CHAPTER THREE : CR-SUBMANIFOLDS OF A NEARLY-KAEHLER MANIFOLD

In this chapter, we will study the properties of a CR-submanifold N of a nearly-Kaehler manifold M. Most of the theorems here have already proven to be true for a CR-submanifold of a Kaehler manifold (see chapter two). Here, we found that with a few alterations, they are also true for a CR-submanifold of a nearly-Kaehler manifold.

3.1 Introduction

Let M be a nearly-Kaehler manifold. Then, for any $X, Y \in \Gamma(TM)$, we have

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0 \qquad (3.1)$$

where $\bar{\nabla}$ is the Levi-Civita connection on M. Let N be a CR-submanifold on M. Similarly, as in chapter two, we denote by ∇, the induced Levi-Civita connection on N and P and Q as the projections on the distributions D and $D^{\perp}$ respectively.

As in Chapter Two, for any $\xi \in \Gamma(T^{\perp}N)$, we put

$$J\xi = B\xi + C\xi$$

where $B\xi \in \Gamma(D^{\perp})$ and $C\xi \in \Gamma(T^{\perp}N)$.

So, for any $X, Y \in \Gamma(TN)$,

$$Jh(X,Y) = Bh(X,Y) + Ch(X,Y) \qquad (3.2)$$

Since M is a nearly-Kaehler manifold, we have

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0$$

$$\bar{\nabla}_X JY - J\bar{\nabla}_X Y + \bar{\nabla}_Y JX - J\bar{\nabla}_Y X = 0$$

$$\bar{\nabla}_X JY + \bar{\nabla}_Y JX = J\bar{\nabla}_X Y + J\bar{\nabla}_Y X \qquad (3.3)$$

The Gauss formula gives us

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and
$$\overline{\nabla}_Y X = \nabla_Y X + h(X,Y)$$

By combining the two equations above, we obtain

$$\overline{\nabla}_X Y + \overline{\nabla}_Y X = \nabla_X Y + \nabla_Y X + 2h(X,Y)$$

Applying J to both sides of the equation above, we have

$$J\overline{\nabla}_X Y + J\overline{\nabla}_Y X = J\nabla_X Y + J\nabla_Y X + 2Jh(X,Y) \tag{3.4}$$

By using equations (3.2) and (3.3), equation (3.4) becomes

$$\overline{\nabla}_X JY + \overline{\nabla}_Y JX - J\nabla_X Y - J\nabla_Y X = 2Bh(X,Y) + 2Ch(X,Y)$$

Thus, we obtain

$$\overline{\nabla}_X JPY + \overline{\nabla}_X JQY + \overline{\nabla}_Y JPX + \overline{\nabla}_Y JQX - JP\nabla_X Y - JQ\nabla_X Y - JP\nabla_Y X - JQ\nabla_Y X$$
$$= 2Bh(X,Y) + 2Ch(X,Y)$$

By applying the Gauss and Weingarten formulas, we have

$$\nabla_X JPY + h(X,JPY) + \nabla_X^{\perp} JQY - A_{JQY}X + \nabla_Y JPX + h(Y,JPX) + \nabla_Y^{\perp} JQX$$
$$- A_{JQX}Y - JP\nabla_X Y - JQ\nabla_X Y - JP\nabla_Y X - JQ\nabla_Y X$$
$$= 2Bh(X,Y) + 2Ch(X,Y) \tag{3.5}$$

Therefore, we have the following equations:

$$P(\nabla_X JPY) - P(A_{JQY}X) + P(\nabla_Y JPX) - P(A_{JQX}Y)$$
$$= JP\nabla_X Y + JP\nabla_Y X \tag{3.6}$$

$$Q(\nabla_X JPY) - Q(A_{JQY}X) + Q(\nabla_Y JPX) - Q(A_{JQX}Y) = 2Bh(X,Y) \tag{3.7}$$

and

$$h(X,JPY) + \nabla^{\perp}_X JQY + h(Y,JPX) + \nabla^{\perp}_Y JQX = JQ\nabla_X Y + JQ\nabla_Y X + 2Ch(X,Y) \tag{3.8}$$

Equations (3.7) and (3.8) will be used in the proof of proposition 3.6 and 3.7.

3.2 Integrability of Distributions of a CR-submanifold of a Nearly-Kaehler Manifold.

In this section, we will discuss the integrability condition for the distributions D and $D^{\perp}$ of a CR-submanifold of a nearly-Kaehler manifold M. We will also discuss necessary and sufficient conditions for the leaves of D and $D^{\perp}$ to be totally geodesic in N and M. We start off with a proposition where N is not necessarily a CR-submanifold.

Proposition 3.1

Let N be a submanifold of a nearly-Kaehler manifold M and D_X the holomorphic subspace of $T_X N$, with constant dimension. If the distribution D is integrable then

$$h(X,JY) = h(JX,Y) \tag{3.9}$$

for any $X, Y \in \Gamma(D)$.

