and others in the same discipline (Goldin & Shteingold, 2001; Kosslyn, 1989; Logan & Greenlees, 2008; Presmeg, 1986). Alcock and Simpson's characteristics of visualisers were developed based on the cognitive activities of the students to identify their mathematical behaviour and their self-perception as learners. Park and Kim (2007) developed their visual reasoning theory based on how architecture students performed their sketches. The encoding and

decoding processes were incorporated in the three main processes of seeing, imagining and drawing outlined by Park and Kim (2007).

Table 4.2: Comparison of theories/models/frameworks on visual reasoning for points of convergence

	Lowrie et al. (2011)	Alcock & Simpson (2004)	Park & Kim (2007)
Name	Theory on representation	Characteristics of visualisers	Visual reasoning model
Main purpose	Improved understanding	Improved understanding	Understanding phenomena
Research framework	Theoretical	Conceptual – a set of local theories	Conceptual
Approach	Empirical	Cognitive	Empirical
Process/Concept	Encoding & decoding process	Mathematical behaviour & perception as learner	Visual analysis (seeing) Synthesis (imaging) Modelling (drawing)
Method	Open-based tasks	Open-based tasks Semi-structured interview	Open-based tasks
Evidence	Content analysis	Interview protocols	Content analysis
Conclusion	Graphic representations greatly impact students' understanding and teaching practices	Theory on how students' own beliefs on learning relate to their tendency to visualize	The characteristics of tasks are related to the components of visual reasoning

(b) Taxonomy of skills on encoding and decoding process among theories/models/ frameworks

The three theories, models and frameworks taken into consideration for the encoding process were analysed to understand the phenomena when students need to introduce diagrams, graphs or any visual representations to assist them explaining or solving mathematical problems (Table 4.3). The theories, models or frameworks were broken into analytic, geometric and harmonic types. The analytic type refers to students who favour the algebraic or logical modes as compared to visual modes while the geometric type indicates students who choose to use visual such as diagrams or graphs

rather than the algebraic expressions or calculations to explain solutions. The harmonic type denotes those students who have the flexibility to swap from one mode to the other.

The three theories, models and frameworks referred to for the decoding process were conceptualised to understand how individuals make use of visual tools such as graphs or diagrams to help them solving mathematical problems (Table 4.4). The theories, models and frameworks were broken into elementary, intermediate and advance levels of ability to read and interpret graphs, diagrams or other visual representations together with their reasoning. These are to indicate their understanding on the relationships between the rules and concepts and their graphical or visual representations.

In summary the encoding process is able to determine the students' mathematical visibility or their preference in the method that they adopt when need to solve mathematical word problems while the decoding process can be used to described the students' graph-based reasoning or how they make use of the graphs, diagrams or visual representations provided for them in order to solve mathematical problems.

The initial framework was prepared based on the document analysis on the selected theories, models and frameworks which involved the 3-stage processes of planning of synthesis, synthesis and refinement of synthesis. For the encoding process, students were expected to either draw graph in order to represent their solution to the problems or they may end up with an algebraic methods through the use and manipulation of formulae. On the other hand, for the decoding process where they have to make use of the graphs provided for solving the tasks in the items, the students were expected to present either correct or incorrect solution based on their reading the graphs and extracting or interpreting correct information from the graphs.

Table 4.3

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Table 4.4

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Table 4.5: The initial framework for assessing visual reasoning

Visual reasoning process	Category	Description
Encoding	Draw correct graph	Correct graph to solve and represent the solutions
	Draw incorrect graph	Incorrect graph to solve and represent the solutions
	Algebraic method	Algebraic manipulation to solve the problems
	No answer / Not attempted	Left the item un-attempted – no graphs or any algebraic solutions.
Decoding	Correct solution	Correct solution based on the graph
	Incorrect solution	Incorrect solution based on the graph
	No answer / Not attempted	Left the item un-attempted.

4.2.2 Stage 2 : Refining the framework

The content validity of the framework was determined by a panel of seven experts through a focus group discussion. The experts were from various areas :

- 2 on mathematical content (functions and derivative) (MC1, MC2)
- 2 on visual reasoning (VR1, VR2)
- 2 on Cartesian graph (CG1, CG2)
- 1 on mathematics educations (problem solving) (ME)

The experts did not only examined and confirmed each category of both the encoding and decoding processes, but they had also systematically scrutinised the framework in parallel with the proposed instruments to ensure that it fully reflects the potential solution methods by the students. The technique of assessing, through both the encoding and decoding processes, the conceptual ideas on functions and derivatives, how the concepts come together and how they are used and understood were taken into

consideration in order to structure a visual setting of reasoning. Questions were set to guide the experts on the topics of discussion (Appendix G). The summary of their responses are as listed in Table 4.6.

The first question requested for some ideas on visual reasoning related to mathematics. Four of the experts referred *visual* in mathematics to be tasks or information on non-word problems while the other three of the experts treated them as those tasks or questions that are posted in other forms than algebraic expressions or numbers. All of them categorised *visual* in mathematics to be other than both texts and numbers, such as graphs, diagrams, images, pictures or any 2-dimensional or 3-dimensional geometrical figures. In terms of how they employed *visual* to reason their mathematical understanding, four of them made use of the information provided in the graphs while the other three would draw, or at least sketch, graphs related to the problems in the contexts.

The second question sought the experts' opinions on the use of Cartesian graphs in the learning of functions and derivatives. Three of the experts admitted that students did not make use or draw graphs as their solution or parts of the method in solving mathematical word problems while the others stated that students would refer to graphs if only they have strong understanding on the graphs or the relationships between the algebraic and graphical representations. Six of the experts asserted that students would relate the mathematical concepts to their graph representations through the properties of graphs and functions. Six of the experts agreed that it is possible for the students to achieve the correct solutions when employing graphs although most of them would struggle throughout.

Table 4.6: Responses from the experts in the focus group discussion

Question	Responses	Expert	No. (%)
1(a)	Not word problems	MC2, VR1, VR2, ME	4 (57)
	Other than algebraic expressions / numbers	MC1, CG1, CG2	3 (43)
1(b)	Graphs (all types)	MC2, VR1, CG1, CG2	4 (57)
	Diagrams / images / pictures	MC1, ME	2 (29)
	Geometry	VR2	1 (14)
1(c)	Use information in graphs / diagrams	MC1, VR1, VR2, CG1	4 (57)
	Draw related graphs / diagrams	MC2, CG2, ME	3 (43)
2(a)	No, need more exposure	VR2, CG2, ME	3 (43)
	Yes, if understand graphs	VR1, CG1	2 (29)
	Maybe, depending on understanding the relationship	MC1, MC2	2 (29)
2(b)	Understand properties of graphs	VR1, VR2, CG1	3 (43)
	Understand properties of functions	MC1, CG2, ME	3 (43)
	Knowledge on slope	MC2	1 (14)
2(c)	Yes, but mostly with struggle	MC2, VR1, VR2, CG1, CG2, ME	6 (86)
	Yes if strong basic knowledge on functions and derivatives	MC1	1 (14)
3(a)	Understand the relationships between algebraic/symbolic & graph	MC2, VR1, VR2, CG1, CG2	5 (71)
	Understand the relationship between variables	MC1, ME	2 (29)
3(b)	Strong understanding on relationships between functions/derivatives & graphs	MC1, MC2, VR1, VR2, CG2, ME	6 (86)
	With help from algebraic expressions or equations (if given)	CG1	1 (14)
3(c)	Looking at patterns of graphs	MC1, MC2, VR1, ME	4 (57)
	Understand properties of graph of functions	VR2, CG1, CG2	3 (43)

The third question needed for the experts to analyse on how the students would read and interpret data or information that are embedded in graphs. All of them (except one) agreed that students need to comprehend the relationships between the algebraic expressions and their graphical representations or between the independent and dependent variables in order to be able to read and interpret graphs efficiently. When extracting information that are not shown on graphs and when interpolating or extrapolating the graphs for hidden information or specific patterns of the characteristics on the functions, again, almost all of them highlighted that students must have very strong knowledge on the relationships between the functions and their derivatives and between the algebraic expressions and graphical representations.

Comments based on Question 4 to refine the framework were gathered for improvement :

1) Encoding :

- a. The addition of *with correct solution* and *with incorrect solution* to each of the categories *Draw correct graph*, *Draw incorrect graph* and *Algebraic solution*.
- b. Elaboration on the *No answer/Not attempted* to indicate the possible skills and knowledge of the students
- c. The use of consistent terminologies among *Draw correct graph/Draw incorrect graph* and *Algebraic method* to *Correct/Incorrect* and *No graph* since students may produce any other method than algebraic manipulations.
- d. Further elaborations on the descriptions for all categories

2) Decoding :

- a. The inclusion of *with valid reason*, *with invalid reason* and *with no reason* to the category of *Correct solution*

- b. The inclusion of *invalid reason* and *no reason* to the category Incorrect solution.
- c. Elaboration on the *No answer/Not attempted* to indicate the possible skills and knowledge of the students
- d. The elaborations on the descriptions for all categories

The refined framework is as shown in Table 4.7.

4.2.3 Stage 3 : Development of the final framework

The development of the final framework employed a Delphi method of 3-round emailing to experts for comments and feedbacks. The experts were 50 lecturers from various public and private institutions who have at least five years of experience in teaching differential calculus at pre-university and university levels. 40 of them were from local institutions while the other 10 were from international institutions. The questionnaire that was emailed to the experts consisted of 10 items (Likert scale) and an open-ended question intending to seek clarity on the categories, their flow and logical sequence together with the grammatical and spelling errors, if any. The measurement on the Likert scale were defined as Unsatisfactory, Poor, Satisfactory, Good and Outstanding (Appendix H).

The details of the numbers of experts responding to the questionnaire in the 3-round emailing are as shown in Table 4.8. In the first round, two locals and one international expert did not respond to the request, giving a 94% rate of return. After the analysis of the responses from 47 experts, another three of them were dismissed due to their responses being outliers, one of them was responding almost all *Poor* while the other two of them assigned almost all *Outstanding*. In Round 2 of the emails, 42 experts returned their feedbacks resulting in 95% rate of return. One of the experts was treated

Table 4.7: The refined framework for assessing visual reasoning

Visual reasoning process	Category	Description
Encoding	Correct graph with correct solution	Produces correct graph to solve and represent the solutions and managed to arrive to the correct solution
	Correct graph with incorrect solution	Produces correct graph to solve and represent the solutions but did not manage to arrive to the correct solution
	Incorrect graph with correct solution	Produces incorrect graph to solve and represent the solutions and managed to arrive to the correct solution based on the wrong graphs.
	Incorrect graph with incorrect solution	Produces incorrect graph to solve and represent the solutions and did not manage to arrive to the correct solution
	No graph with correct solution	Produces no graph to solve and represent the solutions and managed to arrive to the correct solution
	No graph with incorrect solution	Produces no graph at all to solve and represent the solutions and did not manage to arrive to the correct solution
	No answer / Not attempted	Left the item un-attempted – no graphs or any algebraic solutions.
Decoding	Correct solution with valid reason	Produces correct solution based on the graph and managed to provide valid reason(s) to arrive to the correct solution
	Correct solution with invalid reason	Produces correct solution based on the graph but did not manage to provide valid reason(s) to arrive to the correct solution
	Correct solution with no reason	Produces correct solution based on the graph but did not manage to provide any reason(s) to arrive to the correct solution
	Incorrect solution with invalid reason / no reason	Produces incorrect solution based on the graph and did not manage to provide valid reason(s) to arrive to the correct solution
	No answer / Not attempted	Left the item un-attempted.

as outlier and being dismissed for assigning all items as *Outstanding*. In Round 3, three of the experts did not return the feedbacks resulting in an 86% rate of return. The details of the responses to each item in the questionnaire throughout all the three round of emailing are as displayed in Table 4.9, Table 4.10 and Table 4.11.

On the overall, as can be seen in Table 4.9, at least 90% of the experts agreed with Satisfactory and Good for all items describing the clarity on the framework. Minor adjustments were made based on the proposal from the open-ended questions :

- 1) The term *solution* in the *Description* column for the *Encoding* process was proposed to be replaced by *explain* or *describe*
- 2) The additional description of *Solutions may differ from the original solutions set* for the *Incorrect graph with correct solution* to enhance explanation.
- 3) The spelling of *unattempted* was re-spelled as *un-attempted*.
- 4) The inclusion of *s* for possible pluralism in the word *reason*.

The analysis for Round 2 is as shown in Table 4.10. All experts were at least satisfied with the refined framework although one of them answered all Outstanding. It was assumed that he/she did not really evaluate the framework thoroughly or he/she might had assumed that the framework was totally refined.

The analysis for Round 3 is as displayed in Table 4.11. The responses were fairly distributed between the *Satisfactory* and *Good*. All of them did not find any spelling or grammatical error in the framework.

Based on the final feedbacks from the experts and the final refinement process, the final framework that can be used to assess the visual reasoning ability of pre-university students is as shown in Table 4.12.

Table 4.8

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Table 4.9

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Table 4.10

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Table 4.11

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Table 4.12: The final framework for assessing visual reasoning

Visual reasoning process	Category	Description
Encoding	Correct graph with correct solution	Produces correct graph to explain and represent the solutions and managed to arrive to the correct solution
	Correct graph with incorrect solution	Produces correct graph to explain and represent the solutions but did not manage to arrive to the correct solution
	Incorrect graph with correct solution	Produces incorrect graph to explain and represent the solutions and managed to arrive to the correct solution based on the wrong graphs. Solutions may differ from the original solutions set.
	Incorrect graph with incorrect solution	Produces incorrect graph to explain and represent the solutions and did not manage to arrive to the correct solution
	No graph with correct solution	Produces no graph to explain and represent the solutions and managed to arrive to the correct solution
	No graph with incorrect solution	Produces no graph at all to explain and represent the solutions and did not manage to arrive to the correct solution
	No answer / Not attempted	Left the item un-attempted – no graphs or any algebraic solutions.
Decoding	Correct solution with valid reason	Produces correct solution based on the graph and managed to provide valid reason(s) to arrive to the correct solution
	Correct solution with invalid reason	Produces correct solution based on the graph but did not manage to provide valid reason(s) to arrive to the correct solution
	Correct solution with no reason	Produces correct solution based on the graph but did not manage to provide any valid reason(s) to arrive to the correct solution
	Incorrect solution with invalid reason / no reason	Produces incorrect solution based on the graph and did not manage to provide valid reason(s) to arrive to the correct solution
	No answer / Not attempted	Left the item un-attempted.

The final framework was then sent to an expert, an international professor, for final validation (Appendix I). It is named Visual Reasoning over Graph (VR-G), an in-depth assessment of how graphs of functions and derivatives and their concepts must be *constructed* and *interpreted* for the students to use within the contexts in the curriculum.

It is principally used in categorising students' encoding and decoding ability. In this study, constructs for students' visual reasoning ability when using graph to solve mathematical problems on functions and derivatives were extracted from the above-mentioned frameworks. The items in the instruments are conceptualized based on the content domain but enhanced according to the knowledge and scheme.

The term 'correct graph' refers to the students being able to produce a complete and effective graphs that are characterized by labelling of the axes, scales and function(s). The complete graphs that are constructed accurately and neatly reflects understanding on the conceptual knowledge and the relationship between symbolic or algebraic representations and their visual representations on the graphs. The term 'incorrect graph' refers to students demonstrating limited understanding of graphing and some understanding on the relationship between symbolic or algebraic representations and their visual representation on the graphs.

The term 'valid reason' refers to the students being able to infer on the relationships between the properties of functions or/and derivatives, being able to integrate contextual knowledge, and understand the purpose of the information displayed in the graphs. On the other hand, 'no valid reason' refers to students that has no or appropriate engagement with the context where he/she may understand the single or basic elements of graph reading, at the same time serves as an indicator of employing memorization techniques or procedural knowledge.

4.3 Usage levels of visual representations

4.3.1 Frequencies and percentages

The instrument Visual Representation Usage Level (VRUL) was a Likert type test consisting of 17 questions with four different categories to measure :

- 1) the students' usage levels on using graphs or diagrams in their daily learning behaviour
- 2) the students' view on the usefulness of graphs or diagrams in solving mathematical problems
- 3) the students' difficulty of the use of graphs and diagrams in solving mathematical problems
- 4) the teacher's behaviours in using graphs or diagrams in solving mathematical problems.

The usage levels of visual representations were judged based on 5-point scales (1=Not at all, 2=Slightly, 3=Moderately, 4=Very much and 5=Definitely). The analysis on the items in each category was rearranged based on the descending order of their mean scores.

4.3.1.1 Analysis on the usage levels on using graphs or diagrams in their daily learning behaviour

The results of the survey on the students' usage levels on graphs or diagrams in their daily learning behaviour are as shown in Table 4.13. The items received mean scores that range between 3.40 and 4.16. The findings show that the rate of students who answered *Definitely* or *Very much* to pay attention to the use of graphs or diagrams for solving mathematical word problems that their teachers shows on the board during class was about 84%. Less than 4% either *Slightly* or *Not at all* paid attention to the use of graphs or diagrams that their teachers showed on the board during class. About 54.12% of the students were *Definitely* or *Very much* tried to copy the way their teacher uses graphs or diagrams to solve mathematical word problems. The total rate of students who responded *Definitely* and *Very much* for trying to use the kinds of graphs or

Table 4.13

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diagrams shown in the textbooks or by their teachers was almost 55%. Almost 68% of the students were *Definitely* and *Very much* to solve other similar mathematical problems were almost 54.12% and 67.22% respectively while 41.75% of the students gave similar answers for using graphs or diagrams in solving mathematical problems. For all items, less than 20% of the students responded as *Slightly* or *Not at all* indicating their massive use of graphs or diagrams when solving mathematical problems. Figure 4.2 depicts graphically the responses from the students on their usage levels in using graphs or diagrams in their daily learning behaviour.

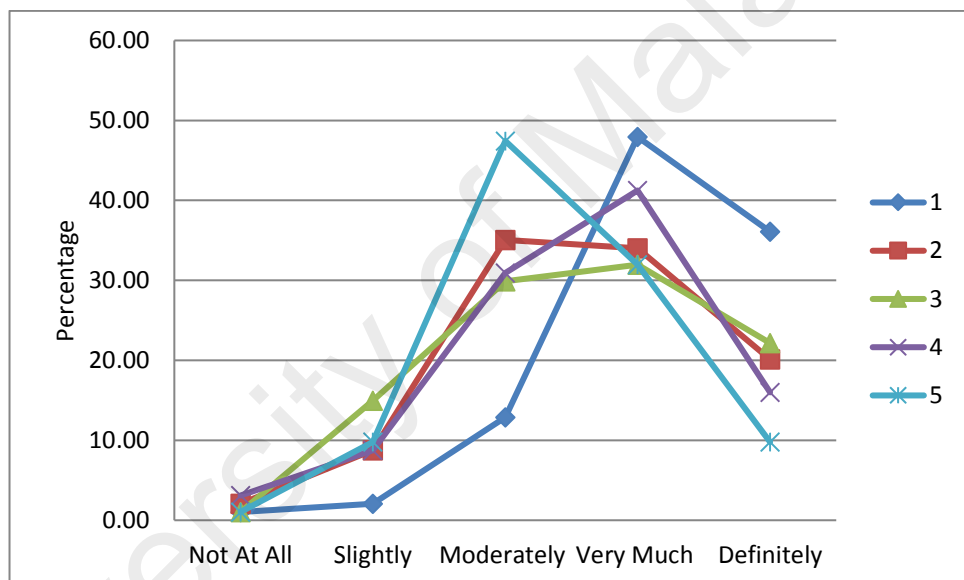


Figure 4.2: The usage levels in using graphs or diagrams in daily learning behaviour

4.3.1.2 Analysis on the usefulness of graphs or diagrams in solving mathematical problems

Table 4.14 displays the results of the survey on the students' views on the usefulness of graphs or diagrams in solving mathematical problems. The mean scores for all items range between 4.12 and 3.96, indicating positive views on the benefit of using graphs or diagrams to assist them in solving mathematical problems. The finding

Table 4.14

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also identified that about 78% of the students said that the use of graphs or diagrams *Definitely* or *Very much* helpful to efficiently solving mathematical problems. Similarly, about 71% of the students admitted that they were *Definitely* or *Very much* confident that it was good to use graphs or diagrams to solve mathematical problems. About 71% of them admitted that graphs and diagrams were *Definitely* or *Very much* help them figuring out how to solve the mathematical problems respectively. On the other hand, less than 7% responded either *Not at all* or *Slightly* for all items in this category. These indicate that students do treat graphs or diagrams as being very useful tools in guiding them to solve mathematical problems. Figure 4.3 depicts graphically the responses from the students of their views on the usefulness of graphs or diagrams in solving mathematical problems.

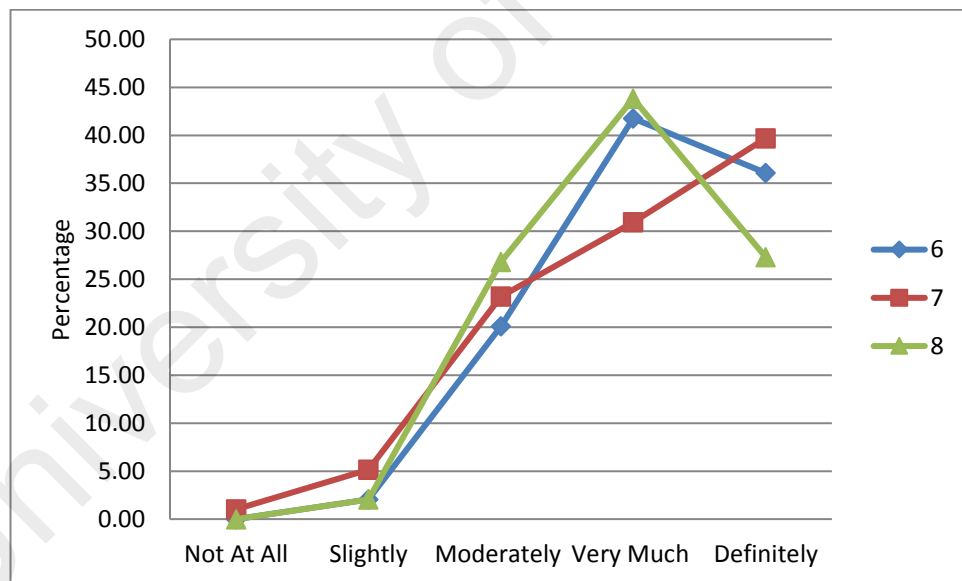


Figure 4.3: The usefulness on using graphs or diagrams in solving mathematical problems

4.3.1.3 Analysis on the difficulty on the use of graphs or diagrams in solving mathematical problems

The results of the survey on the students' difficulties when using graphs or diagrams in solving mathematical problems are as shown in Table 4.15. The mean scores for all items ranged from 2.99 to 3.37. The overall finding shows that less than 45% of the students in all items in this category responded to *Definitely* or *Very much*. Only about 43% of the students admitted that they knew how to construct graphs or diagrams for solving mathematical word problems while about 10% said that they were *Slightly* or *Not at all* knew how to construct them. About 40% of the students found that it was easy to use graphs or diagram to solve mathematical word problems and only about 12% did not find that it easy to do so. Similarly, only about 31% actually knew the kinds of graphs or diagrams that were helpful in solving different kinds of mathematical word problems while about 55% of the students found that it was *Moderately* easy to draw graphs or diagrams by themselves for solving mathematical word problems. The rest of about 14% of the students were either *Slightly* or *Not at all* knew which kinds of graphs or diagrams to suit different mathematical word problems. A smaller portion of about 21% of the students was confident to easily draw the graphs or diagrams by themselves and about 24% of the students were *Slightly* or *Not at all* found it easy to sketch or draw them. For all the items in this category, approximately half of the students responded as *Moderately*. These percentages indicate that students did face various types of difficulties when using, constructing or even identifying different graphs or diagrams for different mathematical problems. Figure 4.4 depicts graphically the responses from the students on the difficulties in dealing with graphs or diagrams in solving mathematical problems.

Table 4.15

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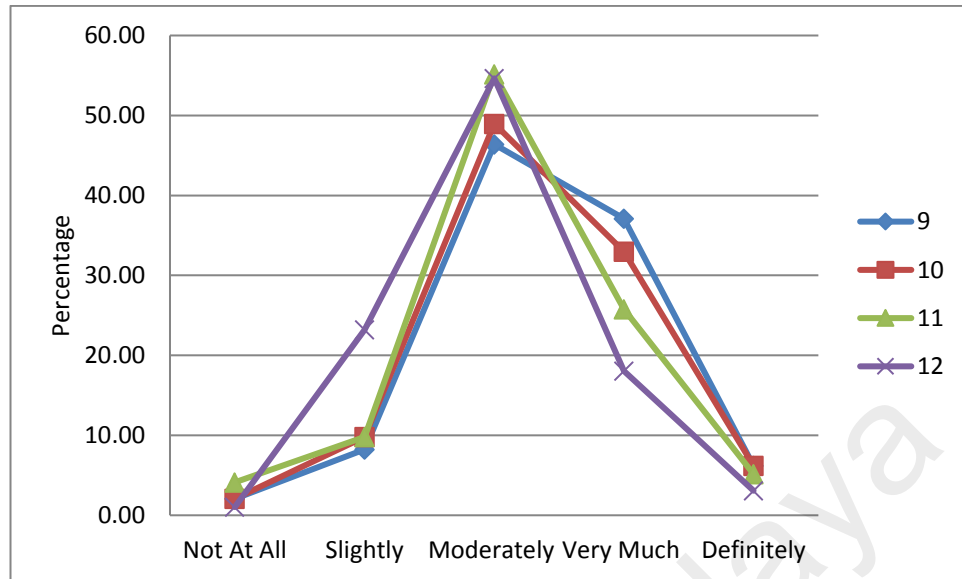


Figure 4.4: The difficulty in using graphs or diagrams in solving mathematical problems

4.3.1.4 Analysis on the teacher’s behaviours in using graphs or diagrams in solving mathematical problems

The results of the survey on the teachers’ behaviour in using graphs or diagrams in solving mathematical word problems are as shown in Table 4.16. The mean scores of the items ranged from 3.88 to 4.22. It can be seen that the majorities of the students were in the *Definitely* or *Very much* levels. About 80% of the students regarded their teachers as *Definitely* or *Very much* used the graphs or diagrams to efficiently solve mathematical problems. About 77% of the students agreed that their teachers use graphs or diagrams to explain on how to solve mathematical word problems. About 74% of the students agreed that the graphs or diagrams that their teachers used to show on how to solve mathematical problems *Definitely* or *Very much* helped them to understand how those problems can be solved and consequently approximately 71% of the students said that their teachers actually taught them how to use graphs or diagram to solve mathematical word problems. Lastly, about 67% of the students said that they were

Definitely or *Very much* told and encouraged by their teachers to use graphs or diagrams in solving mathematical words problems. On the overall, less than 10% responded either *Not at all* or *Slightly* for all items in this category. These indicate that the teachers were making positive use of graphs or diagrams in their teaching in order to assist the students to understand mathematical concepts and solve mathematical word problems. Figure 4.5 depicts graphically the responses from the students of their perception on their teacher's behaviours in using graphs or diagrams in solving mathematical problems.

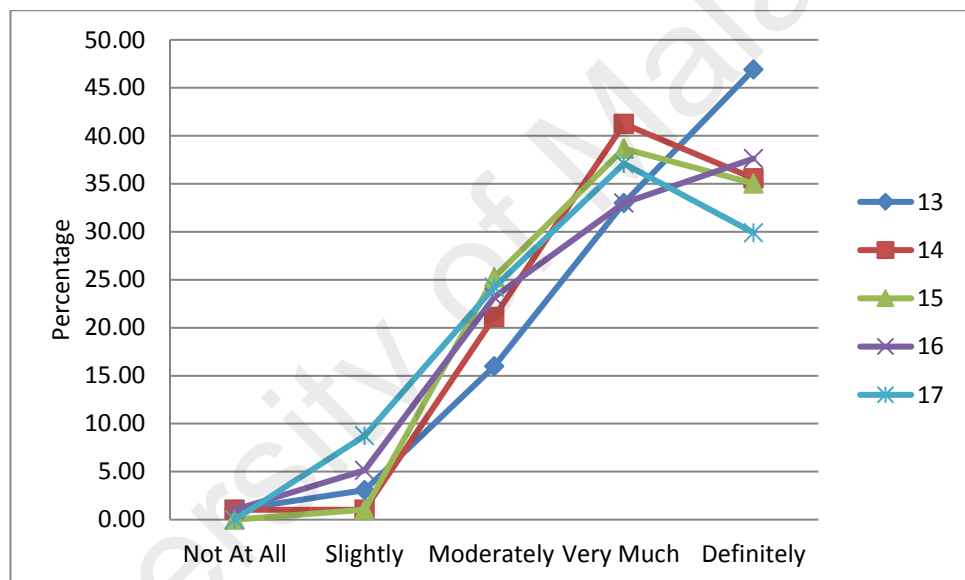


Figure 4.5: The teachers' behaviour in using graphs or diagrams in solving mathematical problems

Table 4.16

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4.3.2 Analysis on VRUL based on gender, race and major

The detail analysis on the mean and standard deviation for each category for the gender, race and major are as given in Appendix J. The male students have higher mean values as compared to the female students while the Chinese outperformed the other races with higher mean values in almost all items. The Engineering students had also shown higher means values in almost all items indicating their positivity in the usage level of graphs and diagrams in their daily learning and solving of mathematical problems.

4.3.3 Correlations among the categories in VRUL

The purpose of the correlation analysis for the categories in VRUL was to investigate if students' usage level of graphs or diagrams for one category is correlated to the other category. The relationships among the categories, are as presented in Table 4.17.

Table 4.17: Correlation among the overall VRUL and the categories in VRUL

Category	1	2	3	4	5	Mean	SD
1 Overall VRUL	-						
2 Daily behaviour	.85*	-				3.67	0.96
3 Usefulness	.71**	.54**	-			4.04	0.86
4 Difficulty	.78**	.60**	.49**	-		3.22	0.84
5 Teachers' behaviour	.79**	.50**	.34	.47**	-	4.06	0.89

Note. N = 190; VRUL = Visual Representative Usage Level ;

*p < .05, **p < .01

Positive and strong Pearson correlation values between 0.71 and 0.85 were observed between the VRUL and all its categories. These indicate that the students with high usage level of graphs or diagrams in the overall teaching and learning of mathematics had also made use of graphs or diagrams in their learning behaviour, were positive on their usefulness, faced less difficulties and positive usage of graphs or diagrams by their teachers in the teaching of mathematical problem solving. The

Pearson correlations values of between 0.34 and 0.60 among the categories are also positive, although they exhibited weaker relationships. All relationships are also statistically significant to each other except for the relationship between the students' view on the usefulness of the graphs and diagrams in solving mathematical problems and their teacher's usage level of graphs and diagrams in their teaching of mathematical problem solving ($r = 0.38$, $N = 194$, $p = 0.74$).

