CHAPTER ONE : DEFINITIONS AND PRELIMINARIES

1.1 Differentiable manifolds and tensor fields

Before we begin with this first section, we shall give some basic notations which will be used throughout this dissertation.

We denote the real line by $\mathbb{R}$ and the m-dimensional Euclidean space by $\mathbb{R}^m$, that is,

$$\mathbb{R}^m = \{ x = (x_1,\dots,x_m) : x_i \in \mathbb{R},\ i = 1,\dots,m \}$$

We shall use $\mathbb{C}$ to denote the complex number field and $\mathbb{C}^m$ to denote the complex m-space, that is,

$$\mathbb{C}^m = \{ (z_1,\dots,z_m) : z_i \in \mathbb{C} ,\ i = 1,\dots m \}$$

The function $u_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ defined by

$$u_i(x) = x_i$$

where $x = (x_1,\dots,x_m) \in \mathbb{R}^m$, is called the *ith coordinate function* on $\mathbb{R}^m$. For an open set $U \subset \mathbb{R}^k$ and if $f : U \longrightarrow \mathbb{R}^m$, then we let

$$f_i = u_i \circ f$$

where f_i is called the *ith component function* of f.

Let $U \subset \mathbb{R}^n$ be open and $f:U \longrightarrow \mathbb{R}$. For a non-negative integer k, f is *differentiable of class C^k* on U (or f is C^k) if all the partial derivatives up to order k exist and are continuous. In particular, f is C^0 if f is continuous. If $f:U \longrightarrow \mathbb{R}^m$, then f is *differentiable of class C^k* if all its component functions are of class C^k. f is said to be C^∞ if it is C^k for all $k \geq 0$. We shall restrict our attention solely to the case of class C^{∞} , so by differentiable, we shall always mean differentiable of class C^{∞}. The terminology *smooth* is also used to indicate differentiability of class C^{∞}.

Let U and V be vector spaces. The *Cartesian product* of U and V is defined as the set $U \times V = \{(u,v) : u \in U \text{ and } v \in V\}$.

An *m-dimensional differentiable manifold* is a paracompact, second countable topological space M such that

a) M is a Hausdorff space and each point p of M has a neighborhood that is homeomorphic to an open set of $\mathbb{R}^m$.

b) there exist a collection of coordinate systems $\{(U_\alpha, \phi_\alpha)\}$ where ϕ_α is a homeomorphism of a connected open set $U_\alpha \subset M$ onto an open subset of $\mathbb{R}^m$ satisfying the following properties:

i) $M = \bigcup_\alpha U_\alpha$

ii) for all α,β the mapping

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

is differentiable.

iii) The collection $\{(U_\alpha,\phi_\alpha)\}$ is maximal with respect to ii).

The collection of coordinate systems $\{(U_\alpha,\phi_\alpha)\}$ defines the differentiable structure of the manifold M. The mapping ϕ_α is called the *coordinate map* and the functions $x_j = u_j \circ \phi_\alpha$, for $j = 1, \ldots m$, are called the *coordinate functions*. The open set U_α is called a *coordinate neighborhood* of p, for all $p \in U_\alpha$. The pair (U_α,ϕ_α) is also known as a *chart* of M.

In order to define a complex manifold of complex dimension m, we replace $\mathbb{R}^m$ in the definition of a differentiable manifold by the m-dimensional complex space $\mathbb{C}^m$. The condition b) is replaced by the condition that the m coordinate functions of $\phi_\alpha \circ \phi_\beta^{-1}(p)$ should be holomorphic functions of the complex coordinates of p.

Let N and M be two manifolds. A mapping $f : N \longrightarrow M$ is said to be *differentiable*, if for every chart (U_i, ϕ_i) of N and every chart (V_j, ψ_j) of M such that $f(U_i) \subset V_j$, the mapping

$$\psi_j \circ f \circ \phi_i^{-1} : \phi_i(U_i) \longrightarrow \psi_j(V_j)$$

is differentiable. A differentiable function on M is a differentiable mapping of M into R. We denote the set of all differentiable function as $F(M)$.

