CHAPTER 1

INTRODUCTION

1.1 Overview of Thesis

Optimal control is an important branch of mathematics, and has been widely applied in a number of fields, including engineering, economics, environment and management. Historically, after more than three hundred years of evolution, optimal control theory has been formulated as an extension of the calculus of variations. The calculus of variations is much harder than standard calculus because finding the optimal form of an entire function is more difficult than finding the optimal value of a variable.

As most real-world problems are too complex to allow for an analytical solution, computational algorithms are inevitably used to solve optimal control problems. As a result, several successful families of algorithms have been developed over the years. The formulation of an optimal control problem requires several steps: the class of admissible controls, the mathematical description of the system to be controlled, the specification of a performance criterion, and the statement of physical constraints that should be satisfied. The objective of optimal control is to determine an optimal open-loop control $u^*(t)$ or an optimal feedback control $u^*(x,t)$ that forces the system to satisfy physical system constraints and at the same time minimizes or maximizes a performance index.

Physical systems are inherently nonlinear in nature. However, nonlinear systems are difficult to analyze mathematically. The typical approach is to linearize the system around some operating point and analyze the resulting linear system. If the motion of the system does not satisfy the superposition principle, then the linear model of the system becomes invalid. Therefore considering the full nonlinear model of the system is desirable. One of two approaches is typically adopted to address the inherent mathematical difficulty of nonlinear system. The first approach is to utilize specific properties of the system to develop specific control laws that perform well for that system. The drawback of this approach is that the results may not be applicable to any other system. The second approach is to develop tools for general classes of nonlinear systems. The drawback of this approach is that these tools will usually result in conservative designs because they do not exploit specific characteristics of the system under design. Having a number of design tools from which to draw is necessary to address any particular problem. Relatively few design tools for nonlinear systems exist. Therefore, one of our objectives is to develop a feedback synthesis method for a general class of nonlinear systems.

Generally, solutions of optimal control problems, except for the simplest cases, are carried out numerically. Therefore, numerical methods and algorithms for solving optimal control problems have evolved significantly over the past fifty years. Most early methods were based on finding a solution that satisfies either Euler-Lagrange equations, which are the necessary conditions of optimality, or the Hamilton-Jacobi-Bellman (HJB) equation, which is a sufficient condition of optimality. These methods are called indirect methods.

Optimal control of nonlinear systems is one of the most challenging and difficult subjects in control theory. The nonlinear optimal control problem can be reduced to the Hamilton-Jacobi-Bellman partial differential equation, but due to difficulties in its solution, this is not a practical approach. Instead, the search for nonlinear control schemes has generally been approached on less ambitious grounds than requiring the exact solution to the Hamilton-Jacobi-Bellman partial differential equation. In fact, even the problem of stabilizing a nonlinear system remains a challenging task. Lyapunov theory, a successful and widely used tool for stability analysis of nonlinear systems, is a century old. Despite having existed for a long time, systematic methods for obtaining Lyapunov functions for general nonlinear systems are still nonexistent. Nevertheless, the ideas presented by Lyapunov nearly a century ago continue to be used and exploited extensively in the modern theory of control for nonlinear systems. One notably successful use of the Lyapunov methodology is the concept of a control Lyapunov function (CLF), the idea of which is to first choose a function that can be made into a Lyapunov function for the closed-loop system by choosing appropriate control actions. The HJB equation provides a global control law in the form of a state feedback controller. Unfortunately, it involves the solution of a partial differential equation (PDE), which is in general computationally intractable. This single fact is largely the reason for the existence of the discipline of nonlinear optimal control. Hence, from one point of view, nonlinear optimal control can be thought of as the development of computationally tractable sub-optimal solutions to the optimal control problem. This explanation is attractive from a pedagogical viewpoint because it provides a natural justification for the close relationship between many popular approaches and the HJB equation. The following important aspects of the HJB solution should be highlighted for clarity: (1) Closed loop: The resulting solution is a state feedback control law. (2) Global: The solution provides the optimal control trajectory from every initial condition. Hence, it solves the optimal control problem for every initial condition all at once. (3) Sufficient: The solution of the HJB equation provides a sufficient condition for the solution to the corresponding optimal control problem.

Optimal control problems without constraints can be solved successfully by using most direct and indirect techniques. However, inequality constraints often generate analytical and computational difficulties. Thus, researchers aim to solve constrained optimal control problems with numerical methods. The direct method is widely used to solve nonlinear optimal control problems. It obtains an optimal solution by directly minimizing the constrained performance index. Furthermore, this method converts the optimal control problem into a mathematical programming problem by using either the discretization or the parameterization technique. Parameterization methods are classified into three types: state, control, and state control. Direct methods were used to obtain an open-loop solution of optimal control problems.

With regard to the parameterization method, a significant amount of published papers are based on either control parameterization or state parameterization. These two approaches have some drawbacks, such as the following: In the control parameterization case, the system state equations need to be integrated, which is a computationally expensive. In the state parameterization case, this approach has not been used extensively because applying it to general optimal control problems is difficult. This difficulty is due to the fact that it is unclear which state variables to be parameterized in case of unequal number of state variables and control variables. Control-state parameterization is a third type of parameterization. The use of this approach has been limited so far because the optimal control problem is reduced to a large mathematical programming problem, i.e., it has a large number of unknown parameters and equality constraints. With the development of computers with high speed and efficient algorithm over the last few decades, it has become possible to solve complicated problems in a reasonable amount of time. Therefore the second goal of this thesis is to apply control-state parameterization to general constrained optimal control problems with finite time horizon by using orthogonal wavelets.

Most of the time, orthogonal functions are used to solve dynamic systems. Among the orthogonal functions, numerical method based on wavelet is a relatively new mathematical tools for solving integral and differential equations. Numerical solutions of these equations have been discussed in many papers, which basically fall either in the class of spectral Galerkin and collocation methods or finite element and finite difference methods. Compared with other mathematical tools, wavelet analysis has captured the attention of mathematicians' because it has obtained positive results in the field of signal and image processing. The most interesting features of wavelet is that its basis function, which is localized in space or time coexists with localization in frequency. The basis functions are usually orthogonal and compactly supported, which allow us to better represent functions with sharp spikes or edges, than other bases. These features result in sparse transformation in wavelet domain for non-stationary signals that contributes to fast algorithms; these are some of the desired properties in numerical analysis. Haar wavelet is the simplest orthogonal wavelet with a compact support. In our work, we considered the method of Beard et al. (2000) to successively approximate the solution of HJB equation. Instead of using the Galerkin method with polynomial basis, we used collocation method with Haar wavelet basis to solve the generalized Hamilton-Jacobi-Bellman (GHJB) equation. Galerkin's method requires the computation of multidimensional integrals which makes the method impractical for higher-order systems. The main advantage of using collocation method in general is that the computational burden of solving the GHJB equation is reduced to matrix computation only. Our new successive Haar wavelet collocation method is used to solve linear and nonlinear optimal control problems. In the process of establishing the method we have to define new operational matrices of integration for a chosen stabilizing domain and new operational matrix for the product of two dimensional Haar wavelet functions.

Another goal of this thesis is to solve the constrained nonlinear optimal control problem by converting it directly, with the use of control-state parameterization via Haar wavelets basis into a sequence of quadratic programming problems. This approach has two advantages: first linear and nonlinear optimal control problems can be solved uniformly, and second guessing nominal trajectories, which we need to convert the nonlinear optimal control problem into a sequence of linear quadratic optimal control problems, is easier than guessing the parameters of these trajectories, which we need to solve the nonlinear mathematical programming problem.

Many classical inventory models emphasize the single-item model. However, such models are seldom applied in the real world. Hence, multi-item inventory models are more realistic than single-item models. In multi-item models, the second item in an inventory favours the demand for the first and vice-versa. The final goal of this thesis is to optimize the control of the multi-item production-inventory model with stock-dependent deterioration rates and deterioration due to self-contact and the presence of the other stock by using the direct method.

1.2 Motivation

1. Although the necessary and sufficient conditions for optimality have already been derived, they are useful only for finding analytical solutions for quite restricted cases. If we assume full-state knowledge and if the optimal control problem is a linear-quadratic, then the optimal control is a linear feedback of the state, which is obtained by solving a matrix Riccati equation. However, if the system is nonlinear, then the optimal control is a state feedback function, which depends on the solution to the HJB equation. HJB equation is a nonlinear partial differential equation that is usually difficult to solve analytically.

2. Historically orthogonal bases are related with differential equations, including partial differential equations. Recently, orthogonal basis with compact support, such as Daubechies wavelet, have been used successfully in signal and image processing.

In addition, the availability of fast transform makes orthogonal basis attractive as a computational tool. Haar wavelet which is a piecewise function, is the simplest orthogonal wavelet with a compact support. Thus, studying where this Haar wavelet can be used to solve ordinary and partial differential equation is an interesting task. Haar wavelet is not continuous. Therefore, the highest derivatives that appear in the differential equations are first expanded by using Haar wavelet basis. Lower-order derivatives and the solutions can then be obtained easily by using Haar operational matrix of integration. The main ideas of using Haar wavelet operational matrix is to convert partial differential equations into matrix equations that can be solved easily by using MATLAB.

3. The following questions need to be addressed: If we are given an initial stabilizing control, how do we improve the closed-loop performance of this control?. Does a simple method of computing the improved control law exist?. A solution to these questions bridges the problems of finding a stable control law and finding the optimal control. For nonlinear systems, the optimal control problem is reduced to the solution of the HJB equation; this equation is difficult to solve. Thus, researchers have looked for methods of approximating its solution with a numerical method. For example, Beard (1995) used Galerkin method with polynomials basis to solve the above problem. We will use collocation method with Haar wavelet to solve the problem. Using Haar wavelet method that deals with matrices is much simpler than polynomial integration in Galerkin method.

1.3 Scope of the Study

The work throughout this study is concerned with quadratic optimal control (QOC) problems that are associated with both finite and infinite time horizon of minimizing a performance index. We will address the following related control problems:

- The **infinite-time horizon problem,** where the system equations are assumed to be linear and nonlinear and the optimization index is over an infinite time interval.
- The **finite-time horizon problem,** where the system equations are assumed to be constrained linear and nonlinear time-varying and the optimal index is over a finite time interval.

The main focus of this study is to establish two methods, which are the indirect and the direct methods to solve the nonlinear optimal control problem. In the process of establishing the methods, we have derived some new operational matrices of integration for a chosen domain and a new operational matrix for the product of two dimensions Haar wavelet functions.

We further our study by utilizing Lyapunov functions for the feedback system. A Lyapunov function is a generalized energy function of the state and is usually suggested by the physics of the problem. With the use of Lyapunov theory, finding a stabilizing control for a particular system is often possible.

However, the numerical stability and error analysis of both proposed numerical methods are not mathematically proven. A comparison with the analytical solution given by others is conducted to justify the accuracy of these numerical results.

1.4 Research Objectives

The following are the objectives of this research:

1. Derive new formulas of two dimensions Haar wavelet operational matrices for partial integration for a chosen interval $[-\tau, \tau)$.
2. Derive a new formula for Haar wavelet operational matrix for the product of two dimensional Haar wavelet functions.
3. Establish a numerical algorithm for solving GHJB equation by using Haar wavelet operational matrices and Haar wavelet collocation method.
4. Solve HJB equation iteratively by using GHJB equation.
5. Establish a novel feedback control method of solving optimal control problems with quadratic performance index subject to nonlinear affine control system with infinite time horizon.
6. Propose a new numerical method for solving constrained nonlinear optimal control problem with finite time horizon by using quasilinearization technique and Haar wavelet operational matrix to convert the nonlinear optimal control problem into a quadratic programming problem.
7. Apply the proposed method in (6) to practical problems such as optimization of the control of nonlinear optimal control of a multi-item production-inventory model with stock-dependent deterioration rates, deterioration due to self-contact, and the presence of the other stock.
8. Develop MATLAB programs for solving infinite time nonlinear optimal control problems and finite time constrained nonlinear optimal control problems.

1.5 Organization of the Thesis

This thesis consists of seven chapters, including this chapter, and is organized as follows:

In Chapter 2, we present an overview of the operational matrix in general. We list a few well-known orthogonal functions that have been used to derive the operational matrix. Next, we narrow it down to a specific orthogonal function namely Haar basis function. The selection of this orthogonal function will be justified by presenting its advantages over that of other orthogonal functions. We present a few advantages of this orthogonal function to justify our selection of the Haar wavelet function. We further discuss our main problem of solving the optimal control problem. At the end of this chapter, we examine the multi-item production-inventory model.

In Chapter 3, we illustrate the mathematical background of Haar wavelets which are needed to understand the concepts that are introduced in the remainder of this thesis. Most studies define Haar wavelet and its operational matrix within the interval [0, 1). We derive Haar wavelet operational matrix which could cater to the Haar series beyond the interval $[0,1)$. The remainder of the thesis presents the difficulties encountered while solving the nonlinear optimal control problems and the solutions to these difficulties as well as provide the reader with sufficient contexts to understand certain related concepts. In particular, we derive some new formulas for Haar wavelet operational matrices in higher dimensions of integration for a chosen interval $[-\tau,\tau)$ and new formulae for Haar Wavelet operational matrix for the product of two dimensional Haar wavelet functions. A general formula of Haar wavelet collocation point's matrix with two variables is derived, which is another motivation behind developing a novel feedback control algorithm described in Chapter 4.

In Chapter 4, a novel method is introduced to solve the HJB equation, which appears in the formulation of the nonlinear control system with quadratic cost functional and infinite time horizon. This method is a numerical technique, which is based on the combination of Haar wavelets operational matrices and successive GHJB equation, to improve the closed-loop performance of stabilizing controls and reduces the problem of solving a nonlinear HJB equation to that of solving the corresponding GHJB equation. The solution to the GHJB equation converges uniformly to the solution of the HJB equation, which is in the form of the gradient of the Lyapunov function $\nabla \mathbf{V}(x)$. In order to determine the Lyapunov function from the resulting solution of the linear system equation. A new method is proposed in this chapter to integrate the gradient of the Lyapunov function using variable gradient method. A number of numerical examples for optimal control problems are given to justify the proposed method.

In Chapter 5, an efficient new algorithm is proposed to solve nonlinear optimal control problems with a finite time horizon under inequality constraints. In this technique we parameterize both the states and the controls by using Haar wavelet functions and Haar wavelet operational matrix. The nonlinear optimal control problem is converted into a quadratic programming problem through quasilinearization iterative technique. The inequality constraints for trajectory variables are transformed into quadratic programming constraints by using the Haar wavelet collocation method. The quadratic programming problem with linear inequality constraints is then solved by using standard QP solver.

In Chapter 6, the proposed method in Chapter 5 is applied to optimize the control of the multi-item production-inventory model with stock-dependent deterioration rates and deterioration due to self-contact and the presence of the other stock. Four different types of demand rates, namely, constant, linear, logistic, and periodic demand rates. The solution to the model is discussed numerically and displayed graphically. By enhancing the resolution of the Haar wavelet, we can improve the accuracy of the states, controls, and cost. Simulation results were compared with those obtained by another researcher's work.

Chapter 7, summarizes the overall works and contributions of the study to the indirect method of nonlinear optimal control problems with an infinite time horizon and the direct method for constrained nonlinear optimal control problems with a finite time horizon. Some recommendations for future work are proposed.

CHAPTER 2

LITERATURE REVIEW

Operational matrix method has received considerable attention from many scholars for solving dynamical system analysis (Sinha and Butcher, 1997), system identification (Dosthosseini et al., 2010), numerical solution of integral and differential equations (Lepik, 2005; Kilicman and Zhour, 2007) and optimal control problem (Mohan and Kar, 2005; Endow, 1989; Karimi, 2006). The operational matrix method mainly involves casting a differential or integral equation into a corresponding matrix equation. The approach is based on converting the underlying differential equations into integral equations through integration of operators and approximating the functions involved in the equation by truncated orthogonal series. An operation of integral operator is converted by an operational matrix. To have a better view of the operational matrix method, let us consider the integral property of function vector $\Phi(x)$ in the following approximation:

$$\int_0^x \Phi(\tau)d\tau = \mathbf{P}\,\Phi(x), \tag{2.1}$$

where

$$\Phi(x) = [\phi_0(x) \quad \phi_0(x) \quad \cdots \quad \phi_{m-1}(x)]^{\mathbf{T}} \tag{2.2}$$

in which the elements $\phi_0(x) \quad \phi_1(x) \quad \cdots \quad \phi_{m-1}(x)$ are the orthogonal basis functions in the Hilbert space $L^2(\Re)$. The operational matrix $\mathbf{P}$ is an $m \times m$ constant matrix and behaves as an integrator (Cheng et al., 1977; Irfan and Kapoor, 2011) and can be uniquely determined on the basis of the particular orthogonal functions, $\phi_i(x)$.

At present, large number of literature derive operational matrix from different orthogonal functions. Orthogonal basis functions that have been given special attention are Walsh function (Chen and Hsiao, 1975), block pulse function (Chi-Hsu, 1983), cosine-sine and exponential function (Paraskevopoulos, 1987), normalized Bernstein polynomials (Singh et al., 2009), linear Legendre mother wavelets (Khellat and Yousefi, 2006), Chebyshev wavelet (Babolian and Fattahzadeh, 2007) and Haar wavelet (Gu and Jiang, 1996; Chen and Hsiao, 1997).

Chen and Hsiao (1975) derived Walsh operational matrix for performing integration and solving generalized state equations. Paraskevopoulos (1987) showed the operational matrix relationship between Fourier sine-cosine series and Fourier exponential series expansion. Babolian and Fattahzadeh (2007) obtained Chebyshev operational matrix for integration in general, and for finding continuous and discontinuous solutions of Volterra type integral equations. All of these numerical computations share a number of advantages. One of the advantages is the possibility of finding the solution using only matrix manipulation rather than performing integration or differentiation in a conventional ways. Another advantage is that the matrices can be transformed into a sparse matrix and a small number of significant coefficients (Hariharan and Kannan, 2011), which is important factor for reducing computation time. Nonetheless, the advantage remains, when a large matrix is involved, whereby large computer storage space and a huge number of arithmetic operations are required (Lepik and Tamme, 2004).

In this study, we are going to work with the Haar wavelet basis function and Haar wavelet operational matrices to approximation functions and integrating functions respectively. Haar wavelet has a few advantages compared with other wavelet functions. Haar wavelet is the oldest and the simplest wavelet function and it is one example of an orthogonal function (Burrus et al., 1998). Haar wavelet bases has compact support, which means that the Haar wavelet vanishes outside of a finite interval and allow us to represent functions with sharp spikes or edges, better than other bases. The admired properties of Haar wavelet orthogonal functions in numerical computation include the following: the sparse representation for piecewise constant function, fast transformation, and the possibility of implementing a fast algorithm in matrix (Shahsavaran, 2011). Faster matrix transformation can be achieved through the expansion of Haar series than the expansion of Walsh series for the same amount of terms required for computation because the resolution order by Haar expansion is less than that by Walsh expansion (Khuri, 1994). Haar wavelet operational matrix for the integral of Haar wavelets is always positive definite. Hence Haar wavelet operational matrix inverses are always available. This property of Haar wavelets makes this method computer oriented because no singularities are involved in the computation (Chen and Hsiao, 1997). This factor gives an additional advantage to the proposed numerical method which is discussed in Chapter 4.

Recently, Haar wavelets have been applied to signal and image processing in communication research and physics research and have been proven to be excellent mathematical tools (Nievergelt, 1999). It has been applied to a wide range of application such as in system analysis (Chen and Hsiao, 1999), and numerical solutions of nonlinear integral equations (Aziz and Islam, 2013; Islam et al., 2014; Aziz et al., 2014), numerical solutions of integro-differential equations (Islam et al., 2013), boundary-value problems (Islam et al., 2010; Islam et al., 2011; Fazal et al., 2011; Aziz et al., 2013) and optimal control problems (Swaidan and Hussin, 2013). The first attempt at using the Haar basis function for solving differential equations was conducted by Chen and Hsiao (1997), who were the first to derive the Haar operational matrix for integrals and brought the application of Haar analysis into dynamic systems. Chen and Hsiao

(1997) applied their proposed method to solve the state equations of lumped and distributed-parameter linear systems based on the Haar wavelet. Hsiao (1997) constructed the new Haar product matrix and coefficient matrix, which have been applied to various problems, such as the state analysis of linear time-delayed systems. The main characteristic of this technique is its capability to convert differential equations into algebraic equations. Thus, solution identification and optimization procedures are either reduced or simplified. Lepik (2005, 2007a, b) used the Haar wavelet method to solve ordinary and partial differential equations (PDE). Lepik (2011) solved PDE with two-dimensional Haar wavelets. Islam et al. (2013) solved parabolic PDE using Haar and Legendre wavelets. In the present study, we derived a new Haar wavelet operational matrix of integration for one dimension on the interval $[-\tau, \tau)$ and some new Haar wavelet operational matrices for integration with two-dimensional Haar wavelet basis in the interval $[-\tau, \tau)$. Finally, we constructed a new algorithm for the operational matrix for product of two-dimensional Haar wavelet functions by extending the work of Hsiao (1997).

The solution to optimal control problems has been an important research subject for hundreds of years. The derivation of necessary and sufficient conditions for optimality is useful for obtaining an analytic solution for a restricted case (Kirk, 1970). However, computational methods for solving optimal control problems had not been attempted until the advent of modern computers. Even with modern computers, the numerical solutions of optimal control problems are not easily obtained (Diehl, 2011).

Computational methods for solving optimal control problems have evolved significantly since Pontryagin and his students presented their well-known maximum principle (Sussmann and Willems, 1996). Unless the system equations of the problem at hand are simple, along with the cost function and the constraints, numerical methods must be used to solve optimal control problems. With the development of economical, high-speed computers over the past few decades, solving complicated problems in a reasonable amount of time has become possible (Diehl, 2011).

Presenting a survey of numerical methods in the field of optimal control problems is a daunting task. Perhaps the most difficult aspect is restricting the scope of the survey to permit a meaningful discussion within a few pages only. With this objective, we shall focus on two types of numerical methods. These methods are labelled as direct methods and indirect methods.

Indirect methods transform the problem into another form before proceeding with the solution. Indirect methods can be grouped into two categories, namely, Bellman's dynamic programming method and Pontryagin's maximum principle. Bellman pioneered the work in dynamic programming, thus leading to sufficient conditions for optimality by using the Hamilton-Jacobi-Bellman (HJB) equation. HJB equation is a first-order PDE that is used for deriving a nonlinear optimal feedback control law. Pontryagin's maximum principle is used to determine the necessary conditions for the existence of an optimum. Pontryagin's maximum principle converts the original optimal control problem into a boundary value problem, which can be solved analytically or numerically by using well-known techniques for differential equations (Kirk, 1970; Ranta, 2004).

The determination of the optimal feedback control law has been one of the main problems in modern control theory (Ho, 2005). If we assume full-state knowledge, if the dynamic system is linear, and if the objective function is quadratic, the optimal control problem is a linear feedback of the state that is obtained by solving a matrix Riccati equation (Bryson, 2002). However, if the system is nonlinear, then the optimal control problem is a state feedback law that depends on the solution to HJB equation. The HJB equation is a nonlinear PDE whose solution is difficult to obtain even in simple cases. Therefore, a practical method of approximating the solution to the HJB equation is highly preferred. The discretization of state space and time yields finite element approximations, but these approaches become intractable as the dimension of the state becomes large (Falcone, 1987). Other series approximations have also been applied to obtain global approximations, but these approaches have achieved only limited success because of the difficulty of solving higher-order terms in the approximation (Garrard et al., 1992).

With regard to deriving approximate solutions to the HJB equation, an interesting quote is found in Merriam (1964): “pertinent methods of approximation must satisfy two properties. First, the approximation must converge uniformly to the optimum control system with increasing complexity of the approximation. Second, when the approximation is truncated at any degree of complexity, the resulting control system must be stable without unwanted limit cycles.”

Successive approximation, which is sometimes called “iteration in policy space,” was first used in the context of the HJB equation by Bellman (1957) to argue the existence of smooth solutions to the HJB equation. The basic idea of successive approximation is to solve a differential equation by establishing a reasonable initial guess to the solution and then updating this guess on the basis of the error that it produces. The method of successive approximation was originally introduced by Bellman. This method was first applied to optimal control problems by Rekasius (1964) who used the idea of successively computing sub-optimal control problems for linear systems with non-quadratic performance criteria. In Leake and Liu (1967), the method of successive approximations is used to derive an algorithm for computing the solution to the HJB equation by computing the solution to a sequence of linear PDEs given by the generalized-Hamilton-Jacobi-Bellman (GHJB) equation. Leake and Liu (1967) were the first to analyze the successive algorithm. The ideas of successive approximation were placed on a sound theoretical foundation by Saridis and Lee (1979). The authors used successive approximation to achieve a design algorithm that improves the performance of an initial stabilizing control. This method is shown to monotonically converge pointwise to the optimal solution, that is, to the solution of the HJB equation. Our work is based on this method which will be explained in Chapter 4.

The successive Galerkin approximation (SGA) technique has recently been introduced as a technique for approximating the HJB equation. Beard et al. (1997) introduced the Galerkin approximation method for solving the GHJB equation to approximate the solution of the HJB equation successively. Given an arbitrary stabilizing control law for a nonlinear system, the solution to the GHJB equation associated with stabilizing control is a Lyapunov function for the system and is equal to the cost function. Their method can be used to improve the performance of the feedback control laws by repeating this process until a successive approximation algorithm that uniformly approximates the HJB equation is obtained. Beard et al. (1997) showed that constructing solutions to the GHJB equation, such that the control derived from its solution is in feedback form, is difficult.

The GHJB is solved by Beard et al. (1997), who used the Galerkin approximation method. The problem with this method is that it only yields an average performance because it attempts to fit its basis functions to some large regions of the state space. The Galerkin method requires the computation of multidimensional integrals. This computational burden makes the method impractical for higher-order systems. Notably, the nonlinear optimal control function is only a function of the local solution to the HJB equation. This realization leads to a unique approach for approximating local solutions to the HJB equation (Curtis and Beard, 2001). However, the computational complexity is still high, although it may be decreased by using the structure of the SGA algorithm (Beard and Mclain, 1998). Another attempt to reduce the computational load of the SGA method has been proposed recently by Curtis and Beard (2001) who devised a collocation method for solving the GHJB locally. Their idea is based on the observation that the optimal control problem is only a function of the local/current state. Thus, the GHJB equation is only solved approximately at a set of discrete points around the current state. Mizuno and Fujimoto (2008) proposed a new approximation to the HJB equation, which is used in nonlinear optimal control problems and showed that the HJB equation is effectively solved by the Galerkin spectral method with Chebyshev polynomials on the basis of successive approximation.

In Chapter 4, we considered the method of Beard et al. (1997) to approximate the solution of the HJB equation successively. Instead of using the Galerkin method with polynomial basis, we will use the collocation method with the Haar wavelet basis to solve the GHJB equation. The Galerkin method requires the computation of multidimensional integral, thus making the method impractical for higher-order systems (Curtis and Beard, 2001). Generally, the main advantage of using the collocation method is that the computational burden of solving the GHJB equation is reduced to matrix computation only.

The significance of the approximation approach of Saridis and Lee (1979) is that any initial control is successively improved and that the control law at any iteration has a guaranteed (sub-optimal) performance index. Beard et al. (1995) applied Saridis's successive approximation theory to the finite-time optimal control problem. The result is an iterative scheme that successively improves any initial control law and ultimately converges to the to the optimal state feedback control. Thereafter, the solution of a nonlinear Riccati equation is replaced by the successive solution to a linear Lyapunov equation.

Beeler et al. (2000) conducted a comparison study of five different computational methods for solving nonlinear optimal control problems and investigated the performance of these methods on several test problems. Beeler et al. (2000) provided recommendations as to which feedback control method can be best used under various conditions.

Park and Tsiotras (2003) proposed a successive wavelet collocation algorithm that uses interpolating wavelets to iteratively solve the GHJB equation and corresponding optimal control law. They however consider problems in one dimension.

Vadali and Sharma (2006) obtained a closed-form solution of the HJB equation by expanding the value function as a power series in terms of the state and constant Lagrange multipliers. Although higher-order approximations can be possibly obtained by using series expansion solutions, this process is time-consuming and the improvement of the performance is not guaranteed (Bando and Yamakawa, 2010).

Hamilton's principle is an alternative formulation of the differential equations of a dynamic system and states that the trajectory between two specified states at two specified times is an extremum of the action integral (Arnold, 1989). Motivated by this observation, Bando and Yamakawa (2010) solved Lambert's problem, namely, the two-point boundary value problem for Keplerian motion, by minimizing the action integral. Lambert's problem is viewed as an optimal control problem by replacing kinetic energy with a quadratic performance index of the control input such that the initial velocity is determined as the optimal control problem. Thereafter, the solution is obtained by the successive approximation of the HJB equation on the basis of the expansion of the value function in the Chebyshev series with unknown coefficients.

Kafash et al. (2013) used the variational iteration method for optimal control problems. The optimal control problems are transferred to the HJB equation. Thereafter, the basic variational iteration method is applied to construct a nonlinear optimal feedback control law. By using this method, the control and state variables can be approximated as a function of time.

