CHAPTER FOUR : CR-SUBMANIFOLDS OF A QUASI-KAEHLER MANIFOLD

4.1 Introduction

Let M be a quasi-Kaehler manifold. Then for any $X, Y \in \Gamma(TM)$, we have

$$(\overline{\nabla}_X J)Y + (\overline{\nabla}_{JX} J)JY = 0$$

where $\overline{\nabla}$ is the Levi-Civita connection on M. As in the previous chapters, we let N be a CR-submanifold of M with ∇ as its Levi-Civita connection. We let P and Q be the projection morphisms on the distributions D and $D^{\perp}$ respectively. Hence, from the equation above, when $X \in \Gamma(D)$ and $Y \in \Gamma(TN)$, we have

$$\overline{\nabla}_X JY - J\overline{\nabla}_X Y - \overline{\nabla}_{JX} Y - J\overline{\nabla}_{JX} JY = 0$$

$$\overline{\nabla}_X JPY + \overline{\nabla}_X JQY - J\overline{\nabla}_X Y - \overline{\nabla}_{JX} Y - J\overline{\nabla}_{JX} JPY - J\overline{\nabla}_{JX} JQY = 0$$

By using the Gauss and Weingarten formulas, we have

$$\nabla_X JPY + h(X,JPY) + \nabla^{\perp}_X JQY - A_{JQY}X - J(\nabla_X Y + h(X,Y)) - \nabla_{JX} Y$$
$$- h(JX,Y) - J(\nabla_{JX} JPY + h(JX,JPY)) - J(-A_{JQY}JX + \nabla^{\perp}_{JX} JQY)$$
$$= 0$$

which gives us,

$$\nabla_X JPY + h(X,JPY) + \nabla^{\perp}_X JQY - A_{JQY}X - JP\nabla_X Y - JQ\nabla_X Y$$
$$- Bh(X,Y) - Ch(X,Y) - P\nabla_{JX} Y - Q\nabla_{JX} Y - h(JX,Y) - J\nabla_{JX} JPY$$
$$- Bh(JX,JPY) - Ch(JX,JPY) + JA_{JQY}JX - B\nabla^{\perp}_{JX} JQY - C\nabla^{\perp}_{JX} JQY$$
$$= 0$$

By comparing the tangential and normal parts, we have

$$P(\nabla_X JPY) - P(A_{JQY}X) - JP\nabla_X Y - P\nabla_{JX}Y - JP\nabla_{JX}JPY + JP(A_{JQY}JX) = 0 \tag{4.1}$$

$$Q(\nabla_X JPY) - Q(A_{JQY}X) - Bh(X,Y) - Q(\nabla_{JX}Y) - Bh(JX,JPY) - B(\nabla^{\perp}_{JX}JQY) = 0 \tag{4.2}$$

$$h(X,JPY) + \nabla^{\perp}_X JQY - JQ\nabla_X Y - Ch(X,Y) - h(JX,Y) - JQ\nabla_{JX}JPY - Ch(JX,JPY) + JQ(A_{JQY}JX) - C\nabla^{\perp}_{JX}JQY = 0 \tag{4.3}$$

4.2 Integrability of the Holomorphic Distribution of a CR-submanifold of a Quasi-Kaehler Manifold.

In this section, we will discuss the integrability conditions of the holomorphic distribution D of a CR-submanifold of a quasi-Kaehler manifold M. The following proposition is a generalization of proposition 3.1 of chapter three (see p. 50).

Proposition 4.1

Let N be a submanifold of a quasi-Kaehler manifold M and D_x the holomorphic subspace of T_xN, with constant dimension. If the distribution D is integrable, then

$$h(X,JY) = h(JX,Y)$$

for any $X, Y \in \Gamma(D)$.

Proof:

Let N′ be an integral submanifold of D and let ∇′ be the

Levi-Civita connection with respect to N′, ∇ the Levi-Civita connection with respect to N and $\overline{\nabla}$ the Levi-Civita connection with respect to M. Also, let h′ be the second fundamental form of N′ in N and $\overline{h}$ the second fundamental form of N′ in M. As usual, we let h be the second fundamental form of N in M.