Let M be an m-dimensional manifold and p be a point of M. Functions f and g define on an open set containing p are said to be equivalent if they agree on some neighborhood of p. The set of equivalent classes is denoted by $\tilde{F}_p$. We note that $\tilde{F}_p$ is an algebra. The *tangent vector* v at a point $p \in M$ is defined to be a linear derivation of $\tilde{F}_p$, that is, for all $\tilde{f}$, $\tilde{g} \in \tilde{F}_p$ and $r \in \mathbb{R}$,

$$\text{i) } v(\tilde{f} + r\tilde{g}) = v(\tilde{f}) + rv(\tilde{g})$$

$$\text{ii) } v(\tilde{f}\tilde{g}) = \tilde{f}(p)v(\tilde{g}) + \tilde{g}(p)v(\tilde{f})$$

The set of tangent vectors at p is called the *tangent space* of M at p and is denoted by T_pM. It can be shown that T_pM is an m-dimensional vector space. The dual space of T_pM will be denoted by T_p^*M.

In practise, we shall treat tangent vectors as operators on functions. If f is a differentiable function defined on a neighborhood of p, and $v \in T_pM$, we define

$$v(f) = v(\tilde{f}), \qquad \text{for } \tilde{f} \in \tilde{F}_p$$

Hence, $v(f) = v(g)$ whenever f and g agree on a neighborhood of p.

Let (U,ϕ) be a coordinate system with coordinate functions $x_1,\ldots,x_m$ and let $p \in U$. For each $i = 1,\ldots m$, we define a mapping $\left.\frac{\partial}{\partial x_i}\right|_p$ as

$$\left(\left.\frac{\partial}{\partial x_i}\right|_p \right)(f) = \left.\frac{\partial(f \circ \phi \quad)}{\partial u_i}\right|_{\phi(p)}$$

for each differentible function f defined on a neighborhood of p. Clearly, for f and g belonging to the same equivalence class in $\tilde{F}_p$, we have

$$\left.\frac{\partial}{\partial x_i}(f)\right|_p = \left.\frac{\partial}{\partial x_i}(g)\right|_p$$

It is easy to see that $(\partial/\partial x_i)|_p$ defines a tangent vector at p and $\{(\partial/\partial x_i)|_p : i = 1,\ldots m\}$ forms a basis of T_pM.

Let $TM = \bigcup_{p\in M} T_pM$ and $T^*M = \bigcup_{p\in M} T^*_pM$. It can be shown that TM and T*M are 2m-dimensional manifolds. TM and T*M are known respectively as the *tangent bundle* and *cotangent bundle* (for definition of vector bundle, see p. 5)

A vector field X on a manifold is an assignment of a vector X_p to every point $p \in M$. If f is a differentiable function on M, then Xf is a function on M defined by

$$(Xf)(p) = X_pf$$

A vector field X is differentiable if Xf is differentiable for every differentiable function f.

Next we let E and M be any arbitrary manifolds and π be a differentiable mapping of E onto M. The manifold E is called the *vector bundle* over M under the projection π if the following conditions are satisfied :

i) $\pi^{-1}(p)$ is a real vector space called the *fibre above* p and each $\pi^{-1}(p)$ is isomorphic to $\mathbb{R}^k$, for some fixed k.

ii) for each $p \in M$, there exist an open neighborhood U of p such that the mapping $\phi : U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U)$ is a diffeomorphism satisfying the commutative diagram below

$$\begin{array}{ccc} U \times \mathbb{R}^k & \xrightarrow{\ \phi\ } & \pi^{-1}(U) \\ & \searrow^{\pi_o} & \downarrow \pi \\ & & U \end{array}$$

where $\pi \circ \phi = \pi_o$.

In the case when E = TM, the fibre above p is just T_pM.

