CHAPTER TWO : CR-SUBMANIFOLDS

2.1 Introduction

Let M be an m-dimensional almost Hermitian manifold with almost complex structure J and with Hermitian metric g and N be an n-dimensional Riemannian manifold isometrically immersed in M.

N is called a *(almost) complex (holomorphic) submanifold* of M if T_xN is invariant by J, that is

$$J(T_xN) = T_xN, \qquad \text{for each } x \in N$$

N is called a *totally real (anti-invariant) submanifold* of M if

$$J(T_xN) \subset T_x^{\perp}N, \qquad \text{for each } x \in N$$

These two classes of submanifolds have been studied extensively in the last decade. For instance, results on the geometry of totally real submanifolds can be found in Yano-Kon [25] and a survey on the geometry of complex submanifolds can be found in Ogiue [19]. Later, Bejancu [1] introduced a study on a submanifold which generalizes both a complex submanifold and a totally real submanifold. This new class of submanifolds, situated between the above two classes is called the CR-submanifolds.

Definition

N is a *CR-submanifold* of M if there exists a differentiable distribution

$$D : x \longrightarrow D_x \subset T_xN$$

on N that satisfies the following conditions:

(i) D is holomorphic, that is,

$$J(D_x) = D_x , \qquad \text{for each } x \in N$$

(ii) the complementary orthogonal distribution

$$D^\perp : x \longrightarrow D_x^\perp \subset T_xN$$

is anti-invariant, that is,

$$J(D_x^\perp) \subset T_x^\perp N , \qquad \text{for each } x \in N$$

We let p be the complex dimension of the distribution D and q the real dimension of the distribution $D^\perp$. When p=0, a CR-submanifold becomes a totally real submanifold and when q=0, a CR-submanifold becomes a complex submanifold. A *proper CR-submanifold* is a CR-submanifold which is neither a complex submanifold nor a totally real submanifold.

2.2 Examples of CR-submanifolds

In this section, we give an elaboration of the proof of two examples which are found in Blair-Chen [7].

Example 1:

Any real hypersurface N of M is a CR-submanifold.

Proof :

For any point $x \in M$, let $\{X_1, JX_1, \ldots, X_{m-1}, JX_{m-1}, Z, JZ\}$ be a basis of T_xM, where $Z \in \Gamma(T_x^\perp N)$ and $\{X_1, JX_1, \ldots, X_{m-1}, JX_{m-1}, JZ\}$ is a basis of T_xN. We define a distribution D_x such that D_x be spanned by $\{X_1, JX_1, \ldots, X_{m-1}, JX_{m-1}\}$ and the complementary distribution, $D_x^\perp$ be spanned by {JZ}. Hence, we obtain

$$J(D_x) = D_x$$

and $$J(D_x^\perp) = \{ J(JZ) = -Z,\ Z \in \Gamma(T^\perp N) \} \subset T_x^\perp N$$

This proves that the hypersurface N of M is a CR-submanifold.

QED

Example 2:

Before we begin with example 2, we first give a definition of an anti-holomorphic submanifold.

Definition

A submanifold N of an almost Hermitian manifold M is *anti-holomorphic* if

$$J(T_x^\perp N) \subset T_x N, \qquad \text{for each } x \in N$$

We proceed to prove the following:

(1) If N is an anti-holomorphic submanifold of an almost Hermitian manifold M, then

$$\dim N \geq \frac{1}{2} \dim M$$

(2) Let N be an anti-holomorphic submanifold of an almost Hermitian manifold M.

i) If $\dim N = \frac{1}{2} \dim M$, then N is a CR-submanifold with

$$D_x^\perp = T_x N \qquad \text{and} \qquad D_x = \{0\}$$

In this case, N is not only a CR-submanifold but also a totally real submanifold.

ii) If $\dim N > \frac{1}{2}\dim M$, then N is a CR-submanifold with

$$D_x^{\perp} = J(T_x^{\perp}N) \qquad \text{and} \qquad D_x = (\ J(T_x^{\perp}N)\ )^{\perp\!\!\perp}$$

where $(\ J(T_x^{\perp}N)\ )^{\perp\!\!\perp} = \{\ X \in T_xN : g(X,Y) = 0,\ Y \in J(T_x^{\perp}N)\}$.