Since M is a quasi-Kaehler manifold, we have

$$(\overline{\nabla}_X J)Y + (\overline{\nabla}_{JX} J)JY = 0$$

$$\overline{\nabla}_X JY - J\overline{\nabla}_X Y - \overline{\nabla}_{JX} Y - J\overline{\nabla}_{JX} JY = 0$$

By using the Gauss formula, we have

$$\nabla'_X JY + \overline{h}(X,JY) - J(\nabla'_X Y + \overline{h}(X,Y)) - \nabla'_{JX} Y - \overline{h}(JX,Y) - J(\nabla'_{JX} JY + \overline{h}(JX,JY)) = 0$$

which gives us

$$\nabla'_X JY - \nabla'_{JX} Y - J(\nabla'_X Y + \nabla'_{JX} JY) + \overline{h}(X,JY) - \overline{h}(JX,Y) - J(\overline{h}(X,Y) + \overline{h}(JX,JY)) = 0$$

Since D is holomorphic, $J(\nabla'_X Y + \nabla'_{JX} JY) \in \Gamma(D)$. Thus, by comparing the tangential and normal part of the above equation, we obtain

$$\nabla'_X JY - \nabla'_{JX} JY - J(\nabla'_X Y + \nabla'_{JX} JY) = 0 \qquad (4.4)$$

and

$$\overline{h}(X,JY) - \overline{h}(JX,Y) - J(\overline{h}(X,Y) + \overline{h}(JY,JX)) = 0 \qquad (4.5)$$

By interchanging X with Y in equation (4.5), this gives us

$$\overline{h}(Y,JX) - \overline{h}(JY,X) - J(\overline{h}(Y,X) + \overline{h}(JY,JX)) = 0 \qquad (4.6)$$

Hence, equations (4.5) and (4.6) gives us

$$\bar{h}(X,JY) - \bar{h}(JX,Y) - \bar{h}(Y,JX) + \bar{h}(JY,X) = 0$$

that is, $$\bar{h}(X,JY) = \bar{h}(JX,Y) \quad (4.7)$$

Since $\bar{h} = h' + h$, we obtain

$$h'(JX,Y) + h(JX,Y) = h'(X,JY) + h(X,JY)$$

that is, $h(JX,Y) - h(X,JY) = h'(X,JY) - h'(JX,Y)$

The left hand side is normal to N in M and the right hand side is tangent to N. Hence, the proposition is proved.

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From the proof of proposition 4.1, we have the following corollary.

Corollary 4.1

Let N be a submanifold of a quasi-Kaehler manifold M and D_x the holomorphic subspace of T_xN, with constant dimension. If D is integrable, the integral submanifold of D is a quasi-Kaehler submanifold of M.

Proof:

Taking account of equation (4.4), we have

$$(\nabla'_X J)Y + (\nabla'_{JX} J)JY = 0$$

Therefore, N′ is a quasi-Kaehler submanifold of M.

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Kon-Tan [16] proved an integrability condition for the distribution D of a CR-submanifold of a quasi-Kaehler manifold. We begin with the following lemma.

<u>Lemma 4.1</u> (Kon-Tan [16])

Let N be a CR-submanifold of a quasi-Kaehler manifold M. Then

i) $$[J,J](X,Y) = 2(\overline{\nabla}_X J)JY - 2(\overline{\nabla}_Y J)JX$$
$$= 2J(\overline{\nabla}_Y J)X - 2J(\overline{\nabla}_X J)Y$$
for any X, $Y \in \Gamma(TM)$.

ii) $$[JX,Y] + [X,JY] = \frac{1}{2} J[J,J](X,Y) + J[X,Y] + \nabla_{JX}Y - \nabla_{JY}X + h(JX,Y) - h(X,JY)$$
for any X, $Y \in \Gamma(D)$.

iii) $$h(X,JY) - h(JX,Y) = \frac{1}{2} J[J,J](X,Y) + J[X,Y] + \nabla_Y JX - \nabla_X JY$$
for any X, $Y \in \Gamma(D)$.