4.4 Mathematical Visuality Test (MVT)

4.4.1 Frequencies and percentages

This section describes the descriptive analysis on the responses by the students in the Mathematical Visuality Test (MVT). The test consisted of five sentential or non-graph tasks. To examine the distribution of the responses from the students, the frequencies and percentages were computed for each part of all the items for their mathematical visuality. Students are categorised as *visual* if they introduced or make used of graphs, regardless whether they managed to draw the correct graphs or not, to help them explaining and solving the problems. The categories of encoding process that were listed under visual are Correct graph with correct solution (CGCS), Correct graph with incorrect solution (CGIS), Incorrect graph with correct solution (IGCS) and Incorrect graph with incorrect solution (IGIS). On the other hand, the categories of encoding that were listed under non-visual are No graph with correct solution (NGCS) and No graph with incorrect solution (NGIS). Those students who did not attempt or answer the tasks were calculated separately. The sample of student's work solution that was assigned with IGIS is as in Figure 4.6. The student drew incorrect graph to represent the average rate of change and provided a wrong description of rate of change.

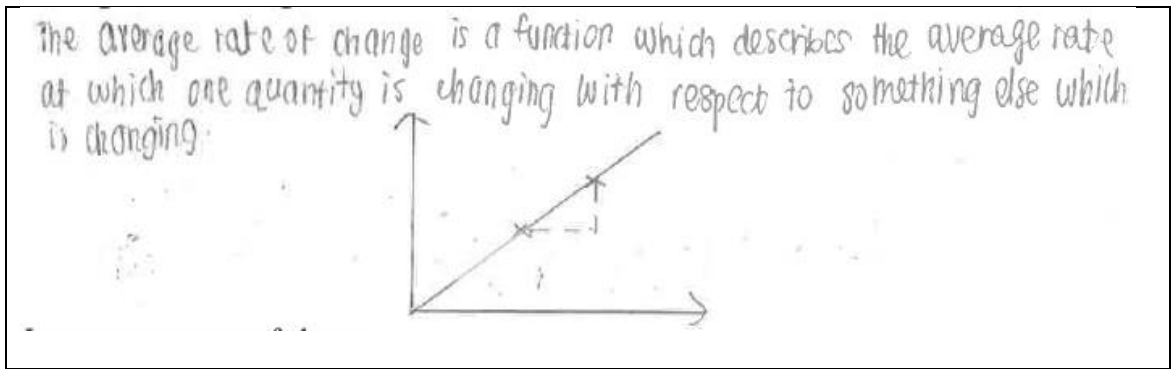


Figure 4.6: Sample of student’s work that was assigned to IGIS

Figure 4.7, shows the sample of student’s work solution that was assigned with IGCS. The student did not manage to draw the correct graph to explain the idea of the relationship among limit, chord and tangent but had provided a correct wrong description of the concepts.

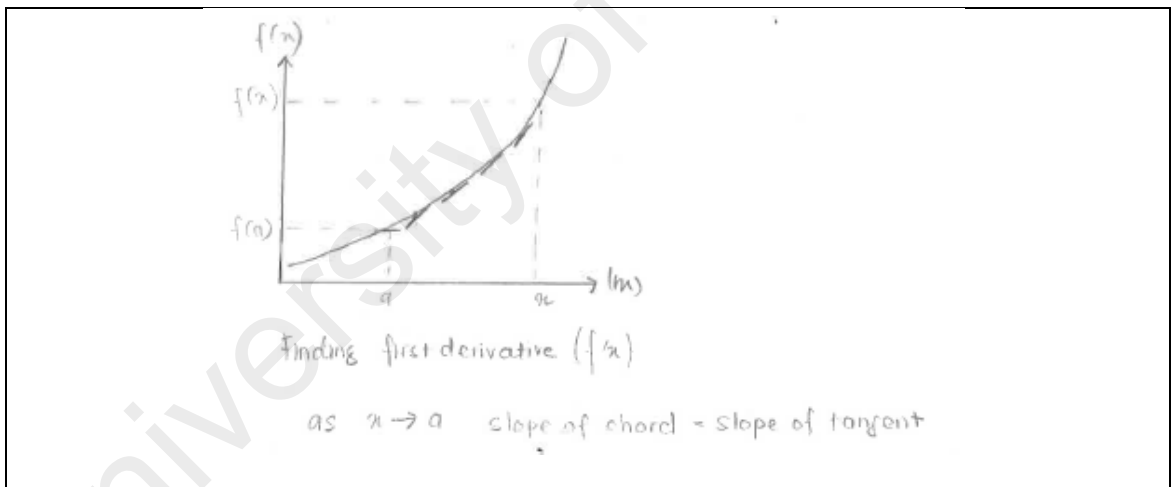


Figure 4.7: Sample of student’s work that was assigned to IGCS

4.4.1.1 Analysis on the mathematical visuality for item 1

Table 4.18 displays the distribution of visual and non-visual category for item 1 and Figure 4.8 illustrates the detail distribution on the encoding process for all parts of item 1. The distribution shows that, for all the three parts, less than 34% of the students employed graphs to express their solutions, regardless whether they managed to come to the correct solutions or not. Only a small portion of at most 13% of the students

managed to sketch the correct graphs and consequently obtained the correct solutions. Less than 15% of the students sketched incorrect graphs. Majority, between 53% to 72%, of the students did not employ graphs to solve the problems but approximately 24% to 44% of them were able to come to the correct answers. Approximately 11% to 14% of the students did not attempt to solve the problems.

Table 4.18: The analysis on the Mathematical Visuality for item 1

Item	Visual	Non-visual	Not answered / attempted	Mean	Standard deviation
1a	24.2	63.9	11.9	2.11	1.69
1b	33.6	52.5	13.9	2.37	1.93
1c	16.5	71.6	11.9	1.92	1.43

Note : The figures represent the percentages of the distribution

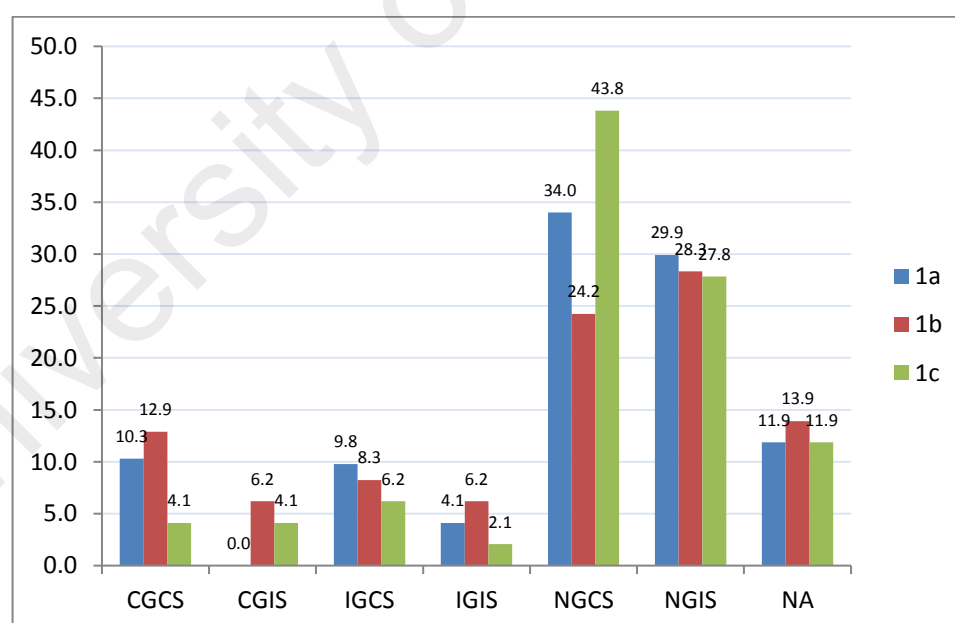


Figure 4.8: Distribution of the encoding process for item 1 of the MVT

4.4.1.2 Analysis on the mathematical visuality for item 2

Table 4.19 displays the distribution of visual and non-visual category for item 2 and Figure 4.9 illustrates the detail distribution on the encoding process for all parts in item 2. The distribution shows that, for both parts of the visual and non-visual, less than 22% of the students employed graphs to express their solutions, regardless whether they managed to come to the correct solutions or not. Only a small portion of at most 10% of the students managed to sketch the correct and consequently obtained the correct solutions. Between 50% to 64% of the students did not turn to graphs as the solution method with some portions of approximately 28% of them had abled to arrive to the correct answers. About 20% of the students did not attempt to solve both problems in item 2.

Table 4.19: The analysis on the Mathematical Visuality for item 2

Item	Visual	Non-visual	Not answered / attempted	Mean	Standard deviation
2a	14.9	64.0	21.1	1.64	1.53
2b	31.0	50.5	18.6	2.15	1.83

Note : The figures represent the percentages of the distribution

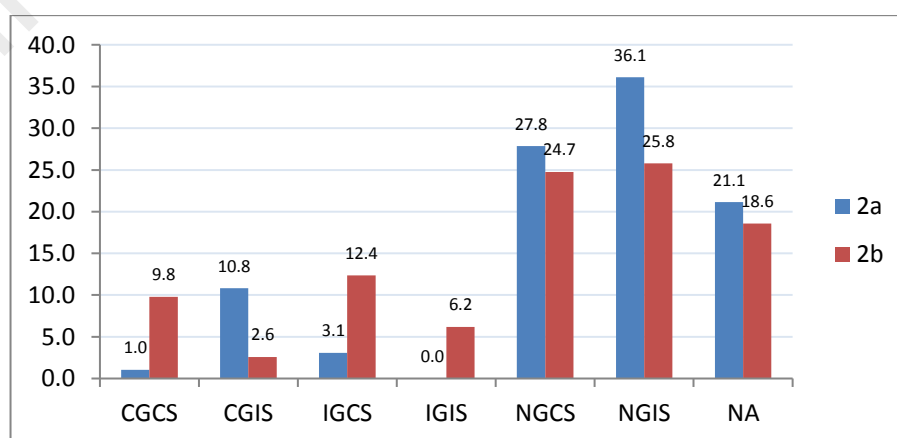


Figure 4.9: Distribution of the encoding process for item 2 of the MVT

4.4.1.3 Analysis on the mathematical visuality for item 3

Table 4.20: The analysis on the Mathematical Visuality for item 3

Item	Visual	Non-visual	Not answered / attempted	Mean	Standard deviation
3a	31.4	60.3	8.3	2.80	1.95
3b	31.4	60.3	8.3	2.77	1.97
3c	24.2	75.8	0.0	2.88	1.60
3d	31.4	66.0	2.6	2.97	1.79
3e	31.5	59.3	9.3	2.75	2.02
3f	31.4	54.2	14.4	2.64	2.07
3g	34.9	57.8	11.3	2.59	2.00
3h(i)	46.4	49.7	4.1	3.30	2.01
3h(ii)	30.9	54.7	14.4	2.55	2.06
3i	37.2	49.5	13.4	2.88	2.06

Note : The figures represent the percentages of the distribution

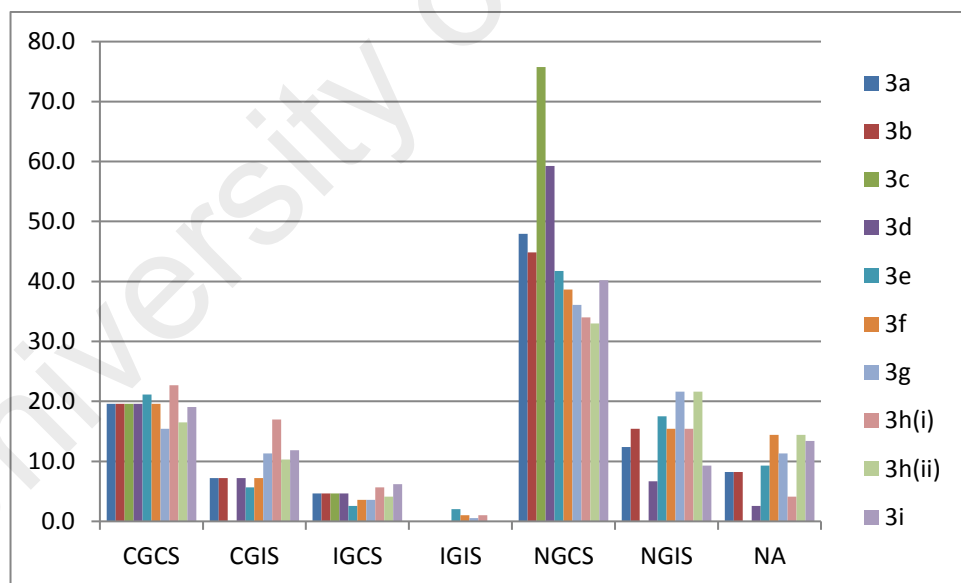


Figure 4.10: Distribution of the encoding process for item 3 of the MVT

Table 4.20 displays the distribution of visual and non-visual category for item 3 and Figure 4.10 illustrates the detail distribution on the encoding process for all parts in item 3. Besides item 3h(i) that had received the highest means score ($M = 3.30$, $SD = 2.01$),

the rest of the items scored means of between 2.00 and 3.00. The distribution also shows that, except for item 3(h)(i) with approximately 46% and item 3(i) with approximately 37%, all the other items received approximately 24% to 32% of the students tried to employ graphs to express their solutions regardless whether they managed to come to the correct solutions or not. Only a portion of less than 23% sketched the correct graphs and consequently obtained the correct solutions. On the other hand, all items had less than 10% of the students who sketched incorrect graphs. At least 50% of the students did not turn to graphs as the solution method although between 33% and 76% of the students managed to come to the correct answers. Less than 15% of the students did not attempt to solve the problems.

4.4.1.4 Analysis on the mathematical visuality for item 4

Table 4.21 displays the distribution of visual and non-visual category for item 4 and Figure 4.11 illustrates the detail distribution on the encoding process for all parts in item 4. The mean score for item 4 was 2.51 with the standard deviation of 1.62. The distribution shows that about 30% of the students employed graphs to express their solutions but only a small portion of approximately 12% of the students managed to sketch the correct graphs of the situation together with the correct description. About 18% of the students sketched incorrect graphs. The other of approximately 62% of the students did not turn to graphs as the solution method but approximately 52% of the students were still able to come to the correct answers. A small portion of about 8% of the students did not attempt to solve the problems.

Table 4.21: The analysis on the Mathematical Visuality for item 4

Item	Visual	Non-visual	Not answered / attempted	Mean	Standard deviation
4	29.9	62.4	7.7	2.51	1.62

Note : The figures represent the percentages of the distribution

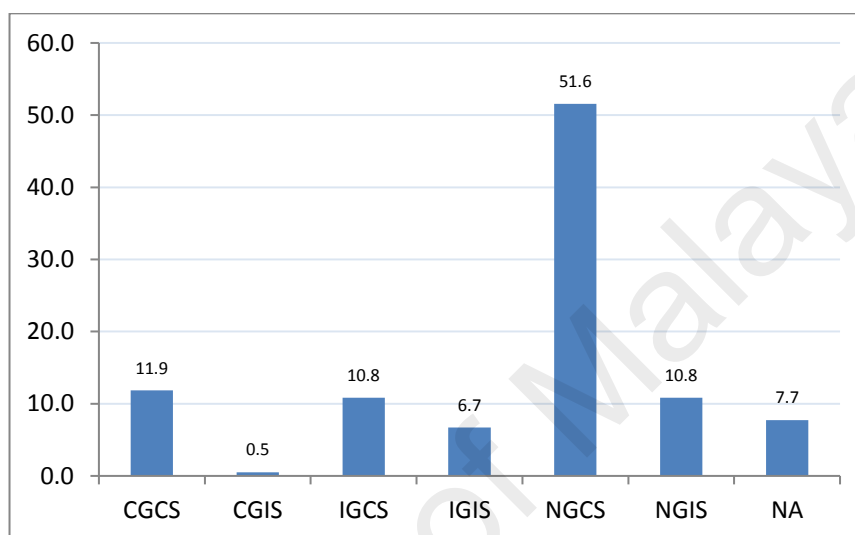


Figure 4.11: Distribution of the encoding process for item 4 of the MVT

4.4.1.5 Analysis on the mathematical visuality for item 5

Table 4.22 displays the distribution of visual and non-visual category for item 5 and Figure 4.12 illustrates the detail distribution on the encoding process for all parts in item 5. Surprisingly, the distribution shows that none of the students made use of graphs, in all the tasks, to solve the problems. Besides approximately 8% of the students who did not attempt questions 5(b) and 5(c), the rest of the students employed algebraic manipulations to solve the tasks. For question 5(a), approximately 98% of the students did not make use of graphs to solve the problem.

Table 4.22: The analysis on the Mathematical Visuality for item 5

Item	Visual	Non-visual	Not answered / attempted	Mean	Standard deviation
5a	0	97.9	2.1	1.81	0.44
5b	0	92.8	7.2	1.71	0.59
5c	0	92.8	7.2	1.38	0.62

Note : The figures represent the percentages of the distribution

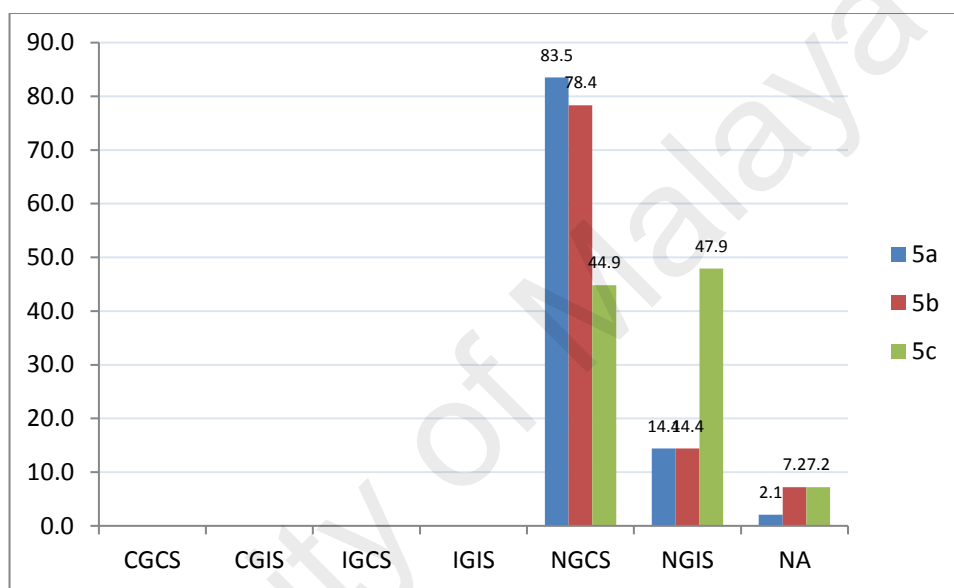


Figure 4.12: Distribution of the encoding process for item 5 of the MVT

4.4.2 Analysis on visuality measure

A visuality measure for each student was determined through the total scores that they had managed to obtain based on the point allocated for each category as shown in Table 4.23. Thus, each student who drew correct graphs and managed to obtain the correct solutions for all the 19 sub-items (i.e. the five items including their parts) would be allocated a visuality measure of 114 points while students who correctly solved the tasks correctly without sketching any graph would be allocated a visuality measure of at most 38. The percentage of students with their appropriate visuality measures are as shown in Table 4.19. From the analysis, it can be seen that more than half (56.7%) of

the students are in the non-visual category which indicate that they prefer to use the algebraic method in solving problems on functions and derivatives. Smaller portions of approximately 27% of the students showed their preference in using graphs as tools to solve problems. Another of approximately 17% of the students were categorised as partially visual. They exhibited a mixture modes of visual and non-visual. Some of the students came out with both visual and non-visual methods which indicate their in-confidence in using the graphical methods.

Table 4.23: Distribution of mathematical visuality measure for the MVT

Visuality measure	Category	Visuality Score	Percentage (%)
Visual	CGCS	57 – 114	26.8
	CGIS		
	IGCS		
	IGIS		
Partially visual	-	39 – 56	16.5
Non-visual	NGCS	0 – 38	56.7
	NGIS		
	NA		

Further analysis using the Chi Square test was done to compare the significance on the proportions of the categories of measures; Visual, Partially visual and Non-visual. The visuality measure was found to be statistically significantly, $\chi^2_{(2)} = 50.15 > 5.991$, $p < 0.05$. Therefore, there is a difference among the percentage of students in the mathematical visuality measures.

4.4.3 Analysis on visuality measure based on gender, race and major

The detail analysis on the percentages for each category of the visuality measures for the gender, race and major are as given in Appendix J. The majority of the students, regardless their gender, race and major, were in the non-visual category. This indicate that most of the students

were still adopting the procedural or memorization techniques and were very comfortable with the algebraic manipulation.

4.5 Graph Reasoning Test (GRT)

4.5.1 Frequencies and percentages

This section describes the descriptive analysis of the responses by the students in the Graph Reasoning Test (GRT). The test consists of five graph-accompanied tasks. Students must make use of the graphs in order to solve the tasks assigned to them. In other words, they need to look for or interpret the information displayed on the graphs or search for the information that were hidden in between the graphs. To examine the distribution of the responses from the students, the frequencies and percentages were computed for each part of all the items for their graph reasoning ability. Analysis were based on the three scales of decoding process: read the graph, read between the graph and read beyond the graph. Frequencies and percentages were calculated for the *Correct* and *Incorrect* solutions for each decoding scale that indicate their capability to read and interpret graphs as visual tools. The categories of the decoding process that were listed under *Correct* are : Correct solution with valid reason (CSVR), Correct solution with invalid reason (CSIR), Correct solution with no reason (CSNR) while Incorrect solution with invalid reason or no reason (ISINR) was categorised under *Incorrect*. Those students who did not attempt or answer the tasks were calculated separately. The sample of student's work solution that was assigned with CSIR is as in Figure 4.13. The student managed to interpret the gradient of the tangent to equal to zero but had produced incorrect concepts to reason it. Figure 4.14 illustrate the work solution of a student that was assigned ISINR, where the student did not managed to obtained the correct answer and had also produced an incorrect reason to justify it.

\Rightarrow tangent of $g'(n) = f'(n) = 0$.
Thus the tangents are parallel to each other.

Figure 4.13: Sample of student's work that was assigned to ICSIR

at point c, tangent of $f(x)$ changed from increases at increasing rate to increase at decreasing rate.
while tangent of $g(x)$ changed from decreases at decreasing rate to increases at increasing rate.
Point c may be the inflection point for both graph.

Figure 4.14: Sample of student's work that was assigned to ISINR

4.5.1.1 Read the graph

Reading the graph required the students to directly see the information on the graphs without doing any calculation or interpretation. As shown in Table 4.24, the mean scores for all the items range between 2.12 and 3.87. At least 67% of the students managed to get the correct answers regardless whether they had provided valid or invalid reasons or failed to provide any reasons to support the solutions. All of the students managed to get the correct answers for items 1(a)(i) and 1(b)(i). Approximately 64% and 88% of the students managed to provide valid reasons for their solutions. This indicates that students were very good in reading the information shown in the graphs. On the overall, less than 10% of the students did not attempt the tasks.

Table 4.24: The analysis on the items for the decoding scale : Read the graph

Item	Correct	Incorrect	Not answered / attempted	Mean	Standard deviation
1(a)(i)	100.00	0	0	3.29	0.96
1(b)(i)	100.00	0	0	3.87	0.38
2(a)(i)	86.60	10.31	3.09	3.35	1.19
3(b)(i)(1)	67.63	29.38	3.09	2.14	1.17
3(b)(i)(2)	86.60	10.31	3.09	2.52	1.13
3(b)(i)(3)	75.26	15.46	9.28	3.07	1.48
3(b)(ii)(1)	93.81	4.64	1.55	3.52	0.91
4(a)(i)	89.69	10.31	0	3.16	1.09
4(a)(ii)	81.44	13.92	4.64	2.86	1.30
5(b)(ii)	73.72	21.13	5.15	2.12	1.09

Note : The figures represent the percentages of the distribution

4.5.1.2 Read between the graph

Reading between the graph required the students to make some relationships among the information shown in the graphs in order to arrive to another meaning of the solutions. As shown in Table 4.25, the mean scores for all the items range between 1.72 and 3.52. Except for three items, 2(a)(ii), 2(b)(i) and 5(b)(i), that scored correctly between 39% to 57%, the rest of the items were correctly managed by a range of 79% to 97% of the students. All of the students managed to answer item 1(d)(i) correctly with about 72% of them managed to provide valid reason that shows their understanding. The results of the analysis indicate that the students had quite a strong foundation in functions and derivatives. Most of them managed to make use of the information displayed in the graphs in order to interpret or calculate related information. On the overall, less than 15% of the students did not attempt the tasks.

Table 4.25: The analysis on the items for the decoding scale: Read between the graph

Item	Correct	Incorrect	Not answered / attempted	Mean	Standard deviation
1(a)(ii)	85.05	7.22	7.73	3.03	1.32
1(b)(ii)	81.44	9.28	9.28	2.76	1.37
1(b)(iii)	96.91	3.09	0	3.48	0.89
1(c)(i)	93.30	3.61	3.09	3.49	1.04
1(d)(i)	90.21	0	9.79	3.27	1.30
1(d)(ii)	83.51	2.06	14.43	1.74	0.77
2(a)(ii)	43.82	49.48	6.70	2.04	1.43
2(b)(i)	56.70	36.60	6.70	1.77	1.04
3(b)(ii)(2)	78.35	17.01	4.64	2.15	0.98
4(b)	85.05	8.76	6.19	2.87	1.31
5(a)	93.30	6.70	0	3.52	0.96
5(b)(i)	38.66	55.67	5.67	1.97	1.40

Note : The figures represent the percentages of the distribution

4.5.1.3 Read beyond the graph

Reading beyond the graph required the students to interpolate or extrapolate the graphs to see the shape or patterns of the graphs. Students are required to possess a strong knowledge on functions and derivatives in order to read beyond the graphs. As shown in Table 4.26, the mean scores for all the items range between 1.18 and 3.09. A mixture of percentages that range of between 30% and 76% of the students managed to get the correct answers regardless whether they had provided valid or invalid reasons or failed to provide any reasons to support the solutions. Subsequently, a lower range of 9% to 43% of the students produced incorrect solutions. At most 27% of the students did not try out the questions. The results of the analysis indicate that as the reading of hidden information are getting more complex, less percentage students were able to arrive to the correct answers and these indicate a lower ability of their visual reasoning skill.

Table 4.26: The analysis on the items for the decoding scale : Read beyond the graph

Item	Correct	Incorrect	Not answered / attempted	Mean	Standard deviation
1(c)(ii)	71.65	9.79	18.56	2.38	1.55
1(e)	43.30	31.44	25.26	1.18	0.81
2(b)(ii)	59.79	19.59	20.62	1.77	1.30
3(a)	76.29	21.13	2.58	3.09	1.34
4(c)	74.23	22.16	3.61	2.14	1.10
5(b)(iii)	30.93	42.78	26.29	1.45	1.34
5(b)(iv)	63.92	9.79	26.29	2.34	1.72

Note : The figures represent the percentages of the distribution

4.5.2 Correlations among the overall GRT and the decoding scales

The GRT comprised of three scales that make up the constructs of the decoding processes in visual reasoning, i.e. reading the graph, reading between the graph, and reading beyond the graph. The correlation analysis based on the scales in GRT was to investigate if students' abilities in the decoding process or extracting information from the given graphs in the overall GRT is correlated to each of the scales and also if one scale is correlated to the other scale. The relationships between the overall GRT and each of the scale and also among the scales are as presented in Table 4.27.

Table 4.27: Correlation among the overall GRT and the decoding scales

Category	1	2	3	4	Mean	SD
1. Overall GRT	-				2.63	0.92
2. Reading the graph	.98	-			2.99	1.23
3. Reading between graph	.99	.96	-		2.67	1.35
4. Reading beyond graph	.99	.91	.92	-	2.05	1.46

Note. N = 190; GRT = Graph Reasoning Test; $p < .01$

Very strong and positive correlations with values of at least 0.98 were observed between the overall GRT and the scales and also among the scales. These indicate that the students who are able to decode the graphs by just reading the information from the graphs would also be able to read between and beyond the graphs in order to solve the tasks on functions and derivatives. The relationships among the scales are also positive, and they exhibited very strong relationships with values of at least 0.91. All relationships are also statistically significant to each other as observed from $p < .01$ values.

Further analysis using the Chi Square test was done to compare the significance on the proportions of the decoding level; Reading the graph, Reading between graph and Reading beyond graph. The decoding levels were found to be statistically significantly, $\chi^2_{(2)} = 13.1 > 5.991$, $p < 0.05$. Therefore, there are differences among the percentage of students in their ability to decode information displayed or hidden in the graphs.

4.5.3 Analysis on visual reasoning ability based on gender, race and major

The detail analysis on the percentages for each category of the visual reasoning ability for the gender, race and major are as given in Appendix J. The majority of the students, regardless their gender, race and major, performed very well when they need to just read out the information straight from the graph. Smaller percentages of students were able to read and interpret the relationships among the information given for the functions and their derivative while only very small portions, in each category, were able to further interpolate or forecast on the hidden information not displayed in the graph.

4.6 Correlation among the results of the instruments

Scatterplot was used to illustrate the relationships among the students' responses to the three instruments, the VRUL, the MVT and the GRT. Three main characteristics are used to explain the correlations between any two variables: the shape, the direction and the magnitude of the scatterplot. Correlation coefficient refers to the covariant statistical measure between any two variables that indicates the strength and direction of their linear relationship, while the trend of the relationship can be seen from the distribution of the scattered points. The shape of the scatterplot depicts the trend of the relationships. The magnitude refers to the strength of the relationship of the variables and it is represented by the number 0 to 1.00. A '0' indicates no relationship exist between the two variables while the value '1' denotes a perfect linear relationship. The nearer a value towards '0' or '1', indicates the weaker or stronger respectively, the relationship between the two variables.

Figure 4.15 illustrate the relationship between the means for VRUL and the means of MVT. Three patterns of positive correlations are observed. The regions with higher VRUL (> 4.5) tend to have higher MVT than the regions with lower VRUL. Among the regions with lower VRUL (between 4.5 and 5), a small difference in VRUL reflects a significant different increase in the MVT. On the other hand, in the regions with the lowest VRUL (< 4.5) a difference in VRUL produces relatively small increase in the MVT.

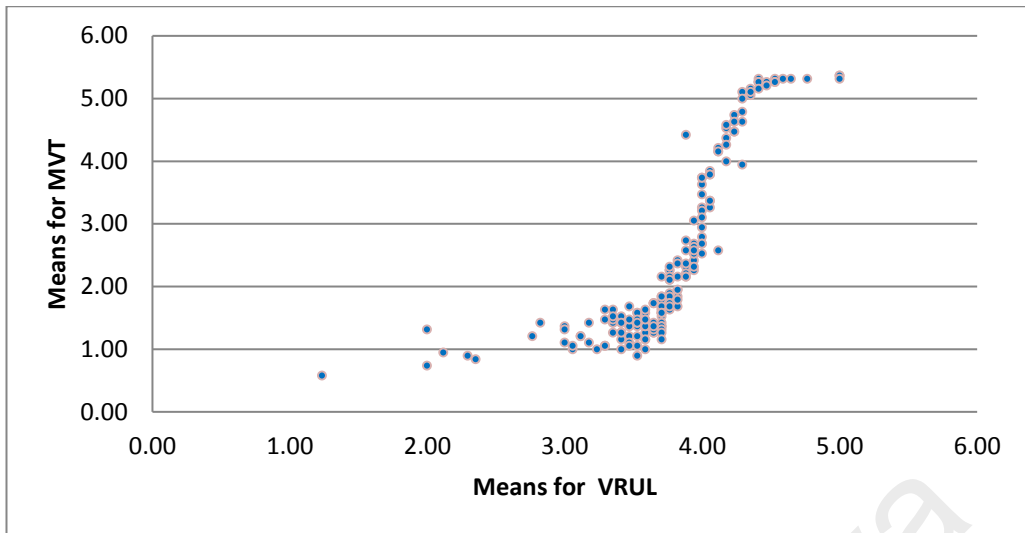


Figure 4.15: Means for VRUL against means for MVT

Figure 4.16 illustrate the relationship between the means for VRUL and the means of GRT. Three patterns of positive correlations are also observed. The regions with higher VRUL (> 4.3) tend to have higher GRT than the regions with lower VRUL. Among the regions with lower VRUL (between 3 and 4.3), a small difference in VRUL reflects a significant different increase in the GRT. On the other hand, in the regions with the lowest VRUL (< 3) a difference in VRUL produces relatively small increase in the MVT.