Proof of (1) :

From the definition of an anti-holomorphic submanifold of M, it follows that

$$\dim (J(T_x^{\perp}N)) \le \dim T_xN$$

Since J is a endormorphism, thus we obtain

$$\dim (T_x^{\perp}N) \le \dim T_xN \tag{2.1}$$

By taking account of (2.1) and $T_xM = T_xN \oplus T_x^{\perp}N$,

$$\dim T_xM = \dim T_xN + \dim T_x^{\perp}N$$

$$\le 2\dim T_xN \quad , \qquad \text{by (2.1)}$$

Therefore $\quad \dim T_xN \ge \frac{1}{2}\dim T_xM$

Since $\dim N = \dim T_xN$ and $\dim M = \dim T_xM$, thus

$$\dim N \ge \frac{1}{2}\dim M$$

QED

Proof of (2) :

Suppose $\dim N = \frac{1}{2}\dim M$. Therefore,

$$2\dim T_xN = \dim T_xM$$

$$= \dim T_xN + \dim T_x^{\perp}N$$

Thus, $\quad \dim T_xN = \dim T_x^{\perp}N$

Since J is a endormorphims, therefore

$$\dim T_x^\perp N = \dim J(T_x N)$$

that is, $$J(T_x N) = T_x^\perp N$$

Hence, $$J(T_x N) \subset T_x^\perp N$$

We let $D_x = \{0\}$ and $D_x^\perp = T_x N$. Then, N is not only a CR-submanifold but also a totally real submanifold of M with

$$T_x M = T_x N \oplus J(T_x N)$$

Next, suppose $\dim N > \frac{1}{2} \dim M$. We now prove that D and $D^\perp$, as defined, is holomorphic and anti-invariant respectively. We note that

$$J(D_x^\perp) = J(J(T_x^\perp N)) = T_x^\perp N$$

This means that, $J(D_x^\perp) \subset T_x^\perp N$. Therefore $D^\perp$ is anti-invariant.

For any $X \in (J(T_x^\perp N))^{\shortparallel}$, we have

$$JX = X_1 + X_2 + X_3$$

where $X_1 \in J(T_x^\perp N)$, $X_2 \in (J(T_x^\perp N))^{\shortparallel}$ and $X_3 \in T_x^\perp N$

We also note that, for any $X' \in T_x^\perp N$, we have

$$\begin{aligned} 0 &= g(X,X') \\ &= g(JX,JX') \\ &= g(X_1,JX') + g(X_2,JX') + g(X_3,JX') \\ &= g(X_1,JX'), \end{aligned}$$

since $g(X_2,JX') = g(X_3,JX') = 0$. Therefore, $X_1 = 0$.

We also have,

$$g(X_3,X') = g(JX - X_1 - X_2 , X'),$$

$$= g(JX - X_2\ ,\ X'), \qquad \text{since } X_1 = 0$$
$$= g(JX,X') - g(X_2,X')$$
$$= g(JX,X')\ , \qquad \text{since } g(X_2,X') = 0$$

Since $J(J(T_x^{\perp}N)) = T_x^{\perp}N$, so for $X' \in T_x^{\perp}N$, there exist $X'' \in J(T_x^{\perp}N)$ such that $X' = JX''$. Therefore

$$g(X_3,X') = g(JX,X')$$
$$= g(JX,JX'')$$
$$= g(X,X'')$$
$$= 0$$

This implies that $X_3 = 0$. Hence, for $X \in (J(T_x^{\perp}N))^{\perp\!\!\perp}$,

$$JX = X_2$$

where $X_2 \in (J(T_x^{\perp}N))^{\perp\!\!\perp}$. This proves that D is holomorphic and thus, N is a CR-submanifold of M with

$$D_x = (J(T_x^{\perp}N))^{\perp\!\!\perp} \qquad \text{and} \qquad D_x^{\perp} = J(T_x^{\perp}N)$$

QED

2.3 Characterization of a CR-submanifold

Let N be an arbitrary Riemannian manifold isometrically immersed in an almost Hermitian manifold M.

For each vector field $X \in \Gamma(TN)$, we put

$$JX = \phi X + \omega X \tag{2.2}$$

where ϕX is the tangent part and ωX is the normal part of JX.

Similarly, for each vector field $\xi \in \Gamma(T^{\perp}N)$, we put

$$J\xi = B\xi + C\xi \tag{2.3}$$

where $B\xi$ is the tangent part and $C\xi$ is the normal part of $J\xi$.

We begin this section with a theorem found in Bejancu [3] which characterizes a CR-submanifold of an almost Hermitian manifold M.

Theorem 2.1 (Bejancu [3], p. 21)

The submanifold N of M is a CR-submanifold if and only if

$$\text{rank } (\phi) = \text{constant}$$

and

$$\omega \circ \phi = 0$$

In this section, we give a proposition which is a slight improvement of the theorem given above. The idea of the proof of the proposition here is almost similar to a theorem found in Yano-Kon [26].