A *cross section* is a mapping $\psi : M \longrightarrow E$ such that

$$\pi\psi(p) = p, \qquad \text{for all } p \in M.$$

The set of all cross section is denoted by $\Gamma(E)$. We note that

i) for $\psi_1, \psi_2 \in \Gamma(E)$ and $p \in M$,

$$(\psi_1 + \psi_2)(p) = \psi_1(p) + \psi_2(p)$$

ii) for $\psi \in \Gamma(E)$ and a function $f \in F(M)$,

$$(f\psi)(p) = f(p)\psi(p)$$

Therefore, $\Gamma(E)$ forms a module over the ring $F(M)$. We note that in the case when E = TM, a cross section X defines a vector field on M.

Let U and V be finite dimensional vector spaces and F(U,V) be the free vector space over $\mathbb{R}$ whose generators are the points of $U \times V$. Hence, F(U,V) consists of all finite linear combinations of pairs (u,v) with $u \in U$ and $v \in V$. Let W(U,V) be the subspace of F(U,V) generated by the set of all elements of F(U,V) of the form

$$(u_1+ u_2,v) - (u_1,v) - (u_2,v) \ , \ (u,v_1+ v_2) - (u,v_1) - (u, v_2)$$
$$(ru,v) - r(u,v) \ , \ (u,rv) - r(u,v)$$

where u_1, u_2, $u \in U$ and v_1, v_2, $v \in V$ and $r \in \mathbb{R}$. The *tensor product* of U and V is defined as the quotient space F(U,V)/W(U,V) and is denoted by $U \otimes V$.

The *contravariant tensor space of degree r* for a vector space U, $T^r(U)$ is defined as $U \otimes \ldots \otimes U$ (r times tensor product), whereas the *covariant tensor space of degree s* for a vector space U, $T_s(U)$ is defined as $U^* \otimes \ldots \otimes U^*$ (s times tensor product) where U^* is the dual vector space of U. We note that $T^1(U) = U$ and $T_1(U) = U^*$. We set $T^o = R = T_o$. The *tensor space of type (r,s)* of a vector space U, T^r_s is defined as $U \otimes \ldots \otimes U \otimes U^* \otimes \ldots \otimes U^*$ (r times tensor product of U and s times tensor product of U^*). It can be shown that $T^r_s(M) = \bigcup_{p \in M} T^r_s(T_pM)$ is a vector bundle over M with fibre $\pi^{-1}(p) = T^r_s(T_pM)$. A *tensor field of type (r,s)* is just a cross section of $T^r_s(M)$.

If $\phi : N \longrightarrow M$ is differentiable, the *differential* of f at a point p is the mapping $(\phi_*)_p : T_pN \longrightarrow T_{\phi(p)}M$ defined by

$$(\phi_*)_p(X_p)(f) = X_p(f \circ \phi)$$

where $X_p \in T_pN$ and $f \in F(M)$.

ϕ is called an *immersion* if ϕ_* is non-singular at each $p \in N$, that is, $(\phi_*)_p(T_pN) \subset T_{\phi(p)}M$ is n-dimensional. The pair (N,ϕ) is called a *submanifold* of M if ϕ is a one-to-one immersion. The map ϕ is an *imbedding* if ϕ is a one-to-one immersion which is also a homeomorphism, that is, ϕ is a map into $\phi(N)$ with its relative topology. The map ϕ is a *diffeomorphism* if ϕ maps N one-to-one onto M and the inverse map ϕ^{-1} is C^∞.

1.2 Linear connections on a manifold.

We begin this section with a definition of a linear connection on a manifold M. Let M be a real m-dimensional connected differentiable manifold. A linear connection on M is a mapping

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM) , \quad (X,Y) \longrightarrow \nabla_X Y$$

which satisfies the following conditions:

i) $\nabla_{fX + Y} Z = f\nabla_X Z + \nabla_Y Z$

ii) $\nabla_X(fY + Z) = f\nabla_X Y + (Xf)Y + \nabla_X Z$

for any $f \in F(M)$ and X, Y, $Z \in \Gamma(TM)$. The operator ∇_X is called the covariant derivative with respect to X.