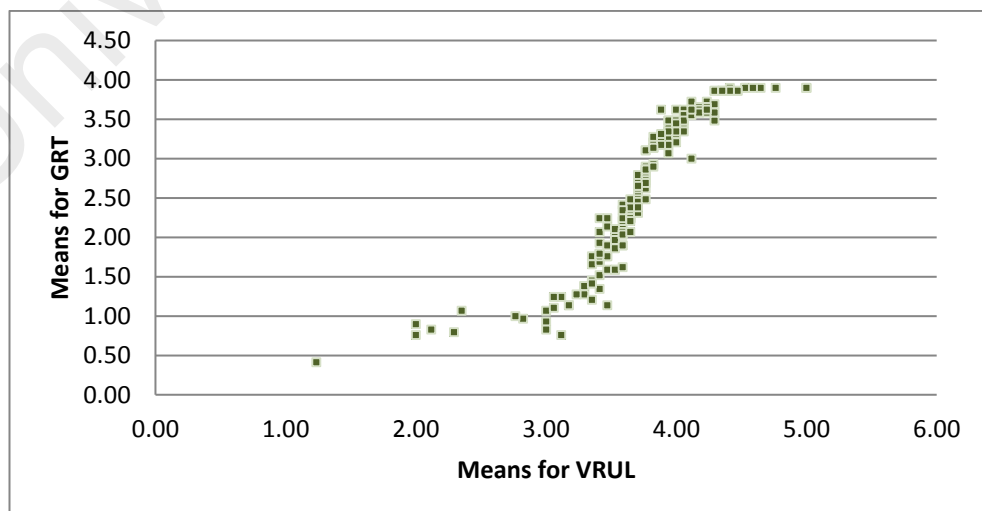


Figure 4.16: Means for VRUL against means for GRT

Figure 4.17 illustrate the relationship between the means for MVT and the means of GRT. Two distinct patterns of positive correlations are observed. The upper regions with higher VMT (between 2.5 and 5.5) tend to have higher MVT than the regions with lower VRUL and increases slowly, i.e. a difference in VRUL resulted in a small different increase in the GRT. On the other hand, in the regions with the lower VMT (< 2.5) the increment looks more proportionate, a different in the MVT produces approximately an equal increase in the GRT.

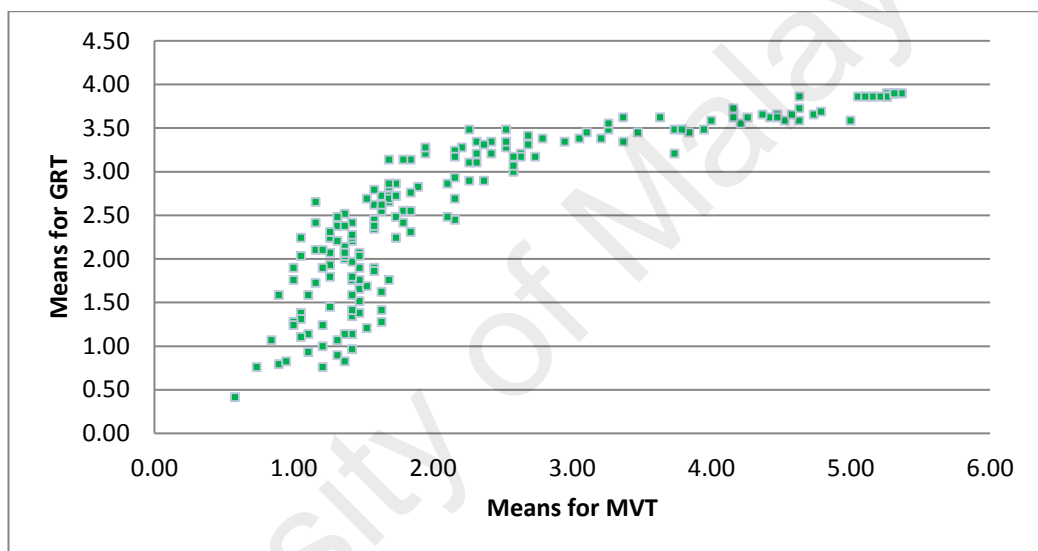


Figure 4.17: Means for MVT against means for GRT

The scatterplots were subjected to linear regression, the best line to pass through all data points and be used to predict or forecast related values. The positive or negative value of the correlation coefficient indicates the directions of the relationship. Positive correlation indicates that the values of the dependent variable are increasing as the values of the independent variable are increasing. Similarly, the negative correlation indicates that as the values of the dependent variable will decrease as the values of the independent variable are increasing. Table 4.28 displays the algebraic properties of each pair of the relationship together with their respective coefficient of correlation, r .

In each of the equation, for example, in the equation $y = 2.21x - 5.87$, the value 2.21 indicates that for every additional point in the VRUL as independent variable, it is expected that the dependent variable to increase by an average of 2.21. The straight line, if to be drawn in each scatterplots, shows the same information. Moving to the left or to the right along the x -axis by an amount that represents one unit change in the VRUL, the fitted line rises (or falls) by 2.21 unit points. However, these VRUL and MVT values were obtained from the pre-university students at one college. Therefore, the relationship is only valid within these intervals of the data range. No prediction is to be made outside the data range.

Table 4.28: The linear regression and correlation coefficients among the VRUL, MVT and GRT

Variable		Linear regression	Coefficient of Correlation (r)
Independent	Dependent		
VRUL	MVT	$y = 2.21x - 5.87$	0.803
VRUL	GRT	$y = 0.55x + 1.31$	0.897
MVT	GRT	$y = 1.6321x - 3.46$	0.838

Another statistical measure, r^2 , the coefficient of determination, takes the values between 0 and 1.00, measures how close the data are to the fitted regression line. It is possible to have a low, r^2 due to two reasons. Firstly, this study attempted to predict human behaviour. Typical value of r^2 is less than 0.5. Humans are just harder to predict as compared to the physical process. Secondly, low r^2 values are only problematic when we need to do precise prediction. Another matter to note is that the intercepts do not make sense in the real world situations. For example, it is not reasonable for the students' mathematical ability to be a 0 or negative when their usage level is 0.

4.7 Analysis on misconceptions and difficulties

4.7.1 Mathematical Visuality

This section describes the errors performed by the students when solving tasks in assessing their mathematical visuality. Since the aim of the tasks is to seek students' preference in using graphs, this section will focus on the 'incorrect' graphs constructed and the conceptual knowledge applied by the students. Therefore, the errors carried out by the students were extracted and analysed based on the graphs sketched, mathematical reasons and worked solutions provided in the 'Incorrect Graph Correct Solution (IGCS)' and 'Incorrect Graph Incorrect Solution (IGIS)'.

Table 4.29 illustrates the analysis on the errors performed by the students in solving problems in the MVT. About approximately 8% - 15% of the students drew wrong graphs or wrong straight lines for various parts of item 1 while approximately 34% to 41% of the students performed incorrect solutions when solving tasks in item 1. Their reasons such as 'the gradient of the graphs is the same at any point of the graph', 'directly proportional', 'the gradient of the graph at particular point' show that the students were able to visualize the situations on gradients of functions but they had expressed them in algebraic forms. Those with vague or lack of understanding on the concepts of derivatives defined constant rate of change as 'horizontal line' or 'gradient = 0', average rate of change as 'overall gradient of the graph' or 'total rate of change divided by total time' and instantaneous rate of change as 'remain constant without moving' or 'when time equal to zero'.

Table 4.29: Distribution of errors for the Mathematical Visuality Test

Item	Incorrect graphs		Incorrect solutions	
	f	(%)	f	(%)
1(a)	27	(13.91)	66	(34.02)
1(b)	28	(14.44)	79	(40.72)
1(c)	16	(8.25)	66	(34.02)
2(a)	6	(3.09)	91	(46.91)
2(b)	36	(18.56)	67	(34.54)
3(a)	9	(4.64)	38	(19.59)
3(b)	9	(4.64)	44	(22.64)
3(c)	9	(4.64)	0	(0.00)
3(d)	9	(4.64)	27	(13.92)
3(e)	9	(4.64)	49	(25.26)
3(f)	9	(4.64)	46	(23.71)
3(g)	8	(4.13)	65	(33.51)
3(h)(i)	13	(6.70)	65	(33.51)
3(h)(ii)	8	(4.12)	62	(31.96)
3(i)	12	(6.19)	41	(21.14)
4	34	(17.52)	35	(18.04)
5(a)	0	(0.00)	28	(14.43)
5(b)	0	(0.00)	28	(14.43)
5(c)	0	(0.00)	93	(47.94)

About 3% of the students drew wrong straight lines while about 47% of them performed incorrect solutions for item 2(a). A portion of 18.56% of the students constructed wrong graphs while some 34.54% of the students gave incorrect solutions for item 2(b). As with the results in item 1, analysis for Item 2 shows that the majority of the students again opted to describe their solutions in written form instead of sketching graphs. Those who chose to draw graphs for the formula $\frac{f(x)-f(a)}{x-a}$ sketched straight lines which pass through the origin to illustrate the slope of the function between two points as in Figure 4.18. The sketching indicates that students know what the formula represent but confined their definition of function to straight lines only. More than 80% of the students who drew graphs for the formula $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ understood what the formula represents but again failed to illustrate the accurate situation of chord becoming tangent as the coordinate of x approaches the coordinate a .

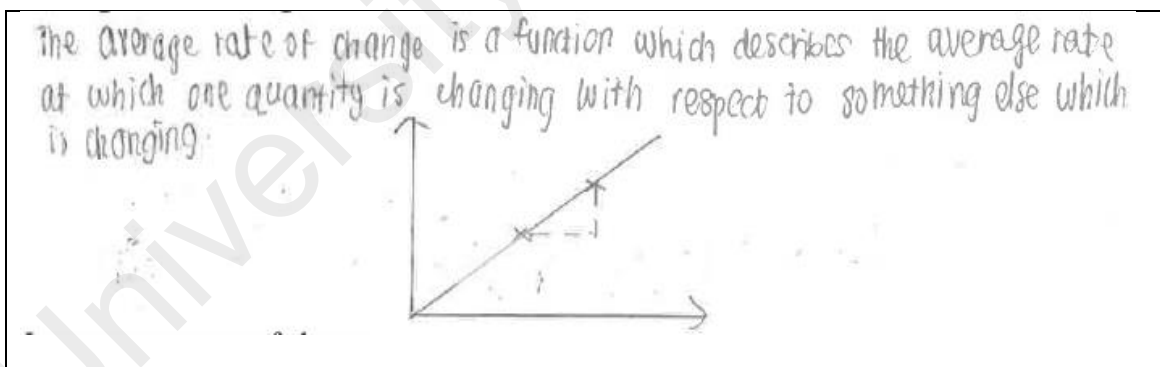


Figure 4.18: Sample of wrong graph sketched and wrong definition and explanation provided by students for item 1 of MVT.

Students who did not sketched any graph for the solution but managed to produce correct definition of the formula indicate some knowledge on understanding the relationships between the chords to tangent and the slopes of chords to the slopes of

tangents but they were reluctant to turn to graphs to express their workings. Their solutions such as ‘the slope of chord between 2 points which are ‘ $(x, f(x))$ and $(a, f(a))$ ’ and ‘the rate of change of a chord’ for $\frac{f(x)-f(a)}{x-a}$ and ‘gradient of the tangent at $(a, f(a))$ when x approaching a ’ and ‘the instantaneous rate of change’ for $\lim_{x \rightarrow a} \frac{f(x)-f(a)}{x-a}$ indicate their ability to visualize the concepts. Those with vague or lack of understanding on the concepts of chords, limits and tangents defined both formulae as ‘the gradient at certain point’, ‘the difference in the function $f(x)$ at ‘ $x = x_1$ and $x = a$ ’ or ‘it involves two rate of changes’ followed by ‘the rate of change approaches infinity’, ‘replace $x = a$, $\frac{f(x)-f(a)}{a-a} = 0$ ’ or ‘under the limit of x until a , $f(x=0) = 0$, therefore a is the root’.

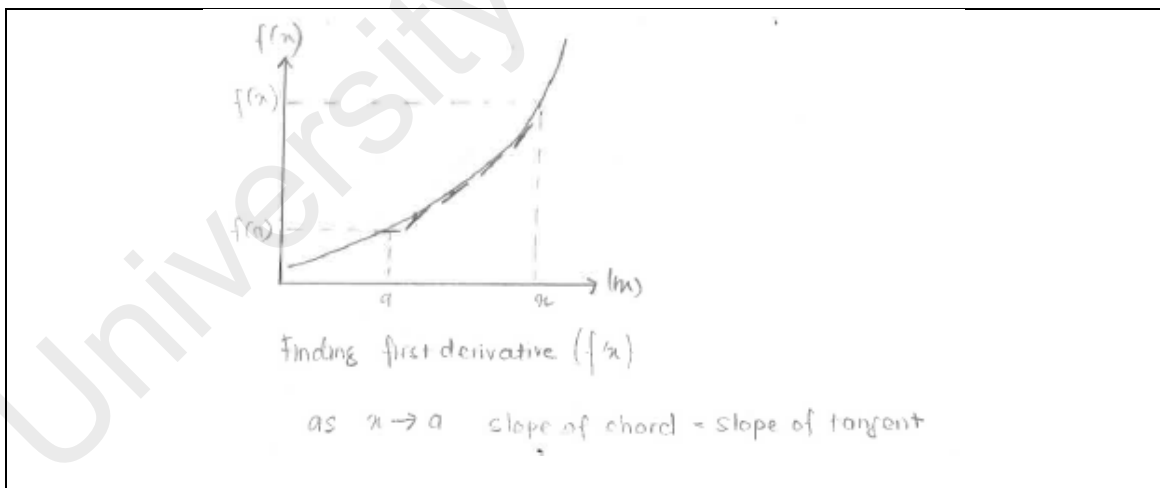


Figure 4.19: Sample of wrong graph sketched but with correct description and explanation provided by students for item 2 of MVT

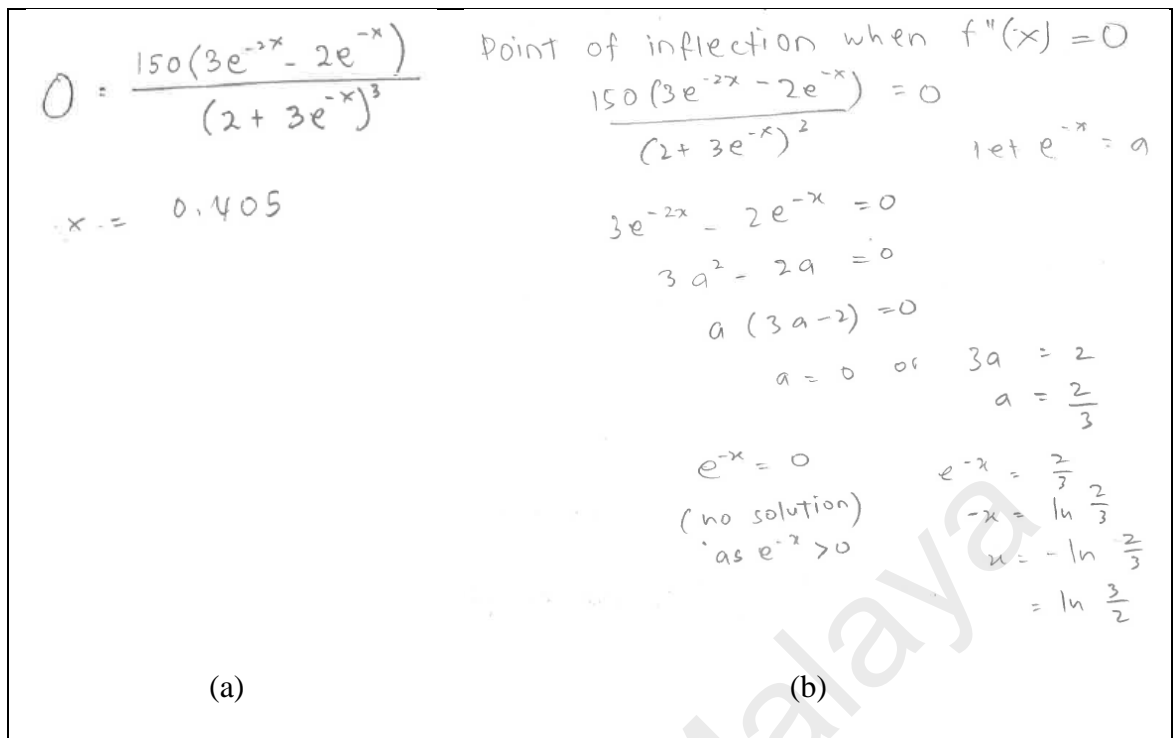


Figure 4.20: Samples of solutions provided by students for item 3(h)(ii) of the MVT

Approximately between 4% to 7% of the students produced wrong graphs and less than 34% of the students performed incorrect solutions for various parts of items 3. Some of them analysed the graph of the function while some use the ‘Y=’, and ‘ZOOM’ or ‘WINDOW’ functions of the graphic calculator to sketch and adjust respectively the graph of the first derivative function in order to obtain the sign of the derivative function and further made decision on the behaviour of the function. About the same number of students use graphics calculator to sketch the graphs of $y = f'(x)$ and use the ‘TRACE’ function in the graphic calculator to identify the zero of the graph for the inflection point while the majority still solve the algebraic equation $y = f''(x) = 0$ for the point of inflexion as shown in Figure 4.20.

For those who sketched wrong graphs, had actually made mistakes in keying in the function which had resulted in $f(x) = \frac{50}{2} + 3e^{-x}$ instead of $f(x) = \frac{50}{2 + 3e^{-x}}$ which in turn resulted in decreasing exponential function. They, however, managed to arrive to

the correct respective answers based on their wrongly sketched graphs. Students who did not sketch any graph for the solution but managed to produce correct solutions indicate that they did understand the basic properties and characteristics of graphs. Their ‘starting’ solutions such as ‘for e^{-x} , x can take any real number’ and ‘ $e^{-x} > 0$ for all x ’ for the domain and range respectively, ‘ $x = 0$ ’ and ‘ $y = 0$ ’ for the y -intercept and x -intercept respectively, ‘ $2 + 3^{-x} = 0$ ’, and substituting $x = \pm\infty$ for analysis of vertical asymptote, suggest that they are familiar with the ‘conditions’ for the particular situations. The expression such as ‘ $f(x)$ is increasing’ does indicate that students did visualize the situations but were expressing them in written form. Those with vague or lack of understanding of the properties and characteristics of graphs produced ‘the exponential graph will not pass through the origin as it has a horizontal asymptote $y = 0$ ’, ‘gradient of $f(x)$ ’ ‘because $f(x)$ is a logistic function’ and ‘the value of $y=f(x)$ is always increasing until it reach vertical asymptote’ for various parts of the question.

A total of about 18% of the students made errors in sketching graphs to help the solution processes while approximately 18% of the students performed incorrect solutions when solving item 4. Some of the students who sketched correct graph for item 4 seem to take both conditions, $\frac{du}{dt} < 0$ and $\frac{d^2u}{d^2t} > 0$ separately. They seem to be very well-versed with $\frac{dy}{dx} < 0$ or $\frac{dy}{dx} > 0$ for the function to be decreasing or increasing respectively while $\frac{d^2u}{d^2t} > 0$ or $\frac{d^2u}{d^2t} < 0$ for the function to be convex or concave respectively. Students who did not sketch any graph for the solution but managed to produce correct definition and explanation indicate that they did understand the basic ideas and relationship of the first and the second derivative to their functions. Their solutions such as ‘decreasing’, ‘decreasing at increasing rate’, ‘convex shape’ or ‘has a minimum point’ show that the students visualize the situations but were expressing

them in written form. Those with vague or lack of understanding on the concepts of derivatives explained the stated conditions as ‘the rate of change of unemployed people is decreasing’, ‘there will be a minimum point as the shape of the graph will be convex’, ‘the number of unemployed is decreasing at increasing rate’ or ‘shape of graph is > 0 , positive function’.

The analysis shows that the majority of the students prefer to describe the situation in words rather than to illustrate them in graphical form. From the students’ work, it can be seen that they are still either confused or did not understand the relationship among the function, the first derivative and the second derivative. Those who drew correct graphs made wrong interpretation of the rate of change while some that drew incorrect graphs continued to misinterpret the situation wrongly as shown in Figure 4.21.

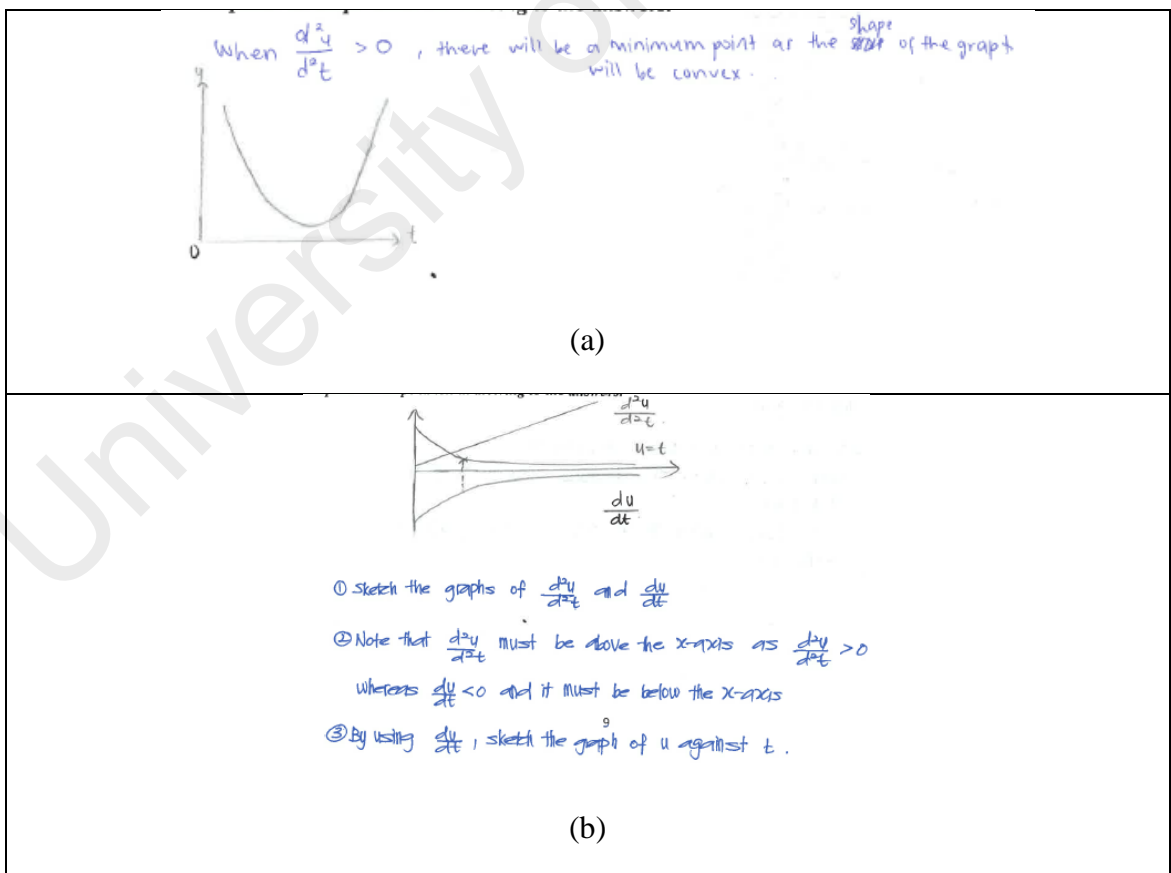


Figure 4.21: Samples of wrong graphs sketched and wrong definition and explanation provided by students for item 4 of the MVT.

None of the students performed any errors in sketching the graphs since no one actually use graph to help them solve the problem for item 5. About 14% to 48% of the students made errors in their worked solutions. The analysis shows that all of the students who attempted the problem were reluctant to use graphs in solving this real-life situation. This shows that they are very convenient with differentiating and solving the function algebraically since the majority of them managed to arrive to the correct solutions. By the way, some of the students did draw the sign diagrams of $A'(t)$ and $A''(t)$ for Item 5(b) and 5(c) to determine the required intervals as shown in Figure 4.22.

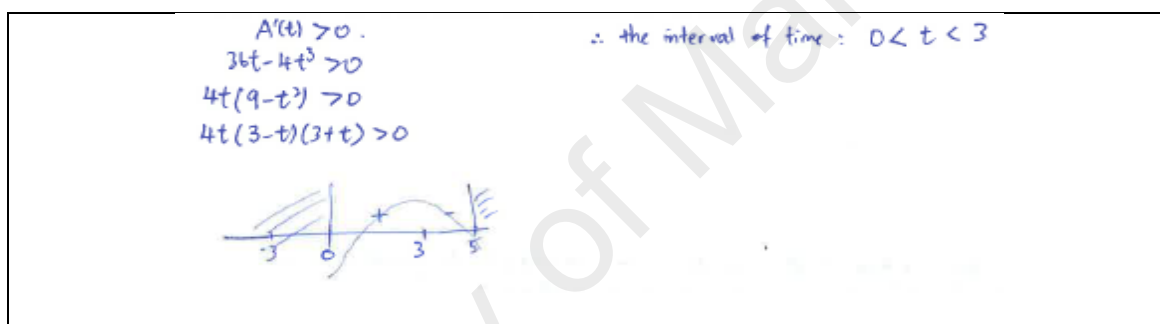


Figure 4.22: Sample of sign diagram drawn by students for item 5 of the MVT.

4.7.2 Graph Reasoning

This section describes the errors performed by the students when solving tasks in the Graph Reasoning Test. Since the aim of the tasks is to assess students' used of graphs to solve derivative problems, this section focused on the 'invalid' reasons provided by the students and also the 'incorrect' conceptual knowledge applied by the students. Therefore, the errors carried out by the students were extracted and analysed based on the mathematical reasons and worked solutions provided in the 'Correct Solution Invalid Reason (CSIR)' and 'Incorrect Solution Invalid or No Reason (ISINR)'.

Table 4.30 illustrates the analysis on the errors performed by the students for items in GRT. Less than 11% of the students provided invalid reasons for their worked

solutions and, except for item 1(e) with about 31%, less than 10% of the students performed incorrect solution for various parts of item 1. The analysis shows that students have were able to relate some of the basic and simple functional or symbolic notation to the graphical forms. Most of the students faced no problem in locating the symbol $f(4)$ as the y -coordinate of the graphs but some struggled with the meaning of $\frac{f(3)-f(1)}{3-1}$. They drew tangent lines at points between $x=1$ and $x=3$ instead (Figure 4.23). Most of the students managed to see that the function is an increasing function and therefore able to analyse the required comparisons of the position of the points, the steepness of the slopes of the chords and tangents. Some with no reasons made sketches of the lines while others must have done some visualization on the respective lines in order to arrive to the correct solutions and with reasons such as ‘the higher location’, ‘value of y is higher’, ‘ $f(x)$ is increasing in value’, ‘positive slope’, ‘slope of the tangent decrease’, ‘tangent is steeper’. Many students were very familiar with the functional notation representing ‘tangent’ and therefore had no problem in illustrating $f'(1)$ on the graph but quite a number of them cannot represent $\frac{f(x)-f(1)}{x-1}$ on the graph. Quite a majority of the student were unable to write the relationship between $f'(1)$ and $\frac{f(x)-f(1)}{x-1}$ indicating their weakness in understanding the basic formulation of the derivatives.

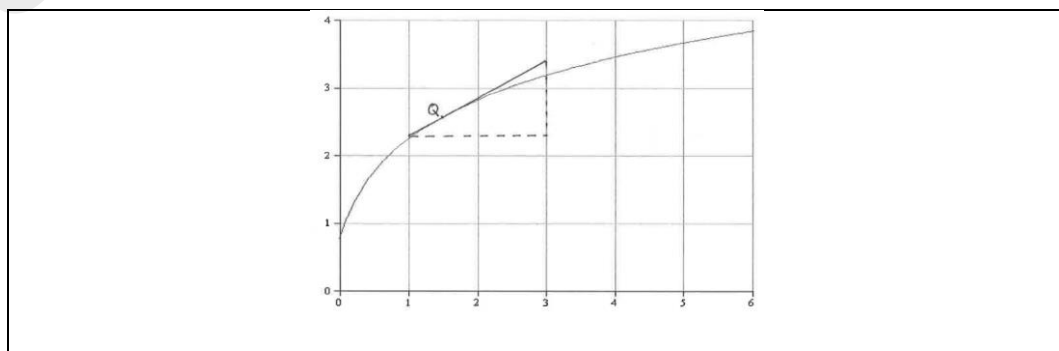


Figure 4.23: Sample of wrong chord drawn by student for item 1 of the GRT.

Table 4.30: Distribution of errors for the Visual Reasoning

Item	Invalid reasons		Incorrect solutions	
	f	(%)	f	(%)
1(a)(i)	0	(0.00)	0	(0.00)
1(a)(ii)	20	(10.31)	14	(7.22)
1(b)(i)	20	(10.31)	0	(0.00)
1(b)(ii)	20	(10.31)	18	(9.28)
1(b)(iii)	12	(6.19)	6	(3.09)
1(c)(i)	6	(3.09)	7	(3.61)
1(c)(ii)	2	(1.03)	19	(9.79)
1(d)(i)	4	(2.06)	0	(0.00)
1(d)(ii)	10	(5.15)	4	(2.06)
1(e)	0	(0.00)	61	(31.44)
2(a)(i)	5	(2.58)	20	(10.31)
2(a)(ii)	9	(4.64)	96	(49.48)
2(b)(i)	11	(5.67)	71	(36.60)
2(b)(ii)	7	(3.61)	38	(19.59)
3(a)	7	(3.61)	41	(21.13)
3(b)(i)(1)	7	(3.61)	57	(29.38)
3(b)(i)(2)	11	(5.67)	20	(10.31)
3(b)(i)(3)	2	(1.03)	30	(15.46)
3(b)(ii)(1)	34	(17.53)	9	(4.64)
3(b)(ii)(2)	38	(19.59)	33	(17.01)
4(a)(i)	21	(10.82)	20	(10.31)
4(a)(ii)	18	(9.28)	27	(13.92)
4(b)	6	(3.09)	17	(8.76)
4(c)	2	(1.03)	43	(22.16)
5(a)	6	(3.09)	13	(6.70)
5(b)(i)	20	(10.31)	108	(55.67)
5(b)(ii)	17	(8.76)	41	(21.13)
5(b)(iii)	36	(18.56)	83	(42.78)
5(b)(iv)	3	(1.55)	19	(9.79)

About 3% to 6% of the students provided invalid reasons for their worked solutions and about 10% - 50% of the students performed incorrect solution for all four parts of item 2. The analysis shows that the majority of the students were able to read the coordinate of the point on the graph and rewrote them in functional form although a few of the students misread the coordinates as $g(B) = 5$, indicating that the students understood the concept but had mistook the x -coordinate. Some others misread $g(1.95) = 5.02$ which indicate that the students knew how to read the coordinate but did not realize/know the location of the required point to be either on the graph of the function or on the tangent line. On the other hand, only a small amount of the students managed to relate the symbolic form of $g'(x)$ as the derivative or the instantaneous rate of change at a point and relate it to the slope of the tangent line. Those students with incorrect answers came out with $g'(1.95) = 5.02$ and $g'(2) = 5$ indicating their misreading the data. Other types of unacceptable solutions are such as $g'(2) = 5.02$, $g'(2) = 0$ and $g'(1.95) = 5$. These indicate their weak basic knowledge in the concepts of derivatives and tangent and lead to not able to relate to the ideas graphically. Some invalid reasons provided by the students were 'points are on the tangent', 'stationary points' and negative slope indicating their weakness in the conceptual understanding of derivatives graphically.