Proposition 2.1

The submanifold N of M is a CR-submanifold if and only if

$$\omega \circ \phi = 0$$

Proof:

Suppose N is a CR-submanifold of M. We let P and Q be the projection of TN to D and $D^{\perp}$ respectively. So, for any $X \in \Gamma(TN)$,

$$JX = \phi X + \omega X$$

$$J(PX + QX) = \phi X + \omega X$$

$$JPX + JQX = \phi X + \omega X \qquad (2.4)$$

Since $PX \in \Gamma(D)$ and $QX \in \Gamma(D^{\perp})$, thus we have $JPX \in \Gamma(D)$, that is,

$JPX \in \Gamma(TN)$ and $JQX \in \Gamma(T^{\perp}N)$. Thus, by comparing the tangential and normal parts of (2.4), we have

$$JPX = \phi X \tag{2.5}$$

$$JQX = \omega X \tag{2.6}$$

Therefore

$$\omega \circ \phi(X) = \omega(JPX) = JQ(JPX) = 0, \quad \text{since } JPX \in \Gamma(D).$$

Conversely, suppose $\omega \circ \phi = 0$. We now prove that the rank of ϕ is a constant and therefore, by theorem 2.1, N is a CR-submanifold.

For any $X \in \Gamma(TN)$, we have

$$JX = \phi X + \omega X$$

$$J^2X = J\phi X + J\omega X$$

By applying (2.2), (2.3) and the assumption $\omega \circ \phi = 0$, we have

$$-X = \phi^2 X + B \circ \omega X + C \circ \omega X$$

which gives us

$$-I = \phi^2 + B \circ \omega + C \circ \omega$$

By comparing the tangential and normal parts of the equation above, we have

$$\phi^2 = -I - B \circ \omega \tag{2.7}$$

and

$$C \circ \omega = 0 \tag{2.8}$$

For any $X \in \Gamma(TN)$ and $\xi \in \Gamma(T^{\perp}N)$,

$$\begin{aligned} g(\omega X, \xi) + g(X, B\xi) &= g(JX - \phi X, \xi) + g(X, J\xi - C\xi) \\ &= g(JX, \xi) - g(\phi X, \xi) + g(X, J\xi) - g(X, C\xi) \end{aligned}$$

$$= -g(X, J\xi) + g(X, J\xi)$$

$$= 0$$

Thus, we have $$g(\omega X, \xi) + g(X, B\xi) = 0$$

For $\xi \in \Gamma(T^{\perp}N)$, there exists $\xi' \in \Gamma(T^{\perp}N)$ such that $\xi = C\xi'$. Hence,

$$g(\omega X, C\xi') + g(X, B\circ C\xi') = 0.$$

Therefore, we obtain

$$g(X, B\circ C\xi') = -g(\omega X, C\xi') = -g(\omega X, J\xi' - B\xi')$$

$$= -g(\omega X, J\xi') + g(\omega X, B\xi')$$

$$= g(J\omega X, \xi')$$

$$= g(B\circ\omega X + C\circ\omega X, \xi')$$

$$= 0, \quad \text{since } C\circ\omega = 0.$$

Hence $$B\circ C = 0 \qquad (2.9)$$

Similarly, from (2.3), we obtain $J^2\xi = JB\xi + JC\xi$. By applying (2.2) and (2.3), we obtain

$$-\xi = \phi\circ B\xi + \omega\circ B\xi + B\circ C\xi + C^2\xi$$

$$= \phi\circ B\xi + \omega\circ B\xi + C^2\xi, \qquad \text{by (2.9)}$$

By comparing the tangential part of the equation above,

$$\phi\circ B = 0 \qquad (2.10)$$

From (2.7), $$\phi^2 = -I - B\circ\omega$$

$$\phi^3 = -\phi - \phi\circ B\omega = -\phi, \qquad \text{by (2.10)}$$

Therefore $$\phi^3 + \phi = 0 \qquad (2.11)$$

(2.11) tells us that ϕ define an f-structure on $\Gamma(TN)$ as defined by Yano [24]. Hence, the rank of ϕ is a constant (see Stong [21]) and by theorem 2.1, N is a CR-submanifold.

QED

From the proof of proposition 2.1, we obtain the following proposition.

Proposition 2.2

The submanifold N of an almost Hermitian manifold M is a CR-submanifold if and only if $C\circ\omega = 0$.

Proof:

For any $X \in \Gamma(TN)$, we have

$$JX = \phi X + \omega X$$

By applying J to the equation above and using (2.2) and (2.3), we obtain

$$-I = \phi^2 + \omega\circ\phi + B\circ\omega + C\circ\omega$$

By comparing the normal part,

$$\omega\circ\phi + C\circ\omega = 0$$

Hence, it is clear that N is a CR-submanifold of M if and only if $C\circ\omega = 0$.

QED

Let N be a CR-submanifold of an almost Hermitian manifold M. We made the following observations:

(1) From the proof of proposition 2.1, we observed that

$$D_x = \text{Im } \phi_x, \qquad \text{for each } x \in N$$

and $$\text{rank}(\phi) = \text{constant}$$

(2) For each $x \in N$, we observed that

$$D_x^{\perp} = \text{Im } B_x$$

Proof:

For any $\xi \in \Gamma(T^{\perp}N)$ and $X \in \Gamma(D)$, we have

$$g(B\xi,X) = g(B\xi+C\xi,\ X) = g(J\xi,X) = -g(\xi,JX) = 0.$$

Thus, $B\xi \in \Gamma(D^{\perp})$, which implies that $\text{Im } B_x \subseteq D_x^{\perp}$.