We define the covariant differentiation of a function f with respect to X by

$$\nabla_X f = Xf$$

For any tensor field K of type (0,k) or (1,k), we define the covariant derivative $\nabla_X K$ with respect to X by

$$(\nabla_X K)(X_1,\ldots,X_k) = \nabla_X(K(X_1,\ldots,X_k)) - \sum_{i=1}^{k} K(X_1,\ldots,\nabla_X X_i,\ldots,X_k)$$

for any $X_i \in \Gamma(TM)$, $i = 1,\ldots.k$.

The tensor field K is said to be parallel with respect to the linear connection if $\nabla_X K = 0$ for any $X \in \Gamma(TM)$.

We define the *torsion tensor* T of type (1,2) as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for any X, Y $\in \Gamma(TM)$, where [X,Y] is the *Lie bracket* of vector fields X and Y defined by

$$[X,Y](f) = X(Yf) - Y(Xf), \qquad \text{for any } f \in F(M).$$

A linear connection ∇ with vanishing torsion tensor field is called a *torsion-free connection*.

We define the *curvature tensor* R of type (1,3) as

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for any X, Y, Z $\in \Gamma(TM)$.

A *Riemannian metric* on M is a tensor field g of type (0,2) which satisfies the following

i) $g(X,Y) = g(Y,X)$, for any X, Y $\in \Gamma(TM)$, that is, g is symmetric.

ii) $g(X,X) \geq 0$ for any $X \in \Gamma(TM)$ and $g(X,X) = 0$ if and only if $X = 0$, that is, g is positive definite.

The manifold M endowed with a Riemannian metric g is called a *Riemannian manifold.* The length of a vector X is denoted by $\|X\|$ and it is defined by $\|X\|^2 = g(X,X)$.

Next, we have a well-known theorem which can be found in [27], p. 29.

Theorem 1.1

There exists one and only one linear connection on a Riemannian manifold that satisfies the following conditions:

i) the torsion tensor T vanishes, i.e,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$$

ii) g is parallel, i.e, $\nabla_X g = 0$. Therefore, we have

$$X(g(Y,Z)) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z)$$

for any X, Y, Z $\in$ $\Gamma(TM)$.

The linear connection stated in theorem 1.1 is called the *Riemannian connection* or the *Levi-Civita connection.* It is characterized by

$$2g(\nabla_X Y,Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)$$

for any X, Y, Z $\in \Gamma(TM)$.

Next, we shall define the Riemannian curvature tensor R of type (0,4) by

$$R(X,Y,U,V) = g(R(X,Y)U,V)$$

for any X, Y, U, V $\in \Gamma(TM)$.

The Ricci tensor field is defined by

$$S(X,Y) = \sum_{i=1}^{n}\{g(R(E_i,X)Y,E_i\}$$

where $\{E_1,....E_n\}$ is a local field of orthonormal frames and X, Y $\in \Gamma(TM)$.

The scalar curvature of M is defined by

$$\rho = \sum_{i=1}^{n}\{S(E_i,E_i)\}$$

1.3 Submanifolds of a Riemannian manifold

In this section, we shall give some fundamental results concerning the geometry of submanifolds of a Riemannian manifold. These results will be used throughout this dissertation.

Let N and M be Riemannian manifolds with Riemannian metric g and $\bar{g}$ respectively. A mapping $f : N \longrightarrow M$ is called *isometric* if

$$g(X,Y) = \bar{g}(f_*X,f_*Y) \qquad \text{for any } X,\ Y \in \Gamma(TN)$$

Let N be a submanifold of a Riemannian manifold M and that T_pN has been identified as a subspace of $T_{f(p)}M$). If M has a Riemannian metric $\bar{g}$ and X_p, $Y_p \in T_pN \subset T_{f(p)}M$, then we define

$$g_p(X_p,Y_p) = \bar{g}_{f(p)}(X_p,Y_p)$$

The Riemannian metric g, defined above is called the *Riemannian metric induced by* $\bar{g}$ (or the *induced metric*). To avoid any confusion, we shall denote the metrics on N and M by g.