In the analysis for the second part of item 2, those who managed to obtained correct solutions failed to reason correctly by defining the vertical distance as vaguely as 'the difference between the two functions' and very unacceptable 'functions are equal at $x=a$ and $x=b$ ', 'functions intersect at two points', 'local maxima or minima' and 'maximum gradient'. This shows that the students who managed to read the information displayed on the graphs made errors when needed to go beyond what were displayed. Some samples of incorrect solutions produced by the students were ' $g(x) = f(x)$ ', ' $g(b) - f(a)$ ', ' $g(x) - f(x)$ ', 'maximum gradient at $x=c$ ' and 'tangent is zero'. Some of the

students reasoned the solutions to be related to the ‘area between the curves’ indicating their assumptions that when functions bounded a region, then tasks must relate to area between the curves.

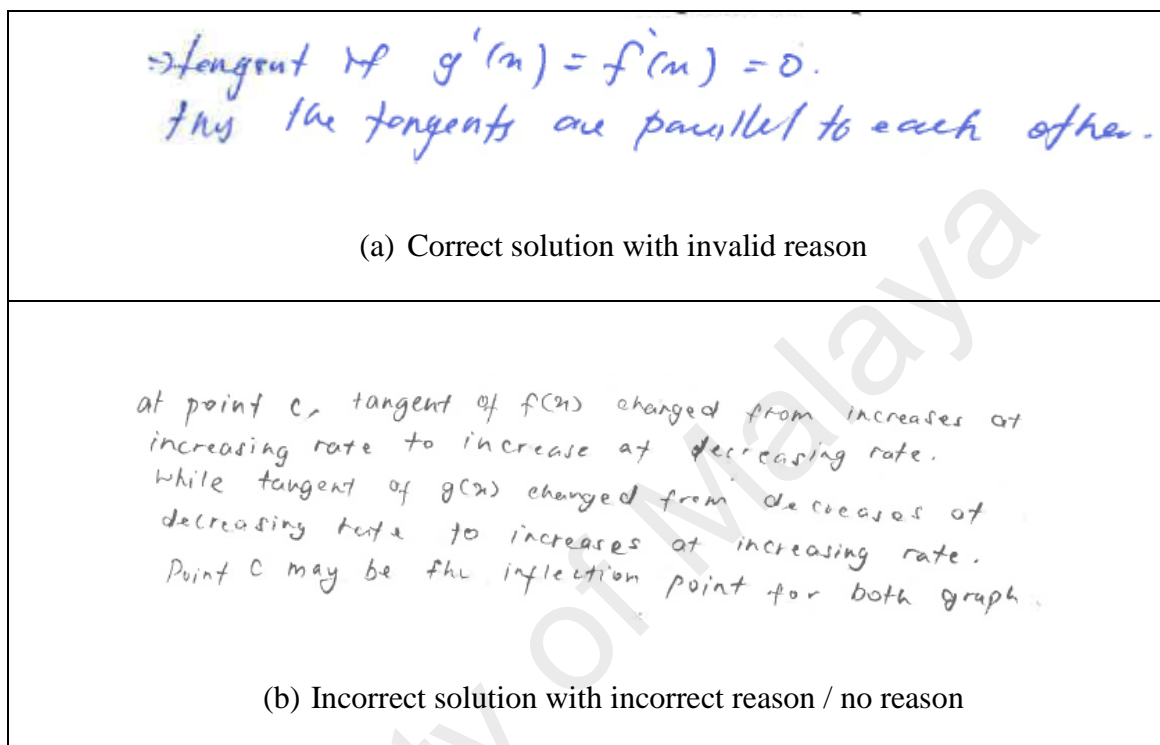


Figure 4.24: Samples of solutions by students for item 2 of the GRT

About 1% to 20% of the students provided invalid reasons for their worked solutions and between 5% to 30% of the students performed incorrect solutions for various parts of item 3, as explained in the next paragraphs.

The analysis shows that errors performed by the students in reasoning were mainly on them assuming that the graph is a quadratic or the shape is a convex (Figure 4.25). Nevertheless they gave the correct reasons from the visual point of argument. Other errors include ‘local minimum at $f(x) = 0$ ’ which indicate their memorization of the standard formula or condition. Some drew wrong sign diagram to represent the signs of the gradient of the functions reflecting their lack of ability to read between the

data, at the same time exhibiting a weak understanding on the concepts of gradients of functions graphically.

In the second part of item 3, students made errors in assuming that vertical and horizontal asymptotes were only indicated by 'dotted lines'. They therefore missed the line $x=0$ as the other vertical asymptote. They reasoned that 'the graph did not touch the x -axis' for choosing $y=0$ as the horizontal axis which is 'seen' from the graph but unfortunately it was not an acceptable answer. Some confused themselves between the horizontal and the vertical asymptotes. The reasons such as 'decreasing function', 'approaching zero', 'negative infinity' and 'decreasing with increasing/decreasing rate' that bring no meaning to the solutions indicated that most of the students memorized the 'standard' or common terminologies with regards to the topic derivatives without understanding them conceptually.

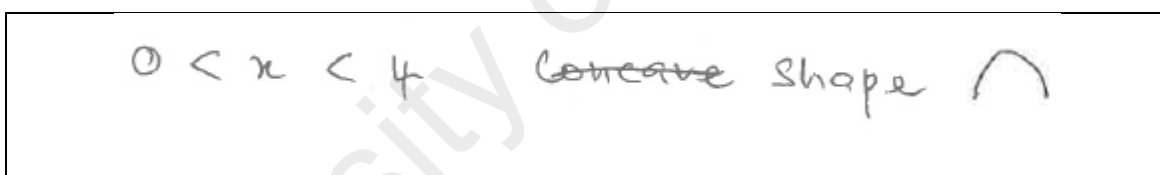


Figure 4.25: Sample of incorrect solution with incorrect reason by students for item 3 of the GRT

Less than 11% of the students provided invalid reasons for their worked solutions and between 10% to 23% of the students performed incorrect solution for all for parts of item 4.

The errors performed by the students, through the analysis on the reasons they provided such as 'below the x -axis', ' $f'(x)$ is decreasing' and ' $f(x)$ is approaching -3 / horizontal asymptote' indicated their ability to read of data and described the properties and behaviours of the function graphically but with lack of conceptual understanding on the topic of derivatives. For item 4(b), those who managed to get the correct solutions

did struggle with providing invalid reasons for their actions such as ‘decreasing’, ‘above the horizontal asymptote’ and ‘convex’. In part 4(c), students performed various errors when sketching the graph of the derivative of the function given. Those who managed to obtain the correct graph were still unable to support their solutions with valid reasons. Theirs were a simple and brief as ‘increasing and decreasing of graph’ and ‘behaviour of graph’. Some of the other students carried out various types of errors in their sketching such as ‘graph passing through the points (0,0) and (4,0)’ and ‘ x is greater than or equal to negative infinity and y is greater than or equal to -3 ’ which indicated their unable to read the properties of the derivatives from the graph of the function. Samples of students’ worked solutions are as shown in Figure 4.26.

Between 1% to 19% of the students provided invalid reasons for their worked solutions and between 6% to 56% of the students performed incorrect solutions for all the five parts of item 5. Item 5 consisted of tasks on the applications onto real-life situations. The analysis on item 5(a) shows that students who, although provide correct answers for the situations, still performed errors in the reasons to accompany their decisions. Various simple explanations were ‘straight line’, ‘different slopes’, ‘horizontal lines’, ‘shape of graph’ and ‘starting at the origin’, again reflects their memorizing of the terms instead of grasping the concepts. Item 5(b) exhibits how students argued their correct descriptions of the ‘rate’ through the shape and hence the specific functions such as the ‘logistics’ or ‘surge’ functions. Some described their reason as simple as ‘shape of the graph’. Other various incorrect descriptions include ‘increasing continuously without bound’ and ‘increase then decrease’. When drawing the sign diagram, some of the students either appointed wrong critical points or drew the sign diagrams of the first derivative instead of for the second derivative. Reading beyond the graph as requested by item 5(b)(iii) to interpret the inflection point to the

population growth seemed to fetch more errors as compared to item 5(b)(iv) that needed the students to relate the horizontal asymptote to the growth pattern (Figure 4.27).

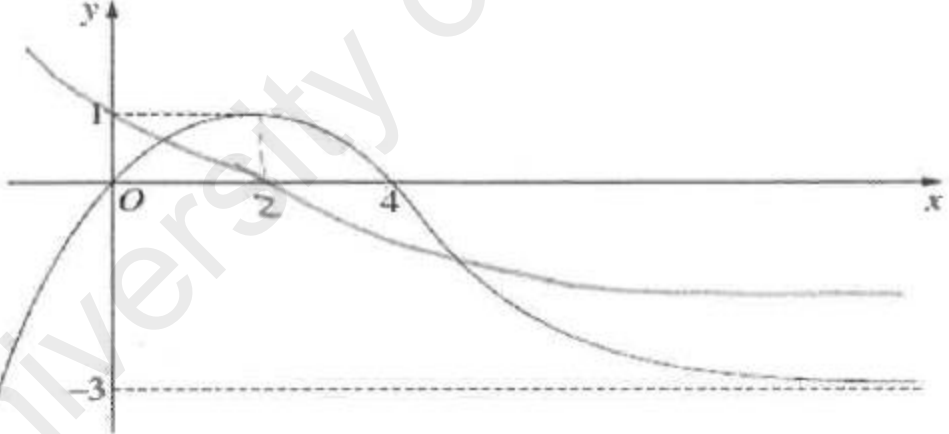
<p style="text-align: center;">find $f(x)$.</p> <p style="text-align: center;">$x = 3$, when $\frac{dy}{dx} = 0$</p> <p style="text-align: center;">(a) Incorrect solution with incorrect reason / no reason</p>
<p style="text-align: center;">$x < 0, x > 4$ below the x-axis</p> <p style="text-align: center;">(b) Incorrect solution with incorrect reason / no reason</p>
<p style="text-align: center;">as $x \rightarrow \infty$ $f'(x) \rightarrow -3$</p> <p style="text-align: center;">(c) Incorrect solution with incorrect reason / no reason</p>
<div style="text-align: center;">  </div> <p style="text-align: center;">(d) Incorrect solution with incorrect reason / no reason</p>

Figure 4.26: Samples of various solutions by students for item 4 of the GRT.

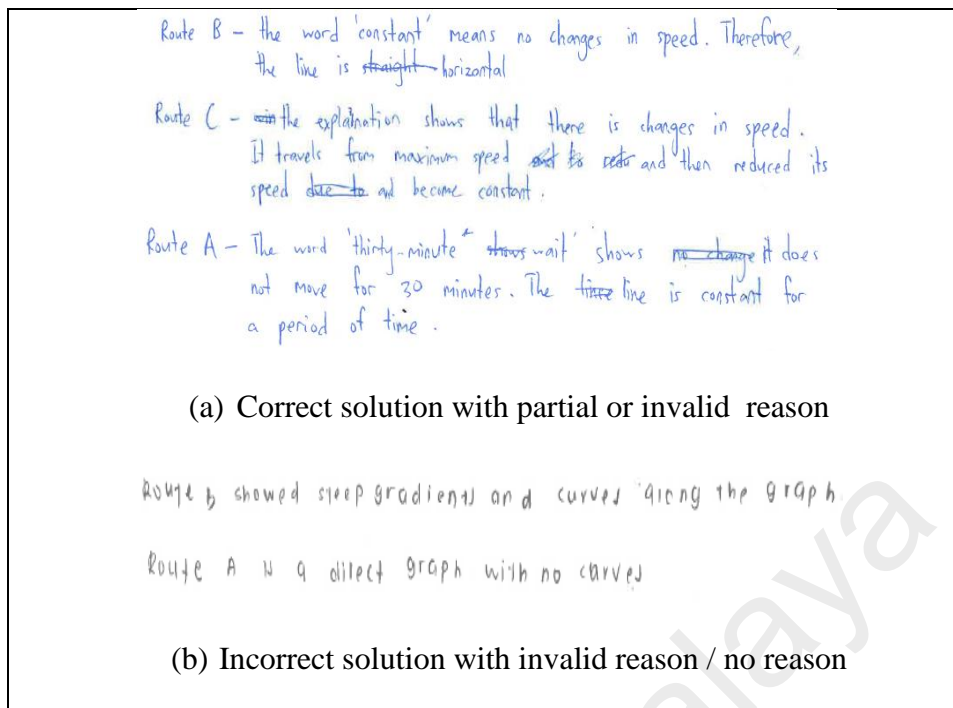


Figure 4.27: Samples of solutions by students for item 5 of the GRT

4.8 Summary

This chapter reports the results on the development of a framework to assess the visual reasoning ability and the quantitative analysis of the visual reasoning ability of pre-university students when dealing with Cartesian graphs to solve problems on functions and derivatives. Using the document analysis on theories, models and frameworks related to visual representation, properties and characteristics of graphs and conceptual knowledge on functions and derivatives, a framework consisted of encoding and decoding processes has been established. The encoding part includes the categories to determine the students' preference in the method that they employ when dealing with tasks that allow them to work either algebraically or graphically. The decoding part encompasses categories that students utilize when they are using graphs as their visual tools to solve mathematical problems.

The descriptive statistics of the students' usage level of visual representation showed that the majority of the students were positive on the usage of graphs and

diagrams for all the four categories : in their daily learning behaviour, on the usefulness and difficulty in solving mathematical problems, and on their teacher's behaviours during the teaching of mathematics. Positive correlations were also identified among the categories. This was followed by the descriptive analysis on the mathematical visuality through the encoding process and the visual reasoning ability through the decoding process. The students can be grouped into three categories of mathematical visuality: visual, partially visual and non-visual. On the other hand, through the analysis of their decoding process, very strong and positive correlation were also observed among the scales that indicate their ability to read the information from the graph directly and to interpret the displayed graph into information.

The correlations among the results of the three instruments were then analysed. It was identified that the three results were positively correlated to each other with coefficient of correlation, r , between 0.803 and 0.897. This is followed by the analysis on the errors performed by the students on both sets of instruments and consequently the identification of their difficulties and misconceptions when dealing with graphs to solve mathematical problems on functions and derivatives. Students were found to perform fundamental, operational and systematic errors. They had also some misconceptions on the use of graphs and faced generic and idiosyncratic types of difficulties. The findings reported in this chapter are further discussed in the next chapter.

CHAPTER 5: MAIN FINDINGS, DISCUSSION AND CONCLUSION

5.1 Introduction

This study investigated the visual form of reasoning through the use of Cartesian graphs in the context of learning and solving problems on functions and derivatives. It was driven in parts by the promise of integrating graphs as visual tools for reasoning and in part propelled by the need to increase understanding in calculus, specifically functions and derivatives, and mathematics among the Malaysian students. The primary purpose was to develop a framework to assess the pre-university students' visual reasoning when solving functions and derivative tasks through the use of Cartesian graphs. The subsequently purposes were to examine their preference to employ graphs, their reasoning ability and the difficulties faced when solving problems on functions and derivative.

The study adopted a descriptive design that collected quantitative data to assess the students' ability to reason visually. The participants were pre-university students who, at the time of data collection, had completed the learning of functions and derivatives and were about to sit for their trial examination and later the final external examination. Three tests, Visual Representation Usage Level, Mathematical Visuality Test and the Graph Reasoning tests were employed to collect data on students.

5.2 Main findings of the study

5.2.1 Development of the framework

A framework to assess the visual reasoning ability of pre-university students when solving mathematical problems on functions and derivatives using Cartesian graphs is as shown in Table 4.12 (reproduced from section 4.2.3). The framework was named

Table 4.12: The final framework for assessing visual reasoning

Visual reasoning process	Category	Description
Encoding	Correct graph with correct solution	Produces correct graph to explain and represent the solutions and managed to arrive to the correct solution
	Correct graph with incorrect solution	Produces correct graph to explain and represent the solutions but did not manage to arrive to the correct solution
	Incorrect graph with correct solution	Produces incorrect graph to explain and represent the solutions and managed to arrive to the correct solution based on the wrong graphs. Solutions may differ from the original solutions set.
	Incorrect graph with incorrect solution	Produces incorrect graph to explain and represent the solutions and did not manage to arrive to the correct solution
	No graph with correct solution	Produces no graph to explain and represent the solutions and managed to arrive to the correct solution
	No graph with incorrect solution	Produces no graph at all to explain and represent the solutions and did not manage to arrive to the correct solution
	No answer / Not attempted	Left the item un-attempted – no graphs or any algebraic solutions.
Decoding	Correct solution with valid reason	Produces correct solution based on the graph and managed to provide valid reason(s) to arrive to the correct solution
	Correct solution with invalid reason	Produces correct solution based on the graph but did not manage to provide valid reason(s) to arrive to the correct solution
	Correct solution with no reason	Produces correct solution based on the graph but did not manage to provide any reason(s) to arrive to the correct solution
	Incorrect solution with invalid reason / no reason	Produces incorrect solution based on the graph and did not manage to provide valid reason(s) to arrive to the correct solution
	No answer / Not attempted	Left the item un-attempted.

Visual Reasoning over Graph (VR-G) and is able to run a thorough assessment of how students construct and interpret Cartesian graphs of functions and derivatives and their concepts. It is primarily used to categorise students based on their encoding and decoding abilities. The encoding process consists of seven categories on how students choose to respond to the mathematical word problems and their competence to produce

the correct graphs. The categories for the encoding process are : Correct graph with correct solution, Correct graph with incorrect solution, Incorrect graph with correct solution, Incorrect graph with incorrect solution, No graph with correct solution, No graph with incorrect solution, and No answer / Not attempted. The decoding process consists of five categories that describe categories on how students read and interpret the information displayed on the Cartesian graphs provided in the problems in order to look for solutions. The categories for the decoding process are : Correct solution with valid reason, Correct solution with invalid reason, Correct solution with no reason, Incorrect solution with invalid reason / no reason, and No answer / Not attempted.

5.2.2 Usage levels of graphs

The usage levels of graphs refer to how students employed graphs or diagrams in their daily learning of mathematics and solving mathematical problems. At least 41% of the students responded to 'Very much' and 'Definitely' for all items for their preference levels in using graphs or diagrams in their daily learning behaviour. The mean scores for the items ranged between 3.40 and 4.16. These show that the students did employ graphs or diagrams when solving mathematical problems. They admitted that they paid attention and even tried to use or to copy the graphs or diagrams shown by their teachers or those used in the textbooks in solving mathematical problems. These indicate that the use of graphs or diagrams (or any visual representations in general) by the teachers in the classrooms or in the textbooks do affect how the students strategies their methods when encountered with similar problems. Less than 4% of the students did not make use of graphs or diagrams either by themselves or those by their teachers or textbooks in their solving mathematical problems.

More than 70.62% of the students responded to 'Very much' and 'Definitely' for items in the category on the usefulness of the graphs or diagrams in solving

mathematical problems. The mean scores for the items ranged between 3.96 and 4.12. These percentages show that the students are positive and mostly assured that the use of graphs and diagrams are beneficial in helping them to solve mathematical problems. Less than 2% of the students of the students did not find the use of graphs or diagrams actually help them to solve mathematical problems.

For the students' difficulty on the use of graphs or diagrams in solving mathematical problems, about 21% to 44% of the students responded to 'Very much' and 'Definitely' in all four items. The mean scores for the items ranged between 2.99 and 3.37. These percentages show that the students faced difficulties in either to construct graphs or diagrams by themselves or to identify different and suitable graphs or diagrams to help them in solving mathematical word problems. Less than 5% of the students admitted that they did not know at all how to construct or use the graphs or diagrams in order to assist them to solve mathematical problems.

About 67% to 80% of the students responded to 'Very much' and 'Definitely' for the category the students' perceptions on their teachers' behaviours in using graphs or diagrams in solving mathematical problems. The mean scores for the items ranged between 3.88 and 4.22. These percentages portray that teachers make use of graphs or diagrams in their teaching for effective learning. They in fact promote and coach their students on the appropriate and correct ways to utilize graphs and diagrams in solving mathematical problems. Less than 2% of the students did not find that their teachers did teach or guide them to use graphs or diagrams to help them in solving mathematical problems. Positive correlations of values between 0.34 and 0.85 were found for the overall questionnaire and the categories and also among the categories of the items.

5.2.3 Mathematical Visuality

The analysis on the results of the MVT shows that the percentages of students who managed to arrive to the correct solutions without sketching any graphs are in the range of 24% to 84%. The students were mostly reluctant to use graphs in solving, defining or explaining the mathematical concepts of functions and derivatives although guides or hints were included for them to sketch graph, for example the word ‘graph’ was mentioned in item 1 to indirectly guide students for the solution method. Smaller range of about 8% to 40% of the students managed to sketch correct graph although some proportions of them did not come to the correct solutions.

Based on the thorough analysis of the students’ worked solutions, it was determined that the students can be categorized into three groups of mathematical visuality: non-visual, partially-visual and visual. The descriptions of the three categories are as shown in Table 5.1.

Table 5.1: Descriptions of the categories for mathematical visuality

Category	Descriptions	Visuality score	% of students
Visual	Produces effective Cartesian graphs <ul style="list-style-type: none"> • Complete labelling of axes, scales and function(s) • Graphs reflect the main concepts of the functions • Include all important data or properties of functions in details 	57–114	26.8
Partially-Visual	Produces incomplete or complete Cartesian graphs followed by algebraic methods <ul style="list-style-type: none"> • A mixture of graphical and algebraic solutions • Does not exhibit confidence in the use of graphs • Partial or incomplete labelling of axes, scales and function(s) • Graphs reflect none to basic concepts of the function or representations • Include some data or properties of functions and derivatives • Apply rules and procedures inappropriately 	39–56	16.5
Non-visual	Not producing any line graphs <ul style="list-style-type: none"> • Solutions are based on algebraic methods 	0–38	56.7

5.2.4 Visual Reasoning

The analysis on the results of the GRT shows that at least 30% of the students managed to arrive to the correct solutions regardless of them providing valid, invalid or no reasons for their solutions methods and steps. This indicates a mixture of students' ability to read, extract and interpret information embedded in graphs. Results also show that as the tasks were getting tougher, where more cognitive loads are needed, the smaller the number of students who were able to accomplish the solutions.

Positive correlations and strong relationships of values between 0.91 and 0.98 were obtained for the overall GRT and the scales and also among the scales of the decoding process. Based on the analysis on the students' worked solutions, it was determined that the students' responses due to their decoding scales can be further detailed as described in Table 5.2.

Table 5.2: Descriptions of the categories for visual reasoning

Category of levels	Descriptions	% correct
Read the graph	<ul style="list-style-type: none"> • Appropriate engagement with the context • Able to recognize the properties of functions and derivatives – understanding of single/basic element and direct graph reading 	67.63 –
	<ul style="list-style-type: none"> • Employ memorization or procedural knowledge • Appropriate use of mathematical terminologies 	100.00
Read between the graph	<ul style="list-style-type: none"> • Able to recognize the relationships between the properties of functions and derivatives • Able to attend to and integrate more than one relevant features and aspects of the displayed information 	38.66 –
	<ul style="list-style-type: none"> • Correct use of mathematical terminologies 	96.91
Read beyond the graph	<ul style="list-style-type: none"> • Able to infer on the relationship between the properties of functions and derivatives • Able to integrate contextual knowledge and understand the purpose of information displayed 	30.93 –
	<ul style="list-style-type: none"> • Advanced visual and mathematical skills • Accurate use of mathematical terminologies and able to interpret subtle aspects of languages 	76.29

The percentages for students who managed to arrive to the correct solutions outnumbered the percentages of students who did not manage to get the correct solutions in both MVT and GRT. This indicates that students know and understand the properties of derivatives and how to solve the problems regardless the method that they used.

5.2.5 Correlations among the instruments

Positive correlations were obtained among the three relationships : 1) the Visual Representation Usage Level and the Mathematical Visuality Test, 2) the Visual Representation Usage Level and the Graph Reasoning Test and 3) the Mathematical Visuality Test and the Graph Reasoning Test. These indicate that students who made use of graphs or diagrams in their daily learning of mathematics will tend to draw graphs to represent and explain their solutions and were able to read and interpret the information that were displayed or hidden in the graphs. It also indicate that those students who made use of graphs as visual tools to represent their solutions were able to read and interpret information in the graphs, either those directly shown or those needed to interpret.

5.2.6 Difficulties and misconceptions

Some of the conceptual issues that cause students to make errors and have difficulties and misconceptions with tasks related to functions and derivatives and the use of graphs are: weak of knowledge on graphing in general, lack of knowledge or practice in graphing derivative functions from graphs of functions that has no algebraic expressions, difficulty in identifying and relating the stationary points, difficulty interpreting critical points from a graphs of derivative functions, focusing primarily on procedural knowledge instead of conceptual knowledge, relying on memorized procedures, creating a short cut or procedure that is not valid, preference for algebraic

approaches to solving problems than graphical approaches and mixing up the attributes of the first and second derivatives .

The students' errors can be grouped into three categories. The descriptions of the three categories are as shown in Table 5.3.

Table 5.3: Descriptions of the categories for errors

Category	Descriptions	% of students
Fundamental error	• Fail to understand or realize the relationship between algebraic and graphical representation involved in the problems	19.6
	• Fail to grasp important principles to solve the problems	13.4
	• Confusion among concepts describing different attributes of the same situation	12.7
Operational error	• Fail to carry out procedural and manipulation processes although had understood the principles engaged	9.8
Subjective errors	• Fail to take into consideration the constraint(s) imposed in the question	25.1
	• Applying the general rule to a specific case	19.4

Note. The category ignores errors performed by the students that due to their carelessness in performing the basic algorithms.

The students' difficulties in dealing with functions and derivatives and graphs can be categorized into three groups. The descriptions of the three categories are as shown in Table 5.4. These categories were developed based on the commonalities among the misconceptions and difficulties faced by the students.

Students expressed their difficulty in providing written explanation on the steps taken or reasons to arrive to the answers. This was caused by their lack of proficiency the English language and can be verified (upon request) from their International English Language Testing System (IELTS) examination results where most of the students

results were in bands 5.5 and 6.0 for the Writing components. They scored higher bands for the other three components, Listening, Speaking and Reading

Table 5.4: Descriptions of the categories for difficulties

Category		Descriptions	% of students	
Non-use of graph	MVT	<ul style="list-style-type: none"> Lack of understanding on the concepts of a graph 	56.7	
		<ul style="list-style-type: none"> Lack of understanding on the graph as a representation 		
Generic difficulties	MVT	<ul style="list-style-type: none"> Constructing unusable graphs 	12.3	
		<ul style="list-style-type: none"> Incorrect or inaccurate representation of quantity 	9.8	
	GRT	<ul style="list-style-type: none"> Misunderstanding or confusing on the written symbols 	4.7	
		<ul style="list-style-type: none"> Misinterpret the properties of line graph 	19.5	
	MVT	<ul style="list-style-type: none"> Weaknesses in identifying specific information from graphs 	8.9	
		<ul style="list-style-type: none"> Graphs viewed inappropriately 	4.9	
	Idiosyncratic difficulties	MVT	<ul style="list-style-type: none"> Lack of precision in the graph 	9.9
			<ul style="list-style-type: none"> Overlooking the constraints imposed in the function 	4.7
GRT		<ul style="list-style-type: none"> Putting parts together to form a whole 	1.9	
		<ul style="list-style-type: none"> Relating the mathematical concepts to the real-life situation 	20.3	
		<ul style="list-style-type: none"> Non-flexible thinking when dealing with non-standard graphs (derivative of function) 	17.3	
		<ul style="list-style-type: none"> Connecting algebraic representations of derivative to graphical forms 	29.1	

5.3 Discussion

5.3.1 Usage Level of Visual Representation

In educational research, Uesaka and Manalo (2007) identified that the Japanese students employed more use of diagrams as compared to the New Zealand cohort when solving mathematical problems although their level of appreciation to the use of diagrams in their daily learning and classroom context were about the same. In this study on the

Malaysian students, the percentages of at least 41.7% of students responded to the 'Very Much' and 'Definitely' for the first category shows that the students have positive perceptions on the usage of graphs and diagrams to help them solving mathematical word problems. The responses given to the first category suggest that the use of graphs or diagrams by their teachers and in their textbooks indicates that visually rich source of teaching and learning materials are able to help, and develop students' interest and problem solving skills (Ball & Ball, 2007; Naidoo, 2007). Emphasizing on the use of graphs or diagrams in classroom environment would help to create patterns for students to employ them when encountered with similar types of tasks. Although the use of graphs or diagrams in no way guarantee the students to produce correct solutions, the resulted percentages do support the benefits of using graphs or diagrams in solving mathematical word problems as concurred by Kissane and Kemp (2011) and Pierce and Stacey (2008). Their studies on enhancing students' understanding on how functions are connected to their derivatives resulted in the implementation of graphic calculator and Computer Aided System helped to support the understanding of the concepts of functions and derivatives.

The high percentages of more than 70% of the students responded to 'Very Much' and 'Definitely' for the category on the usefulness of the graphs or diagrams in solving mathematical problems indicate that the students perceived that using graphs and diagrams is helping them to efficiently solve the problems and as an alternative and a better way of learning as it increased their learning outcomes and success. As agreed by Guler and Ciltas (2011) and Uesaka and Manalo (2011) in their findings, visuals and the visualization process have significant roles in the preparation of the tasks and problems, in guiding the method of solving mathematical problems and at the same time affects the students' cognitive structure (Garderen, 2007).

The high percentages of those who responded to 'Moderately' in the third category indicates that although the students knew that graphs or diagrams do help them in the problem solving they found some degree of difficulty in constructing and using them on their own. This result seems compatible with the findings by Presmeg (1986, 2006) and Uesaka (2002). As Hegarty and Kozhevnikov (1999) regarded graphs as formal representations as compared to some other informal images or graphical forms, they discovered that graphs played more active roles in letting students to focus on relevant details. Owolabi and Adaramati (2015) recently, introduced the Graphic Organizer, a visual representation of text concepts, as one of the instructional strategies in order to help students organize their graphs, information and related concepts embedded in the graphs.