Conversely, for any $Y \in \Gamma(D^{\perp})$, we have,

$$Y = -J(JY) = J(-JY) = B(-JY) + C(-JY)$$

By comparing the tangential part, we obtain,

$$Y = B(-JY),$$

which implies that $D_x^{\perp} \subseteq \text{Im } B_x$. Hence, we conclude that

$$D_x^{\perp} = \text{Im } B_x$$

(3) Let μ be the complementary orthogonal vector bundle of $JD^{\perp}$ in $T^{\perp}N$, that is, $T^{\perp}N = JD^{\perp} \oplus \mu$. We observed that $J(\mu_x) = \mu_x$, $x \in N$.

For any $\xi \in \Gamma(\mu)$ and $Z \in \Gamma(D^{\perp})$,

$$0 = g(J\xi,Z) = g(B\xi + C\xi,Z) = g(B\xi,Z)$$

Thus, $B\xi = 0$, and so, $J\xi = C\xi \in \Gamma(T^{\perp}N)$, for any $\xi \in \Gamma(\mu)$.

We note that, for any $\xi \in \Gamma(\mu)$ and $Z \in \Gamma(D^{\perp})$,

$$\begin{aligned} g(C\xi, JZ) &= g(J\xi, JZ), \qquad \text{since } J\xi = C\xi \\ &= g(\xi, Z) \\ &= 0 \end{aligned}$$

that is, $C\xi \in \Gamma(\mu)$. Therefore, we have

$$J(\mu_x) \subseteq \mu_x, \qquad \text{for any } x \in N$$

Since $J\xi = C\xi \in \Gamma(\mu)$, we have

$$\xi = -JC\xi \in \Gamma(J\mu)$$

that is, $\mu_x \subseteq J(\mu_x)$, for any $x \in N$. Hence, $J(u_x) = \mu_x$.

(4) Finally we observed that ϕ and C define f-structures on TN and $T^{\perp}N$ respectively. By applying ϕ on equation (2.5), we obtain

$$\phi(\phi X) = \phi JPX = JP(JPX) = J(JPX) = -PX$$

Hence, we have, $\qquad \phi^2 = -P$

Thus, $\quad \phi(\phi^2 X) = -\phi(PX) = -JP(PX) = -JPX = -\phi X$, that is,

$$\phi^3 + \phi = 0$$

Similarly, from

$$\begin{aligned} J\xi &= B\xi + C\xi, \qquad \text{for } \xi \in \Gamma(T^{\perp}N) \\ J^2\xi &= JB\xi + JC\xi \\ -\xi &= \phi\circ B\xi + \omega\circ B\xi + B\circ C\xi + C^2\xi \end{aligned}$$

By taking the normal part,

$$C^2\xi + \omega\circ B\xi + \xi = 0$$

$$C^3\xi + C\circ\omega\circ B\xi + C\xi = 0$$

$$C^3 + C = 0, \qquad \text{since } C\circ\omega = 0$$

We summarize our observations in the proposition below:

Proposition 2.3

Let N be a CR-submanifold of an almost Hermitian manifold M. Then, for each $x \in N$,

(i) $D_x = \operatorname{Im} \phi_x$

(ii) $\operatorname{rank} \phi = \text{constant}$

(iii) $D_x^{\perp} = \operatorname{Im} B_x$

(iv) $J(\mu_x) = \mu_x$, that is μ_x is invariant by the almost complex structure J.

(iv) ϕ and C define f-structures on TN and $T^{\perp}N$ respectively.

2.4 Integrability Conditions of Distributions On A CR-submanifold

Let N be a CR-submanifold of an almost Hermitian manifold M. In this section, we give some of the integrability condition of the distributions D and $D^{\perp}$ on a CR-submanifold.

The Nijenhuis tensor field of ϕ is given by

$$[\phi,\phi](X,Y) = [\phi X,\phi X] + \phi^2[X,Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$$

Similarly, for an almost complex structure J, we have

$$[J,J](X,Y) = [JX,JY] + J^2[X,Y] - J[X,JY] - J[JX,Y]$$

for any $X,Y \in \Gamma(D)$.

We first begin with a few theorems found in Bejancu [4] and [3].

Theorem 2.2 (Bejancu [4])

Let N be a CR-submanifold of an almost Hermitian manifold M. For any X, Y $\in \Gamma(D)$, the distribution D is integrable if and only if

$$[J,J](X,Y)^{\perp} = 0 \quad \text{and} \quad Q[\phi,\phi](X,Y) = 0$$

where $[J,J](X,Y)^{\perp}$ is the normal part of $[J,J](X,Y)$ to N.