Proof:

Let N′ be an integral submanifold of D and let ∇' be the Levi-Civita connection with respect to N′, ∇ the Levi-Civita connection with respect to N and $\overline{\nabla}$ the Levi-Civita connection with respect to M. Also, let h′ be the second fumdamental form of N′ in N and $\overline{h}$ the second fundamental form of N′ in M. As in the previous chapters, we let h be the second fundamental form of N in M.

Since M is a nearly-Kaehler manifold,

$$(\overline{\nabla}_X J)Y + (\overline{\nabla}_Y J)X = 0$$

$$\overline{\nabla}_X JY - J\overline{\nabla}_X Y + \overline{\nabla}_Y JX - J\overline{\nabla}_Y X = 0$$

By using the Gauss formula, we obtain

$$\nabla'_X JY + \overline{h}(X,JY) - J(\nabla'_X Y + \overline{h}(X,Y)) + \nabla'_Y JX + \overline{h}(Y,JX) - J(\nabla'_Y X + \overline{h}(Y,X)) = 0$$

$$\nabla'_X JY + \nabla'_Y JX - J(\nabla'_X Y + \nabla'_Y X) + \overline{h}(X,JY) + \overline{h}(Y,JX) - 2J\overline{h}(X,Y) = 0 \quad (3.10)$$

Note that, since D is holomorphic, $J(\nabla'_X Y + \nabla'_Y X) \in \Gamma(D)$. Therefore, by comparing the tangential and normal parts of (3.10), we obtain

$$\nabla'_X JY + \nabla'_Y JX - J(\nabla'_X Y + \nabla'_Y X) = 0 \quad (3.11)$$

and

$$\overline{h}(X,JY) + \overline{h}(Y,JX) - 2J\overline{h}(X,Y) = 0 \quad (3.12)$$

By replacing X by JX in (3.12),

$$\begin{aligned}\overline{h}(JX,JY) - \overline{h}(Y,X) &= 2J\overline{h}(JX,Y)\\ &= 2J(2J\overline{h}(X,Y) - \overline{h}(X,JY)), \quad \text{by (3.12)}\end{aligned}$$

$$= -4\bar{h}(X,Y) - 2J\bar{h}(X,JY)$$

This gives us,

$$\bar{h}(JX,JY) = -3\bar{h}(X,Y) - 2J\bar{h}(X,JY) \tag{3.13}$$

Next, we replace Y by JY in (3.12). Similarly, we obtained

$$-\bar{h}(X,Y) + \bar{h}(JY,JX) = 2J\bar{h}(X,JY),$$

that is,

$$-3\bar{h}(X,Y) + 3\bar{h}(JY,JX) = 6J\bar{h}(X,JY) \tag{3.14}$$

By substituting (3.14) into (3.13), we have

$$\bar{h}(JX,JY) = 6J\bar{h}(X,JY) - 3\bar{h}(JX,JY) - 2J\bar{h}(X,JY)$$

which gives us

$$4\bar{h}(JX,JY) = 4J\bar{h}(X,JY),$$

that is,

$$\bar{h}(JX,JY) = J\bar{h}(X,JY) \tag{3.15}$$

By replacing Y by JY in (3.15), the equation becomes

$$\bar{h}(JX,Y) = J\bar{h}(X,Y) \tag{3.16}$$

Similarly, we have

$$\bar{h}(JY,X) = J\bar{h}(Y,X)$$

Hence,

$$\bar{h}(JX,Y) = \bar{h}(JY,X) \tag{3.17}$$

From the Gauss formula, we have

$$\bar{\nabla}_X Y = \nabla'_X Y + \bar{h}(X,Y)$$

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and

$$\nabla_X Y = \nabla'_X Y + h'(X,Y)$$

By combining the three equations above, we obtain

$$\nabla'_X Y + \bar{h}(X,Y) = \nabla'_X Y + h'(X,Y) + h(X,Y)$$

Hence,

$$\bar{h}(X,Y) = h'(X,Y) + h(X,Y),$$

that is, $$\bar{h} = h' + h \tag{3.18}$$

on $\Gamma(TN')$. By using (3.17) and (3.18), we obtain

$$h'(JX,Y) + h(JX,Y) = h'(X,JY) + h(X,JY)$$

$$h(JX,Y) - h(X,JY) = h'(X,JY) - h'(JX,Y)$$

The left hand side is normal to N in M and the right hand side is tangent to N. Hence (3.9) is proved.

QED

From the proof of proposition 3.1, we have the following corollary.

Corollary 3.1

Let N be a submanifold of a nearly-Kaehler manifold M and D_x the holomorphic subspace of T_xN, with constant dimension. If D is integrable, the integral submanifold of D is a nearly-Kaehler submanifold of M.

Proof:

Taking account of equation (3.11), we obtain

$$(\nabla'_X J)Y + (\nabla'_Y J)X = 0$$

Therefore, N′ is a nearly-Kaehler submanifold of M.

QED

Let N be a CR-submanifold of a nearly-Kaehler manifold M. The following lemma is found in Bejancu [3].