Let ξ be a vector of M at a point p of N that satisfies $g(X,\xi) = 0$, for any vector X of N at p. Then ξ is called a *normal vector* of N in M at p. A unit normal vector field of N in M is also called a *normal section* on N. We denote the vector bundle of all normal vectors of N in M by $T^{\perp}N$. Then the tangent bundle of M, restricted to N is the direct sum of the tangent bundle TN of N and the normal bundle $T^{\perp}N$ of N in M, that is,

$$TM = TN \oplus T^{\perp}N$$

We denote by ∇ and $\overline{\nabla}$ the Levi-Civita connection on N and M respectively.

For any $X, Y \in \Gamma(TN)$, the *Gauss formula* is given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \qquad (1.1)$$

where $h : \Gamma(TN) \times \Gamma(TN) \longrightarrow \Gamma(T^{\perp}N)$ is a normal bundle valued symmetric bilinear form of $\Gamma(TN)$ and is called the *second fundamental form* on N.

For any $X \in \Gamma(TN)$ and $\xi \in \Gamma(T^{\perp}N)$, the *Weingarten formula* is given by

$$\overline{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi \qquad (1.2)$$

where $-A_\xi X$ and $\nabla^{\perp}_X \xi$ are the tangential and normal parts of $\overline{\nabla}_X \xi$ respectively. Hence, for any $\xi \in \Gamma(T^{\perp}N)$, we have a linear operator $A_\xi : \Gamma(TN) \longrightarrow \Gamma(TN)$ that satisfies

$$g(h(X,Y),\xi) = g(A_\xi X, Y) \qquad (1.3)$$

A_ξ is called the *fundamental tensor of Weingarten* with respect to the normal section ξ. The operator $\nabla^{\perp}$ defines a linear connection on the normal bundle $T^{\perp}N$ and is called the *normal connection* on N.

The covariant derivative of h is defined as

$$(\nabla_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \qquad (1.4)$$

for all X, Y, $Z \in \Gamma(TN)$.

By using the Gauss and Weingarten formulas, we obtain

$$\overline{R}(X,Y)Z = R(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y$$
$$+ (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \qquad (1.5)$$

for all X, Y, Z $\in \Gamma(TN)$, where R and $\bar{R}$ are the curvature tensors of N and M respectively. Equation (1.5) then gives us the following equation.

$$g(\bar{R}(X,Y)Z,U) = g(R(X,Y)Z,U) + g(h(X,Z),h(Y,U)) - g(h(Y,Z),h(X,U)) \qquad (1.6)$$

and $$(\bar{R}(X,Y)Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \qquad (1.7)$$

for all X, Y, Z, U $\in \Gamma(TN)$.

Equation (1.6) is called the *Gauss equation* whereas equation (1.7) is known as the *Codazzi equation.*

The curvature tensor $R^{\perp}$ of the normal connection $\nabla^{\perp}$ is defined by

$$R^{\perp}(X,Y)\xi = \nabla^{\perp}_X\nabla^{\perp}_Y\xi - \nabla^{\perp}_Y\nabla^{\perp}_X\xi - \nabla^{\perp}_{[X,Y]}\xi \qquad (1.8)$$

for all X, Y $\in \Gamma(TN)$ and $\xi \in \Gamma(T^{\perp}N)$. For ζ, $\xi \in \Gamma(T^{\perp}N)$ we define

$$[A_{\xi},A_{\zeta}] = A_{\xi}\circ A_{\zeta} - A_{\zeta}\circ A_{\xi} \qquad (1.9)$$

and by using the Gauss and Weingarten formulas, we have

$$g(\bar{R}(X,Y)\xi,\zeta) = g(R^{\perp}(X,Y)\xi,\zeta) + g([A_{\zeta},A_{\xi}]X,Y) \qquad (1.10)$$

for any X, Y $\in \Gamma(TN)$ and ζ, $\xi \in \Gamma(T^{\perp}N)$. Equation (1.10) is called the *Ricci equation.*