The direct method is extensively used to solve nonlinear optimal control problems. The direct method obtains an optimal solution by directly minimizing the constrained performance index. Furthermore, this method converts the optimal control problem into a mathematical programming problem by using either the discretization technique or the parameterization technique (Huntington and Rao, 2008). Parameterizations methods are classified into three types: state parameterizations, control parameterizations, and control-state parameterizations. The control-state parameterization is based on the approximation of the state and control variables by using a sequence of known functions with unknown parameters in the following form:

$$x_i(t) = \sum_{j=0}^{m-1} a_{ij} \Phi_j(t), \qquad i = 1, 2, \cdots, n_1 \tag{2.3}$$

$$u_k(t) = \sum_{j=0}^{m-1} b_{kj} \Phi_j(t), \qquad k = 1, 2, \cdots, n_2 \tag{2.4}$$

where a_{ij} and b_{kj} are unknown parameters and $\Phi_j(t)$ denotes an appropriate set of functions forming the basis of a finite dimension (Spangelo, 1994; Jaddu, 1998).

Many researchers have investigated the theoretical aspects of the inequality constraints of trajectory. Mehra and Davis (1972) noted that the complications in handling trajectory inequality constraints in gradient or conjugate gradient methods are caused by the exclusive use of control variables as independent variables in the search procedure. In response, they presented the generalized gradient technique.

Vlassenbroeck (1988) introduced a numerical technique for solving nonlinear constrained optimal control problems based on Chebyshev series expansion of state and control variables with unknown coefficients. In this method the lengths of the control and state vectors are assumed to be equal. The differential and integral expressions from the system dynamics, performance index, boundary conditions, and other general conditions are converted into algebraic equations. The state inequality constraints are transformed into equality constraints through the use of slack variables. This work was extended previously to nonlinear unconstrained optimal control problems by Vlassenbroeck and Van Dooren (1988). According to Vlassenbroeck (1988), the constrained parameter optimization problem can be converted into an unconstrained problem by using a penalty function technique, thus avoiding the enhancement of the dimensionality of the problem.

Von Stryk and Bulirsch (1992) used a combination of direct and indirect methods for the numerical solution of nonlinear optimal control problems for trajectory optimization in the Apollo capsule. This hybrid approach improves the low accuracy of the direct methods and increases the convergence areas of the indirect methods.

Jaddu (1998) established some numerical methods on the basis of a state parameterization technique with Chebyshev polynomials to solve unconstrained and constrained optimal control problems by using the quasilinearization method. Thereafter, extended this concept to nonlinear optimal control problems with terminal state and control inequality constraints and to simple bounds on state variables (Jaddu, 2002). Yen and Nagurka (1992) proposed the addition of $n-m$ new artificial control variables to the system if the number of control variables is less than the number of state variables. This technique has the following disadvantages: (1) a large number of unknown parameters exist; (2) the original problem is changed. Han et al. (2012) presented a numerical method for solving nonlinear optimal control problems, including terminal state constraints and state and control inequality constraints. The method is based on triangular orthogonal functions. In their method, the state and control inequality constraints are adjoined into the optimization problem by replacing the restrictions inequality constraints of equality by using the auxiliary function. Thereafter, the optimal control problem is converted into algebraic equations by approximating the dynamic systems, performance index, and boundary conditions into triangular orthogonal series. Thus the problem can be easily solved by iterative methods.

Behroozifar and Yousefi (2013) proposed a numerical method for solving the constrained optimal control problems of time-varying singular systems with quadratic performance index. The method is based on Bernstein polynomials. Operational matrices of integration, differentiation, and product are also introduced to reduce the solution of optimal control problems with time-varying singular systems to the solution of algebraic equation sets by using the Lagrange multiplier method. Kafash et al. (2014) reported that the direct method has the potential to calculate continuous control and state variables as functions of time. Kafash et al. (2014) proposed a computational method for solving optimal control problems and the controlled Duffing oscillator on the basis of state parametrization. The state variable is approximated by the Boubaker polynomials. The motion, performance index, and boundary conditions equations are converted into algebraic equations.

Solving the optimal control problem through orthogonal functions, especially Haar wavelets, is an active research area. In fact, Hsiao and Wang, (1999) solved the optimal control problem of linear time-varying systems. On the basis of some useful properties of Haar wavelets, a special product matrix and an operational matrix of integration were used to solve the adjoint equation of optimization. Dai and Cochran (2009) converted optimal control problems into nonlinear programming (NLP) parameters at the collocation points by using a Haar wavelet technique. NLP problems can be solved by using NLP solvers, such as the sparse nonlinear optimizer (SNOPT). Han and Li (2011) presented a numerical method to address nonlinear optimal control problems with terminal state, as well as state and control inequality constraints. This method is based on the quasilinearization and Haar functions. Moreover, the researchers parameterized only the state variables and added artificial controls to equalize the number of state and control variables. In the present study, we do not incorporate artificial variables, but parameterize the state and control variables. Marzban and Razzaghi (2010) presented a numerical method to address constrained and nonlinear optimal control problems. In their method the inequality constraints are integrated into the optimization problem by replacing the restrictions of inequality constraints of equality constraints by using auxiliary function. Although their method is also based on Haar wavelets, it requires a set of necessary conditions. Our method is easier to implement than that of Han (2011) and Marzban (2010) because our method does not required time transformation to the domain time interval $[0,1]$.

Optimal control problems play an important role in a range of application areas including engineering, economics, and inventory (Sethi and Thompson, 2006). The literature on multi-item dynamic inventory models is relatively sparse, because most of the classical studies focused on single-item inventory models. We cite some of the most recent studies to give an idea of the extensive range of optimal control applications in the multi-item production-inventory system. Bhattacharya (2005) proposed a new approach toward a two-item inventory model for deteriorating items with linear-stock dependent demand rate. He derived the necessary criterion for the steady state optimal control problem for optimizing the objective function subjected to the constraints of the ordinary differential equations of the inventory. The multi-item production-inventory system also considers a particular choice of parameters satisfying the aforementioned necessary conditions. Under this choice, the optimal values of control parameters are calculated and the optimal amount of inventories is determined. With respect to the optimal values of the control parameters and optimal inventories, the optimal value of the objective function is obtained. El-Gohary and Elsayed (2008) presented the optimal control problem of a multi-item inventory model with deteriorating items for different types of demand rates and fixed natural deterioration rates. Graian and Essayed (2010) solved the optimal control problem of a multi-item inventory model with deteriorating rates as functions of the inventory levels and time by using the Pontrygin prinnciple. Alshamrani (2012) considered a multi-item inventory model with unknown demand rate coefficients. An adaptive control approach with a nonlinear feedback was applied to track the output of the system toward the inventory goal level. The Lyapunov technique was used to prove the asymptotic stability of the adaptive controlled system. Howevere, we will focus on the problem of El-Gohary and Elsayed (2008) as an application of our proposed method which is presented in Chapter 5.

CHAPTER 3

THE HAAR WAVELET METHOD

3.1 Introduction

The theory of approximation and transformation plays an important role in economics, sciences and engineering. In mathematics, approximation theory is concerned with how functions can best be approximated with simple functions. Moreover, this theory quantitatively characterizes the introduced errors. The objective is to approximate functions as closely as possible to the actual function. The advantage of this technique highlighted through solving complicated mathematics problem (non-linear equations, ordinary differential equation ODE, partial differential equation PDE, among others). In this chapter, we focus on a particular type of function approximation and its properties.

Wavelet theory is a relatively new and emerging area in mathematical research. Wavelets have been applied in the different fields of science and engineering and facilitate the accurate representation of a variety of functions and operators. Orthogonal functions and polynomial series have received considerable attention in terms of addressing various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to a system of algebraic equations, thus simplifying these problems significantly. The approach is based on the conversion of underlying differential equations into integral equations through integration, the approximation of the various functions in the equation using the truncated orthogonal series, and the use of the operational matrix **P** of integration to eliminate integral operations.

The history of Haar wavelet dates back to July 1909. This concept was presented in the inaugural thesis written by Alfred Haar (Haar, 1911). However, the adjective wavelet doesn't appear until around the year 1975. During this period, the concept of wavelet was first pioneered and introduced by Jean Morlet, a French geophysicist who analyzed the backscattered seismic signals carrying information on geological layers (Meyer, 2008). Morlet later collaborated with a Croatian-French physicist named Alexander Grossmann to analyze wavelets. At this point, the term "wavelet" was introduced into the academia for the first time. The French equivalent of this term is "*ondellete*" which means "small wave".

Haar wavelet is a wavelet family or basis that is generated from a sequence of rescaled square wave function series. The fundamental square wave function must be defined to describe the Haar series. Then, the subsequent Haar wavelet functions are generated from this square wave function through translation and dilation processes. Haar wavelet is simple and is the oldest wavelet. This wavelet has compact support, which indicates that the wavelet vanishes beyond of a finite interval. Unfortunately, Haar wavelets are not continuously differentiable, thus limiting its applications somewhat. Haar wavelet is also categorized as an orthogonal function.

In this chapter, the generation of Haar wavelet function, its series expansion, and a one-dimensional matrix for a chosen interval $[\tau_1, \tau_2)$ is introduced in brief . Many studies have defined the operational matrix of Haar wavelet on interval $[0,1)$. We extend the usual defined interval to $[0, \tau)$ and $[-\tau, \tau)$ because the actual problem does not necessarily involve only one dimension. Next, we must define the matrix of Haar wavelet collocation points for two dimensions to establish a method for solving Generalized Hamilton-Jacobi-Bellman (GHJB) equation in this chapter. In addition, we formulate new Haar wavelet operational matrices to integrate the Haar function vectors for two dimensions such as Q_1, Q_2, $\mathbf{E}_1$, and $\mathbf{E}_2$ given the chosen stabilizing domain

$[-\tau,\tau)$. At the end of this chapter, we establish a novel operational matrix for the product of the Haar wavelet functions of two dimensions.

3.2 Haar Wavelet Function

The orthogonal set of the Haar wavelets $h_i(x)$ function is a group of square waves over the interval $[\tau_1,\tau_2)$. These wavelets are defined as follows:

$$h_0(x)=\begin{cases}1, & \tau_1\le x<\tau_2\,,\\ 0, & \text{elsewhere,}\end{cases} \tag{3.1}$$

$$h_1(x)=\begin{cases}1, & \tau_1\le x<\frac{1}{2}(\tau_1+\tau_2)\,,\\ -1, & \frac{1}{2}(\tau_1+\tau_2)\le x<\tau_2\,,\\ 0, & \text{elsewhere,}\end{cases} \tag{3.2}$$

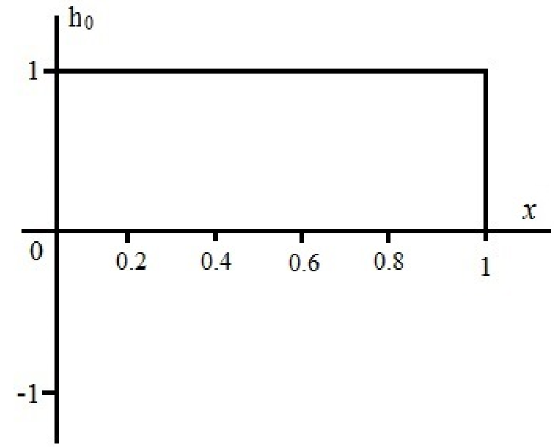
$$h_i(x)=\begin{cases}1, & \tau_1+\frac{2k}{2^j}(\frac{\tau_2-\tau_2}{2})\le x<\tau_1+\frac{(2k+1)}{2^j}(\frac{\tau_2-\tau_1}{2})\,,\\ -1, & \tau_1+\frac{(2k+1)}{2^j}(\frac{\tau_2-\tau_1}{2})\le x<\tau_1+\frac{(2k+2)}{2^j}(\frac{\tau_2-\tau_1}{2}),\\ 0, & \text{elsewhere,}\end{cases} \tag{3.3}$$

where the number of the wavelet is denoted by $i=2^j+k$ (the maximum value is $i=2M$. Here $M=2^J$, where J is the maximal level of resolution); the dilatation parameter $j=0,1,2,\ldots,J$; and the translation parameter $k=0,1,2,\ldots,m-1$ where $m=2^j$. $h_0(x)$ is constant in the interval $[\tau_1,\tau_2)$ and is called the Haar scaling function. $h_1(x)$ is known as the Haar mother wavelet function or the fundamental square wave function.

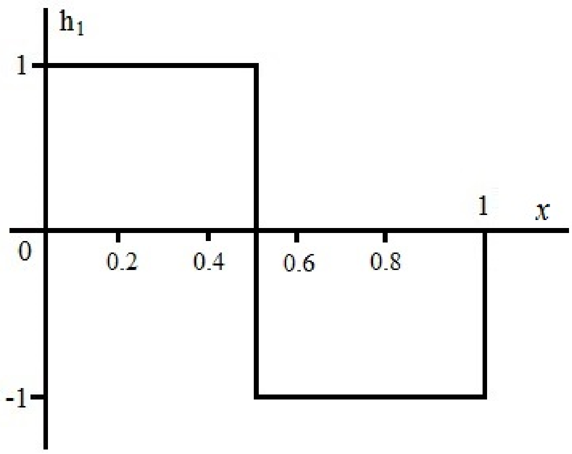
All subsequent Haar wavelet functions are generated from the mother wavelet function $h_1(x)$ through translation and dilation process.

$$h_i(x) = h_1(2^j x - k)\,. \tag{3.4}$$

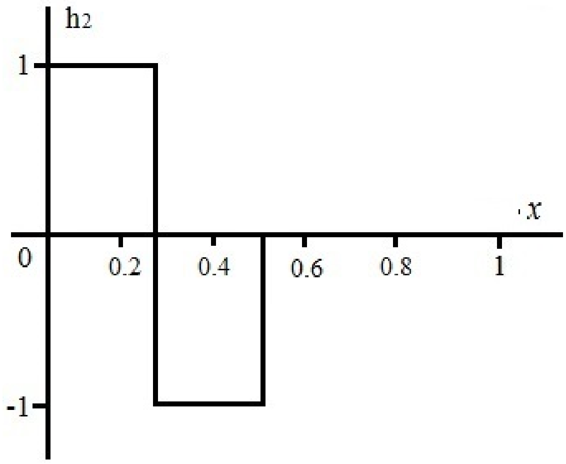
The orthogonal sets of the first four Haar functions ($m = 4$) in the intervals of ($0 \le x < 1$) and ($-1 \le x < 1$) are shown in Figures 3.1 and 3.2, respectively.



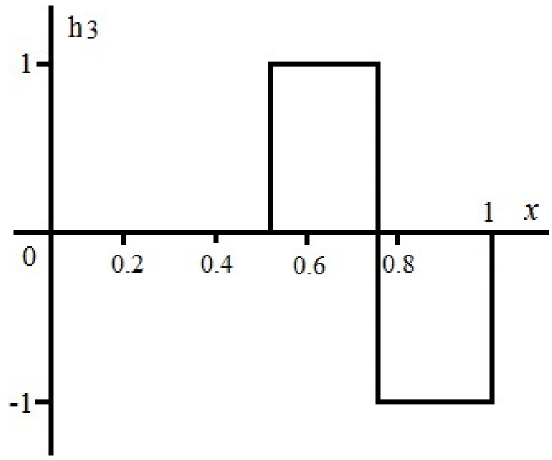
(a) Haar function of $h_0(x)$



(b) Haar function of $h_1(x)$

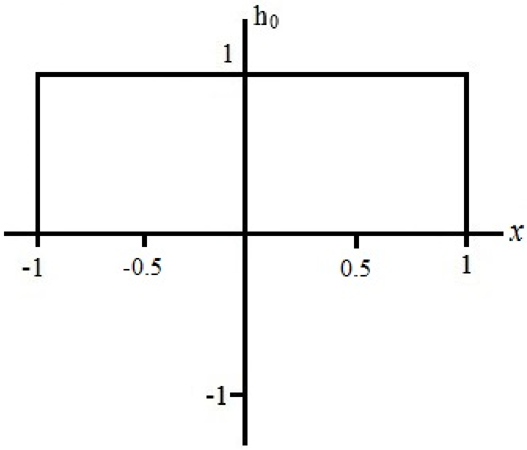


(c) Haar function of $h_2(x)$

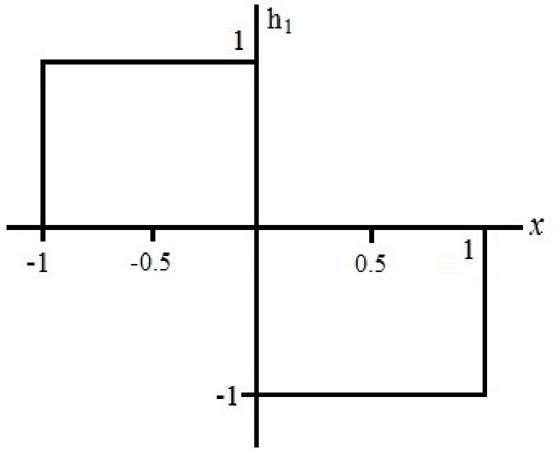


(d) Haar function of $h_3(x)$

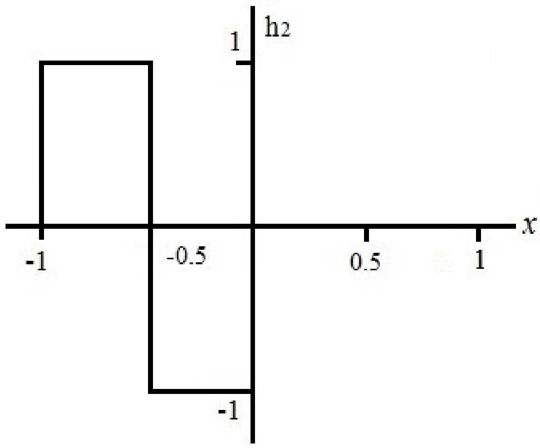
Figure 3.1 First four Haar functions in the interval of ($0 \le x < 1$)



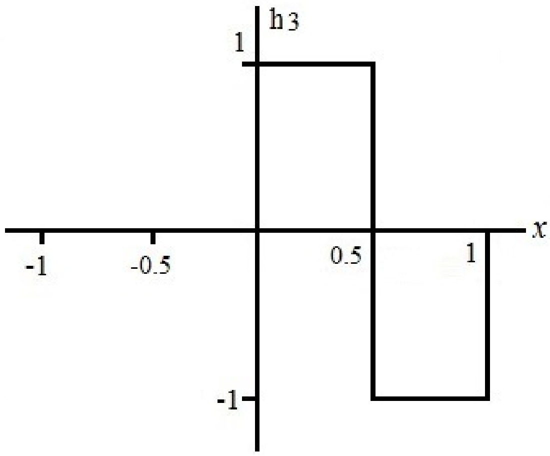
(a) Haar function of $h_0(x)$



(b) Haar function of $h_1(x)$



(c) Haar function of $h_2(x)$



(d) Haar function of $h_3(x)$

Figure 3.2 First four Haar functions in the interval of $(-1 \leq x < 1)$

The value of each Haar wavelet is determined through a couple of constant steps involving opposite signs during the subinterval. This value is zero elsewhere. Therefore, given $p = 2^{-j} + k,$ they have the following relationship:

$$\int_{\tau_1}^{\tau_2} h_p(x)\, h_q(x)\, dx = \begin{cases} (\tau_2 - \tau_1)\, 2^{-j} & , \quad p = q \\ 0 & , \quad p \neq q. \end{cases} \tag{3.5}$$

Eqn. (3.5) can be proven as below

Proof

If $p \neq q$, then we obtain

$$\int_{\tau_1}^{\tau_2} h_p(x) h_q(x) dx = 0\,, \tag{3.6}$$

Since h_p and h_q have disjoint supports if $p \neq q \neq 0$, and sums cancel out if $p \neq q = 0$

If $p = q$, then we obtain

$$\|h_P(x)\|^2 = \langle\, h_p(x)\ , h_p(x) \rangle$$

$$= \int_{\tau_1}^{\tau_2} h_p^2(x)\ dx \tag{3.7}$$

$$= \int_{\tau_1 + (\frac{2k}{2^j})(\frac{\tau_2 - \tau_1}{2})}^{\tau_1 + (\frac{2k+1}{2^j})(\frac{\tau_2 - \tau_1}{2})} dx \ + \int_{\tau_1 + (\frac{2k+1}{2^j})(\frac{\tau_2 - \tau_1}{2})}^{\tau_1 + (\frac{2k+2}{2^j})(\frac{\tau_2 - \tau_1}{2})} dx$$

$$= \left[\tau_1 + (\frac{2k+1}{2^j})(\frac{\tau_2 + \tau_1}{2}) - (\tau_1 + (\frac{2k}{2^j})(\frac{\tau_2 + \tau_1}{2})) \right]$$
$$+ \left[\tau_1 + (\frac{2k+2}{2^j})(\frac{\tau_2 + \tau_1}{2}) - (\tau_1 + (\frac{2k+1}{2^j})(\frac{\tau_2 + \tau_1}{2})) \right]$$

$$= (\tau_2 - \tau_1)\ 2^{-j}\,. \qquad \square \tag{3.8}$$

This relationship shows that Haar wavelet functions are orthogonal to each other and therefore constitute an orthogonal basis. Hence, this relationship facilitates the transformation of any function square interval in the time interval $[\tau_1, \tau_2)$ into a Haar wavelet series.

3.3 Haar Series Expansion

Any function $f \in L^2([\tau_1, \tau_2))$ can be expanded into a Haar series of infinite terms:

$$f(x) = c_0 h_0(x) + c_1 h_1(x) + c_2 h_2(x) + \cdots \ . \tag{3.9}$$

If the function $f(x)$ is approximated as a piecewise constant, then the decomposition in Eqn. (3.9) can be terminated as follows:

$$f(x) \approx f_m(x) = \sum_{i=0}^{m-1} c_i h_i(x) \ . \tag{3.10}$$

where $i = 2^j + k$, $j = 0,1,2,\ldots,\log_2 m$ and $0 \le k < 2^j$. The Haar coefficient c_i can be determined by applying the inner product in Eqn. (3.5).

If $\{h_i(x)\}$ is an orthogonal set of functions on an interval $[\tau_1, \tau_2)$, then a set of coefficients c_i can be determined for which

$$f(x) = c_0 h_0(x) + c_1 h_1(x) + \cdots + c_n h_n(x) + \cdots \ . \tag{3.11}$$

Multiplying Eqn. (3.11) by $h_p(x)$ and integrating the result over the interval $[\tau_1, \tau_2)$ generates

$$\int_{\tau_1}^{\tau_1} f(x)h_p(x)dx = c_0\int_{\tau_1}^{\tau_2} h_0(x)h_p(x)dx + c_1\int_{\tau_1}^{\tau_2} h_1(x)h_p(x)dx + \cdots + c_n\int_{\tau_1}^{\tau_2} h_n(x)h_p(x)dx + \cdots \tag{3.12}$$

$$= c_0 < h_0, h_p > + c_1 < h_1, h_p > + \cdots + c_n < h_n, h_p > + \cdots \tag{3.13}$$

In orthogonality, the value of each term on the right-hand side of the previous equation is zero except when $p = n$. In this case, we obtain

$$\int_{\tau_1}^{\tau_2} f(x)h_n(x)dx = c_n\int_{\tau_1}^{\tau_2} h_n^2(x)dx \ . \tag{3.14}$$

Thus, the required coefficients are

$$c_n = \frac{\int_{\tau_1}^{\tau_2} f(x)h_n(x)dx}{\int_{\tau_1}^{\tau_2} h_n^2(x)dx} \ , \quad n = 0, 1, 2, 3, \dots \ , \tag{3.15}$$

or, we can rewrite this equation as

$$c_n = \frac{\int_{\tau_1}^{\tau_2} f(x)h_n(x)dx}{\left\| h_n(x) \right\|^2} \ , \quad n = 0, 1, 2, 3, \dots \ . \tag{3.16}$$

As per Eqn. (3.5), the norm $\left\| h_n(x) \right\|^2 = \dfrac{(\tau_2 - \tau_1)}{2^j}$; therefore, the Haar wavelet coefficient is determined by

$$c_n = \frac{2^j}{(\tau_2 - \tau_1)}\int_{\tau_1}^{\tau_2} f(x)h_n(x)dx, \quad n = 0, 1, 2, 3, \dots \ . \tag{3.17}$$

Thus, given any function $f(x)$ that is square integrable within interval $\tau_1 \le x < \tau_2$, the Haar wavelet coefficient in Eqn. (3.9) can be determined with as

$$c_i = \frac{2^j}{(\tau_2 - \tau_1)} \int_{\tau_1}^{\tau_2} f(x) h_i(x) dx, \quad i = 0, 1, 2, 3, \dots \; . \tag{3.18}$$

If $f(x)$ is smooth, then approximating $f(x)$ using Haar basis function gives a lower accuracy for a given m (Islam et al., 2013).

If $f(x)$ and $f_m(x)$ in Eqn. (3.10) are the exact and approximate solutions, respectively, then the corresponding errors are defined as follows:

$$e_m(x) = f(x) - f_m(x) \; , \tag{3.19}$$

$$e_m(x) = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k} h_{2^j+k}(x) \; . \tag{3.20}$$

Saeedi et al., (2011), shown that the square of the error norm is

$$\left\| e_m(x) \right\|^2 \le \frac{\ell^2}{3m^2} \, ,$$

where ℓ is the Lipschitz constant. Hence for Haar wavelet approximation, the convergence is of order one, that is,

$$\left\| e_m(x) \right\| = O\left(\frac{1}{m} \right) \; . \tag{3.21}$$

As per Eqn. (3.21), the error is inversely proportional to the level resolution of the Haar wavelet function. This scenario implies that the Haar wavelet approximation method is convergent as m approaches to infinity.

3.4 Matrix of One-dimensional Haar Wavelets

As per Eqn. (3.10), the sum can be written in the following compact matrix form:

$$f(x) \approx \mathbf{c}_m^{\mathbf{T}}\mathbf{h}_m(x), \tag{3.22}$$

where Haar coefficient vector $\mathbf{c}_m^{\mathbf{T}}$ and Haar function vector $\mathbf{h}_m(x)$ are defined as

$$\mathrm{c}_m^{\mathrm{T}} = [c_0 \quad c_1 \quad c_2 \quad \cdots \quad c_{m-1}], \tag{3.23}$$

and

$$\mathbf{h}_m(x) = [\, h_0(x) \quad h_1(x) \quad h_2(x) \;\cdots\; h_{m-1}(x)\,]^{\mathrm{T}}. \tag{3.24}$$

The superscript $\mathbf{T}$ denotes the transpose and the subscript m denotes the dimension of vectors and matrices.

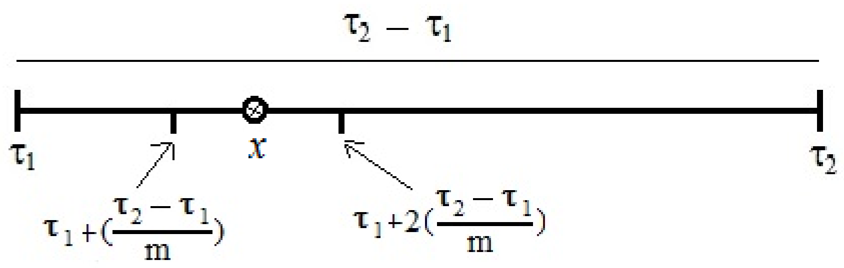


Figure 3.3: Collocation point

The collocation points are defined as follows:

As depicted in Figure (3.3), let x be the middle point between two points in the subinterval $\left[\tau_1 + (\frac{\tau_2-\tau_1}{m}) \;, \tau_1 + 2\,(\frac{\tau_2-\tau_1}{m})\right]$. This point is expressed as

$$x = \frac{1}{2}\left[\tau_1 + (\frac{\tau_2-\tau_1}{m}) + \tau_1 + 2\,(\frac{\tau_2-\tau_1}{m})\right]. \tag{3.25}$$

In general, for $j = 1, 2, 3, \ldots, m$, and $m = 2^J$.

$$x_j = \frac{1}{2}\left[\tau_1 + (j-1)\left(\frac{\tau_2 - \tau_1}{m}\right) + \tau_1 + (j)\left(\frac{\tau_2 - \tau_1}{m}\right) \right]. \tag{3.26}$$

We can simplify Eqn. (3.26) to obtain

$$x_j = \tau_1 + \left(\frac{\tau_2 - \tau_1}{2m}\right)(2j-1) . \tag{3.27}$$

For instance, we can generate four collocation points from Eqn. (3.27) for $j = 1, 2, 3, 4$ in the interval $[-1, 1)$ as follows:

$$x_1 = -\frac{3}{4} \quad , \quad x_2 = -\frac{1}{4} \quad , \quad x_3 = \frac{1}{4} \quad , \quad x_4 = \frac{3}{4}.$$

Haar function vector $\mathbf{h}_m(x)$ can be represented in m square matrix form $\mathbf{H}_m$, the elements of which are given by

$$[\mathbf{H}_m]_{ij} = h_i(x_j), \tag{3.28}$$

for $i = 0, 1, 2, \ldots, m-1$, $j = 1, 2, \ldots, m$ and x_j are the collocation points as defined in Eqn. (3.27).