Theorem 2.3 (Bejancu [3], p.26)

Let N be a CR-submanifold of a Hermitian manifold M. The distribution D is integrable if and only if

$$[\phi,\phi](X,Y) = 0$$

for any $X,Y \in \Gamma(D)$.

Theorem 2.4 (Bejancu [3], p. 26)

Let N be a CR-submanifold of an almost Hermitian manifold M. The distribution $D^{\perp}$ is integrable if and only if

$$[\phi,\phi](X,Y) = 0$$

for any $X,Y \in \Gamma(D^{\perp})$.

The following theorem is found in Kon-Tan [15]. It is an improvement of theorem 2.2.

Theorem 2.5 (Kon-Tan [15])

Let N be a CR-submanifold of an almost Hermitian manifold M. Then the distribution D is integrable if and only if

$$Q[\phi,\phi](X,Y) = 0$$

for any $X,Y \in \Gamma(D)$.

Proof:

If D is integrable, then from theorem 2.2, we have

$$Q[\phi,\phi](X,Y) = 0$$

for any $X,Y \in \Gamma(D)$.

Conversely, observe that

$$[\phi,\phi](X,Y) = [\phi X,\phi Y] - P[X,Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$$

and the last three terms are in $\Gamma(D)$. Thus, by the assumption,

$$\begin{aligned} 0 = Q[\phi,\phi](X,Y) &= Q[\phi X,\phi Y] \\ &= Q[JPX,JPY] \\ &= Q[JX,JY] \end{aligned}$$

for any X, $Y \in \Gamma(D)$. This tells us that $[JX,JY] \in \Gamma(D)$ and so, $[X,Y] \in \Gamma(D)$.

QED

2.5 CR-submanifolds of a Kaehler manifold

In this section, we will give a few results for a CR-submanifold N, of a Kaehler manifold M. Later, we discovered that with a few added conditions, some of the results here could be generalized to a nearly-Kaehler manifold. We will do this in the next chapter.

Let M be a Kaehler manifold. Then for any X, $Y \in \Gamma(TM)$, we have

$$(\overline{\nabla}_X J)Y = 0 \tag{2.12}$$

where $\overline{\nabla}$ is the Levi-Civita connection on M. Let N be a CR-submanifold of M. Similarly, as in the previous section, we let the holomorphic distribution D and the anti-invariant distribution $D^{\perp}$ be defined respectively by the projections P and Q.

For any X, Y $\in$ $\Gamma(TN)$ and taking account of equations (2.12), (2.2) and the Gauss formula, we have

$$\bar{\nabla}_X \phi Y + \bar{\nabla}_X \omega Y - J(\nabla_X Y + h(X,Y)) = 0,$$

where ∇ is the Levi-Civita connection on N. By using the Weingarten and Gauss formula again, the above equation becomes

$$\nabla_X \phi Y + h(X,\phi Y) - A_{\omega Y}X + \nabla^{\perp}_X \omega Y - J\nabla_X Y - Jh(X,Y) = 0$$

and thus,

$$P(\nabla_X \phi Y) + Q(\nabla_X \phi Y) - P(A_{\omega Y}X) - Q(A_{\omega Y}X) - \phi(\nabla_X Y) - \omega(\nabla_X Y) + h(X,\phi Y)$$
$$+ \nabla^{\perp}_X \omega Y - Bh(X,Y) - Ch(X,Y) = 0$$

where ϕ, ω, B and C are as defined in equations (2.2) and (2.3) of section 2.3. By comparing the tangential and normal parts, we obtain the following equations:-

$$P(\nabla_X \phi Y) - P(A_{\omega Y}X) = \phi(\nabla_X Y) \tag{2.13}$$

$$Q(\nabla_X \phi Y) - Q(A_{\omega Y}X) = Bh(X,Y) \tag{2.14}$$

$$h(X,\phi Y) + \nabla^{\perp}_X \omega Y = \omega(\nabla_X Y) + Ch(X,Y) \tag{2.15}$$

Similarly, for any X $\in$ $\Gamma(TN)$ and V $\in$ $\Gamma(T^{\perp}N)$ and taking account of equations (2.12), (2.3) and the Gauss and Weingarten equations, we have,

$$\bar{\nabla}_X BV + \bar{\nabla}_X CV - J(-A_V X + \nabla^{\perp}_X V) = 0$$

$$\nabla_X BV + h(X,BV) - A_{CV}X + \nabla^{\perp}_X CV + J(A_V X) - J(\nabla^{\perp}_X V) = 0$$

and thus,

$$P(\nabla_X BV) + Q(\nabla_X BV) - P(A_{CV}X) - Q(A_{CV}X) + \phi(A_V X) + \omega(A_V X) - B(\nabla^{\perp}_X V)$$
$$- C(\nabla^{\perp}_X V) + h(X,BV) + \nabla^{\perp}_X CV = 0.$$

By comparing the tangential and normal parts, we obtain the following:-

$$P(\nabla_X BV) + \phi(A_V X) = P(A_{CV}X) \quad (2.16)$$

$$Q(\nabla_X BV) = Q(A_{CV}X) + B(\nabla^{\perp}_X V) \quad (2.17)$$

$$h(X,BV) + \nabla^{\perp}_X CV + \omega(A_V X) = C(\nabla^{\perp}_X V) \quad (2.18)$$

We note that equations (2.13) - (2.18) can be found in Bejancu [3], p. 41.