Lemma 3.1:

Let N be a CR-submanifold of a nearly-Kaehler manifold M. Then

$$h(X,JY) - h(JX,Y) = \frac{1}{2}J([J,J](X,Y)) + J[X,Y] + \nabla_Y JX - \nabla_X JY \tag{3.19}$$

for any X, Y $\in \Gamma(D)$.

Proof :

For X, Y $\in \Gamma(D)$ and by using theorem 1.7 of chapter 1, we have

$$\frac{1}{2}J([J,J](X,Y)) + J[X,Y] + \nabla_Y JX - \nabla_X JY$$

$$= -2(\bar{\nabla}_Y J)X + J\bar{\nabla}_X Y - J\bar{\nabla}_Y X + \nabla_Y JX - \nabla_X JY \tag{3.20}$$

$$= -2(\bar{\nabla}_Y JX - J\bar{\nabla}_Y X) + J\bar{\nabla}_X Y - J\bar{\nabla}_Y X + \nabla_Y JX - \nabla_X JY \tag{3.21}$$

Since M is a nearly-Kaehler manifold, we have

$$(\bar{\nabla}_X J)Y = -(\bar{\nabla}_Y J)X$$

Therefore, equation (3.20) can also be written as

$$\frac{1}{2}J([J,J](X,Y)) + J[X,Y] + \nabla_Y JX - \nabla_X JY$$

$$= 2(\bar{\nabla}_X J)Y + J\bar{\nabla}_X Y - J\bar{\nabla}_Y X + \nabla_Y JX - \nabla_X JY$$

$$= 2\bar{\nabla}_X JY - J\bar{\nabla}_X Y - J\bar{\nabla}_Y X + \nabla_Y JX - \nabla_X JY \tag{3.22}$$

By summing up equations (3.21) and (3.22), we obtain

$$2\left(\frac{1}{2}J([J,J](X,Y)) + J[X,Y] + \nabla_Y JX - \nabla_X JY \right)$$

$$= 2(\bar{\nabla}_X JY - \nabla_X JY) - 2(\bar{\nabla}_Y JX - \nabla_Y JX)$$

$$= 2h(X,JY) - 2h(Y,JX)$$

Thus, equation (3.19) is proved.

Sato proved a necessary and sufficient integrability condition for the distribution D. The following version here is found in Bejancu [3]. However, we make a slight change in the proof.

Theorem 3.1

Let N be a CR-submanifold of a nearly-Kaehler manifold M. Then the distribution D is integrable if and only if

$$h(X,JY) = h(JX,Y) \qquad (3.23)$$

and

$$[J,J](X,Y) \in \Gamma(D) \qquad (3.24)$$

for any X, Y $\in \Gamma(D)$.

Proof:

Suppose D is integrable. Then, by proposition 3.1, we have (3.23). Since

$$[J,J](X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$$

therefore, $[J,J](X,Y) \in \Gamma(D)$, for any X, Y $\in \Gamma(D)$.

Conversely, suppose equations (3.23) and (3.24) are satisfied. From lemma (3.1), we have

$$J[X,Y] = \nabla_X JY - \nabla_Y JX - \frac{1}{2}J([J,J](X,Y)) \qquad (3.25)$$

For each $W \in \Gamma(D^{\perp})$, there exists $V \in \Gamma(T^{\perp}N)$ such that $V = JW$. Therefore, by using (3.24) and (3.25), we have

$$g([X,Y],W) = g(J[X,Y],JW) = g(\nabla_X JY - \nabla_Y JX - \frac{1}{2}J([J,J](X,Y)),\ V)$$
$$= g(\nabla_X JY\ ,V) - g(\nabla_Y JX,V) - \frac{1}{2}g(J([J,J](X,Y)),V)$$
$$= 0$$

Hence, $[X,Y] \in \Gamma(D)$ for each $X, Y \in \Gamma(D)$. It follows that D is integrable.

QED

Theorem 3.2 (Urbano [22])

Let N be a CR-submanifold of a nearly-Kaehler manifold M. Then the distribution D is integrable if and only if

$$(\bar{\nabla}_X J)Y \in \Gamma(D)$$

and
$$h(X,JY) = h(JX,Y)$$

for any $X, Y \in \Gamma(D)$.

Proof:

By using theorem 1.7 of chapter 1 and theorem 3.1, theorem 3.2 is clear.

QED

Kon-Tan [15] proved a necessary and sufficient condition for each leaf of D to be totally geodesic in N. Suppose D is integrable. For any $X, Y \in \Gamma(D)$, let

$$\nabla_X Y = \nabla'_X Y + \alpha(X,Y)$$

where $\nabla'_X Y \in \Gamma(D)$ and $\alpha(X,Y) \in \Gamma(D^{\perp})$. Therefore, each leaf of D is totally geodesic in N if and only if

$$\alpha(X,Y) = 0 \text{ or } \nabla_X Y \in \Gamma(D)$$

for any $X, Y \in \Gamma(D)$. Similarly, each leaf of D is totally geodesic in M if and only if

$$\alpha(X,Y) + h(X,Y) = 0 \text{ or } \bar{\nabla}_X Y \in \Gamma(D)$$

for any $X, Y \in \Gamma(D)$.