A normal vector field ξ on N is said to be *parallel* if $\nabla^{\perp}_X\xi = 0$, for any X $\in \Gamma(TN)$. A submanifold N is *totally geodesic* if its second fundamental form vanishes identically, that is, $h = 0$, or equivalently $A_{\xi} = 0$, for any $\xi \in \Gamma(T^{\perp}N)$. If $A_{\xi} = \alpha I$ where α is a differential function and I is the identity morphism on $\Gamma(TN)$, then ξ is called an *umbilical section* on N, or N is said to be *umbilical with respect to* ξ. If the submanifold N is umbilical with respect to every local normal section of N, then N is said to be *totally umbilical*.

Let $\{E_1,\dots E_n\}$ be an orthonormal basis in T_pN, for a point p in N. Then the *trace* of h is given by

$$tr(h) = \sum_{i=1}^{n}\{h(E_i,E_i)\}$$

and it is independent of the basis. We define the *mean curvature* vector of N by

$$H = \frac{1}{n}\,tr(h)$$

If H = 0 on N, then N is called a *minimal submanifold* of M. We note that N is totally umbilical if and only if

$$h(X,Y) = g(X,Y)H$$

for any $X,\ Y \in \Gamma(TN)$.

1.4 Distributions on a manifold

An r-dimensional *distribution* on a manifold M is a mapping D defined on M, which assigns to each point p of M an r-dimensional linear subspace D_p of T_pM. A vector field X is said to belong to D if we have $X_p \in D_p$ for every $p \in M$. We denote this by $X \in \Gamma(D)$. The distribution D is said to be differentiable if for any $p \in M$, there exist r differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighborhood of p. The distributions discussed in this dissertation are supposed to be differentiable of class C^{∞}.

A submanifold N of M is said to be an *integral manifold* of D, if for every $p \in M$, $f_*(T_pN) = D_p$, where f is the imbedding of N into M. This means that D_p coincides with the tangent space to M at p. The distribution D is said to be *integrable* if for every $p \in M$, there exists an integral manifold of D containing p. If there exists no integral manifold of D which properly contains N, then N is called the *maximal integral manifold* or *leaf* of D.

The distribution D is said to be *involutive* if for all $X, Y \in \Gamma(D)$, we have $[X,Y] \in \Gamma(D)$. The following is the classical theorem of Frobenuis which is found in [3], p. 8.

Theorem 1.2

Let D be an involutive distribution on a manifold M. Then D is integrable and through every point $p \in M$, there passes an unique maximal integral manifold of D. Any integral manifold through p is an open submanifold of the maximal one.

Suppose N is a Riemannian manifold endowed with two complementary distributions, that is, $TN = D \oplus D^{\perp}$ and ∇ be its Levi-Civita connection. The distribution D is said to be *parallel* with respect to ∇ if

$$\nabla_X Y \in \Gamma(D)$$

for any $X \in \Gamma(TN)$ and $Y \in \Gamma(D)$.

Similarly, the distribution $D^{\perp}$ is said to be parallel with respect to ∇ if

$$\nabla_X Z \in \Gamma(D^{\perp})$$

for any $X \in \Gamma(TN)$ and $Z \in \Gamma(D^{\perp})$. We then have the following theorems found in Bejancu [3].