Blair-Chen [7] proved the following theorem for the distribution D of a submanifold (not necessary a CR-submanifold) of a Kaehler manifold.

Theorem 2.6 (Blair-Chen [7])

Let N be a submanifold of a Kaehler manifold M and D_X the maximal holomorphic subspace of $T_X N$, with constant dimension. Then the holomorphic distribution D is integrable if and only if the second fundamental form satisfies

$$h(X,JY) = h(JX,Y)$$

for all $X, Y \in \Gamma(D)$.

Therefore, by combining theorem 2.6 with a result found in Bejancu [1], we have

Theorem 2.7 (Blair-Chen [7], Bejancu [1])

Let N be a CR-submanifold of a Kaehler manifold M. Then

i) the distribution $D^{\perp}$ is integrable

ii) the distribution D is integrable if and only if

$$h(X,JY) = h(JX,Y)$$

for all $X,Y \in \Gamma(D)$.

We also have a theorem which is a combination of results obtained from Chen [8] and Bejancu-Kon-Yano [6].

Theorem 2.8 (Chen [8], Bejancu-Kon-Yano [6])

Let N be a CR-submanifold of a Kaehler manifold M. Then

i) the distribution D is integrable and its leaves are totally geodesic in N if and only if

$$g(h(X,Y),JZ) = 0$$

for any $X,\ Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

ii) the distribution D is integrable and its leaves are totally geodesic in M if and only if

$$h(X,Y) = 0$$

for any $X,\ Y \in \Gamma(D)$.

iii) the leaves of $D^{\perp}$ are totally goedesic in N if and only if

$$h(X,Z) \in \Gamma(\mu)$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, where μ is the orthogonal complementary subbundle to $JD^{\perp}$ in $T^{\perp}N$.

Next, we gave some geometrical properties of a totally umbilical CR-submanifold of a Kaehler manifold M. We start off with the following lemmas.

Lemma 2.1

If N is a CR-submanifold of a Kaehler manifold M, then

$$A_{JX}Y = A_{JY}X$$

for all X, Y $\in \Gamma(D^{\perp})$.

Proof :

Since M is a Kaehler manifold, for any X, Y $\in \Gamma(D^{\perp})$,

$$(\bar{\nabla}_X J)Y = 0$$

$$\bar{\nabla}_X JY - J\bar{\nabla}_X Y = 0$$

$$\bar{\nabla}_X JY = J\bar{\nabla}_X Y$$

By applying the Gauss and Weingarten formulas, we obtain

$$-A_{JY}X + \nabla^{\perp}_X JY = J\nabla_X Y + Jh(X,Y) \qquad (2.19)$$

Similarly, we have,

$$-A_{JX}Y + \nabla^{\perp}_Y JX = J\nabla_Y X + Jh(X,Y) \qquad (2.20)$$

Hence, from (2.19) and (2.20), we have the following,

$$-A_{JY}X + \nabla^{\perp}_X JY + A_{JX}Y - \nabla^{\perp}_Y JX = J\nabla_X Y - J\nabla_Y X$$

$$A_{JX}Y - A_{JY}X = J(\nabla_X Y - \nabla_Y X) - \nabla^{\perp}_X JY + \nabla^{\perp}_Y JX$$

$$= J[X,Y] - \nabla^{\perp}_X JY + \nabla^{\perp}_Y JX$$

Since $D^{\perp}$ is integrable, [X,Y] lies in $D^{\perp}$. Therefore J[X,Y] lies in the normal bundle. Hence,

$$A_{JX}Y = A_{JY}X$$

for all X, Y $\in \Gamma(D^{\perp})$.

Lemma 2.2 (Blair-Chen [7])

If N is a totally umbilical CR-submanifold of a Kaehler manifold M, then either $D^{\perp}$ is 1-dimensional or the mean curvature vector H is perpendicular to $JD^{\perp}$.

Proof :

Suppose dim $D^{\perp} > 1$. Since N is totally umbilical, for any unit vector $X \in \Gamma(D^{\perp})$ and perpendicular to a vector $Y \in \Gamma(D^{\perp})$,

$$\begin{aligned} g(h(X,X),JY) &= g(g(X,X)H,JY) \\ &= g(X,X)g(H,JY) \\ &= g(H,JY) \end{aligned}$$

However,

$$\begin{aligned} g(h(X,X),JY) &= g(A_{JY}X,X) \\ &= g(A_{JX}Y,X), \qquad \text{by lemma 2.1} \\ &= g(h(X,Y),JX) \\ &= 0 \end{aligned}$$

Thus, $g(H,JY) = 0$. This tells us that, the mean curvature vector H is perpendicular to $JD^{\perp}$.