For instance, the fourth-order Haar wavelet matrix $\mathbf{H}_4$ in the interval of $[0, 1)$ can be represented in matrix form with the collocation points from Eqn. (3.28) as follows:

$$\mathbf{H}_4 = \begin{bmatrix} h_0(\frac{1}{8}) & h_0(\frac{3}{8}) & h_0(\frac{5}{8}) & h_0(\frac{7}{8}) \\ h_1(\frac{1}{8}) & h_1(\frac{3}{8}) & h_1(\frac{5}{8}) & h_1(\frac{7}{8}) \\ h_2(\frac{1}{8}) & h_2(\frac{3}{8}) & h_2(\frac{5}{8}) & h_2(\frac{7}{8}) \\ h_3(\frac{1}{8}) & h_3(\frac{3}{8}) & h_3(\frac{5}{8}) & h_3(\frac{7}{8}) \end{bmatrix}, \tag{3.29}$$

$$\mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{3.30}$$

When the Haar wavelet matrix is defined as in Eqn. (3.28), then the coefficient $\mathbf{c}_m^{\mathbf{T}}$ in Eqns. (3.17) and (3.22) can be easily obtained as

$$\mathbf{c}_m^{\mathbf{T}} = f_m \quad \mathbf{H}_m^{\mathbf{-1}} , \tag{3.31}$$

where

$$f_m = \left[f(x_1) \quad f(x_2) \quad f(x_3) \quad \cdots \quad f(x_m) \right] . \tag{3.32}$$

In particular, a large Haar wavelet matrix is needed for the solution to the HJB equation method in Chapter 4. Fortunately, $\mathbf{H_m}$ and $\mathbf{H_m^{-1}}$ contains many zeros. As *m* value increases, the element of zeros in the matrix also increases as indicated in Eqn. (3.30). This factor accelerates computation and is one of the reasons for the rapid convergence of the Haar wavelet series. Hsiao (2004) reports that the number of multiplication operations involved in Haar transformation is considerably easier and faster than fast Fourier and Walsh transform.

3.5 Operational Matrix for Integrating One-dimensional Haar Wavelets

The integration of Haar wavelet function $\mathbf{h}_m(x)$ into the specific interval of $[0,\tau)$ can be expanded into a Haar series, that is,

$$\int_0^x \mathbf{h}_m(x)dx \cong \mathbf{P}_m \mathbf{h}_m(x), \tag{3.33}$$

where the $m \times m$ matrix $\mathbf{P}_m$ is called the operational matrix of integration obtain recursively by following as prescribed by Aznam and Hussin (2012).

$$\mathbf{P}_m = \frac{1}{2m}\begin{bmatrix} 2m\mathbf{P}_{m/2} & -\tau\,\mathbf{H}_{m/2} \\ \tau\,\mathbf{H}_{m/2}^{-1} & \mathbf{O}_{m/2} \end{bmatrix} \tag{3.34}$$

The recursive formula above starts with

$$\mathbf{P}_1 = \left[\frac{\tau}{2}\right]. \tag{3.35}$$

The formula in the interval of $[0,1)$ was first presented by Chen and Hsiao (1997).

For example, in order to determine $\mathbf{P}_2$ and $\mathbf{P}_4$ the steps are shown as below.

$$\mathbf{P}_2 = \frac{1}{4}\begin{bmatrix} 4\mathbf{P}_1 & -\tau\,\mathbf{H}_1 \\ \tau\,\mathbf{H}_1^{-1} & \mathbf{0} \end{bmatrix} \tag{3.36}$$

$$= \begin{bmatrix} \mathbf{P}_1 & -\frac{\tau}{4}\mathbf{H}_1 \\ \frac{\tau}{4}\mathbf{H}_1^{-1} & \mathbf{0} \end{bmatrix} \tag{3.37}$$

$$= \begin{bmatrix} \frac{\tau}{2} & -\frac{\tau}{4} \\ \frac{\tau}{4} & \mathbf{0} \end{bmatrix}. \tag{3.38}$$

$\mathbf{P}_4$ can be determined by following the same steps as shown below:

$$\mathbf{P}_4 = \frac{1}{8}\begin{bmatrix} 8\mathbf{P}_2 & -\tau\,\mathbf{H}_2 \\ \tau\,\mathbf{H}_2^{-1} & \mathbf{O}_2 \end{bmatrix} \tag{3.39}$$

$$= \begin{bmatrix} \frac{\tau}{2} & -\frac{\tau}{4} & -\frac{\tau}{8} & -\frac{\tau}{8} \\ \frac{\tau}{4} & 0 & -\frac{\tau}{8} & \frac{\tau}{8} \\ \frac{\tau}{16} & \frac{\tau}{16} & 0 & 0 \\ \frac{\tau}{16} & -\frac{\tau}{16} & 0 & 0 \end{bmatrix} = \frac{\tau}{8}\begin{bmatrix} 4 & -2 & -1 & -1 \\ 2 & 0 & -1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix}. \tag{3.40}$$

In order to solve the nonlinear optimal control problem of infinite time horizon using GHJB equation in the interval $[-\tau,\tau)$ (which is introduced in Chapter 4), it is essential to find the modification formula for operational matrix $\overline{\mathbf{P}}_m$ that covers the entire domain $[-\tau,\tau)$. The integrals of the first four Haar wavelet functions in the interval $[-\tau,\tau)$ are discussed in Section 3.2 and can be expressed as follows:

$$\overline{p}_0(x) = \int_{-\tau}^{x} h_0(t)dt = x+\tau\ , \qquad x \in [-\tau,\tau) \tag{3.41}$$

$$\overline{p}_1(x) = \int_{-\tau}^{x} h_1(t)dt = \begin{cases} x+\tau\,, & -\tau \le x < 0\,, \\ -x+\tau, & 0 \le x < \tau, \end{cases} \tag{3.42}$$

$$\overline{p}_2(x) = \int_{-\tau}^{x} h_2(t)dt = \begin{cases} x+\tau\,, & -\tau \le x < -\frac{\tau}{2}\,, \\ -x & -\frac{\tau}{2} \le x < 0, \end{cases} \tag{3.43}$$

$$\bar{p}_3(x) = \int_{-\tau}^{x} h_3(t)dt = \begin{cases} x, & 0 \le x < \dfrac{\tau}{2}, \\ -x+\tau, & \dfrac{\tau}{2} \le x < \tau \end{cases} \tag{3.44}$$

In general, the integrals of Eqn. (3.3) for $i,= 0,1,2,\ldots,m-1$ in the interval $[-\tau,\tau)$ can be described as below

$$\bar{p}_i(x) = \int_{-\tau}^{x} h_i(t)dt$$

$$= \begin{cases} x-\tau(-1+\dfrac{2k}{2^j}), & \tau(-1+\dfrac{2k}{2^j}) \le x < \tau(-1+\dfrac{(2k+1)}{2^j}), \\ -x-\tau(-1+\dfrac{(2k+2)}{2^j})+2\tau(-1+\dfrac{(2k+1)}{2^j}), & \\ & \tau(-1+\dfrac{(2k+1)}{2^j}) \le x < \tau(-1+\dfrac{(2k+2)}{2^j}), \\ 0, & \text{elsewhere} \end{cases}$$

(3.45)

The Haar operational matrix generated from the recursive formula can be calculated by the following equation:

$$\bar{\mathbf{P}}_m = \frac{1}{2m}\begin{bmatrix} 2m\bar{\mathbf{P}}_{m/2} & -\tau\,\mathbf{H}_{m/2} \\ \tau\,\mathbf{H}^{-1}_{m/2} & \mathbf{O}_{m/2} \end{bmatrix}. \tag{3.46}$$

The recursive formula above [Eqn. (3.46)] begins with

$$\bar{\mathbf{P}}_1 = [\tau]. \tag{3.47}$$

3.6 Operational Matrix of the Product of One-dimensional Haar Wavelet Vectors

The three basic multiplication properties of Haar wavelets are as follows (Hsiao and Wu, 2007):

(i) $h_n(x)h_1(x) = h_n(x)$ for any $n \in \mathrm{N} \cup \{0\}$.

(ii) For any two Haar wavelets $h_n(x)$ and $h_l(x)$ with $n < l$, we obtain

$$h_n(x)h_l(x) = \rho_{nl} h_l(x), \tag{4.48}$$

where

$$\rho_{nl} = h_n(2^{-i}(q+\frac{1}{2})) = \begin{cases} 1, & 2^{i-j}k \le q < 2^{i-j}(k+\frac{1}{2}), \\ -1, & 2^{i-j}(k+\frac{1}{2}) \le q < 2^{i-j}(k+1), \\ 0 & \text{elsewhere}, \end{cases} \tag{3.49}$$

where $n = 2^j + k$, $j \ge 0$, $0 \le k < 2^j$ and $l = 2^i + q$, $i \ge 0$, $0 \le q < 2^i$.

(iii) The square of any Haar wavelet is a block pulse with a magnitude of one during both positive and negative half-waves.

In order to simplify the product of two functions $f(x) = \mathbf{c}_m^{\mathbf{T}}\mathbf{h}_m(x)$ and $g(x) = \mathbf{d}_m^{\mathbf{T}}\mathbf{h}_m(x)$, it is essential to know the product of $\mathbf{h}(x)$ and $\mathbf{h}^{\mathbf{T}}(x)$. The product can be expanded into a Haar series with a Haar coefficient matrix $\mathbf{M}_m$ as follows:

$$f(x)g(x) = \mathbf{d}_m^{\mathbf{T}}\mathbf{h}_m(x)\,\mathbf{h}_m^{\mathbf{T}}(x)\mathbf{c}_m = \mathbf{d}_m^{\mathbf{T}}\mathbf{M}_m(\mathbf{c})\mathbf{h}_m(x), \tag{3.50}$$

where $\mathbf{M}_m(\mathbf{c})$ is a $m \times m$ matrix referred to as the product operational matrix. This matrix was first presented by Hsiao and Wu (2007) as

$$\mathbf{M}_m(\mathbf{c}) = \begin{bmatrix} \mathbf{M}_{m/2} & \mathbf{H}_{\mathbf{m/2}} diag(\mathbf{c}_b) \\ diag(\mathbf{c}_b)\mathbf{H}_{m/2}^{-1} & diag(\mathbf{c}_a^T \mathbf{H}_{m/2}) \end{bmatrix}, \tag{3.51}$$

where $\mathbf{M}_1 = c_0$ and $\mathbf{c}_a = \left[c_0, c_1, \ldots, c_{m/2-1}\right]^{\mathbf{T}}$, $\mathbf{c}_b = \left[c_{(m/2)}, \ldots, c_{m-1}\right]^{\mathbf{T}}$.

In addition, the following formula can be derived from Eqn. (3.33) and can be used to solve problems regarding the nonlinear optimal control problem of finite time horizon. This problem is introduced in Chapter 5. Our calculation method may then be simplified.

$$\mathbf{P}_m \mathbf{h}_m(\tau_2) = \tau_2\, \theta_m, \tag{3.52}$$

where $\theta_m^{\mathbf{T}} = [1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0]$. The proof for Eqn. (3.52) is as follows:

Proof:

$$\begin{aligned}
\mathbf{P}_m\, \mathbf{h}_m(\tau_2) &= \int_0^{\tau_2} \mathbf{h}_m(x)\, dx \\
&= \int_{\mathbf{0}}^{\tau_2} [h_0(x) \quad h_1(x) \quad \cdots \quad h_{m-1}(x)\,]^{\mathbf{T}}\, dx \\
&= \begin{bmatrix} \int_0^{\tau_2} 1 dx & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^{\mathbf{T}} \\
&= \tau_{\mathbf{2}} \theta_m .
\end{aligned}$$

$\square$

3.7 Operational Matrix of the Integral Products of One-dimensional Haar Wavelet Vectors

In this section, the operational matrix of the integral product of two Haar wavelet function vectors are determined in one dimension $\mathbf{h}_m(x)$ and $\mathbf{h^T}_m(x)$ at collocation points x_j. These points on the interval $[\tau_1, \tau_2)$ are defined in Eqn. (3.27). This operational matrix can assist in solving the problem of the nonlinear optimal control of finite time horizon. As mentioned previously. This problem is introduced in Chapter 5.

$$\int_{\tau_1}^{\tau_2} \mathbf{h}_m(x)\mathbf{h}_m^{\mathbf{T}}(x)dx = \mathbf{E}_m \; . \tag{3.53}$$

To calculate the matrix $\mathbf{E}_m$, which is of order $m \times m$, Eqn. (3.24) is first multiplied with its transpose. We obtain

$$\mathbf{h}_m(x)\mathbf{h}_m^{\mathbf{T}}(x) = \begin{bmatrix} h_0(x)h_0(x) & h_0(x)h_1(x) & h_0(x)h_2(x) & \cdots & h_0(x)h_{m-1}(x) \\ h_1(x)h_0(x) & h_1(x)h_1(x) & h_1(x)h_2(x) & \cdots & h_1(x)h_{m-1}(x) \\ h_2(x)h_0(x) & h_2(x)h_1(x) & h_2(x)h_2(x) & \cdots & h_2(x)h_{m-1}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{m-1}(x)h_0(x) & h_{m-1}(x)h_1(x) & h_{m-1}(x)h_2(x) & \cdots & h_{m-1}(x)h_{m-1}(x) \end{bmatrix}. \tag{3.54}$$

By applying the properties of the orthogonal functions of Haar wavelets in Eqn. (3.5) to each element in Eqn. (3.54), we generate

$$\int_{\tau_1}^{\tau_2} \mathbf{h}_m(x)\mathbf{h}_m^{\mathbf{T}}(x)dx =$$

$$= \begin{bmatrix} \int_{\tau_1}^{\tau_2} h_0(x)h_0(x)dx & \int_{\tau_1}^{\tau_2} h_0(x)h_1(x)dx & \int_{\tau_1}^{\tau_2} h_0(x)h_2(x)dx & \cdots & \int_{\tau_1}^{\tau_2} h_0(x)h_{m-1}(x)dx \\ \int_{\tau_1}^{\tau_2} h_1(x)h_0(x)dx & \int_{\tau_1}^{\tau_2} h_1(x)h_1(x)dx & \int_{\tau_1}^{\tau_2} h_1(x)h_2(x)dx & \cdots & \int_{\tau_1}^{\tau_2} h_1(x)h_{m-1}(x)dx \\ \int_{\tau_1}^{\tau_2} h_2(x)h_0(x)dx & \int_{\tau_1}^{\tau_2} h_2(x)h_1(x)dx & \int_{\tau_1}^{\tau_2} h_2(x)h_2(x)dx & \cdots & \int_{\tau_1}^{\tau_2} h_2(x)h_{m-1}(x)dx \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \int_{\tau_1}^{\tau_2} h_{m-1}(x)h_0(x)dx & \int_{\tau_1}^{\tau_2} h_{m-1}(x)h_1(x)dx & \int_{\tau_1}^{\tau_2} h_{m-1}(x)h_2(x)dx & \cdots & \int_{\tau_1}^{\tau_2} h_{m-1}(x)h_{m-1}(x)dx \end{bmatrix} \tag{3.55}$$

$$= \begin{bmatrix} (\tau_2 - \tau_1) & 0 & 0 & \cdots & 0 \\ 0 & (\tau_2 - \tau_1)2^{-0} & 0 & \cdots & 0 \\ 0 & 0 & (\tau_2 - \tau_1)2^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (\tau_2 - \tau_1)2^{-j} \end{bmatrix} \tag{3.56}$$

This equation is generally written as

$$\mathbf{E}_m = (\tau_2 - \tau_1)diag\left[1 \ \underbrace{2^{-0}}_{2^0 \, time} \ \underbrace{2^{-1}2^{-1}}_{2^1 \, times} \ \underbrace{2^{-2}2^{-2}2^{-2}2^{-2}}_{2^2 \, times} \ \cdots \ \underbrace{2^{-j}2^{-j} \cdots \ 2^{-j}}_{2^j \, times}\right], \tag{3.57}$$

where $j = 0, 1, 2, \cdots, \log_2 m - 1$.

For instance, the eighth-order Haar wavelet matrix $\mathbf{E}_8$ in the interval of $[0, 5)$ presented in Chapter 6 can be represented using Eqn. (3.57) as follows:

$$\mathbf{E_8} = 5\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}. \tag{3.58}$$

3.8 Matrix of Two-dimensional Haar Wavelets

The bases of two-dimensional Haar wavelets can be formed by determining the tensor product of two Haar function vectors $\mathbf{h}_n(x)$ and $\mathbf{h}_m(x)$. This product is obtained using the collocation points described in Eqn. (3.27)

$$\mathbf{H}(x_1, x_2) = \mathbf{h}_n(x_1) \otimes \mathbf{h}_m(x_2). \tag{3.59}$$

Let the basis be a vector of two-dimensional Haar wavelet functions:

$$[h_i(x_1)\, h_j(x_2)], \qquad i = 0,1,2, \ldots, n-1, \quad j = 0,1,2, \ldots, m-1. \tag{3.60}$$

Then, the two-dimensional Haar function vector can be expressed as

$$
\mathbf{H}(x_1, x_2) = \begin{bmatrix} h_0(x_1)\, h_0(x_2) \\ h_0(x_1)\, h_1(x_2) \\ \vdots \\ h_0(x_1)\, h_{m-1}(x_2) \\ h_1(x_1)\, h_0(x_2) \\ \vdots \\ h_1(x_1)\, h_{m-1}(x_2) \\ \vdots \\ h_{n-1}(x_1)\, h_0(x_2) \\ \vdots \\ h_{n-1}(x_1)\, h_{m-1}(x_2) \end{bmatrix}. \tag{3.61}
$$

For instance, the fourth-order Haar wavelet matrix for two dimensions $\mathbf{H}_4$ in the interval of $[0,1)\times[0,1)$ can be represented in matrix form as follows:

$$
\mathbf{H_4} \otimes \mathbf{H_4} = \begin{bmatrix} h_0(\frac{1}{8})\mathbf{H_4} & h_0(\frac{3}{8})\mathbf{H_4} & h_0(\frac{5}{8})\mathbf{H_4} & h_0(\frac{7}{8})\mathbf{H_4} \\ h_1(\frac{1}{8})\mathbf{H_4} & h_1(\frac{3}{8})\mathbf{H_4} & h_1(\frac{5}{8})\mathbf{H_4} & h_1(\frac{7}{8})\mathbf{H_4} \\ h_2(\frac{1}{8})\mathbf{H_4} & h_2(\frac{3}{8})\mathbf{H_4} & h_2(\frac{5}{8})\mathbf{H_4} & h_2(\frac{7}{8})\mathbf{H_4} \\ h_3(\frac{1}{8})\mathbf{H_4} & h_3(\frac{3}{8})\mathbf{H_4} & h_3(\frac{5}{8})\mathbf{H_4} & h_3(\frac{7}{8})\mathbf{H_4} \end{bmatrix}. \tag{3.62}
$$

According to (3.59) we obtain

$$
\mathbf{H_4} \otimes \mathbf{H_4} = \begin{bmatrix} (1)\mathbf{H_4} & (1)\mathbf{H_4} & (1)\mathbf{H_4} & (1)\mathbf{H_4} \\ (1)\mathbf{H_4} & (1)\mathbf{H_4} & (-1)\mathbf{H_4} & (-1)\mathbf{H_4} \\ (1)\mathbf{H_4} & (-1)\mathbf{H_4} & (0)\mathbf{H_4} & (0)\mathbf{H_4} \\ (0)\mathbf{H_4} & (0)\mathbf{H_4} & (1)\mathbf{H_4} & (-1)\mathbf{H_4} \end{bmatrix}. \tag{3.63}
$$

Thus,

$$\overline{\mathbf{H}}_{4^2\times 4^2} = \begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 & \mathbf{H}_4 & \mathbf{H}_4 \\ \mathbf{H}_4 & \mathbf{H}_4 & -\mathbf{H}_4 & -\mathbf{H}_4 \\ \mathbf{H}_4 & -\mathbf{H}_4 & \mathbf{O}_4 & \mathbf{O}_4 \\ \mathbf{O}_4 & \mathbf{O}_4 & \mathbf{H}_4 & -\mathbf{H}_4 \end{bmatrix}. \tag{3.64}$$

Finally, we obtain

$$\overline{\mathbf{H}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}, \tag{3.65}$$

where $\overline{\mathbf{H}}$ is the matrix of the collocation points of two-dimensional 16×16 Haar wavelets.

3.9 Approximation of Functions for Two-dimensional Functions

As with one-dimensional functions, any two-dimensional function $f(x_1, x_2)$ in the interval $[-\tau_1, \tau_1) \times [-\tau_2, \tau_2)$ can also be expanded into Haar series through

$$f(x_1, x_2) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} C_{ij} h_i(x_1) h_j(x_2), \tag{3.66}$$

where C_{ij} is the Haar coefficient for two dimensions in the interval of $[-\tau_1, \tau_1) \times [-\tau_2, \tau_2)$. This equation was first presented by Wu (2009) as

$$C_{ij} = m^2 \int_{-\tau_1}^{\tau_1} \int_{-\tau_2}^{\tau_2} f(x_1, x_2) h_i(x_1) h_j(x_2) dx_1 dx_2, \tag{3.67}$$

where $i = 2^{\alpha} + k$, $\alpha \geq 0$, $0 \leq k < 2^{\alpha} - 1$ and $j = 2^{\beta} + q$, $\beta \geq 0$, $0 \leq q < 2^{\beta} - 1$.

Then, Eqn. (3.67) can be decomposed and written as

$$f(x_1, x_2) = \mathbf{h}_n^{\mathbf{T}}(x_1) \mathbf{C}_{nm} \mathbf{h}_m(x_2) \tag{3.68}$$

where $\mathbf{C}_{nm} = \begin{bmatrix} C_{00} & C_{01} & \cdots & C_{0m-1} \\ C_{10} & C_{11} & \cdots & C_{2m-1} \\ \vdots & \vdots & \cdots & \vdots \\ C_{n-10} & C_{n-11} & \cdots & C_{n-1m-1} \end{bmatrix}$ is the now $n \times m$ matrix of the coefficient. To solve the nonlinear optimal control problem of infinite time horizon, which is represented in Chapter 4, the main point that should be determined is $\mathbf{C}_{nm}$.

Let $vec(\mathbf{A})$ denotes the transformation of stacking the column of $\mathbf{A}$ and $\otimes$ represent the Kroneker product operation equation. Then, Eqn. (3.68) can be written in compact form by applying the properties of operation, such as vec, $vec^{\mathbf{T}}(\mathbf{ABC}) = vec^{\mathbf{T}}(\mathbf{B})(\mathbf{A}^{\mathbf{T}} \otimes \mathbf{C})$ (see Appendix A), according to the method prescribed by Brewer (1978)

$$f(x_1, x_2) = vec^{\mathbf{T}}(\mathbf{C})(\mathbf{h}_n(x_1) \otimes \mathbf{h}_m(x_2)), \quad (3.69)$$

where $vec(\mathbf{C}) = [C_{00}\ C_{10}\ \cdots\ C_{n-10}\ C_{01}\ C_{11}\ \cdots\ C_{n-11}\ \cdots\ C_{0m-1}\ C_{1m-1} \cdots C_{n-1m-1}]^{\mathbf{T}}$ is the $nm \times 1$ vector.

When Eqn. (3.59) is applied, Eqn. (3.69) can be written as

$$f(x_1, x_2) = vec^{\mathbf{T}}(\mathbf{C})\,\mathbf{H}(x_1, x_2). \quad (3.70)$$

Subsequently, we assume that $n = m$ and $\tau_1 = \tau_2 = \tau$, so that the operation matrix is square. If the function is known, the coefficient in Eqn. (3.68) can be obtained quite easily by using the Haar wavelet matrix in Eqn. (3.28). This coefficient is determined at the collocation points (x_i, x_j), which are in turn described in Eqn. (3.27) as

$$\mathbf{C}_m = (\mathbf{H}_n^{\mathbf{-1}})^{\mathbf{T}}\ f_m\ \mathbf{H}_m^{\mathbf{-1}}, \quad (3.71)$$

where the matrix function of the elements f_m is given by $f_{i,j} = [f(x_{1i}, x_{2j})]$ at collocation points x_{1i}, x_{2j} for $i, j = 1,2,\cdots,m$.

3.10 Operational Matrices for Two-dimensional Haar Wavelets

To solve GHJB equation, new formulas must be developed for operational matrices. In this section, we obtain the operational matrices for integrating the new function vectors of two-dimensional Haar wavelets in the interval $[-\tau, \tau) \times [-\tau, \tau)$.

3.10.1 Partial Integration of the Function Vectors of Two-dimensional Haar Wavelets with Respect to x_1

Given Eqn. (3.59), we assume that the following form with regard to the integration of two-dimensional Haar wavelets basis on interval $[-\tau,\tau)\times[-\tau,\tau)$.

$$\int_{-\tau}^{x_1}\mathbf{H}(x_1,x_2)dx_1=\int_{-\tau}^{0}\mathbf{H}(x_1,x_2)dx_1+\int_{0}^{x_1}\mathbf{H}(x_1,x_2)dx_1\,. \tag{3.72}$$

Eqn. (3.72) can be rewritten in a new arrangement as follows:

$$\int_{0}^{x_1}\mathbf{H}(x_1,x_2)dx_1=\int_{-\tau}^{x_1}\mathbf{H}(x_1,x_2)dx_1-\int_{-\tau}^{0}\mathbf{H}(x_1,x_2)dx_1\,. \tag{3.73}$$

By incorporating Eqn. (3.60), we can describe the first term on the right-hand side of Eqn. (3.73) as follows:

$$\int_{-\tau}^{x_1}\mathbf{H}(x_1,x_2)dx_1=\begin{bmatrix}\int_{-\tau}^{x_1}h_0(x_1)\,h_0(x_2)dx_1\\ \vdots\\ \int_{-\tau}^{x_1}h_0(x_1)\,h_{m-1}(x_2)dx_1\\ \int_{-\tau}^{x_1}h_1(x_1)\,h_0(x_2)dx_1\\ \vdots\\ \int_{-\tau}^{x_1}h_1(x_1)\,h_{m-1}(x_2)dx_1\\ \int_{-\tau}^{x_1}h_{m-1}(x_1)\,h_0(x_2)dx_1\\ \vdots\\ \int_{-\tau}^{x_1}h_{m-1}(x_1)\,h_{m-1}(x_2)dx_1\end{bmatrix} \tag{3.74}$$

To calculate the Eqn. (3.74), we integrate all of the elements of Haar wavelet function vector $\mathbf{h_m}(x_1)$ with respect to x_1. The terms of the Haar wavelet function, including x_2, are considered as constants. Then, we obtain

$$\int_{-\tau}^{x_1} \mathbf{H}(x_1, x_2)dx_1 = \begin{bmatrix} h_0(x_2)(\sum_{i=0}^{m-1} \overline{p}_{0i} h_i(x_1) \\ \vdots \\ h_{m-1}(x_2)(\sum_{i=0}^{m-1} \overline{p}_{0i} h_i(x_1) \\ h_0(x_2)(\sum_{i=0}^{m-1} \overline{p}_{1i} h_i(x_1) \\ \vdots \\ h_{m-1}(x_2)(\sum_{i=0}^{m-1} \overline{p}_{1i} h_i(x_1) \\ h_0(x_2)(\sum_{i=0}^{m-1} \overline{p}_{m-1i} h_i(x_1) \\ \vdots \\ h_{m-1}(x_2)(\sum_{i=0}^{m-1} \overline{p}_{m-1i} h_i(x_1) \end{bmatrix} \tag{3.75}$$

By simplifying Eqn. (3.75), we generate

$$\int_{-\tau}^{x_1} \mathbf{H}(x_1, x_2)dx_1 = \begin{bmatrix} h_0(x_2) & (\overline{p}_{00}h_0(x_1) + \overline{p}_{01}h_1(x_1) + & \cdots & + \overline{p}_{0m-1}h_{m-1}(x_1)) \\ \vdots & \vdots \quad \vdots & \cdots & \vdots \\ h_{m-1}(x_2) & (\overline{p}_{00}h_0(x_1) + \overline{p}_{01}h_1(x_1) + & \cdots & + \overline{p}_{0m-1}h_{m-1}(x_1)) \\ h_0(x_2) & (\overline{p}_{10}h_0(x_1) + \overline{p}_{11}h_1(x_1) + & \cdots & + \overline{p}_{1m-1}h_{m-1}(x_1)) \\ \vdots & \vdots \quad \vdots & \cdots & \vdots \\ h_{m-1}(x_2) & (\overline{p}_{10}h_0(x_1) + \overline{p}_{11}h_1(x_1) + & \cdots & + \overline{p}_{1m-1}h_{m-1}(x_1)) \\ \vdots & \vdots \quad \vdots & \cdots & \vdots \\ h_0(x_2) & (\overline{p}_{m-10}h_0(x_1) + \overline{p}_{m-11}h_1(x_1) + & \cdots & + \overline{p}_{m-1m-1}h_{m-1}(x_1)) \\ \vdots & \vdots \quad \vdots & \cdots & \vdots \\ h_{m-1}(x_2) & (\overline{p}_{m-10}h_0(x_1) + \overline{p}_{m-11}h_1(x_1) + & \cdots & + \overline{p}_{m-1m-1}h_{m-1}(x_1)) \end{bmatrix} \tag{3.76}$$

By expressing Eqn. (3.76) in matrix form with a two-dimensional Haar wavelet vector, we obtain

$$\int_{-\tau}^{x_1} \mathbf{H}(x_1, x_2)dx_1 =$$

$$\left[\begin{array}{cccc|cccc|c|cccc}
\bar{p}_{00} & 0 & \cdots & 0 & \bar{p}_{01} & 0 & \cdots & 0 & \cdots & \bar{p}_{0m-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & \bar{p}_{00} & 0 & 0 & \cdots & \bar{p}_{01} & \cdots & 0 & 0 & \cdots & \bar{p}_{0m-1} \\
\hline
\bar{p}_{10} & 0 & \cdots & 0 & \bar{p}_{11} & 0 & \cdots & 0 & \cdots & \bar{p}_{1m-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & p_{10} & 0 & 0 & \cdots & p_{11} & \cdots & 0 & 0 & \cdots & \bar{p}_{1m-1} \\
\hline
 & & \vdots & & & & \vdots & & \cdots & & \vdots & & \\
\hline
\bar{p}_{m-10} & 0 & \cdots & 0 & \bar{p}_{m-11} & 0 & \cdots & 0 & \cdots & \bar{p}_{mm} & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & \bar{p}_{m-10} & 0 & 0 & \cdots & \bar{p}_{m-11} & \cdots & 0 & 0 & \cdots & \bar{p}_{mm}
\end{array}\right]
\begin{bmatrix}
h_0(x_1)h_0(x_2) \\
\vdots \\
h_0(x_1)h_{m-1}(x_2) \\
h_1(x_1)h_0(x_2) \\
\vdots \\
h_1(x_1)h_{m-1}(x_2) \\
\vdots \\
h_{m-1}(x_1)h_0(x_2) \\
\vdots \\
h_{m-1}(x_1)h_{m-1}(x_2)
\end{bmatrix}$$

(3.77)

Finally, Eqn. (3.77) can be written as

$$\int_{-\tau}^{x_1} \mathbf{H}(x_1, x_2)\, dx_1 = (\,\overline{\mathbf{P}}_m \otimes \mathbf{I}_m\,)\, \mathbf{H}(x_1, x_2) \tag{3.78}$$

$$= Q_1\, \mathbf{H}(x_1, x_2), \tag{3.79}$$

where Q_1 is the $m^2 \times m^2$ matrix.