Theorem 1.3 (Bejancu [3])

Both distributions D and $D^{\perp}$ are parallel with respect to the Levi-Civita connection ∇ if and only if they are integrable and their leaves are totally geodesic in N

Proof:

Suppose both distributions D and $D^{\perp}$ are parallel with respect to ∇. Thus,

$$[X,Y] = \nabla_X Y - \nabla_Y X \in \Gamma(D)$$

for any X, $Y \in \Gamma(D)$,

and $$[U,V] = \nabla_U V - \nabla_V U \in \Gamma(D^{\perp})$$

for any U, $V \in \Gamma(D^{\perp})$.

Hence, by theorem 1.2, D and $D^{\perp}$ are integrable. Next, we let N′ be a leaf of D and ∇' be the Levi-Civita connection with respect to N′. Then, by the Gauss formula, we have

$$h'(X,Y) = \nabla_X Y - \nabla'_X Y \tag{1.11}$$

for any X, $Y \in \Gamma(TN')$, where h′ is the second fundamental form of N′ on N. Since $\nabla_X Y$, $\nabla'_X Y \in \Gamma(TN')$ and $h'(X,Y) \in \Gamma(D^{\perp})$, by equation (1.11), we obtain h = 0, that is N′ is totally geodesic in N. In a similar way, it follows that each leaf of $D^{\perp}$ is totally geodesic in N.

Conversely, suppose D and $D^{\perp}$ are integrable and their leaves are totally geodesic in N. Then, by using the Gauss formula for the immersions of the leaves of D and $D^{\perp}$ in N, we obtain

$$\nabla_X Y \in \Gamma(D), \quad \text{for any } X,\ Y \in \Gamma(D)$$

and $$\nabla_U V \in \Gamma(D^{\perp}), \quad \text{for any } U,\ V \in \Gamma(D^{\perp})$$

Since g is parallel with respect to ∇, we have

$$g(\nabla_U Y, V) = -g(Y, \nabla_U V) = 0$$

and

$$g(\nabla_X U, Y) = -g(U, \nabla_X Y) = 0$$

for any X, $Y \in \Gamma(D)$ and U, $V \in \Gamma(D^{\perp})$. Hence, both D and $D^{\perp}$ are parallel.

QED

<u>Theorem 1.4</u> (Bejancu [3])

The distribution D is parallel with respect to the Levi-Civita connection ∇ if and only if the complementary orthogonal distribution $D^{\perp}$ is parallel with respect to ∇.

Proof:

Suppose D is parallel. Since g is parallel with respect to ∇, we have

$$g(Y, \nabla_X U) = -g(\nabla_X Y, U) = 0$$

for any $X \in \Gamma(TN)$, $Y \in \Gamma(D)$ and $U \in \Gamma(D^{\perp})$. Therefore, $\nabla_X U \in \Gamma(D^{\perp})$ and thus, $D^{\perp}$ is parallel.

The converse is proved in a similar way.

QED

1.5 Almost Hermitian manifold.

Let M be a real differentiable manifold. A tensor field J of type (1,1) on M is called *an almost complex structure* on M if at every point $p \in M$, J is an endomorphism of the tangent space T_pM such that $J^2 = -I$. A manifold M with a fixed almost complex structure J is called *an almost complex manifold.* It is a well-known fact every almost complex manifold is of even dimension and is orientable (see Kobayashi-Nomizu [10]). A complex manifold M also carries a natural almost complex structure.

Next we define the *torsion tensor field* of type (1,2) of an almost complex structure J by

$$[J,J](X,Y) = [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y]$$

for any $X, Y \in \Gamma(TM)$. The torsion tensor of J is also known as the *Nijenhuis tensor* of J. We shall now state the condition for an almost complex manifold M to be a complex manifold.

Theorem 1.5 (Newlander-Nirenburg [18])

Let M be an almost complex manifold with an almost complex structure J. Then J is a complex structure if and only if J has no torsion.

We state the following integrability theorem of almost complex manifold which is found in [27], p. 113.