QED

Bejancu [5] and Chen [10] proved the following theorem which classifies a totally umbilical CR-submanifold of a Kaehler manifold.

Theorem 2.9 (Bejancu [10] and Chen [5])

Let N be a totally umbilical CR-submanifold of a Kaehler manifold M. Then

i) N is totally geodesic, or

ii) N is totally real, or

iii) the distribution $D^{\perp}$ is one-dimensional.

Proof:

Suppose dim $D^{\perp} > 1$. From lemma 2.1, we have

$$A_{JX}BH = A_{JBH}X \tag{2.21}$$

for all $X \in \Gamma(D^{\perp})$. We note that,

$$\begin{aligned} g(A_{JX}BH, X) &= g(h(BH,X), JX) \\ &= g(g(BH,X)H, JX), \quad \text{since N is totally umbilical} \\ &= g(BH,X)g(H,JX) \end{aligned}$$

Similarly,

$$g(A_{JBH}X, X) = g(X,X)g(H,JBH)$$

Therefore, by using (2.21), we obtain

$$g(X,X)g(H,JBH) = g(BH,X)g(JX,H)$$

or

$$\begin{aligned} -g(X,X)g(JH,BH) &= g(BH,X)g(JX,H) \\ &= 0, \qquad \text{by lemma 2.2} \end{aligned}$$

Since $JH = BH + CH$, we have

$$g(X,X)g(BH,BH) = 0$$

Thus, $BH = 0$. This tells us that $JH \in \Gamma(T^{\perp}N)$.

From (2.16), we have

$$P(A_{CH}Y) = \phi(A_{H}Y) \tag{2.22}$$

for any $Y \in \Gamma(TN)$.

Now, suppose that N is not totally real, that is, dim $D \geq 2$. Then, for $Z \in \Gamma(D)$, we obtain

$$g(PA_{JH}Y,Z) = g(A_{JH}Y,Z) = g(Y,Z)g(JH,H) \tag{2.23}$$

and

$$\begin{aligned} g(\phi(A_H Y),Z) &= g(J(A_H Y),Z) \\ &= -g(A_H Y, JZ) \\ &= -g(Y,JZ)g(H,H) \end{aligned} \tag{2.24}$$

We note that, $g(PA_{JH}Y,Z) = g(PA_{BH + CH}Y,Z)$
$= g(PA_{CH}Y,Z)$, since $BH = 0$

Therefore, by using (2.22), we obtain

$$g(Y,Z)g(JH,H) = -g(Y,JZ)g(H,H) \tag{2.25}$$

or

$$-g(Y,Z)g(H,JH) = -g(Y,JZ)g(H,H) \tag{2.26}$$

By summing (2.25) and (2.26), we have

$$2g(Y,JZ)g(H,H) = 0$$

Therefore, $H = 0$. This tells us that N is totally geodesic and so, the proof is completed.

QED

Let N be a CR-submanifold of an almost Hermitian manifold M. N is a *CR-product* if both the distributions D and $D^{\perp}$ are integrable and N is locally a Riemannian product $N_1 \times N_2$ where N_1 is a leaf of D and N_2 is a leaf of $D^{\perp}$. B.Y.Chen [8] proved that N is a CR-product if and only if $A_{JD^{\perp}}D = 0$. However, we are able to obtain a different proof than the one found in Chen [8]. We first begin with the following lemma.

Lemma 2.3

Let N be a CR-submanifold of a Kaehler manifold M. Then if $A_{JZ}X = 0$, $h(X,Y) \in \Gamma(\mu)$, for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$ and $Y \in \Gamma(TN)$.

Proof:

$$0 = g(A_{JZ}X,Y) = g(h(X,Y),JZ)$$

Thus, $h(X,Y) \in \Gamma(\mu)$, for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$ and $Y \in \Gamma(TN)$.

QED

Theorem 2.10

Let N be a CR-submanifold of a Kaehler manifold M. N is a CR-product if and only if $A_{JZ}X = 0$, for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Proof:

Suppose N is a CR-product. Since D is integrable and its leaves are totally geodesic in N, by theorem 2.8, we have

$$g(A_{JZ}X,Y) = g(h(X,Y)JZ) = 0 \qquad (2.27)$$

for any X, $Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Since $D^{\perp}$ is integrable and its leaves are totally geodesic, by theorem 2.8, we have

$$g(A_{JZ}X,U) = g(h(X,U),JZ) = 0 \qquad (2.28)$$

for any $X \in \Gamma(D)$ and U, $Z \in \Gamma(D^{\perp})$.