Proof:

i) Since the Levi-Civita connection is torsion free, for any X, $Y \in \Gamma(TM)$,

$$\begin{aligned}[J,J](X,Y) &= [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] \\ &= \overline{\nabla}_{JX}JY - \overline{\nabla}_{JY}JX - \overline{\nabla}_X Y + \overline{\nabla}_Y X - J\overline{\nabla}_{JX}Y + J\overline{\nabla}_Y JX - J\overline{\nabla}_X JY \\ &\quad + J\overline{\nabla}_{JY}X \\ &= (\overline{\nabla}_{JX}J)Y - (\overline{\nabla}_{JY}J)X + J(\overline{\nabla}_Y J)X - J(\overline{\nabla}_X J)Y \\ &= -(\overline{\nabla}_{JX}J)J^2Y + (\overline{\nabla}_{JY}J)J^2X - (\overline{\nabla}_Y J)JX + (\overline{\nabla}_X J)JY \\ &= 2(\overline{\nabla}_X J)JY - 2(\overline{\nabla}_Y J)JX, \quad \text{since M is quasi-Kaehler}\end{aligned}$$

$$= 2J(\overline{\nabla}_Y J)X - 2J(\overline{\nabla}_X J)Y$$

ii) For any X, Y $\in \Gamma(D)$,

$$\frac{1}{2}J[J,J](X,Y) + J[X,Y] + \nabla_{JX}Y - \nabla_{JY}X + h(JX,Y) - h(X,JY)$$
$$= \frac{1}{2}J[J,J](X,Y) + J[X,Y] + \overline{\nabla}_{JX}Y - \overline{\nabla}_{JY}X$$
$$= \frac{1}{2}J[J,J](X,Y) + J[X,Y] + \overline{\nabla}_Y JX + [JX,Y] - \overline{\nabla}_X JY - [JY,X]$$
$$= \frac{1}{2}J[J,J](X,Y) + J[X,Y] + [JX,Y] + [X,JY] + (\overline{\nabla}_Y J)X + J\overline{\nabla}_Y X - (\overline{\nabla}_X J)Y - J\overline{\nabla}_X Y$$
$$= \frac{1}{2}J[J,J](X,Y) + [JX,Y] + [X,JY] - J(\ J(\overline{\nabla}_Y J)X - J(\overline{\nabla}_X J)Y\ )$$
$$= [JX,Y] + [X,JY], \qquad \text{using i).}$$

iii) This follows from ii), since ∇ is torsion free.

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The following theorem is found in Kon-Tan [16]. However we gave a slightly different proof here.

Theorem 4.1 (Kon-Tan [16])

Let N be a CR-submanifold of a quasi-Kaehler manifold N. Then the holomorphic distribution D is integrable if and only if

$$h(X,JY) = h(JX,Y)$$

and
$$[J,J](X,Y) \in \Gamma(D)$$

for any X, Y $\in \Gamma(D)$.

Proof:

Suppose D is integrable. Then

$$[J,J](X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] \in \Gamma(D)$$

for any $X,\ Y \in \Gamma(D)$. It follows from proposition 4.1 that

$$h(X,JY) = h(JX,Y)$$

for any $X,\ Y \in \Gamma(D)$.

Conversely, suppose $h(X,JY) = h(JX,Y)$ and $[J,J](X,Y) \in \Gamma(D)$, for any $X,\ Y \in \Gamma(D)$. Then, from lemma 4.1, we have

$$J[X,Y] = -\frac{1}{2}J[J,J](X,Y) + \nabla_X JY - \nabla_Y JX \in \Gamma(TN)$$

It then follows that $[X,Y] \in \Gamma(D)$, for any $X,\ Y \in \Gamma(D)$. Hence, D is integrable.