Between 67% to 80% of the students agreed 'Very Much' and 'Definitely' that their teachers' behaviours in using graphs or diagrams do assist them in solving mathematical word problems. This indicates that teachers' preference in the using of graphs or diagrams in the teaching and learning processes help students to understand the problems better and subsequently able to solve them. This finding is compatible to the finding by Sheehan and Nillas (2010). The student-centred education as being recommended by almost all curriculum developers globally, had urged for learning to be in line with the students' demands and needs (Macini & Gagnon, 2006). Given that graphs and diagrams are considered as highly attention-grabbing, teachers are encouraged to put in efforts and make variations about the layout and arrangement of any written presentation to couple with some graphs or diagrams in order to communicate concepts to students as proposed by Guler & Ciltas, (2011). According to Alcock and Simpson (2009), students may not have any systematic justifications for the variations or alternatives but the outcomes for their choices are likely to positively influence towards the use of graphs or diagrams. Constructing graphs and diagrams

encourages teachers to indirectly expose alternative ways of delivering information through visual representations as additional to written format and subsequently allow students to access this additional information at their own pace. It was observed that students highly appreciated for the teachers to practice, show, coach and encourage the use of graphs and diagrams during the learning sessions. The result seems compatible with the findings of Alcock and Simpson (2009).

NCTM (2000) had also proposed that all mathematical tasks require visual thinking. As discovered by many researchers, visualization ability and visual reasoning skills were positively correlated to mathematical achievements (e.g. Battista, 1990; Clements & Battista, 1992; Diemand-Yauman, Oppenheimer & Vaughan, 2011; Lam et al, 2012; Shah & Freedman, 2011; Theodore, 2010) and are essential elements when solving problems in other important mathematical topics such as geometry, trigonometry and statistics (e.g. Aaron & Herbst, 2015; Cetin, 2015; Grobecker & De Lisi, 2000; Noraini, 2008; Sharma, 2013).

5.3.2 Preference on the Use of Graphs

The overall analysis of the students' worked solutions observed the majority of the students were in the 'No Graph with Correct Solution' (NGCS) and 'No Graph with Incorrect Solution' (NGIS). As concurred by Uesaka et al (2007) and Roskwn (2006), given to students a problem where they can solve by both methods, algebraic or visual, most would prone to opt for the algebraic manipulations instead of using graphs. Upon reading the questions, students tended to revert quickly to algebraic manipulations or they appeared to read or interpret graphs in manners that exhibit their lack of understanding to the underlying concepts of the content domains as suggested by Sharma (2013).

The findings of the study also denote that even though the students were very positive when their teachers demonstrated the use of graphs in the classroom teaching agreeing with the findings by Guler and Ciltas (2011), it was still insufficient to grab their interest or confidence to use graphs, especially spontaneously as their tools, as concurred by the findings from Uesaka and Manalo (2011), either for solving or communicating purposes, when solving problems on functions and derivatives. Teachers are encouraged to create interest by providing opportunities for the students to use graphs in an interactive environments. Those would indirectly allow students to be exposed to more ways of communicating the mathematical ideas and their understanding of the mathematical concepts

Presmeg (2006) proposed that training students to practice visualization or to employ visual method to solve most of the mathematical problems would help them grasp the concepts without undergoing the procedural or algebraic methods. Technological tools such as graphing software or graphing calculators are capable to help illustrate various properties and characteristics of functions and their derivatives such as the intercepts, asymptotes and differentiability at different points visually. Besides the fact that the students owned and had been practising graphic calculator in the classroom to make solving problems and calculations simpler and faster, the analysis shows that the students were, again, mostly reluctant to use the calculator to sketch the graphs in searching for the properties of the function and derivative given (Kissane & Kemp, 2006; Pierce & Stacey, 2008). Those who had used graphic calculator to get the graphs managed to read-off the information correctly, the properties of the graphs such as domain, range, axes intercepts, vertical asymptotes and the behaviour of the graph for the smaller and larger values of the independent variable x . Some of them analysed the graphs of the functions in order to look for their derivatives while some of the students used graphic calculator to sketch the graph of the first

derivative function, checked for the sign of the derivative function and further made decision on the behaviour of the function. About the same number of students use graphics calculator to sketch the graphs of $y = f''(x)$ and 'trace' the zero of the graph for the inflection point while the majority of them still solve the equation $f'(x) = 0$ algebraically to calculate for the point of inflexion.

Base on a thorough analysis on the worked solutions by the students, they can be grouped into three categories: the visual, the partially-visual and the non-visual. The visual students can be distinguished from the other two groups based on the accuracy of them using the mathematical definitions and terminologies, and on the inclination to relate the properties of graphs with the fundamental concepts as discussed by Rivera (2011). The methods they employed to solve the problems revealed their perceptions regarding the usefulness of graphs or diagrams in solving mathematical problems. The way teachings was conducted are not able to clarify the preferences in using or not, graphs among students (Uesaka & Manalo, 2007). On the overall, approximately 56.7% of the students were categorised as non-visual. The non-visual responses consisted of students who did not use graphs, or any other types of diagrams, at all when answering the questions that can be solved by either the algebraic or graphical methods. Although the number of not using graph outnumbered those who did, the number of success and failure to arrive to the solutions were almost equal portraying their understanding on the concepts of functions and reasoning in derivative concepts. Therefore, it can be concluded that there is no such best method of presentation to all students in general.

Students' responses that were categorised as partially-visual exhibited no patterns or trend of working solutions. They were mostly a mixture of algebraic and graphical methods. Some tended to start with one mode and followed by the other mode as a sign of either checking on the correct solutions or not enough confidence on the

earlier method. This finding partially supports that of Alcock and Simpson (2005) and Ducan (2010).

Specifically, these students exhibited hesitance to either refer to graphs or to proceed with the algebraic process as their solution method. Some of them started with sketching some graphs but added the algebraic calculations either to confirm solution or a sign of unsure with the graphical method. One possible explanation for these could rise from teachers that did not emphasize on the use of graphs as tools for solving problems in the classroom. Obviously, students would usually employ the method showed to them in the class. This agrees with the study conducted by Likwambe and Christiansen (2008) where the level of concept images of the derivative of in-service teachers were not in depth and their calculus concepts competencies were mostly at the instrumental level which results for their preference to opt for algebraic method (Booth & Koedinger, 2012).

5.3.3 Graphs as Communication Tools

The present study is within the framework of the on-going literature and discussions about the role of graphs in the problem solving process involving functions and derivatives. Graphs are essential tools for solving most mathematical problems. However, the advantages of graphs are strongly related to the students' knowledge of graphs, their properties and characteristics, and the development of their skills to use the graphs. Students need also to be aware that graphs are dynamic representations. The results of the study suggest that the presence of graphs in the tasks assigned did not increase student's ability in solving the application or non-routine problems. This is evidenced by the many students who were still unable to see through the structure of the problems from the graphs even though similar types of graphs may have been used for similar types of tasks in the classroom practice (e.g. Alacaci et al, 2011; Paoletti, 2004)

Students generally did much better on the reading the data visible on the graphs than tasks that needed them to do extra thinking or to make inference. For students to utilize graphs effectively, they need to go beyond than just reading the graphs. They must be able to interpret and analyse, and inter- or extra-polate the data and information that are displayed in the graphs. In some situations, students may need to refer to or sketch more than one graph as some understanding to solve problems need to evolve through the generation of graphs (Ellis & Grinstead, 2008; Leung & Cheng, 2004). Common difficulty that students faced when reading beyond the graphs was that they were not able to provide answers because information were not there on the graphs. Thus harder thinking and more cognitive load are needed as tasks' complexity increases. The results seem compatible with the findings of Uesaka and Manalo (2011), Biza, (2008) and Sharma (2013). Given the importance of comprehending the concepts of functions and derivatives, tasks in graphical forms should not be avoided just because the students found them difficult to answer or difficult for the teachers to teach and assess.

At the end of each task, students were requested to write or elaborate on the steps taken to solve the problems and to argue on whether the solutions obtained were valid. This is to reflect on their reasoning skills and thinking processes while using graphs. The majority of the students were part or half-way in supplying a complete explanation of their processes. They were far from detail or being precise. The finding seems consistent with finding by Habre and Abboud (2006). Besides teaching the students conceptual knowledge of the subjects, teachers need to help and guide their students to express what they know and understand about functions and derivatives with more calculus terminologies and language. Analysing and comparing the students interpretations of tasks would help to identify the patterns of difficulties which then lead to identifying factors that contribute to good explanations, attend to missing details and

help to develop students' written and communication skills (Ferguson, 2012; Gal, 2004).

Graphic calculators or other technological software emphasize on the graphical representations and the ability to be able to interpret them are essential skills. Since Malaysian students are of limited exposure to the use of technology such as graphic calculator especially at the secondary level, the basic ideas on functions and derivatives were typically introduced in the forms of algebraic expressions and through the definition of first principles respectively. Some students were able to make sense of the approach while others struggled with the symbolic representations. Graphic calculators, in specific, are equipped with the 'zoom' or re-scale function which allows the axes to display functions accordingly. Both the zooming-in and zooming-out processes serve as important activities that lead to the successful of visual reasoning process. 'Zooming in' displays parts of graphs in detail and can help to recall prior knowledge while 'zooming out' exposes the bigger or whole state that is able to stimulate cognitive conflicts which consequently need to be tackled and hence lead to making inferences and conjectures. These processes allow students to concentrate on the critical or required features that determine the properties and relationships of functions and/or their derivatives. As Leung and Cheng's (2004) findings suggested, two critical features in some graphing software that are the catalysts in the visual reasoning processes are : 1) the ability to permit students to view the graph of practically all functions where they do not have to sketch but direct the thinking to 'why' different graphs looks differently and 2) the ability to re-scale the viewing screen allows for graphs to be in different modalities which allows students to observe invariant properties of general graphs.

5.3.4 Ability to employ graphs as visual information

When the students were asked to sketch the derivative of the given graphical function, they seemed to struggle in extending their previous knowledge to the new situations. Some of those who were well equipped with conceptual knowledge, as opposed to the procedural setting, showed competence when completing the tasks with, maybe, little confusion. This situation was observed in studies by Firouzian, (2010), Goerd, (2007), Hacıomeroglu, Aspinwall, and Presmeg (2010) and Hattikudor, Prather, Asquith, Alibali, and Knuth, (2012). Some of the students describe the process or used particular rules to sketch the derivative function but had no idea on how the rules work or their implication.

One of the most common patterns found in the incorrect sketches of the derivative functions was that the derivative curves resembles the original curve for the portion that the values of x get bigger and bigger. This finding correlates with those findings by Kultur et al (2011), Stahley (2011) and Torres and Alarcon (2011) and Yetim (2004). A possible explanation for the students to get most of the answers correct regardless whether they sketched the graph or not could be that they had finished the syllabus and had done a lot of revision and were fully prepared for the trial examination.

The way on how students understand and comprehend graphs is very much related to how they were able to construct and explain them. Researchers and mathematicians recommended presenting data to students using free-response method or open-ended graphical method. Set of axes without labels or scales can be provided for the students to work with and followed by them explaining their works. These will indirectly allow the students to make connections between the data and their visual representations. Another possible way is to provide students with multiple graphs for them to run analysis or comparison, and make decision on which graphs to best represent particular situation or relationship. Rather than taking graphs as mere pictures

of situations or events, this would be able to help students reinforce the role of graphs as visual representations of relationships. Akin strategy would be to guide students to work back and forth between graphs and text accompanying the graphs to enhance the ability to interpret and solving problems.

In item 3(i), approximately 40% of the students managed to obtain the correct answer without drawing any graph. They calculated the condition for inflexion point through setting $f''(x)=0$ instead of sketching the graph of $y = f''(x)$ or $y = f'(x)$. This result aligns with that by Ubuz (2007) where her good and average students were lack of understanding in inflection points due to their unable to visualize the inflection points graphically. These are due to, again, getting used to the procedural method of solving most of the mathematical problems which failed them to see the connection of the fundamental concepts. Their conceptual knowledge includes very strong algebraic skills but very weak graphing skills. Graphically and conceptually, the students should be able to read off the coordinates of any inflection points from both the graphs of the first derivatives or second derivatives.

Most graphs of derivatives sketched by the students in this study suggested that they were somehow unsure if the functions were to be continuous at the vertical asymptotes and how will the derivative functions behave as the independent variable x increases. Typically they dragged the graphs of the derivatives functions to be nearly horizontal elsewhere instead of approaching zero or the x -axis. Some had the graphs to continue to negative infinity for larger values of x . Most of the students were aware of their mistakes but did not manage to see the otherwise due to their cognitive conflicts.

One important characteristic of graphs as visual communication tools is graph literacy. Graph literacy can be regarded as one of the essential elements needed to develop students' competency to understand mathematical ability. In order to use graphs correctly and efficiently, students need to have the capability to encode the word

problems or algebraic expressions into graphical representations and be able to decode a graph to the given word problems or algebraic functions (Hipkins, 2011; Isenberg, Tang & Carpendale, 2008; Lowrie et al., 2011). However, not all students are equipped with these talents naturally. They need to be developed instead. No one specific graph has the same impact on every student and no one specific graph is compatible to every students' ability to visualize and reason. Therefore, it is very crucial that students are exposed to and engaged in, besides the Cartesian graphs and other types of graphs, various types of other visual representations in solving mathematical problems. Development of students' graphical literacy may be attained through a well-designed teaching materials or instructional activities and employ graphs into effective tools for thinking and reasoning.

5.3.5 Misconceptions and difficulty in sketching and employing graphs

Based on the framework outlined in Chapter 4, the encoding and decoding processes can be described in seven and five categories respectively. The descriptive analysis of the students' responses for their encoding processes in the MVT showed that less than 23% of the students were able to successfully presented correct graphs and arrived to the correct solutions. On the other hand, up to 88% of the students showed their capability to read out correct information from the given graphs and provided valid reasons for their worked solutions.

In order to explain the above findings, it is best to describe what the students had acquired through their learning experiences about functions and derivatives. The notion of the functions and derivative appears in stages during both the secondary schooldays and their pre-university levels. This reflects how the teachers had presented the concepts and their applications in the classroom contexts. At first, in coordinate geometry, students learn to calculate the slope from any two points and the slope of a line being

constant throughout the domain. An intuitively obvious idea of this slope is that the line is either increasing or decreasing all the time. Later, in the topic Introductory to Calculus or Calculus, the students are introduced to the concepts of limit and tangent through the use of slope of chord. The ideas of derivative then followed and were proposed by the CDC (2006) to be illustrated by the use of graphs. The concepts of maximum and minimum, rate of change and second derivatives, together with their applications to the real life situations were proposed to be explained with the use of graphic calculators or technological software with the aim that students are able to explore and understand the concepts better.

Analysing the students' answers and work solutions, although they had been exposed to graphs in the classroom context, they were not extremely successful at tackling questions that relate to searching and interpreting information that are not shown in the graphs or that involved application to real life situations and required higher order thinking skills. The students were inclined to read and interpret the graphs in a way that reflect inconsistency to the clear understanding of the concepts of functions and derivative. This suggests that the knowledge on the structure of graphs could contribute to the making sense of graphs.

Although there were less than 11% of errors formed during the direct reading of data from the graphs, there were major problems with the understanding of the terminologies and interpreting the tasks needed to carry out, and the prediction of the contexts in questions. On the other hand, the consistent finding in many studies (e.g. Bautista et al, 2015; DeToffoli & Giardino, 2014; Nelsen, 2006; Proux, 2015; Stumpf & Eliot, 1999; Tappenden, 2005) anticipated two main cause roots: the lack of the use of English language in their daily conversations and in the classroom contexts, and the situation where the data or information that was not visible on the graphs.

A common difficulty that students experienced was the reading beyond the graphs. As mentioned in the errors that they had performed in Chapter 4, again, the missing of data or information was the reason for their not able to predict the subsequent situations, in general. Based on the analysis on the incorrect solutions obtained by the students, as high as 50% of them faced difficulties in describing the relationships between functions and derivatives graphically. The findings are consistent with those findings by Aksoy (2007), Bingolballi (2008), Durmus (2004), Li (2006), Muzangwa and Chifamba (2012), and Shaughnessy and Zawojewski (1999). They reported that students performed much better when dealing with literal reading of data or values shown on graphs as compared to tasks that needed them to infer on situations. In preparing the assessment tasks, Sharma (2013) recommended that specific hints are not to be provided for students to search or interpret the data within the graphs. They should be worded and designed such that students are encouraged to offer reasoning or opinions rather than getting specific solutions or numbers. Due to the tasks designed in open-ended mode, appropriate rubric should also be prepared to guide when assessing the students point of views and reasoning. Sharma had also advised that the highest score of the rubric to contain several criteria such as: response specifically to the data in question, using the correct terminologies and offering sensible assumptions in the light of the data displayed in the graphs.

Up to 42% of the students performed misinterpretation of the data read from the graphs that was caused by them trying to look for familiarity or patterns on the graphs or they were not able to see those patterns. Students justified these patterns in terms of explanations. These appeared even in the instance where attempts to look for patterns did not make any conceptual rational. The students tended to be certain that a pattern must exist and failed to offer any forecasting or extrapolating due to their unable to explore for the pattern. Literature shows that students, even the good ones, struggle

with constructing graphs of functions and their derivatives such as from sign diagrams. They tend to perform better when being provided with the algebraic expressions.

Students' knowledge and beliefs affect the way they encode and remember the graphs they had seen before either in the classroom by the teachers or those in the textbooks. Students may also have some fixed expectations about the information that they can read and depict in graphs. These could lead to interpretation errors. They typically expect the independent variable to be plotted along the horizontal axis and the dependent variable to be plotted along the vertical axis and consequently, the steeper any line is from another line would be taken for granted to represent a faster change of rate. If a graph is to violate from this rule, such that the independent and the dependent variables are to interchange axes, students would have problem to assume that steeper lines would indicate faster rate of change. On another matter, students may make error when interpreting abstract representations of the data in the graphs as symbols to represent real life icon, event or situation. Students may interpret the graph of velocity against time of a car to imply the position of the car along a road. Therefore it is important for the students to be well equipped with knowledge and understanding on the mapping between the gradient and rate of change and hence derivative, in order to avoid the students making interpretation errors.

Students with lack of knowledge on graphs may not have the familiarity on how to map between the visual features of the graphs and their meanings. Their prior knowledge may also play bigger roles in influencing their understanding of graphs, and their properties and characteristics. Using texts to highlight and describe the important points and concepts could help students in the accurate reading and interpretation of the graphs. Segregating categorical information through the use of different colours, for example in comparing the graphs of the functions and their first and second derivatives,

would also help to reduce students' cognitive load in reading, interpreting and understanding the information embedded in graphs.

Teachers and authors of textbooks may unintentionally emphasize the students' preference for their perceptive and intuitive ideas and disregard the conceptual definitions. Students would opt for the methods that are practical and allow them to complete their mathematical tasks and subsequently be able for them to score in the tests and examinations. Students tend to regard the conceptual definitions as irrelevant if they are able to solve problems using repetitive procedures or formats trained daily in the classrooms. Students would search and employ methods that require the minimum effort or cognitive load. Since they were able to solve problems just by memorizing the methods, they may also under-value the informal conceptual definitions as well.

Dissimilarity between algebraic expression and graphs as visual representations is an example of a condition when a student may embrace two equal ideas without noticing the conflict. When students are working using their intuitive mind, without the present of formal or conceptual definitions, they may attend to the same problem presented to them in different forms differently and in a contradictory method. For example, slope of a line between two points, tangent and a line touching a curve at one point as in item 2(a). Apparently, students tend to retain several visualized ideas that they will select to use accordingly based on the nature of the problems.

Some of the students did express that they have problems understanding or comparing some terminologies. They tended to regard the word 'troublesome' and 'difficult' to bring the same meaning. My teaching and classroom experience had also identified that students faced difficulties when responding to instructions between for example 'explain' and 'discuss'. The wordings used in the questions play a crucial role for the students to work with graphs, either when they need to construct them or when they had to read and interpret them. Students may have different connotations for the

words or expressions that teachers consider to be synonymous. Teachers and authors of textbooks need to understand and penetrate into students' intuitive ideas and how they expressed these ideas in order to help design instructional materials that are able to help students sketch and produce inter-related graphs. Although, teachers tend to think that students' informal ideas on conceptual definitions as one of the obstacle for the students to grasp the conceptual understanding, there is some truth in it. No complex or complicated concepts were understood and acquired on the spot. Taking positively, these obstacles are also the building blocks towards the developing of more complex definitions of conceptual ideas. The formal concepts of functions and derivatives graphically must be developed through some processes of seeing, generating, interpreting, transforming, maintaining, drawing and connecting more algebraic and properties and characteristics of graphs. The findings of the visual reasoning process appear to concord with those of Kim and Park (2007). Although the study was not in the field of mathematics, the studies also found that the students underwent the processes outlined in the visual reasoning when using graphs to solve mathematical problems.

An understanding of the concepts of functions and derivatives in one type of representation will not necessarily indicate the understanding of them in the other representations. What important is the ability to be able to encode and decode among the various forms and to effectively read and interpret the problem situations. This finding is in contrast to the finding by Koedinger & Terao (2002) where he discovered that students were more successful in solving algebra problems in in terms of story as compared to those in mathematical equations. Students were found to face difficulties when dealing with quantitative relationships in the form of mathematical symbols. Combining the forms of representations, for example the algebraic, symbolic and the visual forms, the resulted assembled information will contribute to a more

comprehensive and deeper understanding of the essential and underlying functional situations.

Abbey's (2008) study on deriving properties of function from the signs of the derivatives described how the collective of representations of functions formed the foundation of a concept image. Individual student was able to develop various concept images, which are able to exist in both complementary and contradictory ways (Sabella and Redish, 2005). Therefore, the more closely the representations are connected, the more robust and compatible the system of the concept images is. Each representation has their own strength and limitations in different contexts. Having one to complement the other will benefit and facilitate the flexibility of moving and controlling the form of representations in which one needs to work with.

Students' lack of knowledge of the Cartesian graphs raises critical issues in mathematics education. Teaching about how two related variables vary with respect to each other is an essential learning goal and a significant practice in reasoning (Alacaci et al, 2011, Curcio, 1987, Friel et al 2001). In addition, the students' unfamiliarity with Cartesian graphs poses a challenge for their ability to extract and relate important forms of analysis in other mathematical areas and different disciplines such as trigonometry and economics respectively, where Cartesian graph is an indispensable tool for investigations in various fields that embody relations and correlations. Hence, students' limited understanding of Cartesian graphs would severely limit the support they would eventually utilize in any educational activities that require extensive use of inquiry for instructions or explanations. Study by Alacaci et al (2011) on pre-service teacher's understandings of graphs where they were able to recognize various types of graphs and their uses but had limited knowledge on scatterplots and its applications.

Why were the students not well-equipped with the understanding and knowledge of Cartesian graphs? Three possibilities might contribute to the results. Firstly, it may be

that Cartesian graphs require a cognitively more challenging and demanding form of reasoning as compared to the other types of graphs such as bar graphs, Pie charts or histograms. For example, bar graphs require only numerical comparison which is more straightforward form of reasoning than to deduce the types relationships of the variables or the shapes of graphs to interpret the characteristics of the functions or their real life applications. Students who do not have sound reasoning skills might be able to perform well in simpler types of graphs but disappointingly in the others which require the same types of visual reasoning skills.

Secondly, the students might have insufficient instructional exposure to the Cartesian graphs in classroom contexts. This could results from the teaching method that emphasized more on the algebraic manipulation to arrive to the answers or solutions although the secondary education and the contents of the mathematical curriculum and syllabus emphasized on the use of visual and technology to assist students in comprehending graphs. This result is consistent with the analysis done by Alacaci et al (2011) on pre-service teachers who followed the Competency Based Curriculum, a Miami-Dade public school system for secondary level. The system specified that the use and understanding of scatterplots as one of the curriculum goals but the pre-service teachers had either not taught or had not retained and hence were unable to recall the knowledge.

Third, the Cartesian graphs are not used in daily routines, such as in advertisements, magazines or posters as often as how the bar graphs, Pie chart or histograms are utilized. Therefore the students might have little exposure to the applications of Cartesian graphs in the non-classroom contexts.

The students faced three categories of difficulties in generating correct and effective Cartesian graphs to represent the functions and their derivatives: the non-use of graph, the generic difficulties and the idiosyncratic difficulties. Sketching Cartesian

graphs is one of the many methods that students may use to solve most of the mathematical problems. When students failed to continue working algebraically in solving any mathematical problems, they should be encouraged to generate graphs or any other representations such as diagrams. However, from the results of the study, it can be seen that the students did not regard graphs as able to help them in problem solving or have some hesitation and in-confidence in the use of graphs. This can be seen when approximately 16% of the students who had worked with graphs were subsequently accompanied by algebraic process to either complete or to run a check on the solution process. This result is consistent with finding by Diezmann (2000) when her students did not regard diagram as an alternative tool to help solving the problem when they failed with another strategy. Among the main features of graphs is their capacity to make use of scales. Students' refusal to use Cartesian graphs might also stemmed from their competency to read or deal with scales. As can be seen from some the students' worked solutions, quite a number left the axes unlabelled, either the title or the scales on the axes. This indicates that they were not taught of the importance of labelling or the information was not retained in their thinking.

Among the second type of difficulty which was the generic difficulty is when students sketched unusable graphs. These include the graphs that they sketched being too small in order to accommodate all the relevant information, the position of the sketched graphs were such that insufficient surrounding space to extent for extrapolation or forecasting, or the sketched graphs are so untidy to be able to see or locate embedded information in them. This might cause further serious problem where students tended to abandon the work instead of to re-sketch them. Another common error performed by the student under the category of generic difficulty is for the students to represent the quantities of the variables incorrectly. Although this type of

error may be assumed to emerge due to the students' carelessness, they were undetected by the students and eventually led to more complicated process and incorrect solutions.

Cartesian graphs are useful when describing patterns or trends of changes and locating specific points or intervals. Idiosyncratic difficulties were encountered when students were lacked of precision especially in locating the points on the graphs of functions. Students sketching graphs were mostly tend to rush and estimate the positions of intercepts, critical points and even the shapes of graphs. These could lead to them interpreting the steepness of the slopes relatively incorrect. Students always tend to overlook the constraints imposed in the problems. In item 4 of the mathematical visuality, the graphs that approximately 30% of the students drew to represent the rate of change of employee, almost 60% of the students included the negative portion of the horizontal axis which represent time in months and therefore stretched the graphs to the left of the vertical axis. Some of the others, although a small portion of them, extended the number of employees to below the horizontal axis indicating negative number of employees. They completed their works without rationalizing the situations.

Students difficulties and misconceptions in generating Cartesian graphs for functions and their derivatives indicate that, despite their potentials, the strategy to sketch a graph was not an effective spontaneously problem solving tool for many students. While the explanation for all types of difficulties above differ from one student to another, their misconceptions and difficulties were basically relate to the lack of understanding and knowledge on the usability, capability and even limitations of graphs as tools for solving mathematical problems specifically on calculus such as functions and derivatives. Clearly, as encouraged by Maharaj (2013), students must be trained on the use of graphs as problem solving tools. Students should be alerted on which graphs and types of functions are appropriate for different situations, why graphs can be useful in solving problems and how to make use of graphs in solving problems.

Additionally, students need to have the expertise to distinguish graphs from other types of graphical representations and understand their respective purposes and uses. There are massive significant differences between Cartesian graphs and bar graphs, pie charts, and histograms (Riveria, 2011). Surface and group details are generally important in bar graphs, pie charts and histograms while individual features or characteristics and properties of each point are important in Cartesian graphs. Using the term graph synonymously for all different types of graphs will fail the students to distinguish one from the other and their respective functions and purposes and hence lead to misconceptions as the findings by Tishkovskaya and Lancaster (2012) and Watson (2006) when dealing with students employing various graphs in statistics classroom.

Students must also be able to understand the degree of vagueness associated with Cartesian graphs. Cartesian graphs by nature are at times vague representations where embedded information can be seen, read and interpreted in various ways. Nevertheless, what is important is that the organization of the visual information embedded in the graphs portrays the details and structure of the problem. While some general visual representation in the graphs can be useful as basic reminder about the particular points or functions, a focus on representing the details of points or functions can distract students from considering the structure of the problem as the findings by Mc Culloch (2011), and on affects students when using graphic calculator as visual tool to view the properties of functions.

Students must also be able to develop awareness that Cartesian graphs are relatively dynamics representations. Graphs of functions are tangible working space for tracking relationships between interdependent variables of any problem and therefore need to be relatively organized and sufficiently big. As understanding on how the concepts are applied to problems can develop through the sketching of graphs, it is

beneficial to produce some other types of representations to accompany the graphs. Learning through multiple representations is an alternative way how students can acquire the conceptual knowledge. Students are able to work through multiple representations, translate from one representation to another, such as the numerical, algebraic and graphics to access information, and hence, allow their mind to evolve metacognitive thinking. Well-connected knowledge is much easier to be remembered of because there are several routes to access to the solutions. According to Hiebert and Carpenter (1992), “the degree of understanding is determined by the number and the strength of connections. A mathematical idea, procedure or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections” (p.67).

The SACE system, specifically for the mathematical subjects, allows the use of a note sheet to be brought along into the examination time. Students are allowed to prepare two double-sided A4 size note pages which they think that would help them in the examination such as formulae and trigonometric identities. Since the students were very heavy-used of the graphics calculator in the classroom context, which there are many long steps of process for particular operation, the actual purpose of the note sheet was for them to compile the steps needed for the operations on the graphic calculator. Unfortunately, upon checking their note sheets, students jotted down their formulae even the basic formulae such the area and volume of some geometrical figures and objects, sets of steps to solve some of the problems and even specific examples as guides, graphs for particular functions, and samples of analysis or conclusions of solutions. Those who rely very much on the note sheets were less expected to consider alternative methods of solving mathematical problems. For example, they may have the necessary skills for solving problems on functions and derivatives but limited ability to consider alternative methods such as graphical methods due to their heavy reliance on

the note sheets and indirectly from the teachers' notes and textbooks. This can be concluded that the students did have the ideas and concepts of functions and derivatives but were not able to retrieve the knowledge when needed to use them to apply appropriately.

5.4 Limitations of the study

The study was limited to differential calculus at the pre-university (or matriculation) level. Specifically, this study confined itself to the learning of concepts on functions and derivatives, and their applications in the topic of differentiation of the SACE curriculum using graphs as visual tools. Although visual reasoning studies suggested encouraging and positive results in various domain similar conclusion of this study might not generalise to other areas of mathematics teaching and learning.

The study dealt with mostly excellent post-SPM students nationwide who were awarded scholarships to further their studies abroad. It is not clear how the results of the study would generalize to average or weak students. Assessing visual reasoning based on the structured tasks questionnaire might have limited the accessibility of visual reasoning exhibited, as with other technique of assessment on this difficult to measure constructs. Although the validity and reliability of all questionnaires were established, the instruments might not have been sensitive enough to detect marginal changes in thinking, especially those related to broader view of visual reasoning.

This study was also limited by the method of intervention, where a deeper and thorough involvement might minimise possible novelty effects. However, Leung and Cheng (2004) in reviewing the effectiveness of visual reasoning considered the students' written works as ample to determine educational significance in such studies. Nevertheless, the effect of students' written work on identifying types of visual

reasoning exhibited might be enhanced through the observation of physical gestures or interview techniques while solving the mathematical problems.