Theorem 3.3 (Kon-Tan [15])

Let N be a CR-submanifold of nearly-Kaehler manifold M. Suppose D is integrable. Then each leaf of D is a totally geodesic submanifold of N if and only if

$$h(X, JY) = Jh(X, Y) \tag{3.26}$$

for any $X, Y \in \Gamma(D)$.

Proof:

Since D is integrable, by theorem 3.2, we have

$$(\overline{\nabla}_X J)Y = \nabla_X JY + h(X, JY) - J\nabla_X Y - Jh(X, Y) \in \Gamma(D).$$

Therefore, if each leaf of D is totally geodesic in N,

$\nabla_X JY$, $J\nabla_X Y \in \Gamma(D)$ and this gives us $h(X, JY) - Jh(X, Y) = 0$

Hence, $$h(X, JY) = Jh(X, Y).$$

Conversely, for any $X, Y \in \Gamma(D)$, $U \in \Gamma(D^{\perp})$ and since $(\overline{\nabla}_X J)Y \in \Gamma(D)$, we have

$$\begin{aligned} 0 = g((\overline{\nabla}_X J)Y, U) &= g(\nabla_X JY + h(X, JY) - J\nabla_X Y - Jh(X, Y),\ U) \\ &= g(\nabla_X JY, U) - g(J\nabla_X Y, U), \quad \text{by the assumption of (3.26)} \\ &= g(\nabla_X JY, U) \end{aligned}$$

Therefore $\nabla_X JY \in \Gamma(D)$. This tells us that each leaf of D is totally geodesic in N.

QED

Kon-Tan [15] also proved a necessary and sufficient condition for each leaf of D to be totally geodesic in M.

Theorem 3.4 (Kon-Tan [15])

Let N be a CR-submanifold of a nearly-Kaehler manifold M. Suppose D is integrable. Then, each leaf of D is a totally geodesic submanifold of M if and only if

$$h(X,Y) = 0 \tag{3.27}$$

for any X, $Y \in \Gamma(D)$.

Proof:

For any X, $Y \in \Gamma(D)$, we have

$$\overline{\nabla}_X Y = \nabla'_X Y + \alpha(X,Y) + h(X,Y)$$

where $\nabla'_X Y \in \Gamma(D)$, $\alpha(X,Y) \in \Gamma(D^\perp)$ and $h(X,Y) \in \Gamma(T^\perp N)$. If each leaf of D is totally geodesic in M, we have

$$\alpha(X,Y) + h(X,Y) = 0.$$

Thus, $\alpha(X,Y) = 0$ and $h(X,Y) = 0$, for any $X,Y \in \Gamma(D)$.

Conversely, if $h(X,Y) = 0$, then for any X, $Y \in \Gamma(D)$ and $U \in \Gamma(D^\perp)$,

$$\begin{aligned} 0 &= g((\overline{\nabla}_X J)Y,U) = g(\nabla_X JY - J\nabla_X Y,U), \quad \text{by the assumption of (3.27)} \\ &= g(\nabla_X JY,U) + g(\nabla_X Y, JU) \\ &= g(\nabla_X JY,U) \end{aligned}$$

Therefore, $\nabla_X JY \in \Gamma(D)$ for any X, $Y \in \Gamma(D)$. This implies that $\alpha(X,Y) = 0$ and since $h(X,Y) = 0$, this tells us that each leaf of D is totally geodesic in M.

QED

By combining theorem 3.2, theorem 3.3 and theorem 3.4, we have the following theorem which is a generalization of theorem 2.8 ((i) and (ii)) of chapter two.

Theorem 3.5 (Kon-Tan [15])

Let N be a CR-submanifold of a nearly-Kaehler manifold M. Then

(i) the distribution D is integrable and its leaves are totally geodesic in N if and only if

$$h(X,JY) = Jh(X,Y) \text{ and } (\bar{\nabla}_X J)Y \in \Gamma(D)$$

for any X, Y $\in \Gamma(D)$.

(ii) the distribution D is integrable and its leaves are totally geodesic in M if and only if

$$h(X,Y) = 0 \text{ and } (\bar{\nabla}_X J)Y \in \Gamma(D)$$

for any X, Y $\in \Gamma(D)$.

Before we end this section, we discuss the integrability condition for the distribution $D^{\perp}$ of a nearly-Kaehler manifold. We have the following results found in Bejancu [3]. We note that theorem 3.6 is proved by Sato.

Theorem 3.6

Let N be a CR-submanifold of a nearly-Kaehler manifold M. The distribution $D^{\perp}$ is integrable if and only if

$$g(h(U,X),JW) = g(h(W,X),JU) \qquad (3.28)$$

for all U, W $\in \Gamma(D^{\perp})$ and X $\in \Gamma(D)$.

Theorem 3.7 (Bejancu [3], p. 28)

Let N be a CR-submanifold of a nearly-Kaehler manifold M. If $D^\perp$ is integrable, then each leaf of $D^\perp$ is immersed in N as a totally geodesic submanifold if and only if

$$g(h(U,X),JW) = 0 \tag{3.29}$$

for all U, $W \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$.