The second term on the right-hand side of Eqn. (3.73) is written as follows upon its description according to Eqn. (3.60):

$$\int_{-\tau}^{0}\mathbf{H}(x_1,x_2)dx_1=\begin{bmatrix}\int_{-\tau}^{0}h_0(x_1)\,h_0(x_2)dx_1\\ \vdots\\ \int_{-\tau}^{0}h_0(x_1)\,h_{m-1}(x_2)dx_1\\ \int_{-\tau}^{0}h_1(x_1)\,h_0(x_2)dx_1\\ \vdots\\ \int_{-\tau}^{0}h_1(x_1)\,h_{m-1}(x_2)dx_1\\ \vdots\\ \int_{-\tau}^{0}h_{m-1}(x_1)\,h_0(x_2)dx_1\\ \vdots\\ \int_{-\tau}^{0}h_{m-1}(x_1)\,h_{m-1}(x_2)dx_1\end{bmatrix} \tag{3.80}$$

As with Eqn. (3.73). Eqn. (3.80) is integrated into all of the elements of Haar wavelet function vector $\mathbf{h}_m(x_1)$ with respect to x_1. The terms of the Haar wavelet function, including x_2, are considered constants. Then, we obtain

$$\int_{-\tau}^{0}\mathbf{H}(x_1,x_2)dx_1=\begin{bmatrix}h_0(x_2)\int_{-\tau}^{0}h_0(x_1)\,dx_1\\ \vdots\\ h_{m-1}(x_2)\int_{-\tau}^{0}h_0(x_1)\,dx_1\\ h_0(x_2)\int_{-\tau}^{0}h_1(x_1)\,dx_1\\ \vdots\\ h_{m-1}(x_2)\int_{-\tau}^{0}h_1(x_1)\,dx_1\\ \vdots\\ h_0(x_2)\int_{-\tau}^{0}h_{m-1}(x_1)\,dx_1\\ \vdots\\ h_{m-1}(x_2)\int_{-\tau}^{0}h_{m-1}(x_1)\,dx_1\end{bmatrix} \tag{3.81}$$

By integrating Eqn. (3.81) into all of the elements of Haar wavelet function vector $\mathbf{h}_m(x_1)$ with respect to x_1 on interval $[-\tau,0)$, we determine τ only for the Haar functions $h_0(x_1)$ and $h_1(x_1)$ see Figure 3.2. While integration into the remaining functions yields a value of zero, as follows:

$$\int_{-\tau}^{0} \mathbf{H}(x_1, x_2)dx_1 = \begin{bmatrix} h_0(x_2)(\tau) \\ \vdots \\ h_{m-1}(x_2)(\tau) \\ h_0(x_2)(\tau) \\ \vdots \\ h_{m-1}(x_2)(\tau) \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.82}$$

By writing Eqn. (3.82) in matrix form, we generate

$$\int_{-\tau}^{0} \mathbf{H}(x_1, x_2)dx_1 =$$

$$= \tau \left[\begin{array}{ccccc|cccc|c|cccc}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & 0 & \cdots & \vdots & \vdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hdashline
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & 0 & \cdots & \vdots & \vdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hdashline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & 0 & \cdots & \vdots & \vdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hdashline
 & & \vdots & & & & \vdots & & & \cdots & & & \vdots & \\
\hdashline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & 0 & \cdots & \vdots & \vdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array} \right]
\begin{bmatrix}
h_0(x_1)\, h_0(x_2) \\
\vdots \\
h_0(x_1) h_{m-1}(x_2) \\
h_1(x_1) h_0(x_2) \\
\vdots \\
h_1(x_1) h_{m-1}(x_2) \\
\vdots \\
h_{m-1}(x_1) h_0(x_2) \\
\vdots \\
h_{m-1}(x_1) h_{m-1}(x_2)
\end{bmatrix} \tag{3.83}$$

Then, Eqn. (3.83) can be decomposed as

$$\int_{-\tau}^{0} \mathbf{H}(x_1, x_2)dx_1 = \tau\, \mathbf{E_1}\, \mathbf{H}(x_1, x_2)\,, \tag{3.84}$$

where $\mathbf{E_1}$ is the $m^2 \times m^2$ matrix.

For instance, the operational matrix is as follows when $m = 4$:

$$\mathbf{E}_1 = \begin{bmatrix} \mathbf{I_4} & \mathbf{O_4} & \mathbf{O_4} & \mathbf{O_4} \\ \mathbf{I_4} & \mathbf{O_4} & \mathbf{O_4} & \mathbf{O_4} \\ \mathbf{O_4} & \mathbf{O_4} & \mathbf{O_4} & \mathbf{O_4} \\ \mathbf{O_4} & \mathbf{O_4} & \mathbf{O_4} & \mathbf{O_4} \end{bmatrix}, \tag{3.85}$$

where $\mathbf{O}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{I_4}$ is the identity matrix.

In general,

$$\mathbf{E}_1 = \begin{bmatrix} \mathbf{I}_m & \mathbf{O}_m & \mathbf{O_m} & \cdots & \mathbf{O_m} \\ \mathbf{I}_m & \mathbf{O_m} & \mathbf{O_m} & \cdots & \mathbf{O_m} \\ \mathbf{O}_m & \mathbf{O_m} & \mathbf{O_m} & \cdots & \mathbf{O_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O}_m & \mathbf{O_m} & \mathbf{O_m} & \cdots & \mathbf{O_m} \end{bmatrix}. \tag{3.86}$$

Therefore, Eqn. (3.73) can be written as follows by combining Eqns. (3.79) and (3.84):

$$\int_0^{x_1} \mathbf{H}(x_1, x_2) dx_1 = (Q_1 - \tau\, \mathbf{E_1}) \mathbf{H}(x_1, x_2)\ . \tag{3.87}$$

3.10.2 Partial Integration of the Function Vectors of Two-dimensional Haar Wavelets with Respect to x_2

Given Eqn. (3.59), we assume the following form of two-dimensional Haar wavelets basis on the interval $[-\tau, \tau) \times [-\tau, \tau)$:

$$\int_{-\tau}^{x_2} \mathbf{H}(x_1, x_2) dx_2 = \int_{-\tau}^{0} \mathbf{H}(x_1, x_2) dx_2 + \int_0^{x_2} \mathbf{H}(x_1, x_2) dx_2\ . \tag{3.88}$$

Eqn. (3.88) can be rewritten in a new arrangement as follows:

$$\int_0^{x_2} \mathbf{H}(x_1, x_2) dx_2 = \int_{-\tau}^{x_2} \mathbf{H}(x_1, x_2) dx_2 - \int_{-\tau}^{0} \mathbf{H}(x_1, x_2) dx_2\,. \tag{3.89}$$

We can describe the Eqn. (3.89) for the first term on the right-hand side by using the Eqn. (3.60) as follows:

$$\int_{-\tau}^{x_2} \mathbf{H}(x_1, x_2) dx_2 = \begin{bmatrix} \int_{-\tau}^{x_2} h_0(x_1)\, h_0(x_2) dx_2 \\ \vdots \\ \int_{-\tau}^{x_2} h_0(x_1)\, h_{m-1}(x_2) dx_2 \\ \int_{-\tau}^{x_2} h_1(x_1)\, h_0(x_2) dx_2 \\ \vdots \\ \int_{-\tau}^{x_2} h_1(x_1)\, h_{m-1}(x_2) dx_2 \\ \vdots \\ \int_{-\tau}^{x_2} h_{m-1}(x_1)\, h_0(x_2) dx_2 \\ \vdots \\ \int_{-\tau}^{x_2} h_{m-1}(x_1)\, h_{m-1}(x_2) dx_2 \end{bmatrix} \qquad (3.90)$$

To calculate Eqn. (3.90), we simplify the integration of Haar wavelet function vectors $\mathbf{h}_m(x_2)$ with respect to x_2. While the terms of the Haar wavelet function including x_1, are regarded as constants. Then, we obtain

$$\int_{-\tau}^{x_2} \mathbf{H}(x_1, x_2) dx_2 = \begin{bmatrix} h_0(x_1) \int_{-\tau}^{x_2} h_0(x_2)\, dx_2 \\ \vdots \\ h_0(x_1) \int_{-\tau}^{x_2} h_{m-1}(x_2)\, dx_2 \\ h_1(x_1) \int_{-\tau}^{x_2} h_0(x_2)\, dx_2 \\ \vdots \\ h_1(x_1) \int_{-\tau}^{x_2} h_{m-1}(x_2)\, dx_2 \\ \vdots \\ h_{m-1}(x_1) \int_{-\tau}^{x_2} h_0(x_2)\, dx_2 \\ \vdots \\ h_{m-1}(x_1) \int_{-\tau}^{x_2} h_{m-1}(x_2)\, dx_2 \end{bmatrix} \tag{3.91}$$

By integrating Eqn. (3.91), we obtain

$$\int_{-\tau}^{x_1} \mathbf{H}(x_1, x_2) dx_1 = \begin{bmatrix} h_0(x_1)\ (\bar{p}_{00}h_0(x_2) + \bar{p}_{01}h_1(x_2) + & \cdots + \bar{p}_{0m-1}h_{m-1}(x_2)) \\ \vdots \quad \vdots \quad \vdots & \cdots \quad \vdots \\ h_0(x_1)(\bar{p}_{m-10}h_0(x_2) + \bar{p}_{m-11}h_1(x_2) + & \cdots + \bar{p}_{m-1m-1}h_{m-1}(x_2)) \\ h_1(x_1)\ (\bar{p}_{00}h_0(x_2) + \bar{p}_{01}h_1(x_2) + & \cdots + \bar{p}_{0m-1}h_{m-1}(x_2)) \\ \vdots \quad \vdots \quad \vdots & \cdots \quad \vdots \\ h_1(x_2)\ (\bar{p}_{m-10}h_0(x_2) + \bar{p}_{m-11}h_1(x_2) + & \cdots + \bar{p}_{m-1m-1}h_{m-1}(x_2)) \\ \vdots \quad \vdots \quad \vdots & \cdots \quad \vdots \\ h_{m-1}(x_1)\ (\bar{p}_{00}h_0(x_2) + \bar{p}_{01}h_1(x_2) + & \cdots + \bar{p}_{0m-1}h_{m-1}(x_2)) \\ \vdots \quad \vdots \quad \vdots & \cdots \quad \vdots \\ h_{m-1}(x_1)\ (\bar{p}_{m-10}h_0(x_2) + \bar{p}_{m-11}h_1(x_2) + & \cdots + \bar{p}_{m-1m-1}h_{-1m}(x_2)) \end{bmatrix}.$$

$$(3.92)$$

Then, Eqn. (3.92) can be written as

$$\int_{-\tau}^{x_1} \mathbf{H}(x_1, x_2) dx_1 =$$

$$= \left[\begin{array}{cccc|cccc|c|cccc}
\overline{p}_{00} & \overline{p}_{01} & \cdots & \overline{p}_{0m-1} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\overline{p}_{m-10} & \overline{p}_{m-11} & \cdots & \overline{p}_{m-1m-1} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & \overline{p}_{00} & \overline{p}_{01} & \cdots & \overline{p}_{0m-1} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & \overline{p}_{m-10} & \overline{p}_{m-11} & \cdots & \overline{p}_{m-1m-1} & \cdots & 0 & 0 & \cdots & 0 \\
\hline
 & \vdots & & & & \vdots & & & \cdots & & \vdots & & \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \overline{p}_{00} & \overline{p}_{01} & \cdots & \overline{p}_{0m-} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \overline{p}_{m-10} & \overline{p}_{m-11} & \cdots & \overline{p}_{m-1}
\end{array}\right]
\begin{bmatrix}
h_0(x_1)h_0(x_2) \\
\vdots \\
h_0(x_1)h_{m-1}(x_2) \\
h_1(x_1)h_0(x_2) \\
\vdots \\
h_1(x_1)h_{m-1}(x_2) \\
\vdots \\
h_{m-1}(x_1)h_0(x_2) \\
\vdots \\
h_{m-1}(x_1)h_{m-1}(x_2)
\end{bmatrix} \tag{3.93}$$

Eqn. (3.93) can generally be written as

$$\int_{-\tau}^{x_2} \mathbf{H}(x_1, x_2)\, dx_2 = (\mathbf{I_m} \otimes \overline{\mathbf{P}}_{\mathbf{m}})\, \mathbf{H}(x_1, x_2) \tag{3.94}$$

$$= Q_2\, \mathbf{H}(x_1, x_2), \tag{3.95}$$

where Q_2 is the $m^2 \times m^2$ matrix.

The second term on the right-hand side of Eqn. (3.89) is expressed as follows after its description as per Eqn. (3.60):

$$\int_{-\tau}^{0} \mathbf{H}(x_1, x_2) dx_2 = \begin{bmatrix} \int_{-\tau}^{0} h_0(x_1)\, h_0(x_2) dx_2 \\ \int_{-\tau}^{0} h_0(x_1)\, h_1(x_2) dx_2 \\ \vdots \\ \int_{-\tau}^{0} h_0(x_1)\, h_{m-1}(x_2) dx_2 \\ \int_{-\tau}^{0} h_1(x_1)\, h_0(x_2) dx_2 \\ \vdots \\ \int_{-\tau}^{0} h_1(x_1)\, h_{m-1}(x_2) dx_2 \\ \vdots \\ \int_{-\tau}^{0} h_{m-1}(x_1)\, h_0(x_2) dx_2 \\ \int_{-\tau}^{0} h_{m-1}(x_1)\, h_{m-1}(x_2) dx_2 \end{bmatrix} \tag{3.96}$$

By integrating Eqn. (3.96) for all of the elements of Haar wavelet function vector $\mathbf{h}_m(x_2)$ with respect to x_2, we obtain

$$\int_{-\tau}^{0} \mathbf{H}(x_1, x_2) dx_1 = \begin{bmatrix} h_9(x_1)(\tau) \\ h_0(x_1)(\tau) \\ 0 \\ \vdots \\ h_1(x_1)(\tau) \\ h_1(x_1)(\tau) \\ 0 \\ \vdots \\ h_{m-1}(x_1)(\tau) \\ h_{m-1}(x_1)(\tau) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.97}$$

$$
= \tau \left[\begin{array}{cccc|cccc|c|cccc}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
 & & \vdots & & & \vdots & & & \cdots & & \vdots & & \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \vdots & \cdots & \vdots & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array} \right]
\begin{bmatrix}
h_0(x_1)h_0(x_2) \\ \vdots \\ h_0(x_1)h_{m-1}(x_2) \\ h_1(x_1)h_0(x_2) \\ \vdots \\ h_1(x_1)h_{m-1}(x_2) \\ \vdots \\ h_{m-1}(x_1)h_0(x_2) \\ \vdots \\ h_{m-1}(x_1)h_{m-1}(x_2)
\end{bmatrix}
\tag{3.98}
$$

$$
\int_{-\tau}^{0} \mathbf{H}(x_1, x_2)dx_2 = \tau \, \mathbf{E}_2 \, \mathbf{H}(x_1, x_2), \tag{3.99}
$$

For instance, the operational matrix is formulated as follows when $m = 4$:

$$
\mathbf{E}_2 = \begin{bmatrix}
\delta_4 & \mathbf{O}_4 & \mathbf{O}_4 & \mathbf{O}_4 \\
\mathbf{O}_4 & \delta_4 & \mathbf{O}_4 & \mathbf{O}_4 \\
\mathbf{O}_4 & \mathbf{O}_4 & \delta_4 & \mathbf{O}_4 \\
\mathbf{O}_4 & \mathbf{O}_4 & \mathbf{O}_4 & \delta_4
\end{bmatrix}, \tag{3.100}
$$

where $\delta_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general form for $\mathbf{E_2}$ is

$$\mathbf{E}_2 = \begin{bmatrix} \delta_m & \mathbf{O_m} & \mathbf{O_m} & \cdots & \mathbf{O_m} \\ \mathbf{O}_m & \delta_m & \mathbf{O_m} & \cdots & \mathbf{O_m} \\ \mathbf{O}_m & \mathbf{O_m} & \delta_m & \cdots & \mathbf{O_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O}_m & \mathbf{O_m} & \mathbf{O_m} & \cdots & \delta_m \end{bmatrix}. \tag{3.101}$$

Thus, Eqn. (3.89) can be written as follows by combining Eqns. (3.95) and (3.99):

$$\int_0^{x_2} \mathbf{H}(x_1, x_2)dx_2 = (Q_2 - \tau\,\mathbf{E_2})\mathbf{H}(x_1, x_2) \ . \tag{3.102}$$

3.11 Operational Matrix of the Product of Two-dimensional Haar Wavelet Vectors

To solve the GHJB equation described in Chapter 4, we must determine the product of $\mathbf{H}(x_1, x_2)$ and $\mathbf{H^T}(x_1, x_2)$. The product of two functions can be expanded into series of two-dimensional Haar wavelets with Haar coefficient matrix $\mathbf{N(D)}$.

Let

$$f(x_1, x_2) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij} h_i(x_1) h_j(x_2) \ , \tag{3.103}$$

and

$$g(x_1, x_2) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} d_{ij} h_i(x_1) h_j(x_2) \, . \tag{3.104}$$

Eqns. (3.103) and (3.104) can be rewritten in matrix form as

$$f(x_1, x_2) = \mathbf{h^T}(x_1)\,\mathbf{C}\,\mathbf{h}(x_2) \ , \tag{3.105}$$

$$g(x_1, x_2) = \mathbf{h^T}(x_1)\,\mathbf{D}\,\mathbf{h}(x_2) \, , \tag{3.106}$$

where

$$\mathbf{C}=\begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0m-1} \\ c_{10} & c_{11} & \cdots & c_{1m-1} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m-10} & c_{m-11} & \cdots & c_{m-1m-1} \end{bmatrix}$$ and

$$\mathbf{D}=\begin{bmatrix} d_{00} & d_{01} & \cdots & d_{0m-1} \\ d_{10} & d_{11} & \cdots & d_{0m-1} \\ \vdots & \vdots & \cdots & \vdots \\ d_{m-10} & d_{m2} & \cdots & d_{m-1m-1} \end{bmatrix}$$ are known $m \times m$ constant matrices.

Eqns. (3.105) and (3.106) can be rewritten in compact form by applying the *vec* operation (Appendix A),

$$f(x_1, x_2) = vec^T(\mathbf{C})\,(\mathbf{h}(x_1) \otimes \mathbf{h}(x_2)), \tag{3.107}$$

$$g(x_1, x_2) = vec^T(\mathbf{D})\,(\mathbf{h}(x_1) \otimes \mathbf{h}(x_2)), \tag{3.108}$$

where

$vec(\mathbf{C}) = \left[c_{00} \;\; c_{10} \;\cdots\; c_{m-10} \;\; c_{01} \;\; c_{11} \cdots c_{m-11} \;\cdots\; c_{0m-1} \;\; c_{1m-1} \;\ldots\; c_{m-1m-1} \right]^{\mathbf{T}}$ is the $m^2 \times 1$ vector and

$vec(\mathbf{D}) = \left[d_{00} \;\; d_{10} \;\cdots\; d_{m-10} \;\; d_{01} \;\; d_{11} \cdots d_{m-11} \;\cdots\; d_{0m-1} \;\; d_{1m-1} \;\ldots\; d_{m-1m-1} \right]^{\mathbf{T}}$ is the $m^2 \times 1$ vector.

By utilizing Eqn. (3.59), we obtain

$$f(x_1, x_2) = vec^T(\mathbf{C})\mathbf{H}(x_1, x_2)\,. \tag{3.109}$$

$$g(x_1, x_2) = vec^T(\mathbf{D})\,\mathbf{H}(x_1, x_2)\,. \tag{3.110}$$

The product of two functions is determined by

$$f(x_1,x_2)g(x_1,x_2)=vec^T(\mathbf{C})\mathbf{H}(x_1,x_2)vec^T(\mathbf{D})\mathbf{H}(x_1,x_2) \tag{3.111}$$

$$=vec^T(\mathbf{C})\mathbf{H}(x_1,x_2)\mathbf{H}^{\mathbf{T}}(x_1,x_2)vec(\mathbf{D}) \tag{3.112}$$

$$=vec^T(\mathbf{C})\,\mathbf{N(D)H}(x_1,x_2) \tag{3.113}$$

where $\mathbf{N}(\mathbf{D})$ is the $m^2\times m^2$ square matrix and $vec(\mathbf{D})$ is the $m^2\times 1$ vector.

Subsequently, we prove Eqn. (3.113).

Proof

$$f(x_1,x_2)g(x_1,x_2)=vec^T(\mathbf{C})(\mathbf{h}(x_1)\otimes\mathbf{h}(x_2))vec^T(\mathbf{D})(\mathbf{h}(x_1)\otimes\mathbf{h}(x_2)) \tag{3.114}$$

This equation can be rewritten as

$$f(x_1,x_2)g(x_1,x_2)=vec^T(\mathbf{C})(\mathbf{h}(x_1)\otimes\mathbf{h}(x_2))(\mathbf{h}(x_1)\otimes\mathbf{h}(x_2))^{\mathbf{T}}vec(\mathbf{D}) \tag{3.115}$$

When the transpose properties $(\mathbf{A}\otimes\mathbf{B})^{\mathbf{T}}=(\mathbf{A}^{\mathbf{T}}\otimes\mathbf{B}^{\mathbf{T}})$ are applied (see Appendix A.3), then Eqn. (3.115) can written as

$$f(x_1,x_2)g(x_1,x_2)=vec^T(\mathbf{C})(\mathbf{h}(x_1)\otimes\mathbf{h}(x_2))(\mathbf{h}^{\mathbf{T}}(x_1)\otimes\mathbf{h}^{\mathbf{T}}(x_2))vec(\mathbf{D}) \tag{3.116}$$

On the basis of Kronecker product properties, we determine that $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=(\mathbf{AC}\otimes\mathbf{BD})$ (Appendix A.3) as per Zhang and Ding (2013). Eqn. (3.116) can be modified using the following formula:

$$f(x_1,x_2)g(x_1,x_2)=vec^T(\mathbf{C})(\mathbf{h}(x_1)\mathbf{h}^{\mathbf{T}}(x_1))\otimes(\mathbf{h}(x_2)\,\mathbf{h}^{\mathbf{T}}(x_2))vec(\mathbf{D}) \tag{3.117}$$

When the following equation is applied

$$(\mathbf{A}_n \otimes \mathbf{B}_m) = (\mathbf{A}_n \otimes \mathbf{I}_n)(\mathbf{I}_m \otimes \mathbf{B}_m) = (\mathbf{I}_m \otimes \mathbf{B}_m)(\mathbf{A}_n \otimes \mathbf{I}_n) \tag{3.118}$$

then Eqn. (3.117) can be rewritten as follows:

$$f(x_1, x_2)g(x_1, x_2) = vec^T(\mathbf{C})(\mathbf{h}(x_1)\mathbf{h}^{\mathbf{T}}(x_1) \otimes \mathbf{I}_m)(\mathbf{I}_m \otimes \mathbf{h}(x_2)\,\mathbf{h}^{\mathbf{T}}(x_2))vec(\mathbf{D}) \tag{3.119}$$

First, we address the term with variable x_2 in the right-hand side of Eqn. (3.119) as follows:

$$(\mathbf{I}_m \otimes (\mathbf{h}(x_2)\,\mathbf{h}^{\mathbf{T}}(x_2))vec(\mathbf{D}) \tag{3.120}$$

By incorporating Eqn. (3.54), we obtain

$$(\mathbf{I}_m \otimes (\mathbf{h}(x_2)\,\mathbf{h}^{\mathbf{T}}(x_2))vec(\mathbf{D}) =$$

$$\left[\begin{array}{cccc}
h_0(x_2)h_0(x_2)\ \cdots h_0(x_2)h_{m-1}(x_2) & 0\ \cdots 0 & \cdots & 0\ \cdots 0 \\
\vdots \quad \cdots \quad \vdots & \vdots & \cdots & \vdots \\
h_{m-1}(x_2)h_0(x_2)\ \cdots h_{m-1}(x_2)h_{m-1}(x_2) & 0\ \cdots 0 & \cdots & 0\ \cdots 0 \\
0\ \cdots 0 & h_0(x_2)h_0(x_2) \cdots h_0(x_2)h_{m-1}(x_2) & & \\
\vdots & \vdots \quad \cdots \quad \vdots & \cdots & 0\ \cdots 0 \\
0\ \cdots 0 & h_{m-1}(x_2)h_0(x_2) \cdots h_{m-1}(x_2)h_{m-1}(x_2) & \cdots & 0\ \cdots 0 \\
\vdots & \vdots & \vdots & \vdots \\
0\ \cdots 0 & \cdots & & h_0(x_2)h_0(x_2)\ \cdots h_0(x_2)h_{m-1}(x_2) \\
\vdots & \cdots & \vdots & \vdots \quad \cdots \quad \vdots \\
0\ \cdots 0 & \cdots & & h_{m-1}(x_2)h_0(x_2)\ \cdots h_{m-1}(x_2)h_{m-1}(x_2)
\end{array}\right]
\left[\begin{array}{c}
d_{00} \\ \vdots \\ d_{m-10} \\ d_{01} \\ \vdots \\ d_{m-11} \\ \vdots \\ d_{0m-1} \\ \vdots \\ d_{m-1m-1}
\end{array}\right]$$

$$\tag{3.121}$$

By multiplying two matrices on Eqn. (3.121), we generate

$$(\mathbf{I}_m \otimes (\mathbf{h}(x_2)\,\mathbf{h}^{\mathbf{T}}(x_2))vec(\mathbf{D}) =$$

$$\begin{bmatrix} d_{\mathbf{00}}h_0(x_2)h_0(x_2)+\cdots+d_{m-10}h_0(x_2)h_{m-1}(x_2) \quad +0+\cdots+0+\cdots+0+\cdots+0 \\ \vdots \quad \cdots \quad \vdots \quad \vdots \quad \cdots \vdots \quad \cdots \vdots \quad \cdots \quad \vdots \\ d_{\mathbf{00}}h_{m-1}(x_2)h_0(x_2)+\cdots+d_{m-10}h_{m-1}(x_2)h_{m-1}(x_2)+0+\cdots+0+\cdots+0+\cdots+0 \\ 0+\cdots+0+d_{\mathbf{01}}h_0(x_2)h_0(x_2)+\cdots+d_{m-11}h_{m-1}(x_2)h_{m-1}(x_2) \quad +0+\cdots+0+\cdots \\ \vdots \quad \vdots \quad \cdots \quad \vdots \quad \vdots \quad \cdots \quad \vdots \\ 0+\cdots+0+d_{\mathbf{01}}h_{m-1}(x_2)h_0(x_2)+\cdots+d_{m-11}h_{m-1}(x_2)h_{m-1}(x_2)+0+\cdots+0+\cdots+ \\ \vdots \quad \cdots \quad \vdots \quad \vdots \quad \vdots \quad \cdots \vdots \quad \cdots \quad \vdots \\ 0+\cdots+0++0+\cdots+0+\cdots+d_{0m-1}h_0(x_2)h_0(x_2)+ \quad \cdots+d_{m-1m-1}h_0(x_2)h_{m-1}(x_2) \\ \vdots \quad \cdots \quad \vdots \quad \vdots \quad \cdots \quad \vdots \\ 0+\cdots+0++0+\cdots+0+\cdots+d_{0m-1}h_{m-1}(x_2)h_0(x_2)+\cdots+d_{m-1m-1}h_{m-1}(x_2)h_{m-1}(x_2) \end{bmatrix},$$

(3.122)

Eqn. (3.122) can then be written as

$$(\mathbf{I}_m \otimes (\mathbf{h}(x_2)\,\mathbf{h}^{\mathbf{T}}(x_2))vec(\mathbf{D}) = \begin{bmatrix} \sum_{\mathbf{k}=0}^{m-1} d_{k0}h_0(x_2)h_k(x_2) \\ \vdots \\ \sum_{\mathbf{k}=0}^{m-1} d_{k0}h_{m-1}(x_2)h_k(x_2) \\ \sum_{\mathbf{k}=0}^{m-1} d_{k1}h_0(x_2)h_k(x_2) \\ \vdots \\ \sum_{\mathbf{k}=0}^{m-1} d_{k1}h_{m-1}(x_2)h_k(x_2) \\ \vdots \\ \sum_{\mathbf{k}=0}^{m-1} d_{km-1}h_0(x_2)h_k(x_2) \\ \vdots \\ \sum_{\mathbf{k}=0}^{m-1} d_{km-1}h_{m-1}(x_2)h_k(x_2) \end{bmatrix} \qquad (3.123)$$

By applying the one-dimensional Haar wavelet product, which is defined in Eqn. (3.51) $\mathbf{h}(x)\mathbf{h}^{\mathbf{T}}(x)\mathbf{c} = \mathbf{M}(\mathbf{c})\mathbf{h}(x)$, for x_2, we arrive

$$(\mathbf{I}_m \otimes (\mathbf{h}(x_2)\,\mathbf{h}^{\mathbf{T}}(x_2))vec(\mathbf{D}) = \begin{bmatrix} \mathbf{M}(d_0) \\ \mathbf{M}(d_1) \\ \mathbf{M}(d_2) \\ \vdots \\ \mathbf{M}(d_{m-1}) \end{bmatrix} \begin{bmatrix} h_0(x_2) \\ h_1(x_2) \\ h_2(x_2) \\ \vdots \\ h_{m-1}(x_2) \end{bmatrix}. \qquad 3.124)$$

Finally, we have

$$(\mathbf{I}_m \otimes (\mathbf{h}(x_2)\,\mathbf{h}^{\mathbf{T}}(x_2))vec(\mathbf{D}) = \beta(\mathbf{D})\mathbf{h}(x_2)\,, \qquad (3.125)$$

where $\beta(\mathbf{D})$ is a block $m \times 1$ matrix with each block $\mathbf{M}(d_i)$ is of size $m \times m$. $\mathbf{M}(d_i)$ is obtained using Eqn. (3.51) with the i^{th} column of $\mathbf{D}$ as the coefficient vector.