Theorem 1.6

Let M be a real 2m-dimensional almost complex manifold with almost complex structure J. Suppose there exists an open covering $\{U_i\}$ of M satisfying the following conditions: There is a local coordinate system $(x_1,\ldots,x_m,y_1,\ldots y_m)$ on each U_i, such that for each point of U_i,

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \quad \text{and} \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

for $j = 1,\ldots m$.

Then M is a complex manifold.

A *Hermitian metric* on an almost complex manifold M is a Riemannian metric g satisfying

$$g(JX,JY) = g(X,Y),$$

for any $X, Y \in \Gamma(TN)$.

An almost complex manifold endowed with a Hermitian metric is called an *almost Hermitian manifold.* It is easily proved that every almost complex manifold with a Riemannian metric h admits a Hermitian metric g.

The fundamental 2-form Ω of an almost Hermitian manifold M is defined by

$$\Omega(X,Y) = g(X,JY), \qquad \text{for any } X, Y \in \Gamma(TM).$$

We note that

$$\bar{\nabla}_X\Omega(Y,Z) = g(Y,(\nabla_X J)Z)$$

and $$3d\Omega(X,Y,Z) = g((\bar{\nabla}_X J)Y,Z) + g((\bar{\nabla}_Y J)Z,X) + g((\bar{\nabla}_Z J)X,Y)$$

for all $X, Y, Z \in \Gamma(TM)$.

An almost Hermitian manifold M is said to be a *Kaehler manifold* if $(\bar{\nabla}_X J)Y = 0$ for all $X, Y \in \Gamma(TM)$ and an *almost Kaehler* *manifold* if $d\Omega(X, Y, Z) = 0$, for all $X, Y, Z \in \Gamma(TM)$. M is called a *nearly Kaehler manifold* if $(\bar{\nabla}_X J)X = 0$, for any $X \in \Gamma(TM)$. We note that M is a nearly-Kaehler manifold if and only if

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0, \qquad \text{for any } X, Y \in \Gamma(TM).$$

A *quasi-Kaehler manifold* is an almost Hermitian manifold that satisfies the following

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_{JX} J)JY = 0, \qquad \text{for any } X, Y \in \Gamma(TM).$$

Gray [12] proved that every almost Kaehler or nearly Kaehler manifold is necessarily a quasi-Kaehler manifold while a Kaehler manifold is both an almost Kaehler and nearly-Kaehler manifold. The following theorem is found in Bejancu [3].

Theorem 1.7 (Bejancu [3])

Let M be a nearly-Kaehler manifold. Then the Nijenhuis tensor of J is given by

$$[J,J](X,Y) = 4J(\bar{\nabla}_Y J)X$$

for any $X, Y \in \Gamma(TM)$.

Proof:

From the definition of the Nijenhuis tensor and since $\bar{\nabla}$ is a torsion free connection, we obtain

$$[J,J](X,Y) = (\bar{\nabla}_{JX} J)Y - (\bar{\nabla}_{JY} J)X + J((\bar{\nabla}_Y J)X) - J((\bar{\nabla}_X J)Y) \tag{1.12}$$

Since M is nearly-Kaehler, we have

$$(\bar{\nabla}_{JY} J)X = -(\bar{\nabla}_X J)JY = \bar{\nabla}_X Y + J(\bar{\nabla}_X JY) = J((\bar{\nabla}_X J)Y)$$

Therefore (1.12) becomes

$$\begin{aligned}[J,J](X,Y) &= 2(\overline{\nabla}_Y X + J(\overline{\nabla}_Y JX) - \overline{\nabla}_X Y - J(\overline{\nabla}_X JY)) \\ &= 2\{J(\overline{\nabla}_Y JX - J(\overline{\nabla}_Y X)) - J(\overline{\nabla}_X JY - J(\overline{\nabla}_X Y))\} \\ &= 2J((\overline{\nabla}_Y J)X - (\overline{\nabla}_X J)Y) \\ &= 4J((\overline{\nabla}_Y J)X)\end{aligned}$$

QED