3.3 Totally umbilical CR-submanifolds of a nearly-Kaehler manifold.

Most of the results in this section are generalized from the previous results found in section 2.5 of chapter two. For instance, lemma 3.2 (see p. 61) found in this section, is almost similar to lemma 2.2 (see p. 40), whereas the results obtained in theorem 2.9 is also true when M is a nearly-Kaehler manifold.

The definition of a totally umbilical submanifold is as in section 1.3 of chapter one. Firstly, we gave a theorem which is proven by Kon-Tan [17] which says that if N is a totally umbilical CR-submanifold of a nearly-Kaehler manifold M, then $D^\perp$ is integrable.

Theorem 3.8 (Kon-Tan [17])

If N is a totally umbilical CR-submanifold of a nearly-Kaehler manifold M, then $D^\perp$ is integrable and its leaves are totally geodesic in N.

Proof:

Since N is totally umbilical, for any U, $W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$,

$$g(h(U,X),JW) = g(g(U,X)H,JW) = 0$$

Similarly, we also obtain $g(h(W,X),JU) = 0$. Hence, equation (3.28) is satisfied. It follows from theorem 3.6 and 3.7 that $D^{\perp}$ is integrable and its leaves are totally geodesic in N.

QED

Lemma 3.2 (Kon-Tan [17])

Let N be a totally umbilical CR-submanifold of a nearly-Kaehler manifold M. If $\dim D^{\perp} > 1$, then

(i) the mean curvature vector H is perpendicular to $JD^{\perp}$.

(ii) $A_{JZ}W = 0$, for all $Z,W \in \Gamma(D^{\perp})$

Proof :

Since M is nearly-Kaehler, for any Z, $W \in \Gamma(D^{\perp})$,

$$(\bar{\nabla}_Z J)W + (\bar{\nabla}_W J)Z = 0$$

$$\bar{\nabla}_Z JW - J\bar{\nabla}_Z W + \bar{\nabla}_W JZ - J\bar{\nabla}_W Z = 0$$

$$\bar{\nabla}_Z JW - J\bar{\nabla}_Z W = -\bar{\nabla}_W JZ + J\bar{\nabla}_W Z \tag{3.30}$$

By using the Gauss and Weingarten formulas, (3.30) becomes

$$-A_{JW}Z + \nabla^{\perp}_Z JW - J\nabla_Z W - Jh(Z,W) = A_{JZ}W - \nabla^{\perp}_W JZ + J\nabla_W Z + Jh(Z,W)$$

Since N is totally umbilical, we have

$$J(\nabla_W Z + \nabla_Z W) + 2g(Z,W)JH = -A_{JW}Z - A_{JZ}W + \nabla^{\perp}_Z JW + \nabla^{\perp}_W JZ$$

Thus, $g(J(\nabla_W Z + \nabla_Z W), Z) + 2g(Z,W)g(Z,JH)$

$$= - g(A_{JW}Z,Z) - g(A_{JZ}W,Z) + g(\nabla^{\perp}_Z JW,Z) + g(\nabla^{\perp}_W JZ,Z)$$

$$= - g(h(Z,Z),JW) - g(h(Z,W),JZ)$$

$$= \|Z\|^2 g(W,JH) + g(Z,W)g(Z,JH),$$ since N is totally umbilical

Therefore, $g(Z,W)g(Z,JH) = \|Z\|^2 g(W,JH)$, for any $W, Z \in \Gamma(D^{\perp})$.

By interchanging Z and W in the equation above, we obtain

$$g(Z,W)g(W,JH) = \|W\|^2 g(Z,JH)$$

Hence,
$$g(W,JH) = \frac{g(Z,W)^2}{\|Z\|^2\|W\|^2} g(W,JH) \qquad (3.31)$$

If $\dim D^{\perp} > 1$, then for Z not parallel with W,

$$g(Z,W)^2 < \|Z\|^2 \|W\|^2$$

Hence, equation (3.31) shows that $g(W,JH) = 0$, that is, H is perpendicular to $JD^{\perp}$.

Let $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(TN)$. Then

$$\begin{aligned} g(A_{JZ}W,X) &= g(h(W,X),JZ) \\ &= g(W,X)g(H,JZ) \\ &= 0 \end{aligned}$$

Therefore, $A_{JZ}W = 0$ for all $Z,W \in \Gamma(D^{\perp})$.

QED

By using the lemma 3.2, we are able to prove this simple result when N is a totally umbilical anti-holomorphic submanifold of a nearly-Kaehler manifold M.

Proposition 3.2

Let N be a totally umbilical anti-holomorphic submanifold of a nearly-Kaehler manifold M, with $\dim N > \frac{1}{2} \dim M$.

(i) If $\dim D^{\perp} > 1$, then N is totally geodesic.

(ii) If $\dim D^{\perp} = 1$, then N is a hypersurface.