Subsequently, we deal with the term that includes x_1 together with the result of Eqn. (3.125) as follows

$$\mathbf{F}(x_1, x_2) = ((\mathbf{h}(x_1)\,\mathbf{h}^{\mathbf{T}}(x_1) \otimes \mathbf{I}_m)\beta(\mathbf{D})\mathbf{h}(x_2)$$

$$= \mathbf{K}(x_1)\beta(\mathbf{D})\mathbf{h}(x_2)\,, \qquad (3.126)$$

where $\mathbf{K}(x_1)$ is a block $m \times m$ matrix that can be written as follows:

$$\mathbf{K}(x_1) = \begin{bmatrix} K_{00} & K_{01} & K_{02} & \cdots & K_{0m-1} \\ K_{10} & K_{11} & K_{12} & \cdots & K_{1m-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ K_{m-10} & K_{m-11} & K_{m-12} & \cdots & K_{m-1m-1} \end{bmatrix}. \qquad (3.127)$$

Each $(i, j)^{th}$ block is a diagonal matrix, that is,

$$K_{ij} = h_i(x_1)h_j(x_1)\,\mathbf{I}_m\,, \qquad (3.128)$$

for $i, j = 0,1,2,\ldots,m-1$.

Let

$$\beta(\mathbf{D}) = \begin{bmatrix} \beta^{(0)} \\ \beta^{(1)} \\ \vdots \\ \beta^{(m-1)} \end{bmatrix}, \tag{3.129}$$

where $\beta^{(r)} = \begin{bmatrix} \beta^{(r)}_{00} & \beta^{(r)}_{01} & \beta^{(r)}_{02} & \cdots & \beta^{(r)}_{0m-1} \\ \beta^{(r)}_{10} & \beta^{(r)}_{11} & \beta^{(r)}_{12} & \cdots & \beta^{(r)}_{1m-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \beta^{(r)}_{m-10} & \beta^{(r)}_{m-11} & \beta^{(r)}_{m-12} & \cdots & \beta^{(r)}_{m-1m-1} \end{bmatrix}$ for $r = 0,1,2,\ldots,m-1$. (3.130)

Then, by performing the multiplication operation on Eqn. (3.126) as block wise, we have

$$\mathbf{F}(x_1,x_2) = \begin{bmatrix} K_{00}\beta^{(\mathbf{0})} + K_{01}\beta^{(\mathbf{1})} + K_{02}\beta^{(2)} + \cdots + K_{0m-1}\beta^{(m-1)} \\ K_{10}\beta^{(\mathbf{0})} + K_{11}\beta^{(\mathbf{1})} + K_{12}\beta^{(2)} + \cdots + K_{1m-1}\beta^{(m-1)} \\ \vdots \quad \vdots \quad \vdots \quad \cdots \quad \vdots \\ K_{m-10}\beta^{(\mathbf{0})} + K_{m-11}\beta^{(\mathbf{1})} + K_{m-12}\beta^{(2)} + \cdots + K_{m-1m-1}\beta^{(m-1)} \end{bmatrix} \begin{bmatrix} h_0(x_2) \\ h_1(x_2) \\ \vdots \\ h_{m-1}(x_2) \end{bmatrix}. \tag{3.131}$$

By using Eqn. (3.128), then Eqn. (3.131) can be rewritten as follows:

$$\mathbf{F}(x_1,x_2) = \begin{bmatrix} h_0(x_1)h_0(x_1)\beta^{(\mathbf{0})} + h_0(x_1)h_1(x_1)\beta^{(\mathbf{1})} + \cdots + h_0(x_1)h_{m-1}(x_1)\beta^{(m-1)} \\ h_1(x_1)h_0(x_1)\beta^{(\mathbf{0})} + h_1(x_1)h_1(x_1)\beta^{(\mathbf{1})} + \cdots + h_1(x_1)h_{m-1}(x_1)\beta^{(m-1)} \\ \vdots \quad \vdots \quad \cdots \quad \vdots \\ h_{m-1}(x_1)h_0(x_1)\beta^{(\mathbf{0})} + h_{m-1}(x_1)h_1(x_1)\beta^{(\mathbf{1})} + \cdots + h_{m-1}(x_1)h_{m-1}(x_1)\beta^{(m-1)} \end{bmatrix} \begin{bmatrix} h_0(x_2) \\ h_1(x_2) \\ \vdots \\ h_{m-1}(x_2) \end{bmatrix} \tag{3.132}$$

The current definition of $\beta^{(r)}$ as explained in Eqn. (3.130) is used to simplify Eqn. (3.132) as

$$\mathbf{F}(x_1, x_2) =$$

$$\left[\begin{array}{cccc}
\sum_{k=0}^{m-1}\beta_{00}^{k}h_0(x_1)h_k(x_1) & \sum_{k=0}^{m-1}\beta_{01}^{k}h_0(x_1)h_k(x_1) & \cdots & \sum_{k=0}^{m-1}\beta_{0m-1}^{k}h_0(x_1)h_k(x_1) \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{k=0}^{m-1}\beta_{m-10}^{k}h_0(x_1)h_k(x_1) & \sum_{k=0}^{m-1}\beta_{m-11}^{k}h_0(x_1)h_k(x_1) & \cdots & \sum_{k=0}^{m-1}\beta_{m-1m-1}^{k}h_0(x_1)h_k(x_1) \\
\hline
\sum_{k=0}^{m-1}\beta_{00}^{k}h_1(x_1)h_k(x_1) & \sum_{k=0}^{m-1}\beta_{01}^{k}h_1(x_1)h_k(x_1) & \cdots & \sum_{k=0}^{m-1}\beta_{0m-1}^{k}h_1(x_1)h_k(x_1) \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{k=0}^{m-1}\beta_{m-10}^{k}h_1(x_1)h_k(x_1) & \sum_{k=0}^{m-1}\beta_{m-11}^{k}h_1(x_1)h_k(x_1) & \cdots & \sum_{k=0}^{m-1}\beta_{m-1m-1}^{k}h_1(x_1)h_k(x_1) \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
\sum_{k=0}^{m-1}\beta_{00}^{k}h_{m-1}(x_1)h_k(x_1) & \sum_{k=0}^{m-1}\beta_{01}^{k}h_{m-1}(x_1)h_k(x_1) & \cdots & \sum_{k=0}^{m-1}\beta_{0m-1}^{k}h_{m-1}(x_1)h_k(x_1) \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{k=0}^{m-1}\beta_{m-10}^{k}h_{m-1}(x_1)h_k(x_1) & \sum_{k=0}^{m-1}\beta_{m-11}^{k}h_{m-1}(x_1)h_k(x_1) & \cdots & \sum_{k=0}^{m-1}\beta_{m-1m-1}^{k}h_{m-1}(x_1)h_k(x_1)
\end{array}\right]
\begin{bmatrix} h_0(x_2) \\ h_0(x_2) \\ \vdots \\ h_{m-1}(x_2) \end{bmatrix}$$

(3.133)

Finally, we have

$$\mathbf{F}(x_1, x_2) = \begin{bmatrix} N(\beta_{00}) & N(\beta_{01}) & N(\beta_{02}) & \cdots & N(\beta_{0m-1}) \\ N(\beta_{10}) & N(\beta_{11}) & N(\beta_{12}) & \cdots & N(\beta_{1m-1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ N(\beta_{m-10}) & N(\beta_{m-11}) & N(\beta_{m-12}) & \cdots & N(\beta_{m-1m-1}) \end{bmatrix} [\mathbf{h}(x_1) \otimes \mathbf{h}(x_2)].$$

(3.134)

Then, Eqn. (3.134) can be written as

$$\mathbf{F}(x_1, x_2) = \mathbf{N}(\beta(\mathbf{D}))\,\mathbf{H}(x_1, x_2)\,, \tag{3.135}$$

where $\mathbf{N}(\beta(\mathbf{D}))$ is a block $m \times m$ matrix. Each block has a size of $m \times m$, and each $(i, j)^{th}$ block of $\mathbf{N}(\beta_{i\,j})$ is obtained from Eqn. (3.51) using the $vec(\beta_{i\,j})$ column as the coefficient vector.

Finally, Eqn. (3.119) can be written in the compact form

$$f(x_1, x_2)g(x_1, x_2) = vec^{\mathbf{T}}(\mathbf{C})\,\mathbf{N}(\beta(\mathbf{D}))\,\mathbf{H}(x_1, x_2)\,. \tag{3.136}$$

□

3.12 Algorithm to Compute the Operational Matrix of the Product of Two-dimensional Haar Wavelet Vectors

The algorithm to obtain $\mathbf{N(D)}$ is expressed as follows:

Step 1: Let $\mathbf{D}$ be a matrix of $vec(\mathbf{D})$.

Step 2: Compute $\mathbf{M}_{\mathbf{d}_i}$, $i = 1, 2, \ldots, m$ according to Eqn. (3.51) using column $\mathbf{d}_i$ as the coefficient vector.

Step 3: For $i = 1, 2, \ldots, m$, compute $vec(\mathbf{M}_{\mathbf{d}_i})$.

Step 4: Form a large matrix by concatenating all vectors from Step 3; that is, $\mathbf{S} = [vec(\mathbf{M}_{\mathbf{d}_1})\ \ vec(\mathbf{M}_{\mathbf{d}_2})\ \ \ldots vec(\mathbf{M}_{\mathbf{d}_m})]$.

Step 5: For each row k of matrix $\mathbf{S}$, compute $\mathbf{N}_{i,j}$ according to Eqn. (3.51) using row S_{k} as the coefficient vector.

Step 6: Form matrix $\mathbf{N(D)}$ as follows:

$$\mathbf{N(D)} = \begin{bmatrix} \mathbf{N_{11}} & \mathbf{N_{12}} & \cdots & \mathbf{N_{1m}} \\ \mathbf{N_{21}} & \mathbf{N_{22}} & \cdots & \mathbf{N_{2m}} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{N}_{m1} & \mathbf{N_{m2}} & \cdots & \mathbf{N_{mm}} \end{bmatrix}. \qquad (3.131)$$

Step 7: End.

For instance, $f(x_1,x_2)g(x_1,x_2)$ should be computed when $f(x_1,x_2)=x_2$ and $g(x_1,x_2)=-x_1+2x_2$.

First, the functions above are approximated for the Haar wavelet function when $m=2$.

$$f(x_1,x_2)=\begin{bmatrix} h_1(x_1) & h_2(x_1)\end{bmatrix}\begin{bmatrix} 0.5 & -0.25 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} h_1(x_2) \\ h_2(x_2)\end{bmatrix}.$$

$$g(x_1,x_2)=\begin{bmatrix} h_1(x_1) & h_2(x_1)\end{bmatrix}\begin{bmatrix} 0.5 & -0.5 \\ 0.25 & 0 \end{bmatrix}\begin{bmatrix} h_1(x_2) \\ h_2(x_2)\end{bmatrix}.$$

These formulas can be rewritten by using vec as follows:

$$f(x_1,x_2)=\begin{bmatrix} 0.5 & -0.25 & 0 & 0\end{bmatrix}\mathbf{H}(x_1,x_2).$$

$$g(x_1,x_2)=\begin{bmatrix} 0.5 & -0.5 & 0.25 & 0\end{bmatrix}\mathbf{H}(x_1,x_2).$$

Step 1: $\mathbf{D}=\begin{bmatrix} 0.5 & -0.25 \\ 0 & 0 \end{bmatrix}.$

Step 2: $\mathbf{M}_{D_1}=\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$

$$\mathbf{M}_{D_2}=\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

Step 3: $$vec(\mathbf{M}_{\mathbf{d}_1}) = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}, \; vec(\mathbf{M}_{\mathbf{d}_2}) = \begin{bmatrix} -0.25 \\ 0 \\ 0 \\ -0.25 \end{bmatrix}.$$

Step 4: $$\mathbf{S} = \begin{bmatrix} 0.5 & -0.25 \\ 0 & 0 \\ 0 & 0 \\ 0.5 & -0.25 \end{bmatrix}.$$

Step 5: $$\mathbf{N_{11}} = \begin{bmatrix} 0.5 & -0.25 \\ -0.25 & 0.5 \end{bmatrix}, \; \mathbf{N_{12}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{N_{21}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; \mathbf{N_{22}} = \begin{bmatrix} 0.5 & -0.25 \\ -0.25 & 0.5 \end{bmatrix}.$$

Step 6: $$\mathbf{N} = \begin{bmatrix} 0.5 & -0.25 & 0 & 0 \\ -0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & -0.25 \\ 0 & 0 & -0.25 & 0.5 \end{bmatrix}.$$

Therefore,

$$f(x_1, x_2)g(x_1, x_2) = \begin{bmatrix} 0.5 & -0{,}5 & 0.25 & 0 \end{bmatrix} \mathbf{N}\,\mathbf{H}(x_1, x_2).$$

In subsequent equations, we drop subscript m to limit notations if no confusion will be induced.

3.13 Conclusion

Throughout the work in this chapter, the Haar wavelet method is used to derive some new formulas for the operational matrices of Haar wavelets on intervals $[0,\tau)$ and $[-\tau,\tau)$. All of these formulas are important, as highlighted in subsequent chapters. First, the new operational matrices are developed for integrating one-dimensional Haar wavelets on intervals $[0,\tau)$and $[-\tau,\tau)$. Second, a general formula is established for the matrix of the collocation points of Haar wavelets with two variables. Third, an operational matrix is defined for the integral products of one-dimensional Haar wavelet vectors on interval $[\tau_1,\tau_2)$. Fourth, new operational matrices are derived and proved for integrating two-dimensional Haar wavelets basis on interval $[-\tau,\tau)$. Finally, we derive and prove a new algorithm for the operational matrix of the product of two-dimensional Haar wavelet functions. In addition, a general formula is established for approximation the function of two dimensions using Haar wavelets of two functions.

CHAPTER 4

INDIRECT METHOD

NONLINEAR OPTIMAL CONTROL PROBLEMS

4.1 Introduction

In this chapter, a novel method to solve the Hamilton-Jacobi-Bellman (HJB) equation, which appears in the formulation of the nonlinear control system with quadratic cost functional and an infinite time horizon is introduced. This method is a numerical technique that is based on the combination of the Haar wavelets operational matrices and the successive Generalized Hamilton-Jacobi-Bellman (GHJB) equation. This chapter begins with the problem statement.We explain the underlying concept that leads to GHJB equation usage in this work before establishing the numerical method for the nonlinear optimal control problem using the operational matrices of Haar wavelets. The GHJB equation is a first-order linear partial differential equation; thus, it is theoretically easier to solve than the nonlinear first-order hyperbolic partial differential HJB equation. HJB equation is used for constructing a nonlinear optimal feedback control law. There is no general closed form solution to this equation. In this chapter, we show how to approximate the GHJB equation. We then use the successive GHJB equation to improve the closed-loop performance of stabilizing controls and reduce the problem of solving HJB equation to one of solving GHJB equation. Interestingly, when the process of improving the control and solving GHJB equation is iterated, the solution to the GHJB equation converges uniformly to the solution of the HJB equation which in form of the gradient of the Lyapunov function ∇V. Thus, to determine the Lyapunov function from the resultant solution for the linear system equations, we proposed a new method that depends only on the initial and final states using variable gradient method.

The proposed approach is simple and stable. Moreover, it has been tested on linear and nonlinear optimal control problems of infinite time horizon in two-dimensional state space and controls. The numerical results and discussions are provided at the end of this chapter.

4.2 Problem Statement

In this chapter, we consider the following optimal control problem:

The system to be controlled is given by nonlinear differential equations with affine in the control of the form

$$\dot{x} = f(x) + g(x)\, u(x), \qquad x(0) = \mathrm{x}_0, \tag{4.1}$$

where $x(t) \in \Omega$ is the state vector, $u : \Omega \to \Re^{n_2}$ is the control vector, $f : \Omega \to \Re^{n_1}$, and $g : \Omega \to \Re^{n_1 \times n_2}$ are continuously differentiable with respect to all its arguments, $x_0 \in \Omega$ is the initial condition vector, and Ω is the domain of attraction region. System (4.1) is denoted by (f, g).

The problem is determining the optimal control $u^*(x)$ that minimizes the following performance index,

$$J(\mathbf{x}_0, u) = \int_0^\infty (l(x(t) + \|u(x(t))\|_R^2)\, dt \ , \tag{4.2}$$

where $l : \Re^{n_1} \to \Re$ is a positive definite that is called the state penalty function. $l(x)$ is typically a quadratic weighting of the states; $l(x) = x^T \mathrm{Q}\, x$; and $\|u(x)\|_R^2 = u^T \mathbf{R}\, u$ is the control penalty function, where $\mathbf{Q} \in \Re^{n_1 \times n_1}$ is a positive semi-definite matrix and $\mathbf{R} \in \Re^{n_2 \times n_2}$ is a symmetric positive definite matrix. In the case of infinite time horizon, the system equations (f , g) and l, as well as the initial control $u^{(0)}$ are independent of time.

4.3 Generalized Hamilton-Jacobi-Bellman Equation

In this section, we derive the Generalized Hamilton-Jacobi-Bellman partial differential equation solution to the nonlinear optimal control problem with infinite time horizon. This problem is subject to time-invariant dynamics; that is, does not depend on t explicitly. The solution follows from the technique known as dynamic programming, which was popularized by Bellman (1954). We first explain the concept of dynamic programming then apply this concept to the optimal control problem to derive the GHJB partial differential equation.

Dynamic programming is the concept of using the principle of optimality to formulate an optimization problem as a recurrence relation. That is, the remaining sub-problem has precisely the same structure as the previous sub-problem. Thus, a particular optimization problem is solved by studying a family of problems of which the particular problem is a member. The basis for applying the dynamic programming solution to the optimal control problem is the so-called principle of optimality.

4.3.1 Principle of Optimality

The principle of optimality states that if an optimal control is divided into two pieces, then the last piece is itself optimal. The basic assumption underlying this principle is that the system can be characterized by its state $x(t)$ at time t . This assumption completely summarizes the effect of all inputs $u(t)$ prior to time t, thereby facilitating the local characterization of optimality as given in the following formal statement of the principle of optimality (Primbs, 1999).

Definition 4.1: Principle of Optimality

If $u^*(\tau)$ is optimal over the interval $[t, t_f]$, beginning from state $x(t)$, then $u^*(\tau)$ is necessarily optimal over the subinterval $[t+\Delta t, t_f]$, for any Δt such that $t_f - t \geq \Delta t > 0$.

Definition 4.2: Admissible Controls (Beard et al., 1998).

Given the system (f, g), for an infinite-time horizon problem, a control $u: R^{n_1} \to R^{n_2}$ is admissible with respect to the state penalty function l on Ω, which is written $u \in A_l(\Omega)$, if

- u is continuously differentiable on Ω,
- $u(0) = 0$,
- system $\dot{x} = f + gu$ is the Lyapunov stable on Ω,
- and cost function $J(x,u)$ is finite for all $x \in \Omega$, where $J(x,u)$ is given by Eqn. (4.2).

Lemma 4.1: If Ω is compact, f and g are Lipschitz continuous on Ω and $f(0) = 0$, l is a positive definite and monotonically increasing function on Ω, and R is a symmetric positive definite matrix, $u \in \Omega$ then:

- On Ω, there exists a unique continuously differentiable solution $\mathrm{V}(x)$ to the equation $\mathrm{GHJB}(\mathrm{V}; u) = 0$ with boundary conditions $\mathrm{V}(0) = 0$,
- $\mathrm{V}(x)$ is a Lyapunov function for the system (f, g, u) on Ω,
- $\mathrm{GHJB}(\mathrm{V}; u) = 0 \Leftrightarrow \mathrm{V}(x) = J(x)$, where $J(x)$ is the performance index given in Eqn. (4.2).

Proof (see Saridis and Lee, 1979).

We define $\mathrm{V}^*(x_0)$ as the minimum value of the performance index over all admissible trajectories $(x(t), u(t))$, where x starts at x_0:

$$\mathrm{V}^*(x_0) = \min_{u(.)} \int_0^\infty (l(x(t) + \|u(x(t))\|_R^2)\, dt \ , \tag{4.3}$$

subject to

$$\dot{x} = f(x) + g(x)\, u(x), \qquad f(0) = 0, \tag{4.4}$$

Function $\mathbf{V}^* : \Re^{n_1} \to \Re_+ \cup \{\infty\}$ determines the rule that associates an optimal value with each initial point and is called the value function or the Bellman's function of the optimal control problem. An optimal pair (often simply referred to as an "optimal trajectory") is a pair $(x(t), u(t))$ that has a starting point x_0 and achieves the optimal cost $\mathrm{V}^*(x_0)$.

$\mathrm{V}^*(x_0)$ is independent of $u(.)$ precisely because knowledge of the initial state abstractly determines specific control on the basis of the requirement that the control minimizes the performance index. Rather than merely determining the control that minimizes Eqn. (4.4) and for the value of $\mathrm{V}^*(x(t))$ for various x_0, the problem is addressed by evaluating $\mathrm{V}^*(x(t))$ for all $x(t)$, as well as the associated optimal control.

We then apply the principle of optimality. Consider $\mathrm{V}^*(x)$ given by Eqn. (4.3), and let $u[t, \infty]$ be defined as the control function over the interval $[t, \infty)$. Applying the additive properties of integrals and the principle of optimality yields

$$\mathbf{V}^*(x(t)) = \min_{u[t, t+\Delta t]} \left\{ \int_t^{t+\Delta t} (l(x(\tau) + \|u(x(\tau))\|_R^2)\, d\tau + \mathbf{V}^*(x(t+\Delta t)) \right\} . \tag{4.5}$$

That is, the optimal cost at state $x(t)$ is given by the minimum of moving to state $x(t+\Delta t)$ in addition to the optimal cost from $x(t+\Delta t)$. In essence, the problem of determining the optimal control over the interval $[t,\infty)$ is reduced to one of determining the optimal control over the reduced interval $[t, t+\Delta t]$ when the principle of optimality is applied.

When Δt value is small, the integral in Eqn. (4.5) can be approximated with $\mathbf{[}l(x(t)+\|u(x(t))\|_R^2]\Delta t$. The application of a multivariable Taylor-series expansion of $\mathrm{V}^*(x(t+\Delta t))$ for $x(t)$, with $x(t+\Delta t)-x(t)$ approximated by $[f(x(t))+g(x(t))u(t)]\Delta t$ generates

$$\mathbf{V}^*(x) = \min_u \left\{ [l(x)+\|u(t)\|_R^2]\Delta t + \mathbf{V}^*(x) + (\frac{\partial \mathbf{V}^{*\mathbf{T}}(x)}{\partial x}) + [f(x)+g(x)u]\Delta t + O(\Delta t) \right\} \quad (4.6)$$

where $\dfrac{\partial \mathrm{V}^*}{\partial x}$ denotes the gradient of V^* with respect to vector x and $O(\Delta t)$ denotes the higher-order terms in Δt. Cancelling $\mathrm{V}^*(x)$ on both sides and taking the limit as Δt approaching zero yields

$$\min_{\mathbf{u(t)}} \left\{ \mathbf{[}l(x(t)) + \|u(x(t))\|_R^2] + (\frac{\partial \mathbf{V}^{*\mathbf{T}}(x)}{\partial x})[f(x(t)+g(x(t)u(t)] \right\} = 0 \quad (4.7)$$

The boundary condition for this equation is given by $\mathrm{V}^*(0)=0$ where $\mathrm{V}^*(x)$ must be positive for all x (given that it corresponds to the optimal cost that must be positive).

Eqn. (4.7) is one form of the so-called Generalized Hamilton-Jacobi-Bellman (GHJB) equation. Assuming that a unique optimal control u^* exists and is an admissible control, then the optimal cost is given by the solution to the GHJB equation

$$\frac{\partial \mathbf{V}^{*\mathbf{T}}(x)}{\partial x}(f(x)+g(x)u^*(x)) + l(x) + \|u^*(x)\|_R^2 = 0 \quad (4.8)$$

We obtain a new feedback control law from the solution to the GHJB equation Eqn. (4.8) by fixing $\hat{\mathrm{V}}^*$ and by minimizing the Hamiltonian, that is,

$$\hat{u}^* = -\frac{1}{2}\mathbf{R}^{-1} g^{\mathbf{T}}(x)\frac{\partial \mathbf{V}^*(x)}{\partial x} \tag{4.9}$$

Let $\hat{\mathrm{V}}^*$ be the solution to the equation GHJB $(\hat{\mathrm{V}}^*, \hat{u}^*) = 0$ then $\hat{\mathrm{V}}^* \leq \mathrm{V}^*$. But V^* is the optimal cost; therefore, $\hat{\mathrm{V}}^* = \mathrm{V}^*$. The optimal control is unique; thus, $\hat{u}^*$ must be the optimal control. Substituting $\hat{u}^*$ into the GHJB equation generates the HJB equation

$$\mathrm{HJB}(\mathrm{V}^*) = \mathrm{GHJB}(\mathrm{V}^*, -\frac{1}{2}\mathrm{R}^{-1} g^{\mathrm{T}} \frac{\partial \mathrm{V}^*}{\partial x}) = 0\,. \tag{4.10}$$

That is,

$$\mathrm{HJB}(\mathrm{V}^*) = \frac{\partial \mathrm{V}^{*\mathrm{T}}}{\partial x} f - \frac{1}{4}\frac{\partial \mathrm{V}^{*\mathrm{T}}}{\partial x} g\, \mathrm{R}^{-1} g^{\mathrm{T}} \frac{\partial \mathrm{V}^{*\mathrm{T}}}{\partial x} + l = 0\,. \tag{4.11}$$

We can interpret the GHJB equation geometrically. Figure 4.1 (a) illustrates the phase of a two-dimensional, infinite time system. The dotted lines represent the trajectories of the system. The cost at any point x is computed by integrating Eqn. (4.2) along the unique trajectory of the system passing through x. The solid lines in Figure 4.1 (a) are the constant contours of the cost function. This geometrical interpretation suggests an intuitive concept for improving the cost of the system. If we fix the constant cost contours and minimize the action of the system with respect to these contours, then the cost of the system is reduced. For instance, the system in Figure 4.1 (b) costs less than the system in Figure 4.1 (a), as per Beard (1995).