5.5 Implications for Practice

The findings show that the pre-university students have inadequate level of visual reasoning subsequently inflicts great educational implications especially to the mathematics education in Malaysia. Students will need to face greater challenges later in their higher or tertiary level of educations. As a results, many will fear and not being motivated to learn mathematics especially on derivatives or calculus. They will not be able to appreciate the power of calculus and mathematics since their limited level of visual reasoning barred them from exploring many other areas of mathematics and other disciplines. Therefore, limitation notwithstanding, the findings of this study may serve as a guide to the teaching and learning of derivatives using graphs in the classrooms as well as to the development of curriculum at the ministry level and assessing techniques to employ for the educational context, both in the light of how students learn and the learning tools which include technological tools.

5.5.1 Mathematics Teaching and Learning

One of the important implications of the study is on teachers' instructional methods. Students with inadequate level of visual reasoning may be hindered in their effort to develop meaningful understanding on the concepts of functions and derivatives. Therefore, teachers should ensure that students are well-equipped with ability to reason visually for them to grasp the concepts taught in the class. Teachers should pose the students with visual problem setting or reasoning tasks and get them to communicate their thinking while solving the problems. By doing this, students are not only trained to explain their thinking to support their justifications but be able to reflect upon their

reasoning and understanding and communicate their mathematical ideas to others (Noraini, 2006).

In some countries including Malaysia, despite the official inclusion in the national curriculum, the teaching of calculus and mathematical concepts using graphs is progressing very slowly. Among the factors associating to this issue is the role of teachers. The students' ability to sketch and interpret graphs are not spontaneous or primer actions. Therefore, teachers are to play crucial roles in constructing the teaching contexts for students to *see* that graphing is a meaningful and purposeful process of learning. They should be able to guide students through properly directing their inquiries, focusing their attention, encouraging specific initiatives and discouraging some others, provoking meaningful negotiation, maintaining suitable articulation of conceptual matters and activities (Uesaka, & Manalo, 2007; Van de Walle, 2007; Yavuz, 2010). Consequently, teachers could possibly set pedagogical context to situations in which relevant graphical aspects of the mathematical context such as the functions and derivatives are discussed and interrogated, such as queries and issues related to critical analysis of data or the need to generate new and useful information (Wall & Benson, 2009; White & Mesa, 2014).

Mason (1992) discovered that although students may understand and be able to reconstruct the visuals presented by teachers to them, their conception of the visuals may not necessarily match to that of the teachers. Similarly, Bautista et al, (2015) ascertain that those intuitive visual ideas and concepts by experienced and skilled educators and mathematicians may not necessarily be perceived by inexperienced students. In general, students who opted to use visuals in solving mathematical problems confronted four particular difficulties; 1) reading and interpreting the visual inappropriately, 2) inflexible or rigid thinking and reasoning when handing unfamiliar, non-standard or new visuals, 3) set unrelated or uncontrollable visuals, and 4) producing

vague or imprecise visuals (Maharaj, 2013; Orhun, 2012). From the study, it is clear that effective problem solving, which make use visual to reason, depends on the relationships between the graphs and the contexts (the functions and derivatives) and the students' abilities and conceptual knowledge. Therefore, it is important for the teachers to select graphs based on the concepts and functions in the mathematical problem solving and to also take into attention and consideration on the students' preference method when solving the problems. They need to encourage and train the students to develop effective strategies that they are still not competent with implementing. The informational content in the graphs and of the graphs for instructional purposes should be explained so that students are able to retrieve the embedded information. Through the application and employing of visual and visual reasoning theories and frameworks, teachers and educators, even those with massive experiences, can become better equipped to identify, understand, analyse, foresee and manage students with problematic visual misconceptions and difficulties or those with lack of logical didactic solutions (Gal, 2005) that tend to resort to their intuitive actions.

At the early stage of graphs learning, students often do not have the adequate graph knowledge to relate between the functions and graphical properties or meaning. Their thinking and reasoning are also influenced solely by their prior knowledge. Accompanying text to help describing the main point and features or characteristics of the graphs will help students in the reading and interpretation of the data embedded the graphs. Keeping track of what the displayed information is referring to is a cognitively demanding activity. Adding symbols, colours, and labels rather than using legends, will help to reduce the cognitive load.

Learning opportunities and prospects should be broad to boost students' thinking and reasoning and facilitate cognitive transfer. These can be done through the inclusion of graphical languages (Lowrie & Diezmann, 2009) that are also used outside the formal

classroom and mathematical contexts, such as maps, in addition to those that are typically exercised and incorporated into the mathematical curriculum. Similar to what is happening in the mathematics classroom, where the understanding of relationship between functions and graphs are the main emphasis, it may be helpful to have students to explicitly focus on the relationship between the visual representations and their meaning in another contexts.

Students' misconceptions and difficulties in constructing accurate and effective graphs, and reading and interpreting graphs are generally linked to students' insufficient expertise in graph representations. However, the results in this study suggest that effective graph representations of functions and derivatives also depends on the comprehensive mathematical, specifically the calculus, knowledge base which include the sense-making in relation to the real life situations. The graphs that students drew are able to provide the teachers an insight of their weaknesses and strength in the relationship between their mathematical knowledge and their graphs representations. Although graphs are known to support the conceptualization of problems and real life situations, they cannot be used to substitute the lack of any mathematical knowledge. Thus graphs should be regarded as both the reflection on the students' mathematical conceptual knowledge and the representations that stimulate thinking and reasoning on the problem structure. Knowing the students' errors in constructing and reading and interpreting graphs is an important component in guiding instructions for students to draw, and read and interpret graphs.

Therefore, teachers, at all levels of teaching and learning processes, should employ and inspire their students to explore and investigate, and generate their own visual forms since the visual understanding of concepts, objects or processes are more effective and robust and are more inclined to retain in the mind and hence to recall in the longer term as compared to a purely algebraic or non-visual forms (Bell, Wilson,

Higgins, McCoach, 2010; Cunningham, 1994; Riveria, 2011). The findings of this study indicate that students' visual reasoning is at very critical level. Therefore, students should be presented and prepared with various types of graphical exercises in their textbooks, such as those that entails authentic components and related to the real life situations. By doing the exercises, students are able to 'see' that mathematics are part of the daily life and be made aware of the usefulness of the graphs to the real life situations. Furthermore, teaching and learning functions and derivatives and mathematics will be more fun and attractive.

5.5.2 Assessing Techniques

Assessment is an essential component of mathematics education and part of the on-going teaching and learning process. One of the possible reasons many students were not able to show adequate level of visual reasoning is because many teachers focus their assessment tasks on the skills to carry out pen and pencil algorithmic. Uesaka and Manalo (2011) and McMillan (2004) proposed for the assessments tasks to take into consideration the students' cognitive progress and motivation in the learning of functions and derivatives instead of on what they know and can do. Therefore teachers should focus their classroom assessment tasks on assessing students' conceptual understanding and reasoning skills through the use of graphs especially in connecting the calculus ideas through solving problems and applications to real life situations. In fact, teachers should provide students with performance feedback on the use of graphs, from time to time, as this will increase their awareness of their level of visual reasoning and the benefits and effectiveness of using graphs.

Some concerns were expressed by teachers and educators over the students' responses in solving reasoning questions. One of the concerns was the lack of mathematical terminologies and vocabulary for them to thoughtfully reason the solution

methods. In some cases, they provided incomplete solutions or steps while in some others they misused the terminologies wrongly. Thus, some of the goals of mathematical instructions should be designed towards conquering the mathematical terminologies and guiding students to effectively communicate the abstract mathematical concepts and ideas in visual form. Improving visualization and visual reasoning skills in mathematical topics, particularly functions, derivatives and calculus, is essential to assist students to gain better understanding of mathematical concepts. Students should be encouraged to construct the graphs rather than supporting them with lists of algorithms and procedures. The use of graphs in both, the instructions and assessments helps students to explore concepts and ideas and consequently improve their visual reasoning skills. These will help them to understanding and make meaning of mathematics learning. Results of this study indicate that a positive effect of graphs on visual reasoning skills seem most improved when graphs are used in instructions as well as assessment. Thus, teachers may include graphs in topics related to the applications of functions and derivatives especially for students with relatively low visual ability and reasoning skills.

The significant results, from the two sets of instruments distributed to the students namely the MVT and the GRT, in terms of the students' ability to encode and decode data in graphs based on the different settings in which the tasks were presented, proposed that the tasks designed are an important issue to making sure that the students are able to reason in the targeted ways. They should also be aimed so that the students are able to use various forms of arguments when engaged in solving mathematical problems to meet the primary goals of mathematics curriculum and educations. The Malaysian Curriculum Development and Ministry of Education promote and support the introduction of various forms of reasoning including visual reasoning to students starting at the primary levels.

5.5.3 Curriculum Development

Three main elements to be considered in any curriculum development in assessing educational outcomes and performance are the content of the subject, the target recipients and the methods of delivery (Bakker & Gravemeijer, 2004; Stylianides, & Stylianides, 2007; Terwel, van Oers, van Dijk, & van den Eeden, 2009). The questions and tasks set for the examination for the purpose of evaluating students play a very crucial role in conveying the important aspects of teaching and learning for the teachers and students respectively. The questions posed will tell the teachers on what to emphasize in their teaching. If students are exposed to the types that assess visual reasoning, they will begin to realize and begin to develop their visual thinking and reasoning instead of to just master the skills in implementing the algorithmic and procedural knowledge.

The introduction of the reading and interpretation of all types of graphs as a topics in the mathematics curriculum is important and should be implemented from the early years of schooling. Educating students to read and interpret graphs effectively can facilitate them to adapt to the demand of the new approach of communicating information where various graphs (and graphics) are used to represents respective contexts of the daily situations such as in business and engineering. Curriculum developers around the globe emphasize that the teaching of graphs and graphing at the primary levels to start purport the development of knowledge that qualify students to be critical citizens when interpreting graphs of daily situations (Ainley & Monteiro, 2008). They are also advised to go beyond the simple reading and interpreting of graphs or graphical representations such that to be able to interpolate and extrapolate the patterns and use their mathematical knowledge and experience to describe and relate to the real life situations. On the other hand, care has to be taken where the reality of the Malaysian classrooms' settings which are still adopting the conventional teaching contexts and

emphasizing very few sub-skills such as drawing axes, scaling, labelling and plotting points (Stavridou & Kakana, 2008).

Another issue to come to terms with assessment and graphs is related to the use of technology in the classroom teaching and curriculum (Kissane & Kemp, 2011) It is critical to implement consistent learning and assessment environment. Graphics calculators which are rich of graphs, benefit some advantages over computers such as its accessibility. At present, the Malaysian education regarded the use of graphic calculator as merely an optional extra. The implementation of graphics calculators and subsequently the teaching of mathematical concepts through graphs and graphing will steer the curriculum developer to see mathematics syllabus and curriculum through new and fresh lenses. One potential effect is that graphic calculators are able to seize over some longer mathematical procedures. A good example is the identifying the stationary points of any functions. Many students learn this as long and complicated steps especially differentiating rational, surge or logistics functions to determine the specific nature of the stationary points. Students spend a long time and lot of practice to be fluent on such procedures. The use of graphic calculators is able to handle and compensate the routine long procedure and hence allow educators to better recognise aspects of mathematics that are worth attention based on time available in the limited time to complete the syllabus.

Students were very much influenced by the context in their reading and interpreting the information in the graphs. When reading and interpreting graphs in an abstract form, students faced difficulties in applying their read and interpreted information to the real life context, which due to their perception and expectation of the contexts. Therefore by preparing graphs related instructional materials in the context of mathematics and real life situations, students are able to realize that graphs are not just communication and delivery tools but serve as tools to help them think critically.

Textbooks and classroom activities that allow students to be able to translate from one form of representation to another such as texts, tables and symbols, may improve their ability to map or link the visual forms to the quantitative forms and consequently enhance their graph reading and interpretation skills.

Stahley (2011) and Sabella and Redish (2005) proposed that modelling and contextualized exploration of relationships among mathematical concepts to be included as part of the teaching curriculum. The valuable ideas of understanding teaching functions and derivatives are the appreciation for the concepts and the inclusion of their applications into other areas within the mathematics itself and other disciplines outside of the mathematical fields. Teachers must be well equipped with these types of knowledge and understanding not only as preparation to answer queries from students but to stimulate, through the instructional materials, the potentials and powers of functions and derivatives. An important feature in comprehending the mathematical concepts especially those of functions and derivatives is through students' observations and hence recognition of the changes that take places in the surrounding world and also the identification of their relationships to make sense of the concepts.

An area of reform that has been long discussed in education system globally is the use of technology in the mathematical learning specifically the calculus courses. Many topics in mathematics have the characteristics that suggest that technology aided learning environment is among the effective tools to support understanding. The visual aspects of the functions and derivatives, and most of the other mathematics ideas, can be represented and viewed on the graphic calculators and computer screens; the transformational aspects for active implementation, the technical computational aspects and the connected relationships of different representation of the same concepts.

5.6 Recommendations and future directions

To generalize the potential of visual reasoning ability, among pre-university students in learning functions and derivatives, to more calculus and mathematical topics, and different learners necessitates future research. Specifically, research studies that cover more types of visual representatives in various mathematical topics and students of different achievement levels are recommended. This study into assessing and identifying students' reasoning in regards to graphs has initiated possibilities and potentials for future research at a macro-level on students' thinking and to develop more explicit descriptors for each type of visual reasoning. Detailed understanding on students' reasoning and thinking can be obtained with tasks that allow for evidence on students' use of graphs as tools to solve functions and derivatives related problems. Such research would be able to validate the categories of responses outlined above and raise more awareness of the level of mathematical visuality and visual reasoning that need to be considered when planning curriculum and instruction to further develop students' graph comprehension.

This study included only the excellent pre-university students, based on their national examination result. However, visual reasoning is an important tool of solving mathematical problems and the conceptual knowledge on functions, derivatives and calculus are essential at university levels of most of the courses such as engineering and economics. A further study could involve the university students of various courses on their knowledge of various types of graphs used the effects on their understanding of the concepts.

A future study could include larger sample from various programmes (those bound to further their studies in other parts of the world), a mixture of weak, average and good students and to include teachers, either those teaching colleges or those at pre-service institutions.

Mathematical anxiety is a critical aspects in the teaching and learning of mathematics at all levels. The aspects of mathematical anxiety were not included in the study, but may contribute to the students' preference and misconceptions in using graphs as visual tools. A further study may investigate the relationships between mathematical anxiety and the success of mathematics or the use of graphs. .

It is hoped that the findings in this study will generate more interest with respect to data representation and visual reasoning ability that students possess and factors that may impact their learning.

Most cognitive study on specific mathematical areas such as trigonometry and statistics focused on the understanding of graphs rather than on the construction or drawing graphs. Besides the questions on how students from different levels of graphing abilities sketch their graphs and the errors they performed when doing so, it is important to identify if they realize the potential of dealing with misleading graphs and what do they think on the uses and benefits of graphs, both in the classroom's context and real life's context. When students are doing their reading and interpreting of graphs, are they describing the information read and interpreted from the graphs or are they prone to provide explanations.

REFERENCES

- Aaron, W. A., & Herbst, P.G. (2015). Teachers' perceptions of students' mathematical work while making conjectures: an examination of teacher discussions of an animated geometry classroom scenario. *International Journal of STEM Education*, 2(10). doi: 10.1186/s40594-015-0021-0
- Abbey, K. D. (2008). Students' understanding of deriving properties of a function's graph from the sign chart of the first derivative. Unpublished Masters dissertation, the Graduate School, the University of Maine.
- Abdullah, S. (2010). Comprehending the concept of functions. *Procedia Social and Behavioral Sciences*, 8: 281-287.
- Adu-Gyamfi, K., Bosse, M., & Stiff, L. (2012). Lost in translation: Examining translation errors associated with mathematical representations. *School Science & Mathematics*, 112(3): 159-170.
- Ahmad Tarmizi, R., Mohd Ayub, A. F., & Abu Bakar, K. (2010). Effects of Technology Enhanced Teaching on Performance and Cognitive Load in Calculus. *International Journal of Education and Information Technologies*, 4(2): 109-119.
- Ainley, J., & Monteiro, C. (2008) Comparing curricular approaches for statistics in primary school in England and Brasil: A focus on graphing. In Batanero, C., Burrill, G., Reading, C., & Rossman, A. (2008). *Proceedings of the Joint ICMI /IASE Study Teaching Statistics in School Mathematics*. Challenges for Teaching and Teacher Education. Monterrey, México: ICMI &IASE. CD ROM. Available at http://www.ugr.es/~icmi/iase_study/
- Ainsworth, M. (2006). Programs and algorithms of numerical mathematics 13 in honor of Ivo Babuska's 80th birthday: List of participants. In: Jan Chleboun and Karel Segeth and Tomáš Vejchodský (eds.): *Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar*. Prague, May 28-31, 2006. Institute of Mathematics. AS CR, Prague, (pp. 253–257).
- Ainsworth, S. E., & Loizou, A.T. (2003). The effects of self-explaining when learning with text or diagrams. *Cognitive Science*, 27(4): 669-681. doi: 10.1207/s15516709cog2704_5

- Akkoç, H. & Tall, D. (2005). A mismatch between curriculum design and student learning: The case of the function concept. In Hewitt, D., & Noyes, A. (eds). *Proceedings of the sixth British Congress of Mathematics Education held at the University of Warwick*, 1-8. Available at <http://www.bsrlm.org.uk/IPs/ip25-1/>. Accessed 24 September 2012.
- Aksoy, Y. (2007). Impact of computer algebra systems in teaching the concept of derivative. Unpublished doctoral dissertation. *Gazi University University Institute Education Sciences, Ankara*.
- Alacaci, C., Lewis, S., O'Brien, G., & Jiang, Z. (2011). Pre-Service Elementary Teachers' Understanding of Graphs. *Eurasia Journal of Mathematics, Science and Technology Education*, 7(1): 3-14.
- Alcock, L. (2009). Teaching proof to undergraduates: Semantic and syntactic approaches. In Lin, F., Hanna, F.H.G., & de Villiers, M. (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving*. Taipei: National Taiwan Normal University, 1: 227-246.
- Alcock, L., & Simpson, A. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, 57: 1-32.
- Alcock, L., and Simpson, A. (2005). Convergence of sequences and series 2: Interactions between nonvisual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, 58: 77-100.
- Alcock, L., & Simpson, A. (2009). Ideas from mathematics Education: An introduction for mathematicians. MSOR Network. [[Alcock Simpson book.pdf](#)].
- Alkhairusi, H., Kazem, A. M., & Al-Musawai, A. (2011). Knowledge, skills, and attitude of pre-service and in-service teachers in educational measurement. *Asia-Pacific Journal of Techer Education*, 39(2): 113-123.
- Amer, T. S., & Ravindran, S. (2010). The effect of visual illusions on the graphical display of information. *Journal of Information Systems* 24:1, 23-42. Online publication date: 30-Sep-2011.
- Amit, M., & Vinner, S. (1990). Some misconceptions in calculus – anecdotes or the tip of an iceberg? *Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education*, 3-10.

- Anderson-Pence, K. L., Moyer-Packenham, P. S., Westenskow, A., Shumway, J., & Jordan, K. (2014). Relationships between visual static models and students' written solutions to fraction tasks. *International Journal for Mathematics Teaching and Learning*, 15, 1-18.
<http://www.cimt.plymouth.ac.uk/journal/default.htm>
- Aoyama, K. (2007). Investigating a hierarchy of students' interpretations of graphs. *International Electronic Journal of Mathematics Education*, 2(3).
www.iejme.com
- Aparicio, E., Cantoral, R. & Rodríguez-Vásquez, F. (2003). Visualización y tecnología: un enfoque a las aproximaciones sucesivas. In J. R. Delgado (Ed.), *Acta Latinoamericana de Matemática Educativa*, 16(1): 445-457.
- Arcavi, A. (2003). The role of visual representations in the learning of mathematics, *Educational Studies in Mathematics*, 52(3): 215-241.
- Arnheim, R. (1969). *Visual thinking*. Berkeley: University of California Press.
- Arzarello, F., & Robutti, O. (2010). Multimodality in multi-representational environments. *ZDM*, 42: 715-731.
- Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D. & Thomas, K. (2004). A framework for research and curriculum development in undergraduate mathematics education. Wisconsin: University of Wisconsin-Platteville. Barnes, M. (1995). An intuitive approach to calculus. Retrieved from http://www.hsc.csu.edu.au/math/teacher_resources/2384/proofreading/journals/barnes/m_Barnes_Nov_95_.html
- Asiala, M., Brown, A., de Vries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A.Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education, Vol. II* (pp. 1–32). Washington, DC: Conference Board of the Mathematical Sciences.
- Aspinwall, L., & Shaw, K. (2002). Report on calculus: Two contrasting cases. *Mathematics Teacher*, 95(6): 434-439.
- Aspinwall, L., Shaw, K., Edwards, B., & Graham, K. (2002). When visualizing is a barrier to mathematical understanding. *Mathematics Teacher*, 95(9): 714-717.

- Aspinwall, L., Shaw, K., & Presmeg, N. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics*, 33(3): 301-317.
- Baker, G. (2012, March). An introduction to differential equations for scientists and engineers. Unpublished Manuscript. Columbus, OH.
- Bakker, A., & Gravemeijer, K.P.E. (2004). *Learning to reason about distribution*. In Ben-Zvi, D & Garfield, J. (Eds.). *The challenge of developing statistical literacy, reasoning, and thinking* (pp. 147-168). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Balacheff, N., & Gaudin, N. (2010). Modeling students' conceptions: The case of function. *Research in Collegiate Mathematics Education*, 16: 183-211.
- Ball, B., & Ball, D. (2007). Visualising Geometry in *Mathematics Teaching Incorporating Micromath*, 205: 4-5.
- Bardelle, C. (2010). Visual Proofs: An Experiment. *Proceeding of CERME 6, 2009*, Lyon, France.
- Batanero, C., Arteaga, P. & Ruiz, B. (2009). Statistical graphs produced by prospective teachers in comparing two distributions. *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*. Lyon: ERME.
- Battista, M. T. (1990). Spatial visualization and gender differences in high school geometry. *Journal of Research in Mathematics Education*, 21: 47-60.
- Battista, M. T. (2012). *Cognition Based Assessment and Teaching of Geometric Shapes: Building on Students' Reasoning*. Heinemann.
- Bayazit, I., & Aksov, Y. (2010). Connecting representations and mathematical ideas with GeoGebra. *GeoGebra : The New Language for the Third Millennium*, 1(1): 93-106.
- Belenky, D., & Schalk, L. (2014). The effects of idealized and grounded materials on learning, transfer, and interest: An organizing framework for categorizing external knowledge representations. *Educational Psychology Review*, (pp. 27-50).

- Bell, C. A., Wilson, S., Higgins, T., & McCoach, B. (2010). Measuring the effects of professional development on teacher knowledge: The case of Developing Mathematical Ideas. *Journal for Research in Mathematics Education*, 41(5): 479-498.
- Belzile, J. A., & Oberg, G. (2012). Where to begin? Grappling with how to use participant interaction in focus group design. *Qualitative Research*, 12: 459-472.
- Bergqvist, T., & Lithner, J. (2012). Mathematical reasoning in teachers' presentations. *The Journal of Mathematical Behavior*, 31(2): 252– 269.
<http://dx.doi.org/10.1016/j.jmathb.2011.12.002>Bjork
- Berry, J., & Nyman, M. (2003). Promoting students' graphical understanding of calculus. *Journal of Mathematical Behavior*, 22: 481-497.
- Bingölbali, E. (2008). Suggestions for learning difficulties and conceptual understanding of the concept of derivative. (Eds Ozmantar, F.M., Bingölbali, E. & Akkoc, H.). *Mathematical Misconceptions And Solutions*, (pp. 223-252).
- Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, 68: 19-35.
- Bishop, A.J. (1983), Space and Geometry. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematical Concepts and Processes*. Orlando, Flo: Academic Press. (pp. 173-203).
- Biza, I. (2008). Models of students' conceptions about the tangent line. Paper presented at the *11th International Conference on Mathematics Education*, Monterrey, Mexico.
- Biza, I. & Zachariades, T. (2010). First year mathematics undergraduates' settled images of tangent line. *Journal of Mathematical Behavior*, 29(4): 218-229.
- Blackwell, A., & Engelhardt, Y. (2002). A meta-taxonomy for diagram research. In M. Anderson, B. Meyer, & P. Olivier (Eds.), *Diagrammatic representation and reasoning*, (pp.47–64). London: Springer-Verlag.
- Boesen, J., Lithner, J., & Palm, T. (2010). The relation between types of assessment tasks and the mathematical reasoning students use. *Educational Studies in Mathematics*, 75(1): 89–105. <http://dx.doi.org/10.1007/s10649-010-9242-9>

- Booth, J. R., & Koedinger, K. R. (2012). Are diagrams always helpful tools? Developmental and individual differences in the effect of presentation format on student problem solving. *British Journal of Educational Psychology*, 82: 492-511.
- Borgen, K. & Manu, S.S. (2002). What do students really understand? *Journal of Mathematical Behavior*, 21: 151-165.
- Bossé, M. J., Adu-Gyamfi, K., & Cheetham, M. (2011). Translations among mathematical representations: Teacher beliefs and practices. *International Journal of Mathematics Teaching and Learning*.
Retrieved from <http://www.cmit.plymouth.ac.uk/journal/default.htm>.
- Bowen, G. M. & Roth, W. (2002). Why students may not learn to interpret scientific inscriptions. *Research in Science Education*, 32: 303-327.
- Bowen, G. M., & Roth, W. (2003). Graph interpretation practices of science and education majors. *Canadian Journal of Science, Mathematics and Technology Education*, 3: 499-512.
- Bowen, G. M., & Roth, W.-M. (2005). Data and graph interpretation practices among preservice science teachers. *Journal of Research in Science Teaching*, 42(10): 1063-1088.
- Boyce, W. E., & DiPrima, R. C. (2009). *Elementary differential equations and boundary value problems* (8th ed.). New York: Wiley.
- Breen, S., & O'Shea, A. (2011). Designing rich sets of tasks for undergraduate calculus courses. In Dooley, T., Corcoran, D. and Ryan, M. (Eds) *Proceedings of the Fourth Conference on Research in Mathematics Education MEI4*, Dublin, (pp. 82-92).
- Brijlall, D., & Maharaj, A. (2010). An APOS analysis of students' construction of the concept of continuity of a single-valued function. *Proceeding of the 7th Southern Hemisphere Conference on Mathematics and Statistics Teaching and Learning*. Stellenbosch, SAf.
- Brijlall, D., & Maharaj, A. (2014). Exploring support strategies for high school mathematics teachers from underachieving schools. *International Journal of Educational sciences*, 7(1): 99-107.

Brijlall, D., & Ndlovu, Z. (2013). High school learners' mental construction during solving optimisation problems in calculus: a South African case study. *South African Journal of Education*, 33(2): 1-18.

Calder, N. S. (2008). *Processing mathematical thinking through digital pedagogical media: The spreadsheet*. (PhD Thesis, The University of Waikato).

Canham, M. & Hegarty, M. (2010). Effects of knowledge and display design on comprehension of complex graphics. *Learning and Instruction*, 20: 155-166.

Cantoral, R. & Montiel, G. (2001). *Funciones: visualización y pensamiento matemático*. D. F., México: Prentice Hall.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The pre-calculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, 28(2): 113-145.

Carney, R., & Levin, J. R. (2002). Pictorial illustrations still improve students' learning from text. *Educational Psychology Review*, 14(1): 5-26.

Carpenter, P. A. & Shah, P. (1998). A model of the perceptual and conceptual processes in graph comprehension. *Journal of Experimental Psychology: Applied*, 4: 75-100.

Cates, J. (2002). Understanding algebra through graphing calculators. *Learning and Leading with Technology*, 30(2): 33-35.

Cetin, I. (2009). Students' understanding of limit concept: An APOS perspective. Unpublished doctoral dissertation. Northern Cyprus: Middle East Technical University.

Chapman, O. (2013). Mathematical-task knowledge for teaching. *Journal of Mathematics Teacher Education*, 16(1): 1-6. <http://dx.doi.org/10.1007/s10857-013-9234-7>

Chappell, K. & Kilpatrick, K. (2003). Effects of Concept-Based Instruction on Students' Conceptual Understanding and Procedural Knowledge of Calculus. *PRIMUS*, 13(1): 17-37.

- Chazan, D., & Yerushalmy, M. (2003). On appreciating the cognitive complexity of school algebra: Research on algebra learning and directions of curricular change. In J. Kilpatrick, Martin, W., & Schifter, D. (Ed.), *A research companion to principles and standards for school mathematics* (pp. 123-135). Reston, VA: National Council of Teachers of Mathematics.
- Cheng, P. C. H. (2004). Why diagrams are (sometimes) six times easier than words: Benefit beyond locational indexing. In A. Blackwell, K. Marriott, & A. Shimojima (Eds.), *Diagrammatic representation and inference, third international conference, diagrams 2004, LNAI 2980* (pp. 242-254). Heidelberg: Springer.
- Clark, M. & Lovric, M. (2008). Suggestion for a theoretical model for secondary-tertiary transition in mathematics. *Mathematics Education Research Journal*, 20 (2): 25-37.
- Clark, R., Nguyen, F., & Sweller, J. (2006). *Efficiency in learning: Evidence-based guidelines to manage cognitive load*. San Francisco, CA: Pfeiffer.
- Clement, L. (2001). What do students really know about functions? *The National Council of Teachers of Mathematics*, 94(9): 745-748.
- Clements, D. H. & Battista, M. T. (1992). *Geometry and spatial reasoning*. In D. A. Grouws (Eds), *Handbook of research on mathematics teaching and learning*. New York : Macmillan.
- Clements, D. H., & Sarama, J. (2012). Mathematics learning, assessment, and curriculum. In Pianta, R. C., Justice, L., Barnett, S. W. & Sheridan, S. (Eds.), *Handbook of early education* (pp. 217- 239). New York: Guilford.
- Cooley, L., Trigueros M., & Baker B. (2007). Schema thematization: A theoretical framework and an example. *Journal for Research in Mathematics Education*, 38(4): 370 - 392.
- Cooney, T., & Wilson, M. (1996). Teachers' thinking about functions: Historical and research perspectives. In T.A. Romberg & E. Fennema (Eds.), *Integrating research on the graphical representation of functions* (pp. 131-158). Hillside, NJ: Lawrence Erlbaum.

- Cox, R., & Grawemeyer, B. (2003). The mental organisation of external representations. *Proceedings of the European Cognitive Science Conference (EuroCogSci- joint Cognitive Science Society and German Cognitive Science Society conference), Osnabrück, September, 2003*. Available from: <http://www.cs.bath.ac.uk/~bg230/Cox&GrawemeyerEuroCogsci03.pdf>
- Cunningham, S. (1994). Some strategies for using visualisation in mathematics teaching. *Zentralblatt für Didaktik der Mathematik, ZDM, 94* (3): 83-85.
- Curcio, F.R. (1987). Comprehension of mathematical relationships expressed in graphs. *Journal for Research in Mathematics Education, 18*: 382-393.
- Curriculum Development Centre, Ministry of Education Malaysia (2006). *Integrated Curriculum for Secondary School Mathematics Curriculum Specifications*. Kuala Lumpur.
- Curriculum Development Centre, Ministry of Education Malaysia (2010). *Integrated Curriculum for Secondary School Mathematics Curriculum Specifications*. Kuala Lumpur.
- delMas, R., Garfield, J., and Ooms, A. (2005). Using Assessment to Study Students' Difficulty Reading and Interpreting Graphical Representations of Distributions. *Proceedings of the Fourth International Research Forum on Statistical Reasoning, Literacy, and Reasoning (on CD)*. K. Makar (Ed.), Auckland, New Zealand: University of Auckland.
- delos Santos, A. & Thomas, M. (2005). The growth of schematic thinking about derivative. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), *Building Connections: Research, Theory and Practice: Proceedings of the 28th Annual Conference of the Mathematics Education Group of Australasia*. (pp. 377-384). Melbourne, Australia: MERGA.
- Diemand-Yauman, C., Oppenheimer, D. M., & Vaughan, E. B. (2011). Fortune favors the **Bold** and the *Italicized*: Effects of disfluency on educational outcomes, *Cognition, 118*(1).
- Diezmann, C. M. (1999) Assessing diagram quality: Making a difference to representation. In *Proceedings of the 22nd Annual Conference of Mathematics Education Research Group of Australasia*, (pp. 185-191), Adelaide.