Proof:

Suppose $\dim D^{\perp} > 1$. From section 2.2 of chapter 2 (see p.24), $D^{\perp}_x$ is given by $J(T^{\perp}_x N)$, for $x \in N$. Therefore by using lemma 3.2, H is perpendicular to $J(\ J(T^{\perp}_x N)\ )$ which implies that H is perpendicular to $T^{\perp}_x N$. Therefore $H = 0$. Thus, N is totally geodesic.

Suppose $\dim D^{\perp} = 1$. Then, from the definition for $D^{\perp}$ of an anti-holomorphic submanifold, we have

$$\dim (J(T^{\perp}_x N)) = 1$$

which shows that $\dim T^{\perp}_x N = 1$. Since

$$\dim T_x M = \dim T_x N + \dim T^{\perp}_x N$$

therefore, we have

$$m = \dim T_x N + 1$$

Thus, $\dim N = \dim T_x N = m-1$. So, N is a hypersurface.

QED

The following theorem is a generalization of theorem 2.9 of chapter two. It classifies a totally umbilical CR-submanifold of a nearly-Kaehler manifold.

Theorem 3.9 (Kon-Tan [17])

Let N be a totally umbilical CR-submanifold of a nearly-Kaehler manifold M. Then

(i) N is totally geodesic, or

(ii) N is totally real, or

(iii) the distribution $D^{\perp}$ is one-dimensional.

Proof:

Suppose $\dim D^{\perp} > 1$. By taking account of lemma 3.2, we have $H \in \Gamma(\mu)$, where μ is the complementary orthogonal subbundle of $JD^{\perp}$ on $T^{\perp}N$. Since μ is invariant by the almost complex structure J, we have $JH \in \Gamma(\mu)$.

Now, if N is not totally real, then $\dim D \geq 2$. Then, for $X \neq 0$ in $\Gamma(D)$, we obtain

$$\begin{aligned} g(A_{JH}JX,X) &= g(\nabla^{\perp}_{JX}JH - \bar{\nabla}_{JX}JH, X) \\ &= -g(\bar{\nabla}_{JX}JH, X) \\ &= g(JH, \bar{\nabla}_{JX}X), && \text{since } JH \in \Gamma(\mu) \\ &= g(JH, -\bar{\nabla}_{JX}J(JX)) \\ &= g(JH, -(\bar{\nabla}_{JX}J)JX - J\bar{\nabla}_{JX}JX) \\ &= g(JH, -J\bar{\nabla}_{JX}JX), && \text{since } (\bar{\nabla}_{JX}J)JX = 0 \\ &= g(JH, -J\nabla_{JX}JX - Jh(JX,JX)) \\ &= g(JH, -Jh(JX,JX)) \end{aligned}$$

$$= -g(H, h(JX, JX))$$
$$= -g(JX, JX)g(H, H)$$
$$= -g(X, X)g(H, H)$$
$$= -\| X \|^2 \| H \|^2$$

However, $g(A_{JH}JZ, Z) = g(h(JZ, Z), JH)$
$$= g(JZ, Z)g(H, JH)$$
$$= 0, \quad \text{since } g(JZ, Z) = 0$$

Hence, $H = 0$, that is, N is totally geodesic.

QED

3.4 CR-product of a Nearly-Kaehler Manifold

In this section, we obtain a result that is almost similar to theorem 2.10 (see p.43) of chapter two when the manifold M is nearly-Kaehler. Similarly, as in the previous sections, we let N be a CR-submanifold of a nearly-Kaehler manifold M and μ be the complementary orthogonal vector bundle of $JD^{\perp}$ in $T^{\perp}N$. The definition of a CR-product of M is as defined in section 2.5 of chapter two (see p.42). We now begin with a result proven by Sato. Our version here is found in Bejancu [3], p. 33.

Theorem 3.10

Let N be a CR-submanifold of a nearly-Kaehler manifold M. Suppose the following conditions are satisfied:

i) $g(h(X,Y), JZ) = 0$ (3.32)

for any $X \in \Gamma(D)$, $Y \in \Gamma(TN)$ and $Z \in \Gamma(D^{\perp})$

and $\qquad$ ii) $g([J,J](X,Y),W) = 0$ $\qquad$ (3.33)

for any X, $Y \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

Then, N is a CR-product of M.

Proof:

By using (3.32) and the Gauss and Weingarten formulas, for any $X \in \Gamma(D)$ and U, $Z \in \Gamma(D^{\perp})$, we have

$$\begin{aligned}
0 &= g(\bar{\nabla}_U X, JZ) = -g(J\bar{\nabla}_U X, Z) \\
&= g(J\bar{\nabla}_U X - \bar{\nabla}_U JX - \bar{\nabla}_X JU, Z), \quad \text{since N is nearly-Kaehler} \\
&= -g(\bar{\nabla}_U X, JZ) - g(\bar{\nabla}_U JX, Z) - g(\bar{\nabla}_X JU, Z) \\
&= -g(\nabla_U X + h(U,X), JZ) - g(\nabla_U JX + h(U,JX), Z) \\
&\quad -g(-A_{JU}X + \nabla^{\perp}_X JU, Z) \\
&= -g(h(U,X), JZ) - g(\nabla_U JX, Z) + g(A_{JU}X, Z) \\
&= -g(\nabla_U JX, Z) + g(h(X,Z), JU), \qquad \text{by (3.32)} \\
&= -g(\nabla_U JX, Z)
\end{aligned}$$