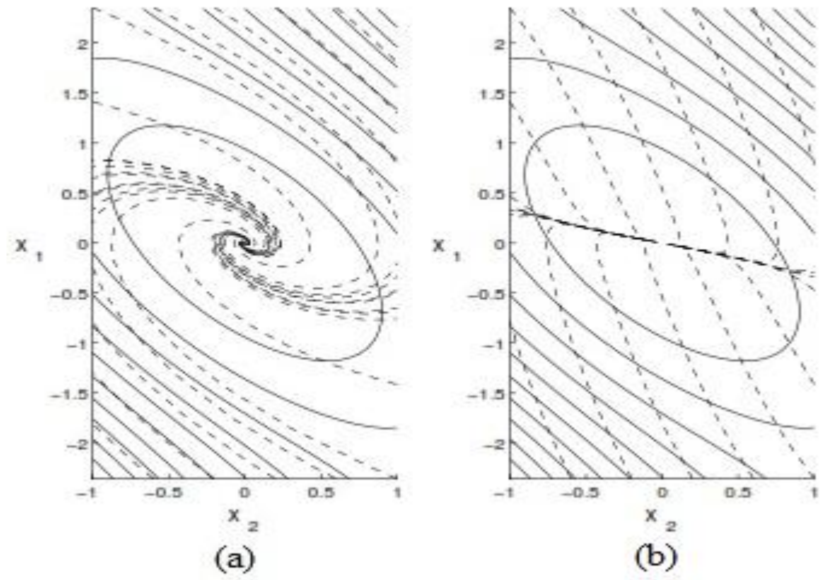


Figure 4.1: Phase flow plotted against lines of constant cost (Beard, 1995)

4.4 Successive Generalized Hamilton-Jacobi-Bellman Equation

The standard optimal control problem involves determining a control to minimize the cost function given in Eqn. (4.2). To effectively pose the problem mathematically, a unique optimal control must exist. This requirement limits the applicability of optimal control theory. In addition, the optimal control is difficult to determine, whereas many controls that are close to optimal may be significantly easier to compute. In this section, we generalize optimal control by considering the problem of improving arbitrary stabilizing control performance. We also show that by iterating the improvement process, we converge uniformly to the optimal control, if it exists. Given an arbitrary control $u(x)$, the performance of the control at $x \in \Omega \subset \Re^{n_1}$ is given by a Lyapunov function for the system Beard et al. (1997).

$$\mathrm{V}(x, u) = \int_0^{\infty} (l(x(t)) + \|u(x(t)\|_R^2)\, dt \ , \tag{4.12}$$

where $l:\Re^{n_{11}} \rightarrow \Re$ is a positive definite and monotonically increasing function on Ω, $\mathbf{R} \in \Re^{n_2 \times n_2}$ is a symmetric and positive definite matrix; $\|u\|_{\mathrm{R}}^2 = u^{\mathrm{T}}\mathrm{R}u$; and $l(x) = x^{\mathrm{T}}\mathrm{Q}\,x$.

The optimal controller in the feedback form is presented as follows as per Beard et al. (1997).

$$u^*(x) = -\frac{1}{2}\mathrm{R}^{-1}g^{\mathrm{T}}(x)\frac{\partial \mathrm{V}^*(x)}{\partial x}, \tag{4.13}$$

where $\mathrm{V}^*(x)$ is the solution to the following HJB equation:

$$\frac{\partial \mathbf{V}^{*T}(x)}{\partial x} f(x) + l(x) - \frac{1}{4}\frac{\partial \mathbf{V}^{*\mathbf{T}}(x)}{\partial x} g(x)R^{-1}g(x)^T \frac{\partial \mathbf{V}^*(x)}{\partial x} = 0, \tag{4.14}$$

under boundary condition $\mathrm{V}^*(0) = 0$; that is $\mathrm{V}(x^*, u^*) \le \mathrm{V}(x, u)$ for all u, $x^*(t)$ is the solution of $\dot{x} = f(x) + g(x)u^*(t)$. Basically, solving the nonlinear partial differential equation (4.14) for the purpose of obtaining $\mathrm{V}^*(x)$ and consequently $u^*(x)$ from Eqn. (4.13) is difficult; instead, the following two linear equations have been iterated using the algorithm proposed by Beard et al. (1997):

$$\frac{\partial \mathrm{V}^{(i)T}(x)}{\partial x}\left(f(x) + g(x)u^{(i)}(x)\right) + l(x) + \left\|u^{(i)}(x)\right\|_R^2 = 0 \tag{4.15}$$

with initial condition $\mathrm{V}^{(\mathrm{i})}(0) = 0$ and

$$u^{(i+1)}(x) = -\frac{1}{2}\mathrm{R}^{-1}g^{\mathrm{T}}(x)\frac{\partial \mathrm{V}^{(i)}(x)}{\partial x} \tag{4.16}$$

In the case of moderate presumptions, Beard et al. (1997) established that the iteration between the GHJB (4.15) and the control (4.16) coincide with the original HJB equation solution (4.14). If we can first determine a stabilizing control $u^{(0)}(x)$, then the performance of this controller can be iteratively enhanced using Eqns. (4.14) and (4.15). Finally the optimal controller can be approximated optimally. Moreover, the controller $u^{(\mathrm{i})}$ is a stable control at each iteration step. In the case of infinite time horizon, the system equations (f , g) and l, as well as the initial control, $u^{(0)}$ are independent of time.

4.4.1 Algorithm of the Successive GHJB Equation

Initial Step: Given an initial control law $u^{(0)}(x)$ that is admissible on Ω , the performance of $u^{(0)}$ on Ω is given by the unique solution $\mathrm{V}^{(0)}(x)$ to

$$\mathrm{GHJB}\left(\mathrm{V}^{(0)}, u^{(0)}\right)(x) = 0 .$$

Set $i = 0$.

***Iterative Step*:** A control law that is admissible on Ω and that improves the performance of $u^{(i)}$ is provided by

$$u^{(i+1)}(x) = -\frac{1}{2}\mathrm{R}^{-1} g^{\mathrm{T}}(x)\frac{\partial \mathrm{V}^{(i)}(x)}{\partial x} . \tag{4.17}$$

The performance of $u^{(i)}$ on Ω is given by the unique solution $\mathrm{V}^{(i)}(x)$ to

$$\mathrm{GHJB}\left(\mathrm{V}^{(\mathrm{i})}, u^{(i)}\right)(x) = 0 . \tag{4.18}$$

Set $i = i + 1$.

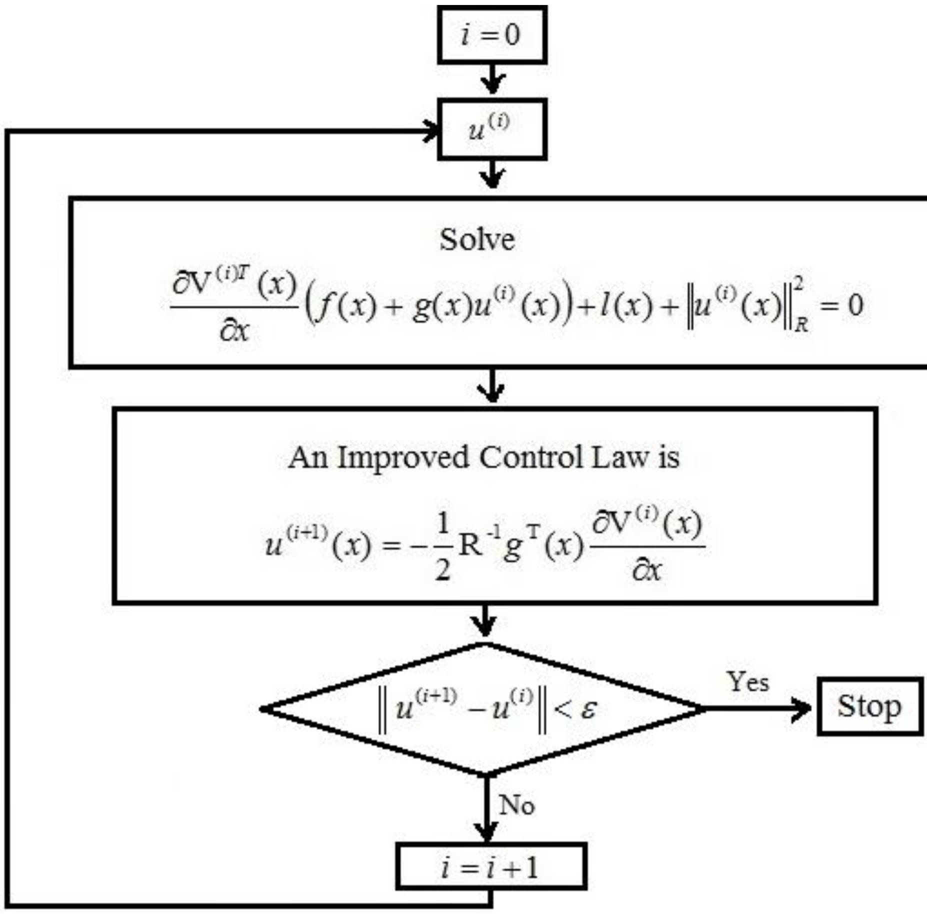


Figure 4.2: Algorithm of the successive Generalized Hamilton-Jacobi Bellman equation

The algorithm is depicted as an image in Figure 4.2. Numerous studies report that the algorithm converges uniformly to the optimal control and to the optimal cost (Mil'shtein, 1964; Leak and Liu, 1967; Saridis and Lee, 1979; Saridis and Wang, 1994). Therefore, we converge uniformly to the solution of the HJB equation by iterating the process of improving an admissible control. According to Beard (1995), previous works failed to note that definite statements can be made regarding the stability region of each successive control law $u^{(i)}$. In particular, $u^{(i)}$ stabilize on the same region as $u^{(0)}$. In fact, the stability region of u^* is the largest possible stabilizing set in $\Re^{n_1}$; that is, an admissible control that can stabilize an initial unstable condition cannot be obtained using u^*.

4.5 Approximation Functions via Haar Wavelets Approximation

The main point in solving the first-order partial differential equation GHJB equation is to approximate the second-order partial derivative of $\mathbf{V}(x_1, x_2)$ by using Haar wavelets on the basis of two dimensions in the interval $[-\tau,\tau)\times[-\tau,\tau)$. We first expand $\dfrac{\partial^2 \mathbf{V}}{\partial x_1 \partial x_2}$ using Eqn. (3.68) as follows:

$$\frac{\partial^2 \mathbf{V}}{\partial x_1 \partial x_2} = \mathbf{H}^{\mathbf{T}}(x_1)\,\omega\,\mathbf{H}(x_2)\,, \tag{4.19}$$

where $\omega_{nm} = \begin{bmatrix} \omega_{00} & \omega_{01} & \cdots & \omega_{0m-1} \\ \omega_{10} & \omega_{11} & \cdots & \omega_{1m-1} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_{m-10} & \omega_{m-11} & \cdots & \omega_{m-1m-1} \end{bmatrix}$ is the $m \times m$ matrix of the unknown coefficients .

When Eqn. (3.70) is utilized, then Eqn. (4.19) can be written as

$$\frac{\partial^2 \mathbf{V}}{\partial x_1 \partial x_2} = vec^{\mathbf{T}}(\omega)\mathbf{H}(x_1, x_2)\,, \tag{4.20}$$

where $vec(\omega) = [\omega_{00}\ \ \omega_{10}\ \cdots\ \omega_{m-10}\ \omega_{01}\ \ \omega_{11}\ \cdots\ \omega_{m-11}\ \cdots\ \omega_{0m-1}\ \ \omega_{1m-1} \cdots \omega_{m-1m-1}]^{\mathbf{T}}$ is a vector $m^2 \times 1$.

The first-order partial derivative can be obtained by integrating Eqn. (4.20) with respect to x_1 and x_2 using (3.87) and (3.102), respectively, we then obtain

$$\frac{\partial \mathbf{V}}{\partial x_1} = vec^{\mathbf{T}}(\omega)(Q_2 - \tau\,\mathbf{E_2})\mathbf{H}(x_1, x_2) + \frac{\partial \mathbf{V}}{\partial x_1}(x_1, 0) \tag{4.21}$$

$$\frac{\partial \mathbf{V}}{\partial x_2} = vec^{\mathbf{T}}(\omega)(Q_1 - \tau\,\mathbf{E_1})\mathbf{H}(x_1, x_2) + \frac{\partial \mathbf{V}}{\partial x_2}(0, x_2), \tag{4.22}$$

where Q_1, Q_2, $\mathbf{E}_1$, and $\mathbf{E}_2$ are the known $m^2 \times m^2$ operational matrices for Haar wavelet functions, $\frac{\partial \mathbf{V}}{\partial x_1}(x_1,0)$, and $\frac{\partial \mathbf{V}}{\partial x_2}(0, x_2)$ are unknown values of the initial condition.

We specify the matrix form for $\frac{\partial \mathbf{V}}{\partial x_1}(x_1,0)$ and $\frac{\partial \mathbf{V}}{\partial x_2}(0, x_2)$ in solving the GHJB equation.

Let

$$s(x_1) = \sum_{i=0}^{m-1} b_i h_i(x_1) \tag{4.23}$$

and

$$g(x_2) = \sum_{j=0}^{m-1} a_j h_j(x_2), \tag{4.24}$$

where b_i and a_j are the Haar coefficients of $s(x_1)$ and $g(x_2)$ respectively. Therefore,

$$s(x_1)g(x_2) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} b_i a_j h_i(x_1) h_j(x_2). \tag{4.25}$$

Eqn. (4.25) implies that a new form of coefficient matrix can be determined for separable functions by multiplying the Haar coefficients b_i and a_j to yield $c_{ij} = b_i a_j$. In matrix form, the coefficient matrix, $\mathbf{C}_m$ for separable functions can be decomposed as

$$\mathbf{C}_m = \begin{bmatrix} b_0 a_0 & b_0 a_1 & \cdots & b_0 a_{m-1} \\ b_1 a_0 & b_1 a_1 & \cdots & b_1 a_{m-1} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m-1} a_0 & b_{m-1} a_1 & \cdots & b_{m-1} a_{m-1} \end{bmatrix}$$

$$= \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{bmatrix} \begin{bmatrix} a_0 & a_1 & \cdots & a_{m-1} \end{bmatrix}$$

$$= \mathbf{b}_m \, \mathbf{a}_m^{\mathbf{T}}, \qquad (4.26)$$

where $\mathbf{a}_m$ and $\mathbf{b}_m$ are column vectors from Eqn. (4.23) and (4.24), respectively.

Eqn. (4.25) can be written in compact form as

$$\mathbf{V}(x_1, x_2) = \mathbf{H}^{\mathbf{T}}(x_1)\,\mathbf{b}_m \mathbf{a}_m^{\mathbf{T}}\,\mathbf{H}(x_2)\,. \qquad (4.27)$$

The concept of separable functions can be utilized to determine the coefficient matrix $\mathbf{C}_m$ for the cases listed below

a) If the function is independent of x_2, then $\mathbf{V}(x_1, x_2) = s(x_1)$. Using Eqn. (4.27), we can express the function as $\mathbf{V}(x_1, x_2) = s(x_1)\,\mathbf{.}1$. Then $s(x_1) = \mathbf{b}_m^{\mathbf{T}}\mathbf{H}(x_1)$ and

$$g(x_2) = 1 = h_0(x_2)$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} h_0(x_2) \\ h_1(x_2) \\ \vdots \\ h_{m-1}(x_2) \end{bmatrix}$$

$$= \theta^{\mathbf{T}}\mathbf{H}(x_2)\,.$$

Therefore

$$\mathbf{V}(x_1, x_2) = s(x_1)\,\mathbf{.}1$$

$$= \mathbf{H}^{\mathbf{T}}(x_1)\,\mathbf{b}_m \theta_m^{\mathbf{T}}\,\mathbf{H}(x_2)\,. \qquad (4.28)$$

Then,

$$\mathbf{C}_m = \mathbf{b}_m \theta_m^{\mathbf{T}}$$

$$= \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\mathbf{C}_m = \begin{bmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ b_{m-1} & 0 & \cdots & 0 \end{bmatrix}.$$

The first column of $\mathbf{C}_m$ alone is nonzero.

b) If the function is independent of x_1, then $\mathbf{V}(x_1, x_2) = g(x_2)$. Using Eqn. (4.27), we can express the function as $\mathbf{V}(x_1, x_2) = 1.\, g(x_2)$. Then $s(x_1) = 1 = \theta^{\mathbf{T}}\mathbf{H}(x_1)$ and $g(x_2) = \mathbf{a}_m^{\mathbf{T}}\mathbf{H}(x_2)$. From these yield

$$\mathbf{V}(x_1, x_2) = 1.\, g(x_2)$$

$$= \mathbf{H}^{\mathbf{T}}(x_1)\,\theta_m\, \mathbf{a}_m^{\mathbf{T}}\, \mathbf{H}(x_2). \tag{4.29}$$

Then,

$$\mathbf{C}_m = \theta_m\, \mathbf{a}_m^{\mathbf{T}}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} a_0 & a_1 & \cdots & a_{m-1} \end{bmatrix}$$

$$\mathbf{C}_m = \begin{bmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The first row of $\mathbf{C}_m$ alone is nonzero.

By utilizing Eqns. (4.28) and (4.29), we can express the terms of the initial conditions in Haar wavelet approximation as follows:

$$\frac{\partial \mathbf{V}}{\partial x_1}(x_1,0) = \mathbf{H}^{\mathbf{T}}(x_1)\,\alpha_m \theta_m^{\mathbf{T}}\,\mathbf{H}(x_2) \tag{4.30}$$

and

$$\frac{\partial \mathbf{V}}{\partial x_2}(0,x_2) = \mathbf{H}^{\mathbf{T}}(x_1)\,\theta_m \beta_m^{\mathbf{T}}\,\mathbf{H}(x_2)\,. \tag{4.31}$$

When Eqn. (3.70) is applied, then Eqns. (4.30) and (4.31) can be written in the following form:

$$\frac{\partial \mathbf{V}}{\partial x_1}(x_1,0) = vec^{\mathbf{T}}(\,\alpha_m \theta_m^{\mathbf{T}})\mathbf{H}(x_1,x_2) \tag{4.32}$$

and

$$\frac{\partial \mathbf{V}}{\partial x_2}(0,x_2) = vec^{\mathbf{T}}(\,\theta_m \beta_m^{\mathbf{T}})\mathbf{H}(x_1,x_2)\,, \tag{4.33}$$

where $vec^{\mathbf{T}}(\alpha\theta^{\mathbf{T}}) = [\alpha_0 \quad \alpha_1 \;\cdots\; \alpha_{m-1} \quad 0 \quad 0 \cdots 0 \cdots 0 \cdots 0\,]$ and $vec^{\mathbf{T}}(\theta\beta^{\mathbf{T}}) = [\beta_0 \quad 0 \cdots 0 \quad \beta_1 \quad 0 \;\cdots\; 0 \quad \cdots\; \beta_{m-1} \;\cdots\; 0\,]$ are $1\times m^2$ in size.

As originally noted by Bellman, (1957), a general fundamental problem with Hamilton-Jacobi based methods is that they all suffer from the curse of dimensionality. In other words, the amount of computation and or memory required to implement the method increases exponentially with the size of the state space. An ideal approximation method is not subject to any of the disadvantages of the methods discussed in previous literatures. In particular, such a method is characterized as follows:

- Low run-time computation and memory requirements.
- Effective handling of the dimensionality problem.
- Guaranteeing that the approximation error approaches zero as the order of approximation increases.

Dimensionality remains an issue, although its effect on the computational requirements can be effectively eliminated. The number of terms in m^{th} order truncation of a complete basis increases exponentially with the size of the state.

To overcome the aforementioned problem, we suggest the following technique to reduce the number of unknown coefficients that are determined for solving the GHJB equation in the following section. This technique uses the properties of vec transform (see A.16). Thus, Eqns. (4.32) and (4.33) can be written as follows:

$$\frac{\partial \mathbf{V}}{\partial x_1}(x_1,0) = vec^{\mathbf{T}}(\alpha)(\mathbf{I} \otimes \theta^{\mathbf{T}})\mathbf{H}(x_1,x_2) \tag{4.34}$$

and

$$\frac{\partial \mathbf{V}}{\partial x_2}(0,x_2) = vec^{\mathbf{T}}(\beta)(\theta^{\mathbf{T}} \otimes \mathbf{I})\mathbf{H}(x_1,x_2)\,, \tag{4.35}$$

where $vec^{\mathbf{T}}(\alpha) = [\alpha_0 \quad \alpha_1 \quad \cdots \quad \alpha_{m-1}]$ and $vec^{\mathbf{T}}(\beta) = [\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{m-1}]$ are m unknown coefficients.

Finally, by substituting Eqns. (4.34) and (4.35) into Eqns. (4.21) and (4.22), respectively, we obtain

$$\frac{\partial \mathbf{V}}{\partial x_1} = vec^{\mathbf{T}}(\omega)(Q_2 - \tau\,\mathbf{E_2})\mathbf{H}(x_1,x_2) + vec^{\mathbf{T}}(\alpha)(\mathbf{I} \otimes \theta^{\mathbf{T}})\mathbf{H}(x_1,x_2) \tag{4.36}$$

$$\frac{\partial \mathbf{V}}{\partial x_2} = vec^{\mathbf{T}}(\omega)(Q_1 - \tau\,\mathbf{E_1})\mathbf{H}(x_1,x_2) + vec^{\mathbf{T}}(\beta)(\theta^{\mathbf{T}} \otimes \mathbf{I})\mathbf{H}(x_1,x_2)\,. \tag{4.37}$$

4.6 Successive Haar Wavelet Collocation Method

The following section describes the successive Haar wavelet collocation method (SHWCM) used to obtain the two-dimensional numerical solution to the HJB equation. An approximate solution to the GHJB equation has been generated in every step of this algorithm.

Eqn. (4.15) has been completely identified; that is, $\frac{\partial \mathrm{V}^{(i)}(x)}{\partial x}$, $\mathrm{V}^{(i)}(x)$, and $u^{(i)}(x)$ can all be approximately expressed in term of Haar wavelets. $\mathrm{V}^{(i)}(x)$, and $u^{(i)}(x)$ approach the optimal solutions $\mathrm{V}^{*}(x)$ and $u^{*}(x)$, respectively, as $i \to \infty$.

We consider the following two-dimensional optimal feedback control problem

$$\min \mathbf{V}(\mathbf{x}_0\,, u) = \int_0^\infty (x^T \mathbf{Q}\, x + u^T \mathbf{R}\, u)\, dt \tag{4.38}$$

subject to the dynamics

$$\dot{x} = f(x) \;+ g(x)\, u(x), \qquad\qquad x(0) = \mathbf{x_0}, \tag{4.39}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; $f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$; $g(x) = \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix}$; and $u : \Omega \to \Re$.

Without loss of generality, $\Omega = [-\tau, \tau) \times [-\tau, \tau)$ is selected as the domain of attraction for convenience. The following equations express the pair that consists of the GHJB equation and the control law:

$$\frac{\partial \mathbf{V}^{(i)\mathbf{T}}(x)}{\partial x}\left(f(x) + g(x) u^{(i)}(x)\right) + x^{\mathbf{T}} \mathbf{Q}\, x + u^{(i)\mathbf{T}} \mathbf{R}\, u^{(i)} \;= 0 \tag{4.40}$$

with initial condition $\mathbf{V}^{(i)}(0) = 0$ and

$$u^{(i+1)}(x) = -\frac{1}{2}\mathbf{R}^{\mathbf{-1}} g^{\mathbf{T}}(x)\frac{\partial \mathbf{V}^{(i)}(x)}{\partial x} . \tag{4.41}$$

If $u^{(0)}(x)$ is initially a stabilizing control in Eqn. (4.39), then the solution to GHJB equation affiliated with $u^{(0)}(x)$, which is derived from Eqn. (4.40), becomes a Lyapunov function for the system. This function is equalized to the cost associated with $u^{(0)}(x)$ as follows:

$$\frac{\partial \mathbf{V}^{(0)\mathbf{T}}(x)}{\partial x}\left(f(x) + g(x)u^{(0)}(x)\right) + x^{\mathbf{T}}\mathbf{Q}\,x + u^{(0)\mathbf{T}}\mathbf{R}\,u^{(0)} = 0 . \tag{4.42}$$

According to Eqn. (3.70), the function approximation equations for $f_1(x) + g_1(x)u^{(0)}(x)$, $f_2(x) + g_2(x)u^{(0)}(x)$, and $x^{\mathbf{T}}\mathbf{Q}\,x + u^{(0)\mathbf{T}}(x)\mathbf{R}\,u^{(0)}(x)$ can be written as

$$f_1(x) + g_1(x)u^{(0)\mathbf{T}}(x) = vec^{\mathbf{T}}(\lambda)\mathbf{H}(x_1, x_2) , \tag{4.43}$$

$$f_2(x) + g_2(x)u^{(0)\mathbf{T}}(x) = vec^{\mathbf{T}}(\mu)\mathbf{H}(x_1, x_2) , \tag{4.44}$$

$$x^{\mathbf{T}}\mathbf{Q}\,x + u^{(0)\mathbf{T}}(x)\mathbf{R}\,u^{(0)}(x) = vec^{\mathbf{T}}(k)\mathbf{H}(x_1, x_2) , \tag{4.45}$$

where $vec^{\mathbf{T}}(\lambda)$, $vec^{\mathbf{T}}(\mu)$, and $vec^{\mathbf{T}}(k)$ are $1\times m^2$ the known coefficient vectors for the Haar wavelet functions that can be calculated from Eqn. (3.71) and $\mathbf{H}(x_1, x_2)$ is $m^2 \times 1$ vector of the known Haar wavelet basis of two dimension functions. Considering that Haar functions are impossible to differentiate because they are not continuous on the domain, and that Eqn. (4.40) involves only first-order derivatives of $\mathbf{V}$, we assume that, the second-order partial derivative of $\mathbf{V}$ exists as follows:

$$\frac{\partial^2 \mathbf{V}(x)}{\partial x_1 \partial x_2} = vec^{\mathbf{T}}(\omega)\mathbf{H}(x_1, x_2) , \tag{4.46}$$

for some coefficient vector $vec^{\mathbf{T}}(\omega)$.

Under the assumption that

$$\frac{\partial^2 \mathbf{V}(x)}{\partial x_1 \partial x_2} = \frac{\partial^2 \mathbf{V}(x)}{\partial x_2 \partial x_1}, \tag{4.47}$$

the first-order partial derivative can be obtained by integrating Eqn. (4.46) on interval $[-\tau, \tau)$ with respect to x_1 and x_2, respectively, as mentioned in Subsection 4.5, as

$$\frac{\partial \mathbf{V}(x_1, x_2)}{\partial x_1} = \omega^{\mathbf{T}}(Q_2 - \tau\, E_{\mathbf{1}})\mathbf{H}(x_1, x_2) + \frac{\partial \mathbf{V}(x_1, 0)}{\partial x_1} \tag{4.48}$$

$$\frac{\partial \mathbf{V}(x_1, x_2)}{\partial x_2} = \omega^{\mathbf{T}}(Q_1 - \tau\, E_{\mathbf{2}})\mathbf{H}(x_1, x_2) + \frac{\partial \mathbf{V}(0, x_2)}{\partial x_1}, \tag{4.49}$$

where Q_1, Q_2, $\mathbf{E}_1$, and $\mathbf{E}_2$ are the $m^2 \times m^2$ known operational matrices for the Haar wavelet functions, and $\frac{\partial \mathbf{V}(x_1, 0)}{\partial x_1}$ and $\frac{\partial \mathbf{V}(0, x_2)}{\partial x_2}$ are unknown initial condition values that are formulated in Eqns. (4.34) and (4.35), which when substituted into Eqns. (4.48) and (4.49), respectively, can be written in Haar wavelet functions as

$$\frac{\partial \mathbf{V}}{\partial x_1} = vec^{\mathbf{T}}(\omega)(Q_2 - \tau\, \mathbf{E}_{\mathbf{2}})\mathbf{H}(x_1, x_2) + vec^{\mathbf{T}}(\alpha)(\mathbf{I} \otimes \theta^{\mathbf{T}})\mathbf{H}(x_1, x_2) \tag{4.50}$$

$$\frac{\partial \mathbf{V}}{\partial x_2} = vec^{\mathbf{T}}(\omega)(Q_1 - \tau\, \mathbf{E}_{\mathbf{1}})\mathbf{H}(x_1, x_2) + vec^{\mathbf{T}}(\beta)(\theta^{\mathbf{T}} \otimes \mathbf{I})\mathbf{H}(x_1, x_2). \tag{4.51}$$

Now, substituting Eqns. (4.43), (4.44), (4.45), (4.50), and (4.51) into Eqn. (4.40), where the term $\frac{\partial \mathbf{V}^{(i)\mathbf{T}}(x)}{\partial x}$ in Eqn. (4.40) can be simplified as $\frac{\partial \mathbf{V}^{(i)\mathbf{T}}(x)}{\partial x} = \begin{pmatrix} \frac{\partial \mathbf{V}}{\partial x_1} & \frac{\partial \mathbf{V}}{\partial x_2} \end{pmatrix}$, we have

$$vec^{\mathbf{T}}(\lambda)\mathbf{H}(x_1,x_2)\left(vec^{\mathbf{T}}(\omega)(Q_2 - \tau\,\mathbf{E_2})\mathbf{H}(x_1,x_2) + vec^{\mathbf{T}}(\alpha)(\mathbf{I}\otimes\theta^{\mathbf{T}})\mathbf{H}(x_1,x_2)\right)$$

$$+ vec^{\mathbf{T}}(\mu)\mathbf{H}(x_1,x_2)\left(vec^{\mathbf{T}}(\omega)(Q_1 - \tau\,\mathbf{E_1})\mathbf{H}(x_1,x_2) + vec^{\mathbf{T}}(\beta)(\theta^{\mathbf{T}}\otimes\mathbf{I})\mathbf{H}(x_1,x_2)\right) \quad (4.52)$$

$$+ vec^{\mathbf{T}}(k)\mathbf{H}(x_1,x_2) = 0\,.$$

Rearranging the Eqn. (4.52), we yield

$$\begin{aligned} & vec^{\mathbf{T}}(\lambda)\mathbf{H}(x_1,x_2)vec^{\mathbf{T}}(\omega)(Q_2 - \tau\,\mathbf{E_2})\mathbf{H}(x_1,x_2) \\ & + vec^{\mathbf{T}}(\lambda)\mathbf{H}(x_1,x_2)vec^{\mathbf{T}}(\alpha)(\mathbf{I}\otimes\theta^{\mathbf{T}})\mathbf{H}(x_1,x_2) \\ & + vec^{\mathbf{T}}(\mu)\mathbf{H}(x_1,x_2)vec^{\mathbf{T}}(\omega)(Q_1 - \tau\,\mathbf{E_1})\mathbf{H}(x_1,x_2) \\ & + vec^{\mathbf{T}}(\mu)\mathbf{H}(x_1,x_2)vec^{\mathbf{T}}(\beta)(\theta^{\mathbf{T}}\otimes\mathbf{I})\mathbf{H}(x_1,x_2) \\ & = -vec^{\mathbf{T}}(k)\mathbf{H}(x_1,x_2) \end{aligned} \quad (4.53)$$

Applying the operational matrix of the product of two dimensions Haar wavelet vectors of the product of two functions in Eqn. (3.113) on (453) we obtain

$$\begin{aligned} & vec^{\mathbf{T}}(\omega)(Q_2 - \tau\,\mathbf{E_2})\mathbf{H}(x_1,x_2)\mathbf{H}^{\mathbf{T}}(x_1,x_2)vec(\lambda) \\ & + vec^{\mathbf{T}}(\alpha)(\mathbf{I}\otimes\theta^{\mathbf{T}})\mathbf{H}(x_1,x_2)\mathbf{H}^{\mathbf{T}}(x_1,x_2)vec(\lambda) \\ & + vec^{\mathbf{T}}(\omega)(Q_1 - \tau\,\mathbf{E_1})\mathbf{H}(x_1,x_2)\mathbf{H}^{\mathbf{T}}(x_1,x_2)vec(\mu) \\ & + vec^{\mathbf{T}}(\beta)(\theta^{\mathbf{T}}\otimes\mathbf{I})\mathbf{H}(x_1,x_2)\mathbf{H}^{\mathbf{T}}(x_1,x_2)vec(\mu) \\ & = -vec^{\mathbf{T}}(k)\mathbf{H}(x_1,x_2)\ . \end{aligned} \quad (4.54)$$

Thus,

$$vec^{\mathbf{T}}(\omega)(Q_2 - \tau\,\mathbf{E_2})\mathbf{N}(\lambda)\mathbf{H}(x_1,x_2) + vec^{\mathbf{T}}(\alpha)(\mathbf{I}\otimes\theta^{\mathbf{T}})\mathbf{N}(\lambda)\mathbf{H}(x_1,x_2)$$

$$+ vec^{\mathbf{T}}(\omega)(Q_1 - \tau\,\mathbf{E_1})\mathbf{N}(\mu)\mathbf{H}(x_1,x_2) + vec^{\mathbf{T}}(\beta)(\theta^{\mathbf{T}}\otimes\mathbf{I})\mathbf{N}(\mu)\mathbf{H}(x_1,x_2) \qquad (4.55)$$

$$= -vec^{\mathbf{T}}(k)\mathbf{H}(x_1,x_2)\,.$$

The vector of two Haar wavelet basis functions $\mathbf{H}(x_1,x_2)$ size $m^2\times 1$ in Eqn. (4.55) is replaced with the matrix of $\overline{\mathbf{H}}$ Haar wavelet collocation points of size $m^2\times m^2$ that is described in Section 3.8.