- Diezmann, C. M., Lowrie, T. J., & Kozak, N. (2007). Essential differences between high and low performers' thinking about graphically-oriented numeracy items. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice. Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia*, Hobart, (pp. 226-235). Sydney: MERGA.
- Dikici, R and İşleyen, T. (2004) Investigation Of Relation And Function Of Some Variables On The Challenges . *Kastamonu Education Journal*, 11(2): 105-116.
- Doerr, H. M., & English, L. D. (2006). Middle grade teachers' learning through students' engagement with modeling tasks. *Journal of Mathematics Teacher Education*, 9: 5–32.
- Dubinsky, E. & Wilson, R. (2013). High school students' understanding of the function concept. *Journal of Mathematical Behaviour*, 32(1): 83-101.
- Durmuş, S. (2004). A Diagnostic Study To Determine Learning Difficulties In Mathematics. *Kastamonu Education Journal*, 12(1): 125-128.
- Eagly, A. H., & Chaiken, S. (2007). The advantages of an inclusive definition of attitude. *Social Cognition*, 25: 582–602.
- Eccles, R., Kapler, T., Harper, R., & Wright, W. (2008). Stories in geotime. *Information Visualization*, 7(1): 3–17.
- Edens, K., & Potter, E. (2008). How students 'unpack' the structure of a word problem: Graphic representations and problem solving. *School Science and Mathematics* , 108 (5): 184-196.
- Elia, I., & Philippou, G. (2004). The function of pictures in problem solving. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME International Conference*, 2: 327-334.
- Ellis, A. & Grinstead, P. (2008). Hidden lessons: How a focus on slope-like properties of quadratic functions encouraged unexpected generalizations. *Journal of Mathematical Behavior*, 27: 277-296.
- Enns, J. T. (2004). *The Thinking Eye, the Seeing Brain: Explorations in visual cognitif*. New York: W.W.Norton & Company.

- Eraslan, A. (2005). *A qualitative study: Algebra honor students' cognitive obstacles as they explore concepts of quadratic functions*. Unpublished doctoral dissertation. Retrieved from ProQuest Dissertations and Theses Database. (3216591).
- Eraslan, A. (2008). The notion of reducing abstraction in quadratic functions. *International Journal of Mathematics Education in Science and Technology*, 39(8): 1051-1060.
- Eraslan, A., Aspinwall, L., Knott, L., and Evitts, T. (2007). Quadratic functions: Students' graphic and analytic representations. *The Mathematics Teacher*, 101(3): 233-237.
- Farnsworth, J., & Boon, B. (2010). Analysing group dynamics within the focus group. *Qualitative Research*, 10: 605-662.
- Fathulla, K., & Hameed, S. (2009), A Framework for Understanding Diagrams in Education. *Journal of Centre of excellence in learning and teaching (CELT)*, 2: ISSN 1758-9258.
- Fazio, R. H. (2007). Attitudes as object-evaluation associations of varying strength. *Social Cognition*, 25: 603–637.
- Ferguson, L. J. (2012). Understanding calculus beyond computations. Unpublished doctoral dissertation, University of Indiana, Bloomington, IN.
- Ferrini-Mundy, J., & Gucler, B. (2009). Discipline-based efforts to enhance undergraduate stem education. *New Directions for Teaching and Learning*, 117: 55-67.
- Font, V., Bolite, J., & Acevedo, J. (2010). Metaphors in mathematics classrooms: Analyzing the dynamic process of teaching and learning of graph functions. *Educational Studies in Mathematics*, 75(2): 131–152.
- Francisco, J.M., & Maher, C.A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior*, 24: 361-372.
- Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics*, 80(1/2): 133–152.

- Friel, S.N., Curcio, F. R., & Bright, G.W. (2001). Making sense of graphs: Critical factors influencing comprehension and instructional implications. *Journal for Research in Mathematics Education*, 32(2): 124-158.
- Firouzian, S.S. (2010). *Correlations between students' multiple ways of thinking about the derivative and their abilities to solve applied derivative problems*. Unpublished masters thesis, University of Missouri, Columbia). Retrieved from <http://umaine.edu/center/files/2011/10/FirouzianThesis.pdf>.
- Fry, E. (1984). A theory of graphs for reading comprehension and writing communication. In Alcaci, C., Lewis, S., O'Brien, G., Jiang, Z. Pre-Service Elementary Teachers' Understandings of Graphs. *Eurasia Journal of Mathematics, Science & Technology Education*, 2011, 7(1): 3-14.
- Gagatsis, A., Elia, I., Panoura, A., Gravvani, K., & Spyrou, P. (2006). An empirical four dimension model for the understanding of function. *International Group for the Psychology of Mathematics Education*, 3: 137-144.
- Gagatsis, A. & Elia, I. (2003). The use of number line in conducting arithmetic operations: A comparative study between Cypriot, Italian and Greek primary pupils. In J. Navotna (Ed.), *Proceedings of the International Symposium in Elementary Mathematics Teaching* (pp. 70-73). Prague, The Czech Republic: Charles University.
- Gal, I. (2004). Statistical literacy: Meanings, components, responsibilities. In J. Garfield & D. Ben-Zvi (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 47-78). Dordrecht, The Netherlands: Kluwer. http://dx.doi.org/10.1007/1-4020-2278-6_3
- Gal, I. (2005). Towards 'probability literacy' for all citizens: building blocks and instructional dilemmas. In G. Jones (Ed.), *Exploring probability in schools: challenges for teaching and learning*, (pp. 39-63). New York: Springer.
- Garderen, V. D. (2007). Teaching Students With LD To Use Diagrams To Solve Mathematical Word Problems. *Journal of Learning Disabilities*, 40(6): 540-553.
- Gattis, M. & Holyoak, K. J. (1996). Mapping conceptual to spatial relations in visual reasoning. *Journal of Experimental Psychology*, 22: 231-239.
- Gerofsky, S. (2010). Mathematical learning and gesture: Character viewpoint and observer viewpoint in students' gestured graphs of functions. *Gesture*, 10(2-3): 321-343.

- Giaquinto, M. (2007). *Visual thinking in mathematics*. NY: Oxford University Press
- Gilbert, J. K. (2005). Visualization: A metacognitive skill in science and science education. In J.K. Gilbert (Ed.), *Visualization in Science Education* (pp. 9-27). Dordrecht: Springer.
- Goerdt, L. S. (2007). The effect of emphasizing multiple representations on calculus students' understanding of the derivative concept. Unpublished doctoral dissertation, Education, Curriculum and Instruction, The University of Minnesota.
- Goldin, G. (2001). Systems of representations and the development of mathematical concepts. In Cuoco, A. A. & Curcio, F. R. (Ed.): *The roles of representation in school mathematics*. Yearbook; 2001. Reston, VA: National council of teachers of mathematics, (pp. 1-23).
- Goldin, G. A. (2004). Representations in school mathematics: A unifying research perspectives. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 275-285). Reston, VA: NCTM.
- Goldin, G. & Kaput, J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397-430). New Jersey: Lawrence Erlbaum.
- Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In Albert A. Cuoco & Frances R. Curcio (Eds.), *The Role of Representation in School Mathematics* (pp.1-23). Reston, VA: National Council of Teachers of Mathematics.
- Gorgorio, N., & Jones, K. (1996). Elements of the Visualisation Process within a Dynamic Geometry Environment. 8th International Congress of Mathematical Education, Spain. (pp. 6).
- Gravemeijer, K., Bowers, J., & Stephan, M. (2003). A hypothetical learning trajectory on measurement and flexible arithmetic. In M. Stephan, J. Bowers, P. Cobb, & K. Gravemeijer (Eds.). *Supporting Students' Development of Measuring Conceptions: Analyzing Students' Learning in Social Context* (pp. 51-66). Reston: NCTM.

- Gray, S. S., Loud, B. J. & Sokolowski, C.P. (2009). Calculus students' use and interpretation of variables; Algebraic vs. Arithmetic Thinking. *Canadian Journal of Science, Mathematics and Technology Education*, 9(2): 59-72.
- Gray, E., & Tall, D. (2001). Relationships between embodied objects and symbolic precepts: An explanatory theory of success and failure in mathematics (3: 65-72). Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education.
- Greenberg, S. (2008). Observing collaboration: Group-centered design. In T. Erickson and D. W. McDonald, editors, *HCI Remixed: Reflections on Works That Have Influenced the HCI Community*, chapter 18, (pp. 111–118). MIT Press, Cambridge, Mass, 2008.
- Guler, G. & Ciltas, A. (2011). The visual representation usage levels of mathematics teachers and students in solving verbal problems. *International Journal of Humanities and Social Science*, 1(11): 145-154.
- Gutierrez, A. (1996). *Visualization in 3-dimensional geometry: In search of a framework*. In: L. Puig & A.
- Haapasalo, L. & Kadjevich, Dj. (2000). Two types of mathematical knowledge and their relation. *Journal für Mathematik-Didaktik*.
- Habre, S., & Abboud, M. (2005). Students conceptual understanding of a function and its derivative in an experimental calculus course. *Journal of Mathematical Behavior*, 25: 57-72.
- Habre, S. & Abboud, M. (2006). Student's conceptual understanding of a function and its derivative in an experimental calculus course. *The Journal of Mathematics Behaviour*, 25(1): 57-72.
- Haciomeroglu, E. S., Aspinwall, L., & Presmeg, N.C. (2010). Contrasting cases of calculus students' understanding of derivative graphs. *Mathematical Thinking and Learning*, 12(2): 152-176.
- Haese & Haese (2010). *Mathematical Studies for Year 12*. Adelaide, Australia: Haese & Haese Publications.

- Hähkiöniemi, M. (2004) Perceptual and symbolic representations as a starting point of the acquisition of the derivative. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (pp. 73-80). Bergen, Norway: PME.
- Hähkiöniemi, M. (2005). The role of different representations in teaching and learning of the derivative through open approach. In E. Pehkonen (Ed.), *Problem solving in mathematics education. Proceedings of the ProMath Meeting, June 30 – July 2, 2004 in Lahti*. University of Helsinki, Department of Applied Sciences of Education. Research Report 261: 71-82.
- Haroz, S., & Ma, K. L. (2006). Natural visualization. In *Proceedings of the European Symposium on Visualization (EuroVis)*, (pp. 43–50), Aire-la-Ville, Switzerland, Eurographics.
- Hattikudur, S., Prather, R., Asquith, P., Alibali, M., & Knuth, E. (2012). Constructing graphical representations: Middle schoolers' intuitions and developing knowledge about slope and y-intercept. *School Science & Mathematics, 112*(4): 230-240.
- Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4): 684-689.
- Herbert, J. (2008). What research says about the NCTM standards. In Kilpatrick, J., Martin, G., & Schifter, D. (Eds), *A Research Companion to Principles and Standards for School Mathematics*, (pp. 5-26). Reston, VA: National Council of Teachers of Mathematics.
- Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 85–97). New York: MPC.
- Hinrichs, U., Schmidt, H., & Carpendale, S. (2008). EMDialog: Bringing information visualization into the museum. *IEEE Transactions on Visualization and Computer Graphics, 14*(6):1181–1188.
- Hipkins, R. (2011). Challenges with graph interpretation: A review of the literature. *Studies in Science Education, 47*(2): 183-219.
- Hitt, F. (Ed.) (2002) Representations and mathematics visualization . Proceedings of the Twenty-Fourth North American Chapter of the International Group for the Psychology of Mathematics Education. Georgia, USA.

- Hodges, T. E., & Conner, E. (2011). Reflections on a technology-rich mathematics classroom. *The Mathematics Teacher*, 104(6): 432-438.
- Hoffer, A. (1977). Mathematics resource project: Geometry and visualization. Palo Alto: Creative Publication.
- Hollebrands, K.F. (2007). The Role of a Dynamic Software Program for Geometry in the Strategies High School Mathematics Students Employ. *Journal for Research in Mathematics Education*, 38 (2): 164–192.
- Holvikivi, J. (2007). Learning styles in engineering education: the quest to improve didactical practice. *European Journal on Engineering Education*, 32(4): 401-408.
- Huang, C. H. (2015). Calculus Students' Visual Thinking of Definite Integral. *American Journal of Educational Research*, 3(4): 476-482.
doi: 10.12691/education-3-4-14.
- Huang, S. L. (2003). Education students' perceptions of computers: A cross-cultural study. *Journal of Educational Computing Research*, 29(4): 451-469.
- Huetinck, L. & Munshin, S. N. (2004). *Teaching mathematics for the 21st century: Methods and activities for grades 6-12* (2nd ed.). Upper Saddle River, NJ: Pearson.
- Huntley, M., & Davis, J. (2008). High school students' approaches to solving algebra problems that are posed symbolically: Results from an interview study. *School Science & Mathematics*, 108(8): 380-388.
- NCERT: Indian National Curriculum Framework (2005). *National Council of Educational Research and Training* (NCERT), India.
- Isenberg, P., Tang, A., & Carpendale, S. (2008). An exploratory study of visual information analysis. In *Proceeding of the Conference on Human Factors in Computing Systems (CHI)*, (pp. 1217–1226), New York, USA, 2008. ACM.
- Ismail, M. B. & Yusof, Z. M. (2010). The effect of individual factors on knowledge sharing quality: an empirical study. *Journal of Organizational Knowledge Management*, 13.
- Jacobson, J. & Turner-Rahman, G. (2007) Visual Intelligence: Bridging the Gap from Visual Literacy to Visual Reasoning, presented at *Visual Literacies: Exploring Critical Issues*, Mansfield College, Oxford, July 3 – 5, 2007.

- Jones, K. (1998), Theoretical Frameworks for the Learning of Geometrical Reasoning. *Proceeding of the British Society for Research into Learning Mathematics*, 18(1&2): 29-34.
- Judson, T. W., & Nishimori, T. (2005). Concepts and skills in high school calculus: An examination of a special case in Japan and the United States. *Journal for Research in Mathematics Education*, 36(1): 24-43.
- Kadunz, G., & Straber, R. (2004). Image-Metaphor-Diagram: Visualisation in Learning Mathematics. *International Group for the Psychology of Mathematics Education*, 28th, Bergen, Norway, July 14-18.
- Kamberelis, G., & Dimitriadis, G. (2013). *Focus groups: From structured interviews to collective conversations*. New York: Taylor & Francis Group.
- Kendal, M., & Stacey, K. (2003). Tracing learning of three representations with the differentiation competency framework. *Mathematics Education Research Journal*, 15,(1): 22- 41.
- Kissane, B. & Kemp, M. (2006). *Mathematics with a graphics calculator: Casio fx-9860G AU*. Perth: Mathematical Association of Western Australia.
- Koedinger, K.R. & Terao, A. (2002). A cognitive task analysis of using pictures to support pre-algebraic reasoning. In C.D. Schunn & W. Gray (Eds.), *Proceedings of the Twenty-Fourth Annual Conference of the Cognitive Science Society*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Koedinger, K.R., Alibali, M. W., & Nathan, M.J. (2008). Trade-offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science: A Multidisciplinary Journal*, 32(2): 366-397.
- Kosslyn, S. M. (1994). *Elements of graph design*. New York: Freeman.
- Kosslyn, S. M. (1989). Understanding charts and graphs. *Applied Cognitive Psychology*, 3(3): 185-225.
- Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Translated from Russian by Teller, J.; Edited by Kilpatrick J. & Wirszup, Chicago: The University of Chicago Press.

- Kultur, M. N., Ozdemir, E., & Konyalioglu, A.C. (2011). Identifying the Learning Difficulties of Freshmen in Mathematics Teacher Training Department in Function Graphs and Derivatives. *International Journal of Humanities and Social Science*, 1(7): Special Issue –June 2011.
<http://www.ijhssnet.com/journal/index/192>.
- Kurikulum Bersepadu Sekolah Menengah (2006). Sukatan Pelajaran Matematik, Bahagian Pembangunan Kurikulum Kementerian Pelajaran Malaysia.
- Lam, H., & Munzner, T. (2008). Increasing the utility of quantitative empirical studies for meta-analysis. In *Proceedings of the CHI Workshop on beyond time and errors: novel evaluation methods for Information Visualization (BELIV)*, (pp. 21–27), New York, USA, ACM.
- Lam, H., Bertini, E., Isenberg, P., Plaisant, C., & Carpendale, S. (2012). Empirical Studies in Information Visualization: Seven Scenarios. *IEEE Transactions on Visualization and Computer Graphics*, 18(9): 1520–1536.
- Lambertus, A.J. (2007) Students' Understanding of the Function Concept: Concept Images and Concept Definitions. Unpublished thesis submitted for Degree of Master of Science. Graduate Faculty of North Carolin State University
- Lassak, M. (2009) Using dynamic graphs to reveal student reasoning. *International Journal of Mathematical Education in Science & Technology*, 40(5): 690-696.
- Lee, K., Khng, K. H., Ng, S. F., & Ng, J. (2013). Longer bars for bigger numbers? Children's usage and understanding of graphical representation of algebraic problems. *Frontline Learning Research*, 1: 81-96.
- Leng, N. W. (2011). Using an advanced graphing calculator in the teaching and learning of calculus. *International Journal of Mathematical Education in Science and Technology*, 42(7): 925-938.
- Leung, A. & Chan, W. A. (2004). Visual reasoning in computational environment : A case of graph sketching. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, 3: 233–240.
- Lezaun, J. (2007). A market of opinions: The political epistemology of focus groups. *The Sociological Review*, 55(2): 130-151.

- Li, X. (2006). Cognitive Analysis of students' errors and misconceptions in variables, equations and functions. *Journal for Research in Mathematics Education*.
- Lim, C. S., & Hwa, T. Y. (2007). *Promoting mathematical thinking in the Malaysian classroom: issues and challenges*. Centre for Research on International Cooperation in Educational Development (CRICED), University of Tsukuba.
- Liu, T. C. (2010). Developing simulation-based computer assisted learning to correct students' statistical misconceptions based on cognitive conflict theory, using 'correlation' as an example. *Educational Technology & Society*, 13 (2): 180–192.
- Liu, Z., & Stasko, J. T. (2010). Mental models, visual reasoning and interaction in information visualization: A top-down perspective. *IEEE Transactions on Visualization & Computer Graphics*, 1: 999-1008.
- Logan, T., & Greenlees, J. (2008). Standardised assessment in mathematics: The tale of two items. In M. Goos, R. Brown & K. Makar (Eds.), *Navigating currents and charting directions. Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia*, 2: 655–658. Brisbane, QLD: MERGA.
- Long, C. (2005). Math's concepts in teaching: procedural and conceptual knowledge. *Phytagoras*, 62: 59-65.
- Lohse, G. L. (1997). The role of working memory on graphical information processing. *Behaviour & Information Technology*, 16: 297-308.
- Lowrie, T. & Diezmann, C. M. (2007) Solving graphics problems: Student performance in junior grades. *The Journal of Educational Research*, 100(6): 369-377.
- Lowrie, T., & Diezmann, C. M. (2011). Solving graphics tasks: Gender differences in middle-school students. *Learning and Instruction*, 21(1): 109–125. doi:10.1016/j.learninstruc.2009.11.005
- Lowrie, T., & Diezmann, C. M. (2009). National numeracy tests: A graphic tells a thousand words. *Australian Journal of Education*, 53(2): 141–158.
- Lowrie, T., Diezmann, C. M., & Kay, R. (2011). The development of the graphic-decoding proficiency instrument. *Evaluation and Research in Education*, 24(4): 285-296.

- Lowrie, T., & Kay, R. (2001). Relationship between Visual and Nonvisual Solution Methods and Difficulty in Elementary Mathematics. *The Journal of Education Research*, 94(4): 248-255.
- Macini, P. & Gagnon, J. C. (2006). Mathematical Instructional Practices and Assessment Accommodations by Secondary Special and General Educators. *Council for Exceptional Children*, 72(2): 217-234.
- Maharaj, A. (2010). An APOS analysis of students' understanding of the concept of a limit of a function. *Pythagoras*, 71: 41-52.
- Maharaj, A. (2013). An APOS analysis of natural science students' understanding of derivatives. *South African Journal of Education*, 33(1): 146-164.
- Maharaj, N., Brijlall, D., & Govender, N. (2008). Pre-service mathematics students' notion of the concept definition of continuity in calculus through collaborative instructional design worksheets. *African Journal of Research in SMT Education*, 12: 93-106.
- Mahir, N. (2010). Students' interpretation of a function associated with a real-life problem from its graph. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 20(5): 392-404.
- Makonye, J.P. (2011). *Learners Mathematical Errors in Introductory Differential Calculus Tasks: A study of misconception in the senior school certificate examinations*. Unpublished masters dissertation, University of Johannesburg, Johannesburg). Retrieved from <https://ujdigispace.uj.ac.za/bitstream/handle/10210/6788/Makonye.pdf?sequence=1>.
- Mancosu, P., Jorgensen, K. F., & Pedersen, S. A. (2005). (Eds.) *Visualization, explanation and reasoning styles in mathematics*. Dordrecht, NL: Springer.
- Markovà, I., Linell, P., Grossen, M., & Orvig, S. A. (2007). *Dialogue in focus groups: Exploring socially shared knowledge*. London: Equinox.
- Marriotti, M.A., Laborde, C., & Falcade, R. (2003). Function and Graph in DGS environment. In N. A.Pateman, B. J. Doherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group of the Psychology of Mathematics Education*, 3: 237-44. Hawaii: PME.

- Mason, J. (1992). Towards a research programme for mental imagery. *Proceedings of the November Conference of BSRLM*. (pp. 24-29).
- Mayer, R. E. (2003). The promise of multimedia learning: Using the same instructional design methods across different media. *Learning and Instruction*, 13:125-139.
- McCulloch, A. W. (2011). Affect and graphing calculator use. *Journal of Mathematical Behavior*, 30(2): 166-179.
- McMillan, J. H. (2004). *Classroom assessment: Principles and practices for effective instruction* (3rd ed.). New York: Pearson.
- Mesa, V. (2007, February). Insights from instructors using textbooks for teaching mathematics. Paper presented at the 10th Annual Conference on Research in Undergraduate Mathematics Education, San Diego, CA.
- Metcalf, D. (2007). *Reducing the digital divide*. *American Librarian*, 38(2): 29.
- Metros, S. E. (2008). The educator's role in preparing visually literate learners. *Theory Into Practice*. 47(2): 102-109.
- Miller, E. K., & Cohen, J. D. (2001). An integrative theory of prefrontal cortex function. *Annual Reviews in Neuroscience* 24: 167-202.
- Ministry of Education (2013). *Malaysian Education Development Plan 2013-2025*. Kuala Lumpur.
- Ministry of Education. (2007). *The New Zealand Curriculum*. Wellington: Learning Media.
- Ministry of Education, Culture, Sports, Science and Technology. (2010). *Junior high school teaching guide for the Japanese Course of Study: Mathematics (Grade 7–9)* (Masami Isoda, trans.). CRICED, University of Tsukuba. Available at <http://globaledresources.com/products.html#Downloads>
- Monteiro, C. & Ainley, J. (2003). Developing Critical Sense in Graphing. *Proceedings of III Conference of the European Society for Research in Mathematics Education*. Available at <http://fibonacci.dm.unipi.it/~didattica/CERME3>

- Monteiro, C. & Ainley, J. (2004). Exploring the Complexity of the Interpretation of media graphs. *Research in Mathematic Education*, 6: 115-128 (BSRLM).
- Monteiro, C. & Ainley, J. (2007). Investigating the interpretation of media graphs among student teachers. *International Electronic Journal of Mathematics Education*, 2 (3): 188-207.
- Monteiro, C. & Ainley, J. (2010). The Interpretation of graphs: Reflecting on contextual aspects. *Alexandria (UFSC)*, 3: 17–30.
- Moore, K. C. (2014). Signals, symbols, and representational activity. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 211-235). Laramie, WY: University of Wyoming.
- Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behaviour*, 32(3): 461-473.
- Moore, K. C., Teuscher, D., & Carlson, M. P. (2011). Exploring shifts in a teacher's key developmental understandings and pedagogical actions. In Wiest, L. R., & Lamberg, T. (Eds.), *Proceedings of the 33rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1673-1681).
- Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In Fukawa-Connelly, T., Infante, N. E., Keene, K., & Zandieh, M. (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education*, (pp. 782-789). Pittsburgh, PA: RUME.
- Moskal, B. M., & Leydens, J. A. (2000). Scoring rubric development: validity and reliability. *Practical Assessment, Research & Evaluation*, 7(10). Retrieved July 1, 2008 from <http://PAREonline.net/getvn.asp?v=2007&n=2010>
- Munez, D., Orrantia, J., & Rosales, J. (2013). The effect of external representations on compare word problems: Supporting mental model construction. *The Journal of Experimental Education*, 81 (3): 337-355.
- Muzangwa J. & Chifamba P. (2012). Analysis of error and misconceptions in the learning of Calculus by undergraduate students. *Acta Didactica Napocensia*, 5(2).

- Natsheh, I., & Karsenty, R. (2014). Exploring the potential role of visual reasoning tasks among inexperienced solvers. *ZDM Mathematics Education*, 46: 109.
doi : 10.1007/s11858-013-0551-1
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics* [Electronic version]. Reston, VA: National Council of Teachers of Mathematics. Retrieved October 20, 2012.
<http://standards.nctm.org/document/chapter3/rep.htm>
- Nemirovsky, R., & Tierney, C. (2001). Children creating ways to represent changing situations: on the development of homogeneous spaces. *Educational Studies in Mathematics*, 45(1-3): 67-102.
- Ng, S. F., & Lee, K. (2009). The model method: Singapore children's tool for representing and solving algebraic word problems. *Journal for Research in Mathematics Education*, 40(3), 282-313.
- Noraini, I. (2006). *Teaching and Learning of Mathematics, Making Sense and Developing Cognitives Ability*. Kuala Lumpur: Utusan Publications & Distributors Sdn. Bhd.
- Noraini, I. (2006). Exploring the Effect of TI-84 Plus on Achievement and Anxiety in Mathematics. *Eurasia Journal of Mathematics, Science and Technology Education*, 2(3): 67-78.
- Noraini, I. N. (2008). *Assessing Mathematical Thinking in Primary Schools in Malaysia*. Shah Alam: McGraw-Hill (Malaysia) Sdn Bhd.
- Noraini, I. (2010). *Penyelidikan dalam Pendidikan*. Malaysia: Mc Graw Hill .
- Novick, L. R. (2006). The importance of both diagrammatic conventions and domain-specific knowledge for diagram literacy in science: The hierarchy as an illustrative case. *Diagrammatic Representative and Inference Lecture Notes in Computer Science*. 4045: 1-11.
- Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and*

teaching in undergraduate mathematics education (pp. 27-42). Washington, D.C.: Mathematical Association of America.

Ontario Ministry of Education (2010). *Growing success: Assessment, evaluation and reporting in Ontario schools. First edition, covering grades 1 to 12*. Toronto: Queen's Printer for Ontario.

Orhun, N. (2012). Graphical understanding in mathematics education: Derivative functions and students' difficulties. *Procedia-Social and Behavioral Sciences*, 55: 679-684.

Orton, A. (1983). Students' Understanding of Differentiation. *Educational Studies in Mathematics*, 14(3): 235-250.

Pantziara, M., Gagatsis, A. & Pitta-Pantazi, D. (2004). The use of diagrams in solving non-routine problems. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, 3: 489-496.

Paoletti, G. (2004). *Writing-to-learn and graph drawing as aids for the integration of text and graphs*. In Rijlaarsdam, G., Van den Bergh, H. & Couzijn, M. (Eds.) *Studies in writing. Volume 14, Effective learning and teaching of writing*, 2nd Edition. Dordrecht: Kluwer Academic Publishers, (pp. 587-598).

Park, J.A. and Kim, Y.S. (2007). Visual reasoning and design processes, *Proceedings of International Conference on Engineering Design*.

Parmjit, S., & White, A. (2006). Unpacking first year university students' mathematical content knowledge through problem solving. *Asian Journal of University Education*, 1(1): 33-56.

Peebles, D. & Cheng, P. (2003). Modelling the effect of task and graphical representation on response latency in a graph reading task. *Human Factors*, 45: 28-46.

Pierce, R. & Stacey, K. (2008). Using pedagogical maps to show the opportunities afforded by CAS for improving the teaching of mathematics. *Australian Senior Mathematics Journal*, 22(1): 6-12.

Pierce, R., Stacey, K., Wander, R., & Ball, L. (2011). The design of lessons using mathematics analysis software to support multiple representations in secondary school mathematics. *Technology, Pedagogy and Education*, 20(1): 95-112.

- Pillay, E. (2008). Grade twelve learners' understanding of the concept of derivative. Unpublished Masters dissertation, University of KwaZulu-Natal, Durban, South Africa.
- Pirolli, P. and Card, S. (2005). The sense-making process and leverage points for analyst technology as identified through cognitive task analysis. *In Proceedings of International Conference on Intelligence*, McLean, VA, USA, 2-4 May 2005.
- Pjanić, K., Liđan, E. & Kurtanović, A. (2015). Visualization of relationship between a function and its derivative. *Eğitim Bilimleri Araştırmaları Dergisi - Journal of Educational Sciences Research*, 5 (1): 205–213. <http://ebad-jesr.com/>
- Plaisant, C. (2004). The challenge of information visualization evaluation. *In Proceedings of the Working Conference on Advanced Visual Interfaces (AVI)*, pages 109–116, New York, USA, 2004. ACM Press.
- Polit, D. F., & Beck, C. T. (2009). *Nursing research: Generating and assessing evidence for nursing practice* (8th Edition). Philadelphia: Wolters Klower/Lippincott Williams & Wilkins.
- Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.
- Presmeg, N. C. (1986). Visualisation and mathematical giftedness. *Educational Studies in Mathematics*, 17: 297-311.
- Presmeg, N. C. (1989). Visualization in multicultural mathematics classrooms. *Focus on Learning Problems in Mathematics*, 11(1 & 2): 17-24.
- Presmeg, N. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, 23: 595 -610.
- Presmeg, N. C. (1993). Mathematics – A bunch of formulas? Interplay of beliefs and problem solving styles. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F.-L. Lin (Eds.), *Proceedings of the 17th PME International Conference*, 3: 57-64.
- Presmeg, N.C. (2006). Research on visualization in learning and teaching mathematics. *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future*. PME 1976-2006. (Eds) Sense Publisher, pp. 205-235.