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Combining theorem 4.1 with lemma 4.1 (i), we have the following theorem.

<u>Theorem 4.2</u> (Kon- Tan [16])

Let N be a CR-submanifold of a quasi-Kaehler manifold M. The holomorphic distribution D is integrable if and only if

$$h(X,JY) = h(JX,Y)$$

and
$$(\overline{\nabla}_X J)Y - (\overline{\nabla}_Y J)X \in \Gamma(D)$$

for any $X,\ Y \in \Gamma(D)$.

We denote by μ the complementary orthogonal subbundle to $J(D^{\perp})$ in $T^{\perp}N$. Then μ is invariant by J, that is, $J(\mu_x) = \mu_x$, for each $x \in N$ (see section 2.3, chapter 2). We obtain the following proposition.

Proposition 4.2

Let N be a CR-submanifold of a quasi-Kaehler manifold M. The condition $h(X,JY) = h(JX,Y)$ is satisfied if and only if

$$g(h(X,JY) - h(JX,Y),\ JZ) = 0$$

for any $X,\ Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Proof:

Suppose for any $X,\ Y \in \Gamma(D)$, $h(X,JY) = h(JX,Y)$. Then, obviously, $g(h(X,JY) - h(JX,Y),\ JZ) = 0$, for any $Z \in \Gamma(D^{\perp})$.

Conversely, suppose $g(h(X,JY) - h(JX,Y)\ ,\ JZ) = 0$. Then for any $\xi \in \Gamma(\mu)$ and by applying lemma 4.1, we obtain

$$\begin{aligned}
&g(h(X,JY) - h(JX,Y),\ \xi)\\
&= g(\tfrac{1}{2}\ J[J,J](X,Y) + J[X,Y] + \nabla_Y JX - \nabla_X JY\ ,\xi\ )\\
&= \tfrac{1}{2}\ g(J[J,J](X,Y),\xi) + g(J[X,Y],\xi) + g(\nabla_Y JX,\xi)\\
&\quad - g(\nabla_X JY,\xi)\\
&= -\tfrac{1}{2}\ g([J,J](X,Y),J\xi) - g([X,Y],J\xi)\\
&= 0
\end{aligned}$$

Therefore, $h(X,JY) = h(JX,Y)$.

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4.3 Minimal Distribution

Let D be a differentiable distribution on a Riemannian manifold with Levi-Civita connection ∇. We write

$$\alpha(X,Y) = (\nabla_X Y)^{\perp}$$

for $X,\ Y \in \Gamma(D)$, where $(\nabla_X Y)^{\perp}$ denotes the component of $\nabla_X Y$ in the orthogonal complementary distribution $D^{\perp}$.

If $\{X_1,\dots,X_r\}$ is an orthonormal basis for D, we define the *mean curvature vector* H^0 of D by

$$H^0 = \frac{1}{r}\sum_{k=1}^{r} \alpha(X_k,X_k)$$

The distribution D is said to be *minimal* if H^0 vanishes identically.

For the holomorphic distribution D, we can find an orthonormal basis of the form $\{X_1,\dots X_s,JX_1,\dots JX_s\}$ where $\dim_{\mathbb{R}} D = r = 2s$, since D is invariant under J. Thus,

$$H^0 = \frac{1}{r}\sum_{k=1}^{s} (\alpha(X_k,X_k) + \alpha(JX_k,JX_k))$$

This shows that D is minimal if $\nabla_{X_k}X_k + \nabla_{JX_k}JX_k$ has no component in $D^{\perp}$, for $k = 1,\dots s$.

The following result is obtained by Kon-Tan [16]. However, we give a slightly different proof here.

Proposition 4.3

Let N be a CR-submanifold of a quasi-Kaehler manifold M. If the holomorphic distribution D is integrable, then each leaf of D is a minimal submanifold in both N and M.