Therefore, by equations (2.27) and (2.28),

$$A_{JZ}X = 0$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Conversely, for any X, $Y \in \Gamma(D)$, we have

$$(\bar{\nabla}_X J)Y = 0$$

$$\bar{\nabla}_X JY - J\bar{\nabla}_X Y = 0$$

By the Gauss formula, we have

$$\nabla_X JY - J\nabla_X Y = -h(X,JY) + Jh(X,Y)$$

Since the left hand side belongs to $TN \oplus JD^{\perp}$, and the right hand side belongs to μ, we have

$$h(X,JY) = Jh(X,Y)$$

that is, $h(X,JY) = h(JX,Y)$.

We note that

$$0 = g(A_{JZ}X,Y) = g(h(X,Y),JZ)$$

for any X, $Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Hence, by theorem 2.8, D is integrable and its leaves are totally geodesic in N.

By theorem 2.7, $D^{\perp}$ is integrable and by lemma 2.3, we have

$$h(X,Z) \in \Gamma(\mu)$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Hence, by theorem 2.8, the leaves of $D^{\perp}$ are totally geodesic in N. N is therefore, a CR-product.

QED

We are able to obtain a similar theorem when M is a nearly-Kaehler manifold. We will do this in chapter 3.

Before we end this section, we gave some geometrical properties of a mixed totally geodesic CR-submanifold N of a Kaehler manifold M. Let μ be the complementary orthogonal vector bundle of $JD^{\perp}$ in $T^{\perp}N$. We begin with the following definitions.

Definition

i) A CR-submanifold is said to be *mixed totally geodesic* if $h(X,Y) = 0$, for any $X \in \Gamma(D)$ and any $Y \in \Gamma(D^{\perp})$.

ii) A normal vector field ξ ($\neq 0$) is said to be a *D-parallel normal section* if $\nabla^{\perp}_{X}\xi = 0$, for each $X \in \Gamma(D)$.

Lemma 2.4

A CR-submanifold N of an almost Hermitian manifold M is mixed totally geodesic if and only if $A_{\xi}X \in \Gamma(D)$, for any $X \in \Gamma(D)$ and $\xi \in \Gamma(T^{\perp}N)$.

Proof :

If N is mixed totally geodesic, then

$$0 = g(h(X,Y),\xi) = g(A_{\xi}X,Y)$$

for $X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp})$, $\xi \in \Gamma(T^{\perp}N)$. Therefore, $A_{\xi}X \in \Gamma(D)$.

Conversely, suppose $A_{\xi}X \in \Gamma(D)$. Let $\{ \xi_1, \ldots\ldots, \xi_{2m-n} \}$ be a local orthonormal basis of $\Gamma(T^{\perp}N)$. Then,

$$0 = g(A_{\xi_i}X,Y) = g(h(X,Y),\xi_i), \qquad 1 \leq i \leq 2m-n$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(T^{\perp}N)$. Since $h(X,Y) \in \Gamma(T^{\perp}N)$, we have

$$h(X,Y) = 0$$

Therefore, N is mixed totally geodesic.

QED

Lemma 2.5 (Bejancu [2])

Let N be a mixed totally geodesic CR-submanifold of a Kaehler manifold M. Then, we have

$$A_{J\xi}X = JA_{\xi}X$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(\mu)$.

Lemma 2.6 (Bejancu [2])

Let N be a mixed totally geodesic CR-submanifold of a Kaehler manifold M. Suppose the distribution D is integrable.Then,

$$JA_{\xi}X = -A_{\xi}JX$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(T^{\perp}N)$.

The following theorem gives a characterization for the parallel normal section which belongs to the normal subbundle $JD^{\perp}$.

Theorem 2.11 (Bejancu [2])

Let N be a mixed totally geodesic CR-submanifold of a Kaehler manifold M. Then the normal section, $\xi \in \Gamma(JD^{\perp})$ is D-parallel if and only if $\nabla_X J\xi \in \Gamma(D)$, for each vector field $X \in \Gamma(D)$.

Definition

The *holomorphic bisectional curvature* for a pair of vector fields (X,Y) on an almost Hermitian manifold M is given by

$$H(X,Y) = \frac{\bar{R}(X,JX;JY,Y)}{g(X,X)g(Y,Y)}$$

By using lemma 2.4 and lemma 2.5, Bejancu [2] proved the following theorem.

Theorem 2.12 (Bejancu [2])

Let N be a mixed totally geodesic CR-submanifold of a Kaehler manifold M. Suppose the distribution D is integrable. If there exists a unit vector field $X \in \Gamma(D)$ such that for all normal sections $\xi \in \Gamma(\mu)$, the holomorphic bisectional curvatures $H(X,\mu)$ are positive, then the normal subbundle μ does not admit D-parallel section.

We note that the theorem above has been proven by B.Y Chen and H.S Lue for complex submanifolds of a Kaehler manifold.