Then, both sides of Eqn. (4.55) are multiplied with the matrix inverse $\overline{\mathbf{H}}^{\mathbf{-1}}$ to remove the term of $\overline{\mathbf{H}}$ and simplifying Eqn. (4.55). Thus, we have

$$\begin{aligned} & vec^{\mathbf{T}}(\omega)\big((Q_2 - \tau\,\mathbf{E_2})\mathbf{N}(\lambda) + (Q_1 - \tau\,\mathbf{E_1})\mathbf{N}(\mu)\big) + vec^{\mathbf{T}}(\alpha)(\mathbf{I}\otimes\theta^{\mathbf{T}})\mathbf{N}(\lambda) \\ & + vec^{\mathbf{T}}(\omega) + vec^{\mathbf{T}}(\beta)(\theta^{\mathbf{T}}\otimes\mathbf{I})\mathbf{N}(\mu) = -vec^{\mathbf{T}}(k) \end{aligned} \qquad (4.56)$$

Next, Eqn. (4.56) is transformed into a standard system of linear equations as follows

$$\begin{bmatrix} vec^{\mathbf{T}}(\omega) & vec^{\mathbf{T}}(\alpha) & vec^{\mathbf{T}}(\beta) \end{bmatrix} \begin{bmatrix} \big((Q_2 - \tau\,\mathbf{E_2})\mathbf{N}(\lambda) + (Q_1 - \tau\,\mathbf{E_1})\mathbf{N}(\mu)\big) \\ (\mathbf{I}\otimes\theta^{\mathbf{T}})\mathbf{N}(\lambda) \\ (\theta^{\mathbf{T}}\otimes\mathbf{I})\mathbf{N}(\mu) \end{bmatrix} = \begin{bmatrix} -vec^{\mathbf{T}}(k) \end{bmatrix}$$

(4.57)

Eqn. (4.57) is a system of underdetermined linear equations with m^2 equations and (m^2+2m) unknown variables that can solve for the unknown vectors $vec^{\mathbf{T}}(\omega)$, $vec^{\mathbf{T}}(\alpha)$, and $vec^{\mathbf{T}}(\beta)$ by using the Moore-Penrose Pseudoinvers solver (Courrieu, 2005), such as pinv() in MATLAB (Hanselman and Littlefield, 2005).

An underdetermined equation is expected, because the Lyapunov function is not unique. The Moore-Penrose solution is the particular solution whose vector 2-norm is minimal.

Using the solution of the GHJB Eqn. (4.40), once we obtain the solution to the unknown parameters $vec^{\mathbf{T}}(\omega)$, $vec^{\mathbf{T}}(\alpha)$ and $vec^{\mathbf{T}}(\beta)$, we substitute these parameters into Eqns. (4.50) and (4.51) to determine $\frac{\partial \mathbf{V}^{(0)}}{\partial x_1}$ and $\frac{\partial \mathbf{V}^{(0)}}{\partial x_2}$ as follows:

$$\frac{\partial \mathbf{V}^{(0)}}{\partial x_1} = vec^{\mathbf{T}}(\omega)(Q_2 - \tau\, \mathbf{E_2})\mathbf{H}(x_1, x_2) + vec^{\mathbf{T}}(\alpha)(\mathbf{I} \otimes \theta^{\mathbf{T}})\mathbf{H}(x_1, x_2) \tag{4.58}$$

$$\frac{\partial \mathbf{V}^{(0)}}{\partial x_2} = vec^{\mathbf{T}}(\omega)(Q_1 - \tau\, \mathbf{E_1})\mathbf{H}(x_1, x_2) + vec^{\mathbf{T}}(\beta)(\theta^{\mathbf{T}} \otimes \mathbf{I})\mathbf{H}(x_1, x_2)\,, \tag{4.59}$$

which can be used to construct a feedback control law $u^{(1)}$ using Eqn. (4.41). Thus, we have

$$u^{(1)}(x) = -\frac{1}{2}\mathbf{R}^{\mathbf{-1}}\left(g_1(x_1, x_2) \quad g_2(x_1, x_2)\right)\begin{pmatrix} \dfrac{\partial \mathbf{V}^{(0)}}{\partial x_1} \\ \dfrac{\partial \mathbf{V}^{(0)}}{\partial x_2} \end{pmatrix} \tag{4.60}$$

which improves the efficiency of $u^{(0)}$. The repetition of this process results in a successive approximation algorithm (SHWCM) that uniformly approximates the Hamilton-Jacobi-Bellman equation.

Let $\mathbf{V}^*(x_1, x_2)$ be the solution to the equation GHJB $(\mathbf{V}^*, u^*) = 0$, then $\mathbf{V}^*$ is the optimal cost. Given that the optimal control is unique, u^* must be the optimal control. Thus, to determine the Lyapunov function $\mathbf{V}^*(x_1, x_2)$ from the solution of linear system equations that satisfy the HJB equation in Eqn. (4.14), we propose a new formula that depends only on the initial and final points and not on the path followed. We calculate the Lyapunov function by using the variable gradient method (Slotine and Li, 1991) to integrate parallel to the axes; this technique can be illustrated as follows:

Given that

$$\mathbf{V}(x) = \int_0^x \nabla \mathbf{V}^{\mathbf{T}} dx \tag{4.61}$$

the Lyapunov function of two dimensions, is given by Slotine and Li (1991)

$$\mathbf{V}^*(x_1, x_2) = \int_0^{x_1} \frac{\partial \mathbf{V}^*}{\partial x_1}(x_1, 0) dx_1 + \int_0^{x_2} \frac{\partial \mathbf{V}^*}{\partial x_2}(x_1, x_2) dx_2 \tag{4.62}$$

where

$$\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1, 0) = \left(vec^{\mathbf{T}}(\omega)(Q_2 - \tau\, \mathbf{E_2}) + vec^{\mathbf{T}}(\alpha\theta^{\mathbf{T}}) \right) \mathbf{H}(x_1, 0) \tag{4.63}$$

and

$$\frac{\partial \mathbf{V}^*}{\partial x_2}(x_1, x_2) = \left(vec^{\mathbf{T}}(\omega)(Q_1 - \tau\, \mathbf{E_1}) + vec^{\mathbf{T}}(\theta\beta^{\mathbf{T}}) \right) \mathbf{H}(x_1, x_2) \tag{4.64}$$

Eqns. (4.63) and (4.64) are known functions that are obtained from the final iteration of the successive GHJB equation in algorithm 4.3.1 that satisfies the stopping criteria for feedback control law when $u^*(x)$ is optimum.

Let

$$vec(\delta) = vec^{\mathbf{T}}(\omega)(Q_2 - \tau\, \mathbf{E_2}) + vec^{\mathbf{T}}(\alpha\theta^{\mathbf{T}}) \ .$$

Then

$$\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1, 0) = vec^{\mathbf{T}}(\delta)\, \mathbf{H}(x_1, 0)\,, \tag{4.65}$$

where $vec(\delta) = \left[\delta_{00} \ \ \delta_{10} \ \cdots \ \delta_{m-10} \ \delta_{01} \ \ \delta_{11} \ \cdots \ \delta_{m-11} \ \cdots \delta_{0m-1} \ \delta_{1m-1} \cdots \delta_{m-1m-1}\right]^{\mathbf{T}}$ is $m^2 \times 1$ constant values corresponding to $vec^{\mathbf{T}}(\omega)(Q_2 - \tau\,\mathbf{E_2}) + vec^{\mathbf{T}}(\alpha\theta^{\mathbf{T}})$. Therefore, Eqn. (4.65) can be rewritten as

$$\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1,0) = vec^{\mathbf{T}}(\eta\theta^{\mathbf{T}})\,\mathbf{H}(x_1,x_2) \tag{4.66}$$

To prove that statement, we start from Eqn. (4.65), which can be rewritten using Eqn. (3.66) as

$$\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1,0) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\delta_{ij}h_i(x_1)h_j(0) \tag{4.67}$$

$$\begin{aligned}\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1,0) = {} & \delta_{00}h_0(x_1)h_0(0) + \delta_{01}h_0(x_1)h_1(0) + \cdots + \delta_{0m-1}h_0(x_1)h_{m-1}(0) \\ & + \delta_{10}h_1(x_1)h_0(0) + \delta_{11}h_1(x_1)h_1(0) + \cdots + \delta_{1m-1}h_1(x_1)h_{m-1}(0) \\ & + \ \cdots \\ & + \delta_{m-10}h_m(x_1)h_0(0) + \delta_{m=11}h_m(x_1)h_1(0) + \cdots + \delta_{m-1m-1}h_{m-1}(x_1)h_{m-1}(0)\end{aligned} \tag{4.68}$$

Substituting the values of the collocation point $x_2 = 0$, $\mathbf{h}(0) = \left[h_0(0) \quad h_1(0) \ \cdots \ h_{m-1}(0)\right]$ into Eqn. (4.68), we have

$$\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1,0)=\delta_{00}h_0(x_1)\gamma_{0(\frac{m}{2}+1)}+\delta_{01}h_0(x_1)\gamma_{1(\frac{m}{2}+1)}+\cdots+\delta_{0m-1}h_0(x_1)\gamma_{m-1(\frac{m}{2}+1)}$$

$$+\delta_{10}h_1(x_1)\gamma_{0(\frac{m}{2}+1)}+\delta_{11}h_1(x_1)\gamma_{1(\frac{m}{2}+1)}+\cdots+\delta_{1m-1}h_1(x_1)\gamma_{m-1(\frac{m}{2}+1)}$$

$$+\ \cdots$$

$$+\delta_{m-10}h_{m-1}(x_1)\gamma_{0(\frac{m}{2}+1)}+\delta_{m-11}h_{m-1}(x_1)\gamma_{1(\frac{m}{2}+1)}+\cdots+\delta_{m-1m-1}h_{m-1}(x_1)\gamma_{m-1(\frac{m}{2}+1)} \tag{4.69}$$

where $\gamma_{0(\frac{m}{2}+1)},\gamma_{1(\frac{m}{2}+1)},\ \cdots\ ,\ \gamma_{m-1(\frac{m}{2}+1)}$ are the elements of the $(\frac{m}{2}+1)$ column in the Haar wavelets collocation points matrix of one dimension.

Now, Eqn. (4.69) can be rewritten as

$$\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1,0)=\begin{bmatrix}\delta_{00} & \delta_{01} & \cdots & \delta_{0m-1}\end{bmatrix}\begin{bmatrix}\gamma_{0(\frac{m}{2}+1)}\\ \gamma_{1(\frac{m}{2}+1)}\\ \vdots\\ \gamma_{m-1(\frac{m}{2}+1)}\end{bmatrix}h_0(x_1)+\begin{bmatrix}\delta_{10} & \delta_{11} & \cdots & \delta_{1m-1}\end{bmatrix}\begin{bmatrix}\gamma_{0(\frac{m}{2}+1)}\\ \gamma_{1(\frac{m}{2}+1)}\\ \vdots\\ \gamma_{m-1(\frac{m}{2}+1)}\end{bmatrix}h_1(x_1)$$

$$+\ \cdots\ +\begin{bmatrix}\delta_{m-10} & \delta_{m-11} & \cdots & \delta_{m-1m-1}\end{bmatrix}\begin{bmatrix}\gamma_{0(\frac{m}{2}+1)}\\ \gamma_{1(\frac{m}{2}+1)}\\ \vdots\\ \gamma_{m-1(\frac{m}{2}+1)}\end{bmatrix}h_{m-1}(x_1) \tag{4.70}$$

Eqn. (4.70) can be rewritten into a compact form as

$$\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1,0)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\delta_{ij}\gamma_j h_i(x_1) \tag{4.71}$$

To simplify Eqn. (4.71),

let $\eta_i = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\delta_{ij}\gamma_j$ be the coefficient values for $h_i(x_1)$ for $i = 0,1,2,\cdots,m-1$

Then, Eqn. (4.71) can be rewritten as

$$\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1,0) = \eta_m^{\mathbf{T}}\mathbf{H}(x_1) \tag{4.72}$$

Eqn. (4.72) is independent of x_2. Therefore, by utilizing Eqn. (4.32), it can be rewritten as

$$\frac{\partial \mathbf{V}^*}{\partial x_1}(x_1,0) = vec^{\mathbf{T}}(\eta\,\theta_m^{\mathbf{T}})\mathbf{H}(x_1,x_2)\,. \tag{4.73}$$

Finally, by substituting Eqn. (4.73) and (4.64) into Eqn. (4.62), we obtain

$$\mathbf{V}^*(x_1,x_2) = \int_0^{x_1} vec(\eta\,\theta_m^{\mathbf{T}})\mathbf{H}(x_1,x_2)dx_1 + \int_0^{x_2}\left(vec^{\mathbf{T}}(\omega)(Q_1 - \tau\,\mathbf{E_1}) + vec^{\mathbf{T}}(\,\theta\beta^{\mathbf{T}}\,)\right)\mathbf{H}(x_1,x_2)dx_2 \tag{4.74}$$

Integrate Eqn. (4.74) by using Eqns. (3.87) and (3.102) on interval $[-\tau,\tau)\times[-\tau,\tau)$, we obtain

$$\begin{aligned}\mathbf{V}^*(x_1,x_2) &= vec^{\mathbf{T}}(\eta\,\theta_m^{\mathbf{T}})(Q_1 - \tau\,\mathbf{E_1})\mathbf{H}(x_1,x_2)\\ &\quad + \left(vec^{\mathbf{T}}(\omega)(Q_1 - \tau\,\mathbf{E_1})(Q_2 - \tau\,\mathbf{E_2}) + vec^{\mathbf{T}}(\,\theta\beta^{\mathbf{T}}\,)(Q_2 - \tau\,\mathbf{E_2})\right)\mathbf{H}(x_1,x_2)\\ &\quad - \mathbf{V}(0,0)\end{aligned} \tag{4.75}$$

where $\mathbf{V}(0,0) = 0$.

4.7 Numerical Results

This section demonstrates the usefulness, efficiency, and accuracy of the Successive Haar Wavelet Collocation Method (SHWCM). For this purpose, we applied the proposed method to solve linear and nonlinear quadratic optimal control problems with infinite time horizon. In particular, five different examples are consider, which are presented in this section. A linear optimal control example is considered first, followed by four nonlinear optimal control examples with one control variables for the three examples and two control variables for one example. All computations were carried out using of MATLAB.

4.7.1 Example 1

Consider the following linear quadratic regulator (LQR):

$$J = \int_0^\infty (x_1^2(t) + u^2(t))\, dt \tag{4.76}$$

subject to

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{4.77}$$

To solve this problem, we take the initial stabilizing control $u^{(0)}(x) = -x_1 - x_2$. Tables (4.1) and (4.2) show the numerical results for $u^{(i)}$ and $\mathbf{V}^{(i)}$, respectively, when $m = 8$ and $x_1 = -\frac{1}{8}$. The iteration is terminated when the difference between two successive controls is less than $\varepsilon = 0.001$, that is, $\| u^{(i+1)} - u^{(i)} \| < \varepsilon$. Subsequently, to display the two dimensional plots, we fix the value for x_1 at $x_1\left(\frac{m}{2}\right) = -\frac{\tau}{m}$ and $x_2 \in [-1, 1)$.

Figure (4.3) shows that for this particular LQR problem, the usage of $m=16$ can adequately approximate the exact optimal feedback control $u^*(x)=-x_1-\sqrt{2}\,x_2$; however, to approximate the cost function we require a higher value of m as shown in Figure (4.4).

Table 4.1: Numerical results $u^{(i)}$ for Example 1 when $m=8$ and $x_1=-\frac{1}{8}$

x_2	$u^{(0)}$	$u^{(1)}$	$u^{(2)}$	$u^{(3)}$	$u^{(4)}$	u_{exact}
$-7/8$	1.0000	1.4463	1.3772	1.3786	1.3793	1.3624
$-5/8$	0.7500	1.0636	1.0114	1.0130	1.0136	1.0089
$-3/8$	0.5000	0.68889	0.6548	0.6548	0.6550	0.6553
$-1/8$	0.2500	0.3135	0.3027	0.3017	0.3015	0.3018
$1/8$	0	-0.0615	-0.0515	-0.0519	-0.0520	-0.0518
$3/8$	-0.2500	-0.4397	-0.4080	-0.4053	-0.4049	-0.4053
$5/8$	-0.5000	-0.8137	-0.7584	-0.7571	-0.7572	-0.7589
$7/8$	-0.7500	-1.1880	-1.1123	-1.1130	-1.1135	-1.1124

Table 4.2: Numerical results $\mathrm{V}^{(i)}$ for Example 1 when $m=8$ and $x_1=-\frac{1}{8}$

x_2	$\mathrm{V}^{(0)}$	$\mathrm{V}^{(1)}$	$\mathrm{V}^{(2)}$	$\mathrm{V}^{(3)}$	$\mathrm{V}_{\text{exact}}$
$-7/8$	0.7051	0.6709	0.6712	0.6714	0.6618
$-5/8$	0.3914	0.3723	0.3722	0.3723	0.3654
$-3/8$	0.1723	0.1640	0.1637	0.1637	0.1574
$-1/8$	0.0470	0.0444	0.0442	0.0441	0.0377
$1/8$	0.0155	0.0130	0.0130	0.0130	0.0065
$3/8$	0.0781	0.0704	0.0701	0.0701	0.0636
$5/8$	0.2348	0.2162	0.2154	0.2153	0.2091
$7/8$	0.4850	0.4500	0.4492	0.4492	0.4431

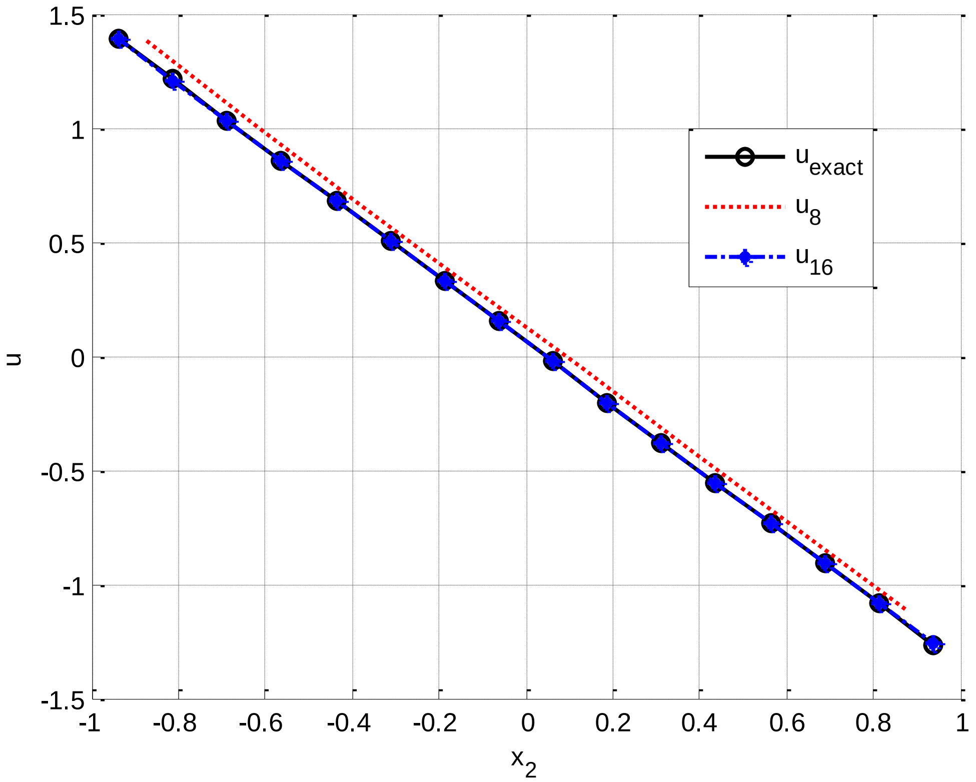


Figure 4.3: Optimal feedback control for Example 1 via the SHWCM with

$m = 8, 16$ **and** $x_1 = -\frac{1}{8}, \ -\frac{1}{16}$**, respectively**

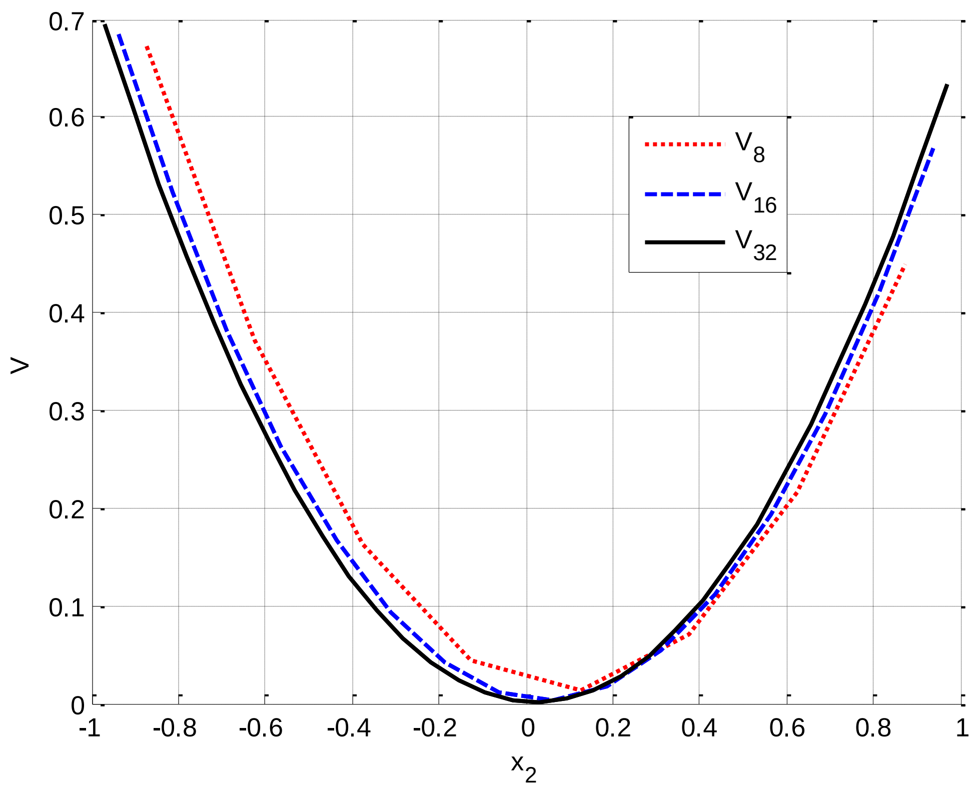


Figure 4.4: Value for cost function for Example 1 via the SHWCM with $m = 8, 16, 32$ and $x_1 = -\frac{1}{8}, -\frac{1}{16}, -\frac{1}{32}$, respectively

4.7.2 Example 2

Consider the following nonlinear one dimension example, as presented by Park and Tsiotras (2003):

$$J = \int_0^\infty (x^{\mathrm{T}} Q\, x + u^{\mathrm{T}} R\, u)dt \tag{4.78}$$

subject to

$$\dot{x} = f(x) + g(x)u \tag{4.79}$$

with the following data: $Q=1$, $R=1$ $f(x)=xe^{-x}$ and $g(x)=e^{-x}$.

For this example, the solution to the HJB equation can be found analytically to be

$\mathbf{V}^*(x)=2(1+\sqrt{2})(xe^{-x}-e^{x}+1)$ with the corresponding optimal control is $u^*(x)=-(1+\sqrt{2})x$. The SHWCM is applied with an initial stabilizing control of $u^{(0)}(x)=-2x$, and the iteration is terminated when the difference between two successive controls is less than $\varepsilon=0.0001$. The results are shown in Figures 4.5 and 4.6, which show the monotonic convergence of the value cost functions and the corresponding control to the optimal one when Haar wavelets resolution levels of $m=8,16$ are used.