Proof:

For any X, $Y \in \Gamma(D)$, we have

$$(\overline{\nabla}_X J)Y + (\overline{\nabla}_{JX} J)JY = 0$$

$$\overline{\nabla}_X JY - J\overline{\nabla}_X Y - \overline{\nabla}_{JX} Y - J\overline{\nabla}_{JX} JY = 0$$

By applying the Gauss formula, we obtain

$$\nabla_X JY + h(X,JY) - J\nabla_X Y - Jh(X,Y) - \nabla_{JX} Y - h(JX,Y) - J\nabla_{JX} JY - Jh(JX,JY) = 0$$

Since D is integrable, the equation above becomes

$$\nabla_X JY - \nabla_{JX} Y - J(\nabla_X Y + \nabla_{JX} JY) = 0$$

that is, $J(\nabla_X Y + \nabla_{JX} JY) = \nabla_{JX} Y - \nabla_X JY$.

Hence, $$\nabla_X Y + \nabla_{JX} JY \in \Gamma(D). \tag{4.8}$$

For any $X \in \Gamma(D)$, we thus obtain

$$\nabla_X X + \nabla_{JX} JX \in \Gamma(D)$$

Therefore, D is minimal in N.

Similarly, by applying the Gauss formula and theorem 4.1, we have

$$\overline{\nabla}_X X + \overline{\nabla}_{JX} JX = \nabla_X X + h(X,X) + \nabla_{JX} JX + h(JX,JX)$$
$$= \nabla_X X + \nabla_{JX} JX \in \Gamma(D)$$

Hence, D is minimal in both N and M.

QED

The following corollary is obtained from the proof of proposition 4.3.

Corollary 4.2

Let N be a CR-submanifold of quasi-Kaehler manifold M. Suppose D is integrable. Then

$$\nabla_X Y + \nabla_{JX} JY \in \Gamma(D)$$

for any X, Y $\in \Gamma(D)$.

Suppose D is integrable. For any X, Y $\in \Gamma(D)$, we write

$$\nabla_X Y = \nabla'_X Y + \alpha(X,Y) \qquad (4.9)$$

where $\nabla'_X Y \in \Gamma(D)$ and $\alpha(X,Y) \in \Gamma(D^{\perp})$. By using proposition 4.3, we obtain the following result.

Corollary 4.3

Let N be a CR-submanifold of a quasi-Kaehler manifold M. Suppose D is integrable. Then

$$\alpha(JX,Y) = \alpha(X,JY)$$

for any X, Y $\in \Gamma(D)$.

Proof:

By using equations (4.8) and (4.9), we obtain

$$\nabla'_X Y + \alpha(X,Y) + \nabla'_{JX} JY + \alpha(JX,JY) \in \Gamma(D)$$

This shows that

$$\alpha(X,Y) = -\alpha(JX,JY)$$

that is,

$$\alpha(JX,Y) = \alpha(X,JY)$$

for any X, Y $\in \Gamma(D)$.

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4.4 Totally Umbilical CR-submanifold of a Quasi-Kaehler Manifold

Let N be a totally umbilical CR-submanifold of a quasi-Kaehler manifold M. In this section, we obtain a necessary and sufficient condition for the distribution $D^{\perp}$ to be integrable. We also obtain a result which shows that the mean curvature vector H lies in $JD^{\perp}$.

We begin with the following proposition.

Proposition 4.4

Let N be a totally umbilical CR-submanifold of a quasi-Kaehler manifold M. Then the distribution $D^{\perp}$ is integrable if and only if

$$g([J,J](V,W),X) = 0$$

for any $V, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$.