Thus, $\nabla_U JX \in \Gamma(D)$, for any $U \in \Gamma(D^{\perp})$, $X \in \Gamma(D)$. $\qquad$ (3.34)

Since $g([J,J](X,Y),W) = 0$, therefore by theorem 1.7 of chapter one, we have

$$g(4J(\bar{\nabla}_Y J)X, W) = 0, \quad \text{for any } X,\ Y \in \Gamma(D),\ W \in \Gamma(D^{\perp})$$

that is, $\qquad g((\bar{\nabla}_Y J)X, JW) = 0$

or $\qquad g(\bar{\nabla}_Y JX - J\bar{\nabla}_Y X, JW) = 0$

By using the Gauss formula, we obtain

$$g(\nabla_Y JX + h(Y,JX) - J\nabla_Y X - Jh(X,Y), JW) = 0$$

$$g(h(Y,JX), JW) - g(\nabla_Y X, W) = 0$$

that is, $\qquad g(\nabla_Y X, W) = 0, \qquad$ by (3.32)

From proposition 3.3, we obtain the following corollary.

Corollary 3.2

Let N be a CR-submanifold of a nearly-Kaehler manifold M. Then if $A_{JZ}X = 0$, $h(X,W) \in \Gamma(\mu)$, for any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$ and $W \in \Gamma(TN)$.

The following proposition is a converse of theorem 3.10.

Proposition 3.4

Let N be a CR-submanifold of a nearly-Kaehler manifold M. Suppose the distribution D and $D^{\perp}$ are integrable and their leaves are totally geodesic. Then

$$A_{JZ}X = 0$$

and

$$g([J,J](X,Y),W) = 0$$

for any X, $Y \in \Gamma(D)$ and Z, $W \in \Gamma(D^{\perp})$.

Proof:

Since $D^{\perp}$ is integrable and its leaves are totally geodesic, we have

$$g(A_{JZ}X,W) = g(h(X,W),JZ) = 0 \qquad (3.36)$$

for any W, $Z \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$.

We note that, for any X, $Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$,

$$\begin{aligned} g(A_{JZ}X,Y) &= g(h(X,Y),JZ) \\ &= -g(Jh(X,Y),Z) \\ &= -g(h(JX,Y),Z), \qquad \text{by theorem 3.3} \\ &= 0 \end{aligned} \qquad (3.37)$$

Therefore (3.36) and (3.37) gives us

$$A_{JZ}X = 0$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Since D is integrable, $[J,J](X,Y) \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$. Hence

$$g([J,J](X,Y),W) = 0$$

for any $W \in \Gamma(D^{\perp})$.

QED

By combining proposition 3.4, proposition 3.3 and theorem 3.10, we have

Proposition 3.5

Let N be a CR-submanifold of a nearly-Kaehler manifold M. N is a CR-product if and only if

$$A_{JZ}X = 0$$

and
$$g([J,J](X,Y),W) = 0$$

for any $X,Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$.

3.5 D-parallel normal section on a CR-submanifold

The main purpose of this section is to obtain a generalization of theorem 2.11 and theorem 2.12 of chapter two (see p. 46 and 47). We will later see that the results in theorem 2.11 is also true when M is a nearly-Kaehler manifold. However, in order to obtain the same results as in theorem 2.12, an extra condition is needed in the hypothesis of the theorem.

The definition for a mixed totally geodesic CR-submanifold, D-parallel normal section and the holomorphic bisectional curvature are as defined in section 2.5, chapter two. (see p. 45 and 47).

We obtain the following propositions.

<u>Proposition 3.6</u>

Let N be a mixed totally geodesic CR-submanifold of a nearly-Kaehler manifold M. Then, for any $X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp})$,

$$\nabla_Y X \in \Gamma(D)$$

Proof:

Taking account of equation (3.7) and by using the assumption that N is a mixed totally geodesic CR-submanifold, we have

$$Q(\nabla_Y JX) = 0$$

Hence, $\nabla_Y JX \in \Gamma(D)$, for any $X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp})$.

QED

The following proposition is a generalization of theorem 2.11 of chapter two.

Proposition 3.7

Let N be a mixed totally geodesic CR-submanifold of a nearly-Kaehler manifold M. Then the normal section $\xi \in \Gamma(JD^{\perp})$ is D-parallel if and only if $\nabla_X J\xi \in \Gamma(D)$ for each $X \in \Gamma(D)$.

Proof :

Let $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$ such that $JY = \xi$. By using equation (3.8) of section 3.1 and the assumption that N is mixed totally geodesic, we have

$$\begin{aligned} \nabla^{\perp}_X\xi &= JQ\nabla_X Y + JQ\nabla_Y X \\ &= JQ\nabla_X Y, \quad \text{by proposition 3.6} \\ &= JQ\nabla_X(-J\xi) = -JQ\nabla_X J\xi \end{aligned}$$

So, ξ is D-parallel if and only if $\nabla_X J\xi \in \Gamma(D)$.