- Ratwani, R.M., Trafton, J. G. & Boehm-Davis, D. A. (2008). Thinking graphically : connecting vision and cognition during graph comprehension. *Journal of Experimental Psychology: Applied*, 14(1): 36-49.
- Rivera, F. (2011). Toward a Visually Oriented School Mathematics Curriculum: Research, Theory, Practice, and Issues. *Books Authored by SJSU Faculty*. Book 2.
- Roddick, C. D. (2001). Differences in Learning Outcomes: Calculus & Mathematica VS. Traditional Calculus. *PRIMUS*, 11(2): 161-184.
- Rodriguez, F., Espinosa, G. M. & Uriza, R. C. (2007). Visualization in iterative processes. *Proceeding of 11th International Congress of Mathematics Education*, Mexico.
- Roorda, G., Vos, P., & Goedhart, M. (2009). Derivatives and applications; development of one student's understanding. *Proceedings of CERME 6*, January 28th-February 1st 2009, Lyon France. Working group 12. Available at www.inrp.fr/editions/cerme6. Accessed 24 October 2013.
- Rosihan, M. A., & Kor, L. K. (2004). Students' Reactions to Learning Mathematics with Graphing Technology. Paper presented at the 2nd National Conference on Graphing Calculators.
- Rosken, B. & Rolka, K. (2006). A Picture worth a 1000 words – the Role of Visualization in Mathematics Learning. In Novotna, J. Morava, H., Krata, M. & Stehlikova, N. (Eds). *Proceedings of 30th Conference of the International Group for Psychology of Mathematics Education*, 4: 457 – 464. Prague: PME. 4-457.
- Roth, W.M. & Jin Lee, Y. (2004). Interpreting unfamiliar graphs: A generative, activity theoretic model. *Educational Studies in Mathematics*, 57: 265-290.
- Rowland, D. R. (2006). Student difficulties with units in differential equations in modelling contexts. *International Journal of Mathematical Education in Science and Technology*, 37 (5): 553-558.
- Ruthven, K., Deaney, R., & Hennessy, S. (2009). Using graphing software to teach about algebraic forms: A study of technology-supported practice in secondary-school mathematics. *Educational Studies in Mathematics*, 71(3): 279–297.
- Ryken, A. (2009). Multiple representations as sites for teacher reflection about mathematics learning. *Journal of Mathematics Teacher Education*, 12(5): 347-364. <http://dx.doi.org/10.1007/s10857-009-9107-2>

Sabella, M., & Redish, E. F. (2005). Knowledge organization and activation in physics problem solving. *American Journal of Physics*, submitted.

Şahin, Z., Yenmez-Aydoğan, A., & Erbaş, A. K. (2015). Understanding the concept of derivative: A case study with mathematical modeling. *EURASIA Journal of Mathematics, Science & Technology Education*, 11(1): 177-188. doi: 10.12973/eurasia.2015.1149a

Sajka, M. (2003). A secondary school student's understanding of the concept of function – a case study. *Educational Studies in Mathematics*, 53: 229-254.

Saraiya, P., North, C., Lam, V., & Duca, K. (2006). An insight-based longitudinal study of visual analytics. *IEEE Transactions on Visualization and Computer Graphics*, 12: 1511-1522.

Shah, P., & Freedman, E. G. (2011). Bar and line graph comprehension: An interaction of top-down and bottom-up processes. *Topics in Cognitive Science*, 3: 560-578.

Shah, P., & Hoeffner, J. (2002). Review of graph comprehension research: Implications for instruction. *Educational Psychology Review* 14(1): 47-69.

Shahrul, A. (2011). Masalah Pembelajaran Matematik Di Kalangan Pelajar Tingkatan Empat Dalam Tajuk Garis Dan Satah Dalam Tiga Matra. Tesis Ijazah Sarjana Muda Sains Serta Pendidikan Matematik (Fizik). Fakulti Pendidikan, Universiti Teknologi Malaysia.

Sharma, S. (2006). High school students interpreting tables and graphs: Implications for research. *International Journal of Science and Mathematics Education*, 4: 241-268.

Sharma, S. (2013). Assessing Students' Understanding of Tables and Graphs: Implications for Teaching and Research. *International Journal of Educational Research and Technology*, 4(4): 51-70.

- Shaughnessy, J. M. (2007). Research on statistics learning and reasoning. In F. K. Lester, Jr Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 957-1009). Reston: The National Council of Teachers of Mathematics.
- Shaughnessy, J. M. & Zawojewski, J. S. (1999). Secondary students' performance on data and chance in the 1996 NAEP. *The Mathematics Teacher*, 92(8): 713-718.
- Sheehan, M., & Nillas, L. (2010). Technology integration in secondary mathematics classrooms: Effect on students' understanding. *Journal of Technology Integration in the Classroom*, 2(3): 67-83.
- Simon, M. A. (1986a). Components of effective use of diagrams in math problem solving. *Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 1-7). East Lansing, MI. (ERIC Document Reproduction Service No. ED 309 924).
- Simon, M. A. (1986b). An external control study of diagram drawing skills for the solution of algebra word problems by novice problem-solvers (figures, heuristics, spatial representations, metacognition. Unpublished doctoral dissertation, University of Massachusetts. Available from Proquest, Paper AAI8701220. <http://scholarworks.umass.edu/dissertations/AAI8701220>
- Singh, R. I. (2007). An investigation into learner understanding of the properties of selected quadrilaterals using manipulatives in a grade eight mathematics class. Master of Education thesis. University of KwaZulu-Natal: Durban.
- Sinnett, S., Spence, C., & Soto-Faraco, S. (2007). Visual dominance and attention: The Colavita effect revisited. *Perception & Psychophysics*, 69(5): 673-686.
- Siyepu, S.W. (2012). Analysis of errors in derivatives of trigonometric functions: A case study in an extended curriculum programme. Unpublished doctoral thesis, The University of the Western Cape, Bellville, South Africa.
- Siyepu, S.W. (2013a). The zone of proximal development in the learning of mathematics. *South African Journal of Education*, 33(2): Art. #714, 13 pages. <http://www.sajournalofeducation.co.za>
- Siyepu, S. W. (2013b). An exploration of students' errors in derivatives in a university of technology. *Journal of Mathematical Behavior*, 32: 577-592.

- Smith, S.P. (2003). Representation in school mathematics: Children's representations of problems. In Kilpatrick, J., Martin, W.G., & Schifter, Deborah (Eds.) *A Research Companion to Principles and Standards for School Mathematics*, (pp. 263-274). Reston, VA: NCTM.
- Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the Early Grades*. Mahwah, NJ: Lawrence Erlbaum Associates/Taylor & Francis Group and National Council of Teachers of Mathematics.
- Sofronas, K. S. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior*, 30(2): 131-148.
- Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? perspectives of 24 experts. *The Journal of Mathematical Behaviour*, 30 (2): 131–148.
- South Australian Certificate of Education. (2014). The Mathematical Studies Curriculum Statement 2014. SACE Board of South Australia, Adelaide, Australia.
- South Australian Certificate of Education. (2015). The Mathematical Studies Curriculum Statement 2015. SACE Board of South Australia, Adelaide, Australia.
- Stahley, J. R. (2011). *Students' qualitative understanding of the relationship between the graph of a function and the graphs of its derivatives*. Unpublished Masters thesis. The University of Maine, USA.
- Stavridou, F., & Kakana, D. (2008). Graphic abilities in relation to mathematical and scientific ability in adolescents. *Educational Research*, 50(1): 75–93.
- Stern, E., Aprea, C., & Ebner, H. G. (2003). Improving cross-content transfer in text processing by means of active graphical representation. *Learning & Instruction*, 13(2): 191-203.
- Stewart, J. (2009). *Calculus* (6th ed). Toronto: Thomson Brooks/Cole.
- Stylianides, A. J. & Stylianides G. J. (2007). Learning mathematics with understanding: A critical consideration of the learning principle in the principles and standards for schools mathematics. *The Montana Mathematics Enthusiast*. 4(1): 103-114.

- Stylianou, D.A., & Silver, E.A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning*, 6(4): 353-387.
- Stylianou, D. A. (2010). Teachers' conceptions of representation in middle school mathematics. *Journal of Mathematics Teacher Education*, 13: 325-343. <http://dx.doi.org/10.1007/s10857-010-9143-y>
- Tall, D. (2010). *A sensible approach to the calculus*. (Presented as a plenary at The National and International Meeting on the Teaching of Calculus, 23–25th September 2010, Puebla, Mexico.) Available at <http://homepages.warwick.ac.uk/staff/David.Tall/downloads.html>. Accessed 24 October 2013.
- Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, 12: 151-169.
- Tallman, M. & Carlson, M. P. (2012). A characterization of calculus I final exams in U.S. colleges and universities. *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (p. 217-226). Portland, OR: Portland State University.
- Tarmizi, R. A. (2010). Visualising students' difficulties in learning calculus. *Proceedings of the International Conference on Mathematics Education Research* (pp. 377-383), Universiti Putra Malaysia.
- Terwel, J., van Oers, B., van Dijk, I., & van den Eeden, P. (2009). Are representations to be provided or generated in primary mathematics education? Effects on transfer. *Educational Research & Evaluation*, 15(1): 25–44.
- Tishkovskaya, S. and Lancaster, G. (2012). Statistical Education in the 21st Century: a Review of Challenges, Teaching Innovations and Strategies for Reform. *Journal of Statistics Education*, 20(2): 1-24.
- Tiwari, T. K. (2007). Computer Graphics as an Instructional Aid in an Introductory Differential Calculus Course. *International Electronic Journal of Mathematics Education*, 2(1): 32-48.
- Torres, S. E. T. & Alarcon, H. (2011). A tutorial-type activity to overcome learning difficulties in understanding graphics in kinematics. *Latin American Journal Physical Education*, 6(1): 285-289.

- Trigueros, M. and Martínez-Planell, R. (2010). Geometrical representations in the learning of two variable functions. *Educational Studies in Mathematics* 73(1): 3-19.
- Toh, T. L. (2009). On in-service mathematics teachers' content knowledge of calculus and related concepts. *The Mathematics Educator*, 12(1): 69-86.
- Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology*, 38(5): 609-637.
- Ubuz, B. & Kirkpınar, B. (2000). Factors contributing to learning of calculus. *Proceeding of the PME*, 24(4): 241-248, Hiroshima, Japan.
- Uesaka, Y. (2002). The relationship between students' use, and beliefs about the effectiveness and difficulties involved in the use of diagrams. *Proceedings of the 44th annual conference of Japanese educational psychology*. Tokyo: The Japanese Association of Educational Psychology. (p. 176).
- Uesaka, Y., & Manalo, E. (2007). Peer instruction as a way of promoting spontaneous use of diagrams when solving math word problems. In D. S. McNamara & J. G. Trafton (Eds.), *Proceedings of the 29th Annual Cognitive Science Society* (pp. 677-682). Austin, TX: Cognitive Science Society.
- Uesaka, Y., & Manalo, E. (2010). Development of a teaching method that promotes the spontaneous use of diagrams in mathematical problem solving: Effects of experience using diagrams as communication tools. *Cognitive Studies*, 17(1): 246-250.
- Uesaka, Y., & Manalo, E. (2011). The effects of peer communication with diagrams on students' math word problem solving processes and outcomes. In L. Carlson, C. Hoelscher, & T. Shipley (Eds.), *Proceedings of the 33rd Annual Conference of the Cognitive Science Society* (pp. 312-317). Austin, TX: Cognitive Science Society.
- Uesaka, Y., Manalo, E., & Ichikawa, S. (2007). What kinds of perceptions and daily learning behaviours promote students' use off diagrams in mathematics problem solving? *Learning and Instructions*, 17: 322-335.

- Uesaka, Y., Manalo, E., & Ichikawa, S. (2010). *The effects of perception of efficacy and diagram construction skills on students' spontaneous use of diagrams when solving math word problems*. Paper presented at the Diagrams 2010 Conference, August 9–11, Portland State University, Portland, Oregon, USA.
- Uygun, T. & Ozdas, U. (2005). Misconceptions and difficulties with the chain rule. The Mathematics Education into the 21st Century Project Universiti Teknologi Malaysia.Reform, Revolution and Paradigm Shifts in Mathematics Education Johor Bahru, Malaysia, Nov 25th – Dec 1st 2005. Retrieved from http://math.unipa.it/~grim/21_project/21_malasya_Uygun209-213_05.pdf
- Uygun, T. & Özdaş, A. (2007). The effect of arrow diagrams on achievement in applying the chain rule. *Primus: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 17: 131-147.
- Van de Walle, J.A. (2007). *Elementary and middle school mathematics: Teaching developmentally*. Boston: Pearson.
- Van Garderen, D. (2006). Spatial Visualization, Visual Imagery, and Mathematical Problem Solving of Students With Varying Abilities. *Journal of Learning Disabilities*. 39 (6): 496–506.
- van Garderen, D. (2004). Reciprocal teaching as a comprehension strategy for understanding mathematical word problems. *Reading and Writing Quarterly*, 20(2): 225–229.
- Vekiri, I. (2002). What Is the Value of Graphical Displays in Learning? *Educational Psychology Review*, 14(3): 261- 312.
- Viholainen, A. (2005). *Relationships Between Informal and Formal Reasoning in the Subject of Derivative*. CERME 4.
- Wall, J.J., & Benson, C.C. (2009). So many graphs, so little time. *Mathematics Teaching in the Middle School*, 15: 82-91.
- Wang, Z.H. (2012). Students' Cognitive Reasoning of Graphs: Characteristics and Progression. *International Journal of Science Education*, 34(13): 2015-2041.
- Watson, J. M. (2006). *Statistical literacy at school: Growth and goals* Mahwah, NJ: Lawrence Erlbaum.

- Watson, J. M., & Callingham R. (2003). Statistical literacy: A complex hierarchical construct. *Statistics Education Research Journal*, 2(2): 3-46.
- Weber, E. (2012). Students' ways of thinking about two-variable functions and rate of change in space. (Ph.D. Dissertation), Arizona State University, USA.
- White, N. J., & Mesa, V. (2014). Describing cognitive orientation of calculus I tasks across different types of coursework. *ZDM The International Journal on Mathematics Education*, 46(4): 675-690. doi: 10.1007/s1185801405889
- Wittmann, M. C. (2006). Using resource graphs to represent conceptual change. *Physical Review Special Topics – Physics Education Research*, 2(020105): 1-17.
- Woleck, K. R. (2001). *Listen to their pictures: An investigation of children's mathematical drawings*. In A. A. Cuoco & F. R. Curcio (Eds.), 2001 Yearbook: The roles of representation in school mathematics (pp. 215-227). Reston, VA: NCTM.
- Woolner, P. (2004). A comparison of a visual-spatial approach and a verbal approach to teaching mathematics. In *Proceedings of the 28th Conference of the International Group for the Psychology of mathematics Education*, 4: 449-456.
- Yavuz, I. (2010). What does a graphical representation mean for students at the beginning of function teaching? *International Journal of Mathematical Education in Science & Technology*, 41(4): 467-485.
- Yerushalmy, M., & Swidan, O. (2012). Signifying the accumulation graph in a dynamic and multi-representation environment. *Educational Studies in Mathematics*, (pp. 1-20). doi: 10.1007/s10649-011-9356-8
- Yetim, S. (2004). Real-Valued Functions Of One Variable Affecting The Concept Of Derivative Analysis Of Some Mathematical Concepts And Analytical Hierarchy Project. *Kastamonu Education Journal*, 12(1), (pp. 137-156).
- Yumus, A. S. (2001). *Levels of reasoning in mathematical concepts among preservice mathematics teachers*. 2nd APED. International Conference in Education Research Understanding Eduactional Issues in the Asia Pacific Region Universals, Uniqueness and Cooperation. Seoul, Korea, (pp. 52-61).

Zandieh, M. J. (2000). A theoretical framework for analysing student understanding of the concept of derivative. In E. Dubinsky, A.H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education IV*. (pp.103-127). Providence, RI: *American Mathematical Society*.

Zazkis, D. (2013). Fostering Students' Understanding of The Connection Between Function and Derivative: A Dynamic Geometry Approach. *Proceedings of the Conference for Research in Undergraduate Mathematics Education*.

Zazkis, D. (2014). Comparing Calculus Students' Representation Use Across Different Settings. *Proceedings of the Conference for Research in Undergraduate Mathematics Education*.

Zimmerman, W. & Cunningham, S. (1991). What is mathematical visualisation? *Visualisation in teaching and learning mathematics* (Eds: W. Zimmerman & S. Cunningham). Washington DC: Mathematical Association of America. (pp. 1-9).

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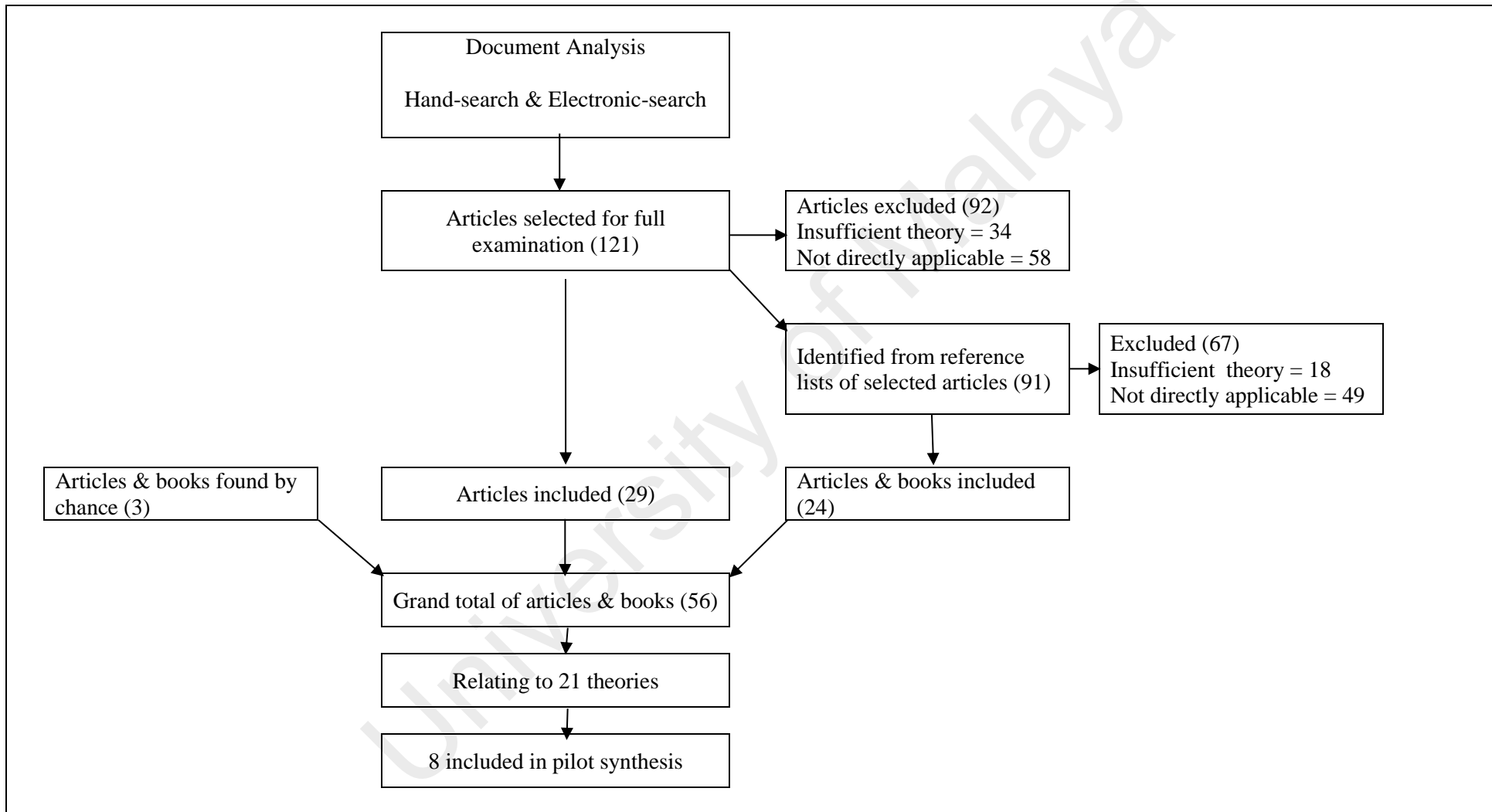


Figure 4.1: Document analysis to locate the framework to assess visual reasoning

Table 4.3 : Taxonomy of skills on decoding process among theories/models/frameworks for points of convergence

	Curcio (1987)	Yumus (2001)	Sharma (2013)
Elementary	<p>Read - recognising components of graphs</p> <p>Describe - speaking the language of graphs</p>	<p>Unable to produce any reasoning</p>	<p>Informal/idiosyncratic - exhibiting characteristics of pre-structural thinking</p> <p>Consistent non-critical - exhibiting characteristics of pre-structural thinking or at most uni-structural thinking</p>
Intermediate	<p>Interpret - understanding relationships among tables, graphs and data</p> <p>Analyse - making sense of graph but avoiding personalization and maintaining an objective stance while reading the graphs</p>	<p>Aware of models, known facts, properties and relationships used as basis of reasoning, but cannot produce any arguments</p> <p>Able to provide reasons although arguments are weak</p>	<p>Consistent non-critical - exhibiting characteristics of uni-structural and multi-structural thinking</p> <p>Early critical - beginning to exhibit characteristics of relational thinking and can attend to more than one relevant aspects of the data and are beginning to integrate the aspects</p>
Advanced	<p>Predict - interpreting information in a graph and answering questions about it</p> <p>Extrapolate - recognising appropriate graphs for a given data set and its context</p>	<p>Able to provide strong arguments to support reasoning</p>	<p>Advanced critical - integrating contextual knowledge that exhibits extended abstract thinking</p>
Strength	<p>Case study realistic</p>	<p>Transformation from instrumental</p>	<p>Evidence gained from a variety of sources</p>

Table 4.4 : Taxonomy of skills on encoding process among theories/models/frameworks for points of convergence

	Krutetskii (1976)	Simon (1986)	Carlson (1998)
	Categories of visual ability	Diagram drawing sub-skills	Mental actions of the co-variation
Analytical type	Individuals who prefer verbal-logical rather than imagery modes when attempting to solve problems	Verbalising what is represented and what needs to be represented	Coordinating the value of one variable with changes in the other Coordinating the amount of change of one variable with changes in the other variable,
Geometric type	Individuals who prefer to use diagrams or images rather than the verbal modes when attempting to solve problems	Creating an integrated diagram that are critical to the conceptualisation of the problem Checking the accuracy of the diagram Labelling completely	Coordinating the average rate-of- change of the function with uniform increments of change in the input variable. Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function
Harmonic type	Individuals who have o tendency to any one of the type in particular.	Representing all relevant information drawing multiple representations are not critical	Coordinating the direction of change of one variable with changes in the other variable
Strength	Case study realistic Influential – early model that stimulated further research	Case study realistic In-depth information	Analysis of co-variational reasoning to a finer degree Guide to enhance structure for teaching and learning activities

Table 4.8 : The distribution of 3-round emailing experts

Experts	MC		VR		CR		ME		Total	
	Loc.	Int.	Loc.	Int.	Loc.	Int.	Loc.	Int.		
Round 1	Out	11	3	4	3	12	2	13	2	50
	Not returned	-	-	1	-	1	1	-	-	3
	In	11	3	3	3	11	1	13	2	47
	Dismissed	1	-	-	-	-	-	2	-	3
Round 2	Out	10	3	3	3	11	1	11	2	44
	Not returned	-	1	-	-	1	-	-	-	2
	In	10	2	3	3	10	1	11	2	42
	Dismissed	-	-	-	-	-	-	1	-	1
Round 3	Out	10	2	3	3	10	1	10	2	41
	Not returned	1	-	-	1	-	-	1	-	3
	In	9	2	3	2	10	1	9	2	38

Note : MC=Mathematical Content, VR=Visual Reasoning, CR=Cartesian Graph, ME=Mathematic Education, Loc.=Local, Int.=International.

Table 4.9 : Analysis on the refined framework – Round 1

No.	Item	Unsatisfactory	Poor	Satisfactory	Good	Outstanding
1.	Describe the information needed	-	1 (2)	31 (66)	12 (26)	3 (6)
2.	Easily understood	-	-	26 (56)	18 (38)	3 (6)
3.	Clear and concise	-	1 (2)	27 (58)	17 (36)	2 (4)
4.	No spelling and grammatical errors	-	3 (6)	2 (4)	40 (86)	2 (4)
5.	Current and valid empirical data	-	2 (4)	39 (83)	5 (11)	1 (2)
6.	All outcomes are covered (encoding)	-	1 (2)	33 (71)	11 (23)	2 (4)
7.	All outcomes are covered (decoding)	-	1 (2)	33 (71)	11 (23)	2 (4)
8.	Statements are consistent	-	1 (2)	37 (79)	6 (13)	3 (6)
9.	The arrangement/flow is logical and clear	-	1 (2)	36 (77)	8 (17)	2 (4)
10.	The framework support visual reasoning ability	-	1 (2)	39 (83)	5 (11)	2 (4)

Table 4.10 : Analysis on the refined framework – Round 2

No.	Item	Unsatisfactory	Poor	Satisfactory	Good	Outstanding
1.	Describe the information needed	-	-	23 (52)	20 (45)	1 (2)
2.	Easily understood	-	-	22 (50)	18 (41)	4 (9)
3.	Clear and concise	-	-	25 (57)	15 (34)	4 (9)
4.	No spelling and grammatical errors	-	-	-	-	44 (100)
5.	Current and valid empirical data	-	-	29 (66)	14 (32)	1 (2)
6.	All outcomes are covered (encoding)	-	-	26 (59)	17 (39)	1 (2)
7.	All outcomes are covered (decoding)	-	-	26 (59)	17 (39)	1 (2)
8.	Statements are consistent	-	-	24 (55)	16 (36)	4 (9)
9.	The arrangement/flow is logical and clear	-	-	26 (59)	15 (34)	3 (7)
10.	The framework support visual reasoning ability	-	-	23 (52)	19 (43)	2 (5)

Table 4.11 : Analysis on the refined framework – Round 3

No.	Item	Unsatisfactory	Poor	Satisfactory	Good	Outstanding
1.	Describe the information needed	-	-	12 (32)	21 (55)	5 (13)
2.	Easily understood	-	-	9 (24)	23 (61)	6 (16)
3.	Clear and concise	-	-	10 (26)	20 (53)	8 (21)
4.	No spelling and grammatical errors	-	-	-	-	38 (100)
5.	Current and valid empirical data	-	-	25 (66)	13 (34)	-
6.	All outcomes are covered (encoding)	-	-	21 (55)	13 (34)	4 (11)
7.	All outcomes are covered (decoding)	-	-	21 (55)	13 (34)	4 (11)
8.	Statements are consistent	-	-	15 (39)	14 (37)	9 (24)
9.	The arrangement/flow is logical and clear	-	-	11 (29)	17 (45)	10 (26)
10.	The framework support visual reasoning ability	-	-	16 (42)	16 (42)	6 (16)

Table 4.13 : The usage levels on using graphs or diagrams in their daily learning behaviour.

Item	Not at all	Slightly	Moderately	Very much	Definitely	Mean	SD
	f (%)	f (%)	f (%)	f (%)	f (%)		
1. Do you pay attention to the use of graphs or diagrams for solving mathematical word problems that your teacher shows on the board during class?	2 (1.03)	4 (2.06)	25 (12.89)	93 (47.94)	70 (36.08)	4.16	0.80
2. Do you try to copy the way your teacher uses graphs or diagrams to solve mathematical words problems?	4 (2.06)	17 (8.76)	68 (35.06)	66 (34.02)	39 (20.10)	3.61	0.97
3. Do you try to use the kinds of graphs or diagrams shown in your textbooks to solve other similar mathematical problems?	2 (1.03)	29 (14.95)	58 (29.90)	62 (31.96)	43 (22.16)	3.59	1.03
4. Do you try to use the kinds of graphs or diagrams shown by your teacher to solve other similar mathematical problems?	6 (3.09)	17 (8.76)	60 (30.93)	80 (41.24)	31 (15.98)	3.58	0.96
5. Do you usually use graphs or diagrams in solving mathematical problems?	2 (1.03)	19 (9.79)	92 (47.43)	62 (31.96)	19 (9.79)	3.40	0.83

Table 4.14 : The usefulness of graphs or diagrams in solving mathematical problems.

Item	Not at all	Slightly	Moderately	Very much	Definitely	Mean	SD
	f (%)	f (%)	f (%)	f (%)	f (%)		
6. Do you think the use of graphs or diagrams is helpful in efficiently solving mathematical word problems?	0 (0)	4 (2.06)	39 (20.10)	81 (41.76)	70 (36.08)	4.12	0.80
7. Do you think it is good to use graphs or diagrams in solving mathematical word problems?	2 (1.03)	10 (5.15)	45 (23.20)	60 (30.93)	77 (39.69)	4.03	0.97
8. Do you think the use of graphs or diagrams helps you figure out how to solve mathematical word problems?	0 (0)	4 (2.06)	52 (26.80)	85 (43.82)	53 (27.32)	3.96	0.79

Table 4.15 : The difficulty of the use of graphs or diagrams in solving mathematical problems.

Item	Not at all	Slightly	Moderately	Very much	Definitely	Mean	SD
	f (%)	f (%)	f (%)	f (%)	f (%)		
9. In general, do you know how to construct graphs or diagrams for solving mathematical word problems?	4 (2.06)	16 (8.25)	90 (46.39)	72 (37.11)	12 (6.19)	3.37	0.81
10. How easy is it for you to use graphs or diagrams in solving mathematical word problems?	4 (2.06)	19 (9.79)	95 (48.97)	64 (33.99)	12 (6.19)	3.31	0.81
11. Do you know what kinds of graphs or diagrams are helpful in solving different kinds of mathematical word problems?	8 (4.12)	19 (9.79)	107 (55.16)	50 (25.78)	10 (5.15)	3.20	0.93
12. How easy is it for you to draw graphs or diagrams by yourself for solving mathematical word problems?	2 (1.03)	45 (23.20)	106 (54.64)	35 (18.04)	6 (3.09)	2.99	0.76

Table 4.16 : The teacher's behaviours in using graphs or diagrams in solving mathematical problems.

Item	Not at all	Slightly	Moderately	Very much	Definitely	Mean	SD
	f (%)	f (%)	f (%)	f (%)	f (%)		
13. Do you think your mathematics teachers use graphs or diagrams to efficiently solve mathematical word problems?	2 (1.03)	6 (3.09)	31 (15.98)	64 (32.99)	91 (46.91)	4.22	0.90
14. Do your mathematics teachers use graphs or diagrams to explain how to solve mathematical word problems?	2 (1.03)	2 (1.03)	41 (21.13)	80 (41.24)	69 (35.57)	4.09	0.83
15. Do the graphs or diagrams that your teachers use to show how to solve mathematical word problems help you to understand how those problems can be solved?	0 (0)	2 (1.03)	49 (25.26)	75 (38.66)	68 (35.05)	4.08	0.80
16. Do your mathematics teachers teach your class how to use graphs or diagrams in solving mathematics word problems?	2 (1.03)	10 (5.15)	45 (23.20)	64 (32.99)	73 (37.63)	4.01	0.95
17. Are you told or encouraged by your mathematic teachers to use graphs or diagrams in solving mathematics word problems?	0 (0)	17 (8.76)	47 (24.23)	72 (37.11)	58 (29.90)	3.88	0.94