Proof:

For any $V, W \in \Gamma(D^{\perp})$, and by applying the result obtained in lemma 4.1, we have

$$\begin{aligned}[J,J](V,W) &= 2J(\bar{\nabla}_W J)V - 2J(\bar{\nabla}_V J)W \\ &= 2J(\bar{\nabla}_W JV - J\bar{\nabla}_W V) - 2J(\bar{\nabla}_V JW - J\bar{\nabla}_V W) \\ &= 2J\bar{\nabla}_W JV + 2\bar{\nabla}_W V - 2J\bar{\nabla}_V JW - 2\bar{\nabla}_V W\end{aligned}$$

By applying the Gauss and Weingarten formulas, we have

$$\begin{aligned}[J,J](V,W) &= 2J(-A_{JV}W + \nabla^{\perp}_W JV) + 2\nabla_W V + 2h(W,V) \\ &\quad - 2J(-A_{JW}V + \nabla^{\perp}_V JW) - 2\nabla_V W - 2h(V,W) \\ &= 2J(-A_{JV}W + \nabla^{\perp}_W JV + A_{JW}V - \nabla^{\perp}_V JW) + 2(\nabla_W V - \nabla_V W) \\ &= 2J(-A_{JV}W + \nabla^{\perp}_W JV + A_{JW}V - \nabla^{\perp}_V JW) + 2[W,V]\end{aligned}$$

Therefore, for any $X \in \Gamma(D)$,

$$\frac{1}{2}\, g([J,J](V,W),X)$$

$$= -g(JA_{JV}W,X) + g(J\nabla^{\perp}_{W}JV,X) + g(JA_{JW}V,X) - g(J\nabla^{\perp}_{V}JW,X) + g([W,V],X)$$

$$= g(A_{JV}W,JX) - g(\nabla^{\perp}_{W}JV,JX) - g(A_{JW}V,JX) + g(\nabla^{\perp}_{V}JW,JX) + g([W,V],X)$$

$$= g(h(W,JX),JV) - g(h(V,JX),JW) + g([W,V],X)$$

$$= g(W,JX)g(H,JV) - g(V,JX)g(H,JW) + g([W,V],X),$$ since N is totally umbilical

$$= g([W,V],X),$$

that is, $\frac{1}{2}\, g([J,J](V,W),X) = g([W,V],X)$. Hence, $D^{\perp}$ is integrable if and only if $g([J,J](V,W),X) = 0$

QED

Next, we have the following proposition.

Proposition 4.5

Let N be a totally umbilical proper CR-submanifold of a quasi-Kaehler manifold M. Then the mean curvature vector $H \in \Gamma(JD^{\perp})$.

Proof:

For any $X, Y \in \Gamma(D)$, we have

$$(\overline{\nabla}_{X}J)Y + (\overline{\nabla}_{JX}J)JY = 0$$

$$\overline{\nabla}_{X}JY - J\overline{\nabla}_{X}Y - \overline{\nabla}_{JX}Y - J\overline{\nabla}_{JX}JY = 0$$

By using the Gauss formula,

$$\nabla_{X}JY + h(X,JY) - J\nabla_{X}Y - Jh(X,Y) - \nabla_{JX}Y - h(JX,Y) - J\nabla_{JX}JY$$

$- Jh(JX,JY) = 0$

Since N is totally umbilical,

$$\nabla_X JY + g(X,JY)H - J\nabla_X Y - Jg(X,Y)H - \nabla_{JX} Y - g(JX,Y)H - J\nabla_{JX} JY - Jg(JX,JY)H = 0$$

Since $-g(JX,Y) = g(X,JY)$ and $g(X,Y) = g(JX,JY)$, thus,

$$\nabla_X JY + 2g(X,JY)H - J\nabla_X Y - 2g(X,Y)JH - \nabla_{JX} Y - J\nabla_{JX} JY = 0$$

For any $\xi \in \Gamma(\mu)$, we have

$$g(\nabla_X JY + 2g(X,JY)H - J\nabla_X Y - 2g(X,Y)JH - \nabla_{JX} Y - J\nabla_{JX} JY, \xi) = 0$$

that is,

$$g(\nabla_X JY,\xi) + 2g(X,JY)g(H,\xi) - g(J\nabla_X Y,\xi) - 2g(X,Y)g(JH,\xi) - g(\nabla_{JX} Y,\xi) - g(J\nabla_{JX} JY,\xi) = 0$$

$$2g(X,JY)g(H,\xi) + g(\nabla_X Y, J\xi) - 2g(X,Y)g(JH,\xi) + g(\nabla_{JX} JY, J\xi) = 0$$