QED

Next, we have several lemmas. We note that lemma 3.3 and lemma 3.5 are almost similar to lemma 2.5 and lemma 2.6 of chapter two respectively (see p. 46).

Lemma 3.3

Let N be a mixed totally geodesic CR-submanifold of a nearly-Kaehler manifold M. Suppose D is integrable.Then

$$JA_{\xi}X = -A_{\xi}JX$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(T^{\perp}N)$.

Proof:

For any $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(T^{\perp}N)$,

$$g(JA_{\xi}X,\ Y) = -g(A_{\xi}X, JY)$$
$$= -g(h(X,JY),\xi)$$
$$= -g(h(JX,Y),\xi), \quad \text{by theorem 3.1.}$$
$$= -g(A_{\xi}JX, Y)$$

Therefore, $g(JA_{\xi}X + A_{\xi}JX, Y) = 0$. By lemma 2.4 (see p. 45), we obtain

$$JA_{\xi}X = -A_{\xi}JX$$

QED

<u>Lemma 3.4</u>

Let N be a mixed totally geodesic CR-submanifold of a nearly-Kaehler manifold M. Suppose D is integrable and its leaves are totally geodesic in N. Then

$$A_{J\xi}X = -A_{\xi}JX$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(\mu)$.

Proof:

For any $X \in \Gamma(D)$, $Y \in \Gamma(D)$ and $\xi \in \Gamma(\mu)$, we have

$$g(h(JX,Y),\xi) = g(Jh(X,Y),\xi), \quad \text{by theorem 3.3}$$
$$= -g(h(X,Y),J\xi)$$
$$= -g(A_{J\xi}X, Y)$$

However, $g(A_{\xi}JX, Y) = g(h(JX,Y),\xi)$. Therefore,

$$g(A_{\xi}JX + A_{J\xi}X, Y) = 0$$

which gives us, $A_{J\xi}X = -A_{\xi}JX$, by lemma 2.4

Lemma 3.5

Let N be a mixed totally geodesic CR-submanifold of a nearly-Kaehler manifold M. Suppose D is integrable and its leaves are totally geodesic in N. Then

$$A_{J\xi}X = JA_{\xi}X$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(\mu)$.

Proof:

From lemma 3.3 and 3.4, we have

$$JA_{\xi}X = -A_{\xi}JX = A_{J\xi}X$$

Therefore $$A_{J\xi}X = JA_{\xi}X$$

for any $X \in \Gamma(D)$ and $\xi \in \Gamma(\mu)$.

QED

From lemma 3.3, lemma 3.4 and lemma 3.5, we are able to prove proposition 3.8 which is almost similar to theorem 2.12 of chapter two.

Proposition 3.8

Let N be a mixed totally geodesic CR-submanifold of a nearly-Kaehler submanifold M. Suppose D is integrable and its leaves are totally geodesic in N. If there exists a unit vector $X \in \Gamma(D)$ such that for all normal sections $\xi \in \Gamma(\mu)$, the holomorphic bisectional curvatures $H(X,\xi)$ are positive, then the normal subbundle μ does not admit D-parallel sections.

Proof:

Suppose ξ is a parallel section of μ. For X, Y $\in \Gamma(D)$, the curvature tensor $R^{\perp}$ of the normal connection $\nabla^{\perp}$ is given by

$$\begin{aligned} R^{\perp}(X,Y)\xi &= \nabla^{\perp}_X\nabla^{\perp}_Y\xi - \nabla^{\perp}_Y\nabla^{\perp}_X\xi - \nabla^{\perp}_{[X,Y]}\xi \\ &= 0 \end{aligned}$$

From the Ricci equation

$$\begin{aligned} \bar{R}(X,Y;J\xi,\xi) &= g(\bar{R}(X,Y)J\xi,\xi) \\ &= g(R^{\perp}(X,Y)J\xi,\xi) - g([A_{J\xi},A_{\xi}](X),Y) \\ &= - g([A_{J\xi},A_{\xi}](X),Y) \\ &= - g(A_{J\xi}\circ A_{\xi}(X) - A_{\xi}\circ A_{J\xi}(X),Y) \\ &= - g(JA_{\xi}A_{\xi}(X) - A_{\xi}JA_{\xi}(X),Y), \quad \text{from lemma 3.5} \\ &= - g(JA_{\xi}A_{\xi}(X) + JA_{\xi}A_{\xi}(X),Y), \quad \text{from lemma 3.3} \\ &= -2g(JA^2_{\xi}X,Y) \end{aligned}$$

Therefore, $\bar{R}(X,JX;J\xi,\xi) = -2g(JA^2_{\xi}X,JX) = -2g(A^2_{\xi}X,X)$. Since $H(X,\xi)$ is positive, therefore

$$0 > 2g(A^2_{\xi}X,X)$$

which is a contradiction because g is positive definite. Thus the proposition is proved.

QED