Therefore, $2g(X,JY)g(H,\xi) + 2g(X,Y)g(H,J\xi) = 0$

For any unit vector $X \in \Gamma(D)$,

$$2g(X,JX)g(H,\xi) + 2g(X,X)g(H,J\xi) = 0$$

Since $g(X,JX) = g(JX,X) = -g(X,JX)$, thus, $g(X,JX) = 0$. Therefore,

$$2g(H,J\xi) = 0$$

It follows that $H \in \Gamma(JD^{\perp})$, since $J\xi \in \Gamma(\mu)$.

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Let M be a nearly-Kaehler manifold. Hence, M is also quasi-Kaehler. It follows from lemma 3.2 of chapter three (see p. 61) and proposition 4.5 that if N is a totally umbilical proper CR-submanifold of a nearly-Kaehler manifold M, with dim $D^{\perp} > 1$, then $H = 0$ and since N is totally umbilical, it follows that $h = 0$. Hence, N is totally geodesic. Therefore, we obtain the following corollary (compare theorem 3.9 of chap. 3, p. 64).

Corollary 4.4

Let N be a totally umbilical proper CR-submanifold of a nearly-Kaehler manifold M. Then N is totally geodesic or the dim $D^{\perp} = 1$.

4.5 Mixed Totally Geodesic CR-submanifold

In this section, we obtain a set of equivalent equations stated in the proposition below.

Proposition 4.6

Let N be a mixed totally geodesic CR-submanifold of a quasi-Kaehler manifold M. Then for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$, the following are equivalent:

i) $\nabla_{JX}Y \in \Gamma(D)$

ii) $B(\nabla^{\perp}_{JX}JY) = 0$

iii) $J(\nabla^{\perp}_{JX}JY) = \nabla^{\perp}_{X}JY$

Proof:

Taking account of equation (4.2), we have

$$Q(\nabla_{JX}Y) + B(\nabla^{\perp}_{JX}JY) = 0$$

$$Q(\nabla_{JX}Y) = -B(\nabla^{\perp}_{JX}JY)$$

Therefore, $\nabla_{JX}Y \in \Gamma(D)$ if and only if $B(\nabla^{\perp}_{JX}JY) = 0$, for any $X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp})$.

Suppose $\nabla_{JX}Y \in \Gamma(D)$. By using equation (4.3), we obtain

$$\nabla^{\perp}_{X}JY - JQ\nabla_{X}Y - C\nabla^{\perp}_{JX}JY = 0$$

$$\nabla^{\perp}_{X}JY - C\nabla^{\perp}_{JX}JY = 0, \qquad \text{since } \nabla_{X}Y \in \Gamma(D)$$

$$\nabla^{\perp}_{X}JY - (J\nabla^{\perp}_{JX}JY - B(\nabla^{\perp}_{JX}JY)) = 0$$

Hence, $J\nabla^{\perp}_{JX}JY = \nabla^{\perp}_{X}JY$, since $B(\nabla^{\perp}_{JX}JY) = 0$, for any $X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp})$.

Suppose $J(\nabla^{\perp}_{JX}JY) = \nabla^{\perp}_{X}JY$. Taking account of equation (4.3), we have

$$\nabla^{\perp}_{X}JY - JQ\nabla_{X}Y - C\nabla^{\perp}_{JX}JY = 0$$

$$\nabla^{\perp}_{X}JY - JQ\nabla_{X}Y - (J\nabla^{\perp}_{JX}JY - B(\nabla^{\perp}_{JX}JY)) = 0$$

$$-JQ\nabla_{X}Y + B(\nabla^{\perp}_{JX}JY) = 0$$

By comparing the tangential part, we obtain $B(\nabla^{\perp}_{JX}JY) = 0$, for any $X \in \Gamma(D)$, $Y \in \Gamma(D^{\perp})$.

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