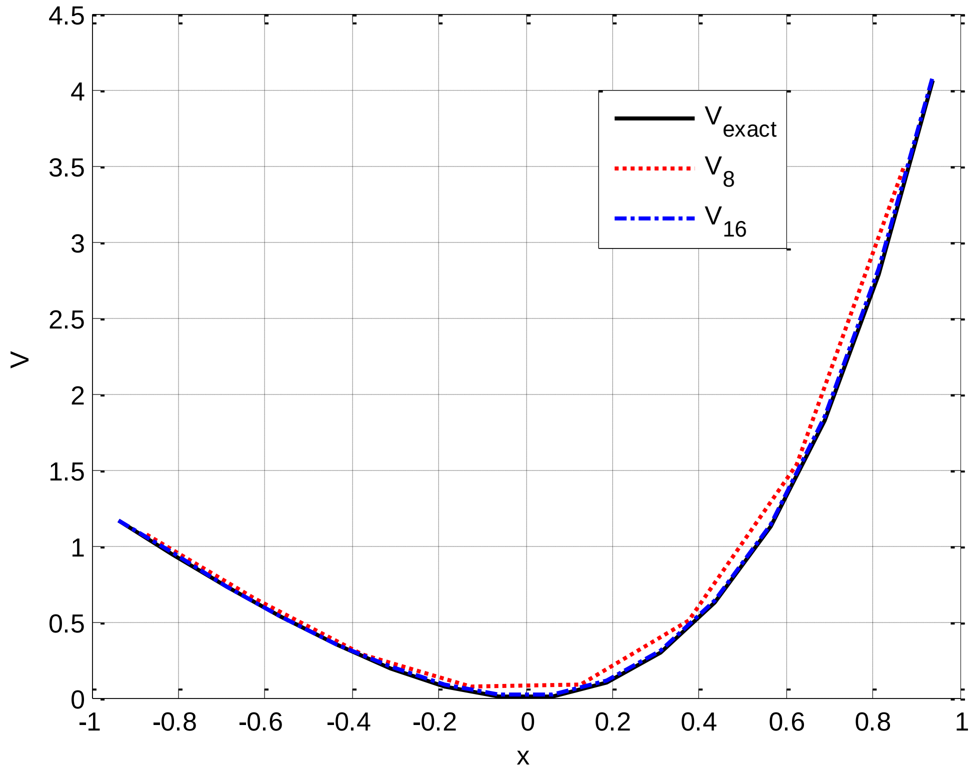


Figure 4.5: Value for cost function for Example 2 via the SHWCM with $m=8,16$

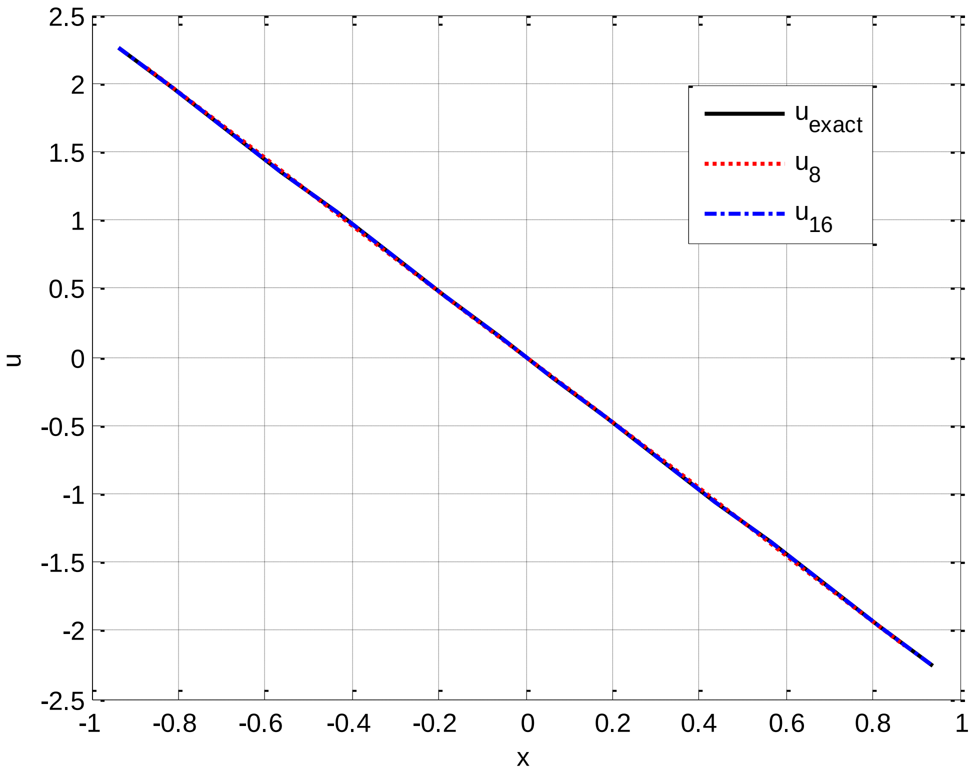


Figure 4.6: Optimal feedback control for Example 2 via the SHWCM with $m = 8, 16$

4.7.3 Example 3

Consider the following nonlinear optimal control problem (Curtis and Beard, 2001):

$$J = \int_0^\infty (x_2^2 + u^2)\, dt \tag{4.80}$$

subject to

$$\dot{x} = \begin{bmatrix} x_2 \\ -x_1\left(\dfrac{\pi}{2} + \tan^{-1}(5x_1)\right) - \dfrac{5x_1^2}{2(1+25x_1^2)} + 4x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u \tag{4.81}$$

The optimum solution for this problem is $u^*(x) = -3x_2$ and $\mathbf{V}^*(x) = x_1^2\left(\frac{\pi}{2} + \tan^{-1}(5x_1)\right) + x_2^2$. To solve this nonlinear optimal control problem, we started with an initial stabilizing control of $u^{(0)}(x) = -1.8x_2$. Figure (4.7) shows the approximate optimal feedback control law $u^*(x)$ for $m = 8$, 16, and 32. The graph for $m = 32$ overlaps with the exact optimal feedback control, and Figure (4.8) shows that the approximate cost function converges with the exact cost function as the resolution is increased.

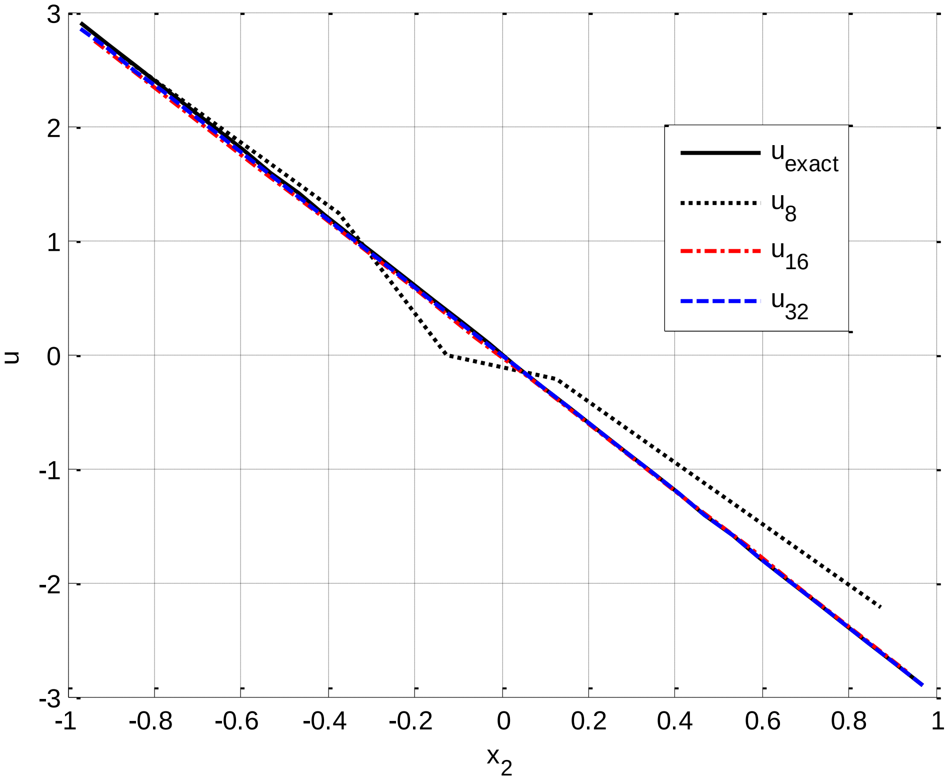


Figure 4.7: Optimal feedback control for Example 3 via the SHWCM with $m = 8, 16, 32$ and $x_1 = -\frac{1}{8}, -\frac{1}{16}, -\frac{1}{32}$, respectively

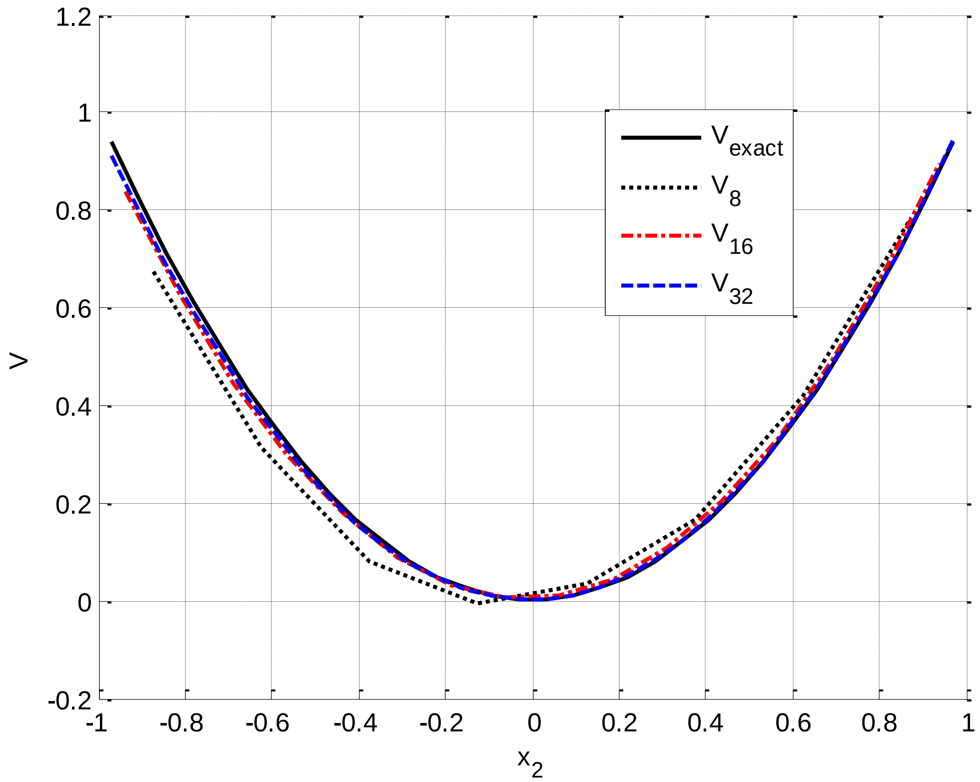


Figure 4.8: Value for cost function for Example 3 via the SHWCM with $m = 8, 16, 32$ and $x_1 = -\frac{1}{8}, -\frac{1}{16}, -\frac{1}{32}$, respectively

In three dimensions plane, Figures (4.9) and (4.10) illustrate the results obtained by proposed numerical method and analytical solutions for the cost function of the nonlinear optimal control problem with $m = 32$. Meanwhile, Figures (4.11) and (4.12) illustrate the results for the approximate and exact solutions for the obtained optimal feedback control, respectively. In this example, the numerical results are obtained within 14 successive controls iterations and a criteria error of $\varepsilon = 0.001$.

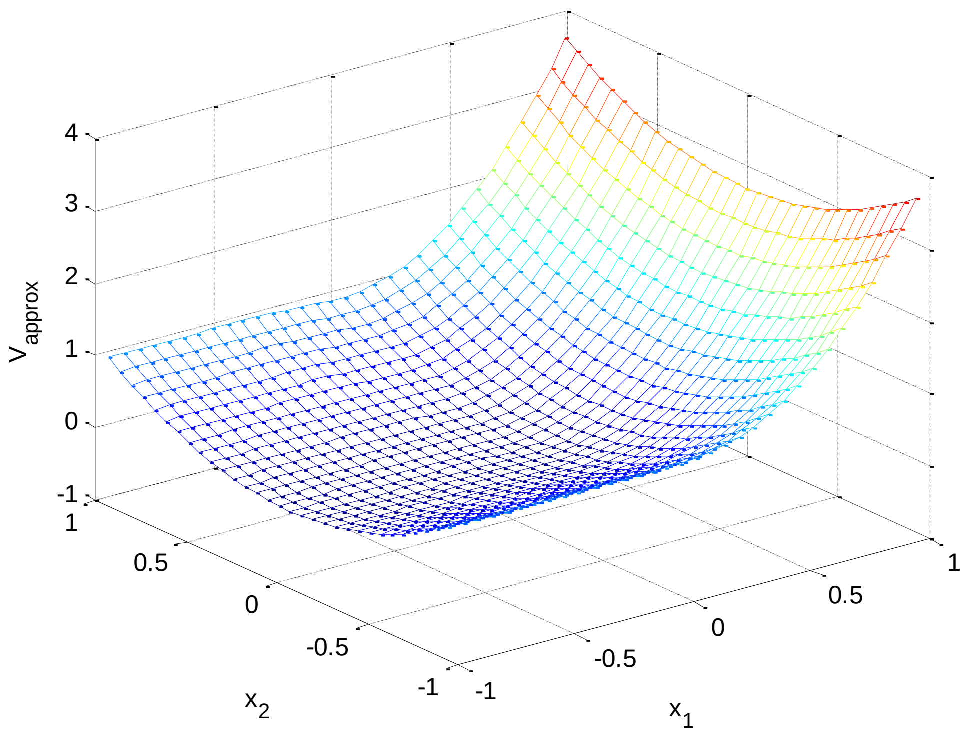


Figure 4.9: Approximate solution for cost function with $m = 32$ and $\varepsilon = 0.001$ for Example 3

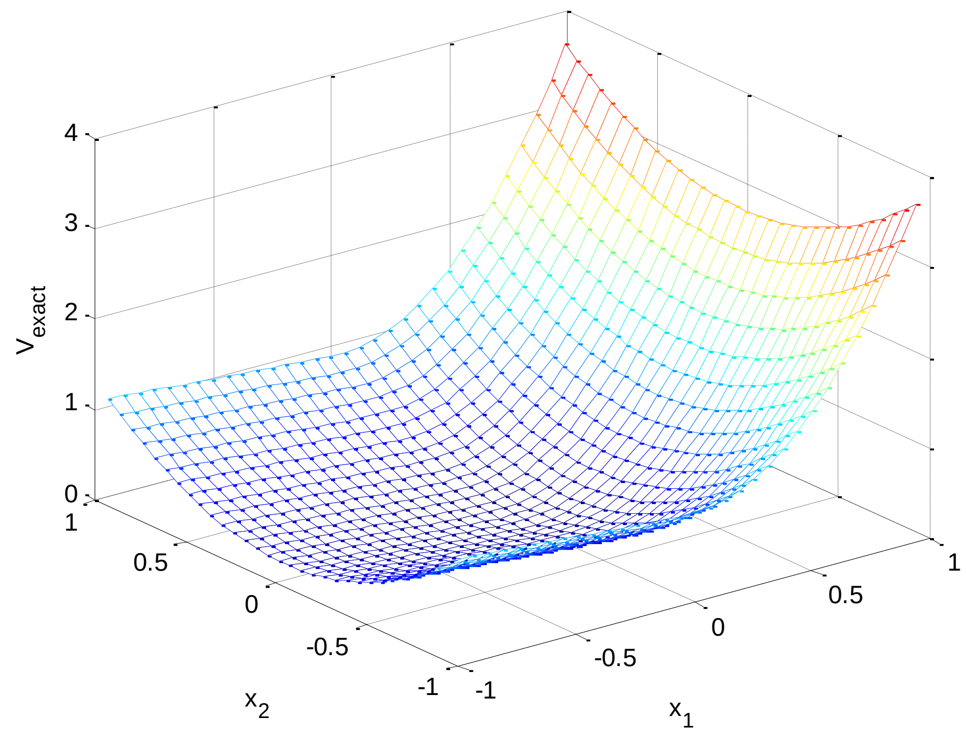


Figure 4.10: Exact solution for cost function with $m = 32$ for Example 3

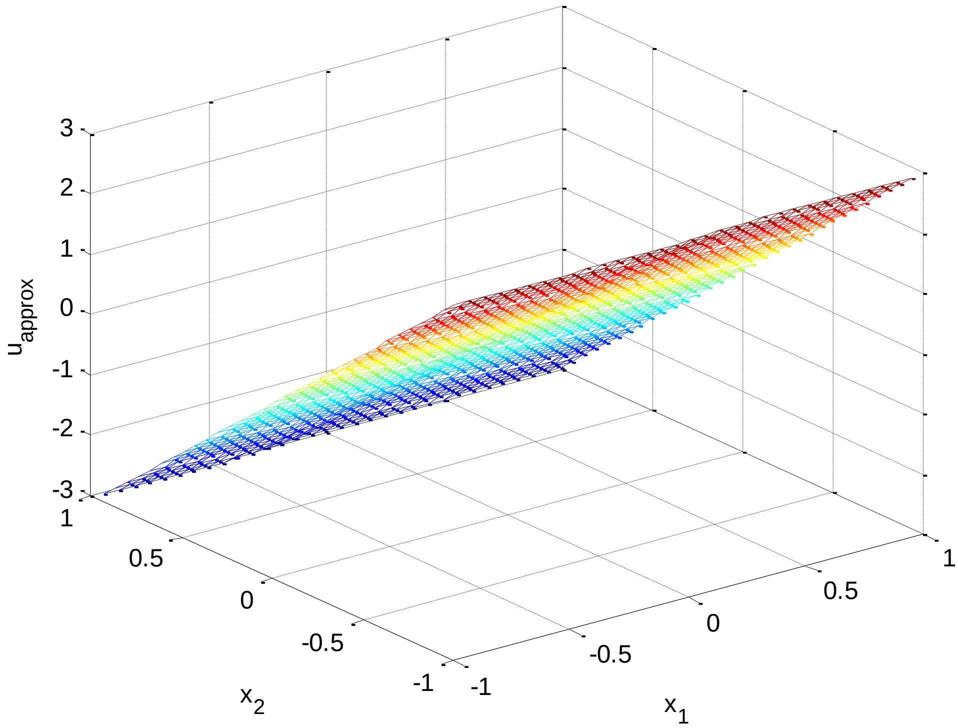


Figure 4.11: Approximate solution for optimal feedback control via the SHWCM with $m = 32$, $\varepsilon = 0.001$ and 14 iterations for Example 3

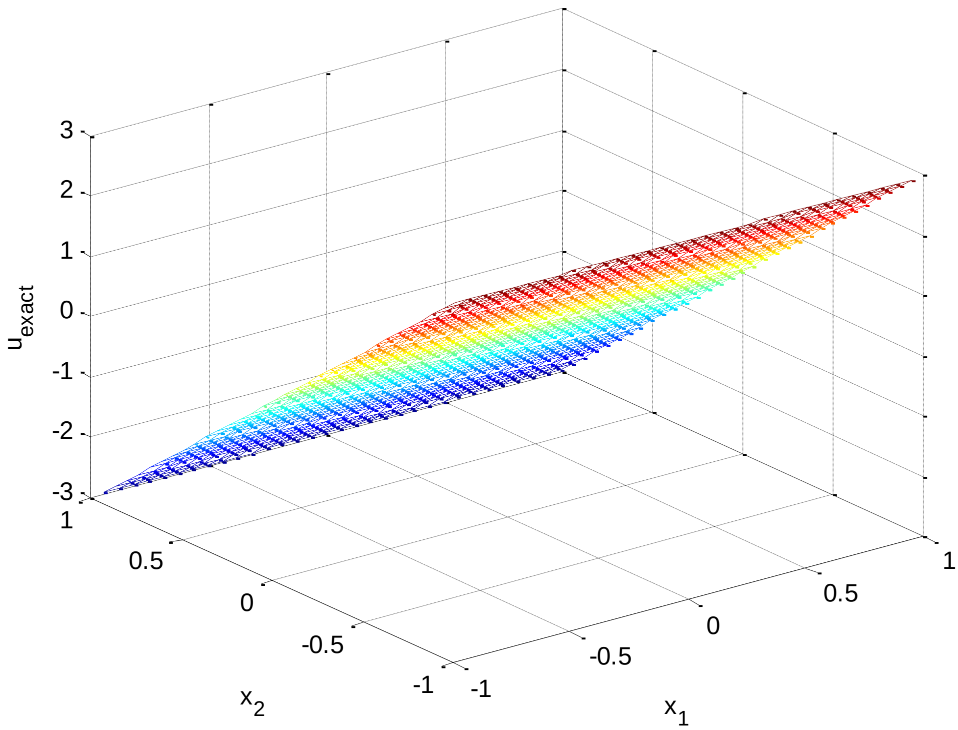


Figure 4.12: Exact solution for optimal feedback control with $m = 32$ for Example 3

4.7.4 Example 4

Consider the following nonlinear optimal control problem (Beard et al., 1997):

$$J = \int_0^\infty (x_1^2 + x_2^2 + u^2)\, dt \tag{4.82}$$

subject to

$$\dot{x} = \begin{bmatrix} -x_1^3 - x_2 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{4.83}$$

The initial stabilizing control $u^{(0)}(x) = 0.4142x_1 - 1.3522x_2$ can be obtained using feedback linearization method as discussed by Isidori (1989). The optimal feedback control and cost function obtained using SHWCM for various Haar wavelet resolutions of $m = 8$, 16, and 32 are illustrated in Figures (4.13) and (4.14), respectively. We believe the SHWCM will yield more accurate results when the Haar wavelet resolution is increased. Figure (4.15) shows the simulation of the system trajectories. Figures (4.13) and (4.14) clearly show that compared with the approximate solutions for the cost function, the approximate solutions for optimal feedback control require lower resolution, than the approximate solutions for the cost function. Nonetheless, more accurate results can be obtained in both cases by increasing the resolution of the Haar wavelet.

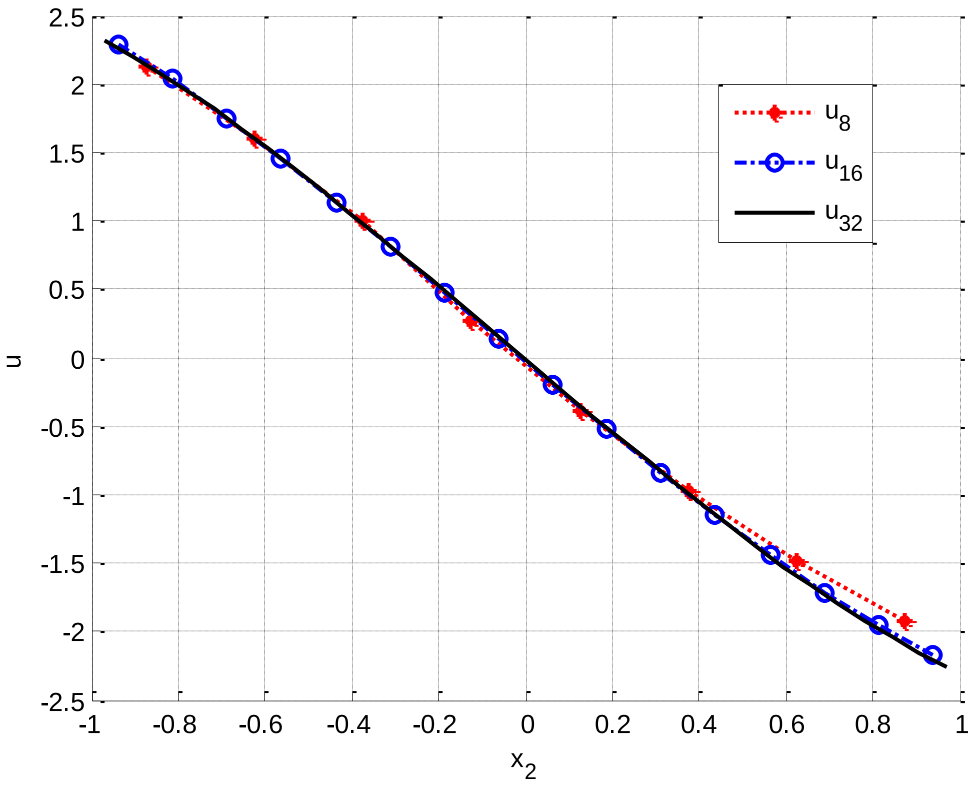


Figure 4.13: Optimal feedback control for Example 4 via the SHWCM with $m = 8, 16, 32$ and $x_1 = -\frac{1}{8}, -\frac{1}{16}, -\frac{1}{32}$, respectively

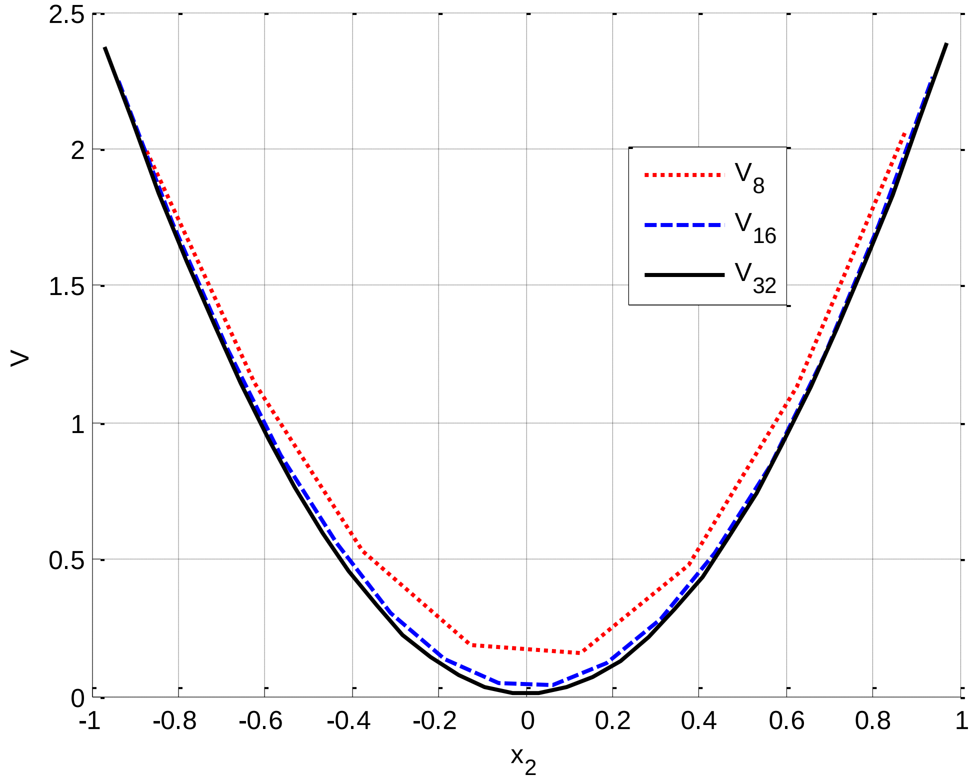


Figure 4.14: Value for cost function for Example 4 via the SHWCM with

$m = 8, 16, 32$ **and** $x_1 = -\frac{1}{8}, -\frac{1}{16}, -\frac{1}{32}$**, respectively**

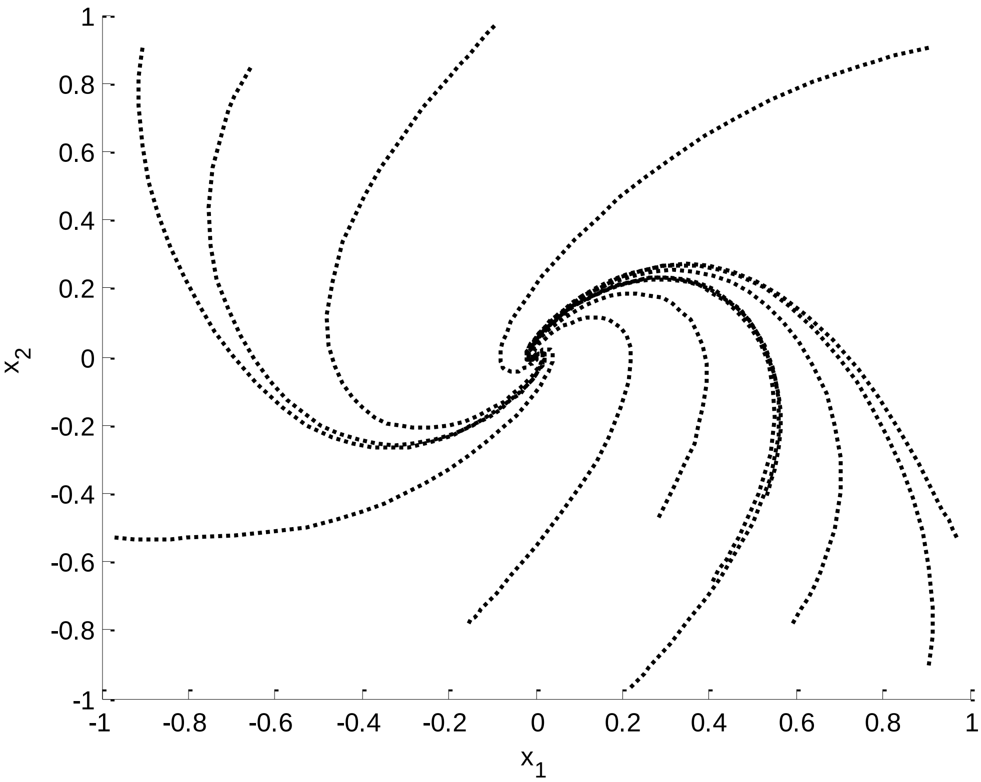


Figure 4.15: Some state trajectories for Example 4

4.7.5 Example 5

Consider the following nonlinear optimal control problem described by Cloutier et al. (1996), which contains two state variables and two control variables. The system, which has cubic nonlinearities in each equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_1^3 + x_2 \\ x_1 + x_1^2 x_2 - x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.84}$$

The cost function to be minimized is

$$J(x_0, u) = \int_0^\infty \left(x^{\mathrm{T}} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} x + u^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \right) dt \tag{4.85}$$

The initial stabilizing control for this example was chosen to be $u_1^{(0)}(x) = -2.5811x_1 - x_2$, $u_2^{(0)}(x) = -x_1 - 0.5811x_2$ and the region $\Omega = [-1,1]\times[-1,1]$. The optimal feedback control and cost function obtained using SHWCM for various Haar wavelet resolutions of $m = 4$,16, 32, and 64 are illustrated in Figures (4.16), (4.17), and (4.18). These figures clearly demonstrate that the SHWCM will be capable of yielding more accurate results when the Haar wavelet resolution is increased.

This problem was reduced in Beeler et al. (2000) and was solved by using five different methods, and they obtained the values for the cost functional $J(x_0, u)$ that are listed in Table 4.3. To compare, our control cost at initial condition $\mathbf{x}_0 = (1,\ 1)^{\mathbf{T}}$ is 4.66185392 when the Haar wavelet resolution $m = 64$ and within $i = 31$ successive controls iteration.

Table 4.3: Numerical comparison of feedback control methodologies in Example 5 at initial condition $\mathbf{x}_0(1,1)$, Beeler et al. (2000)

Numerical methods	Cost
HJB with Power Series Approximation	4.6985
State-Dependent Riccati Equation	4.6929
Interpolation of TPBV Problem Solution	4.6809
Interpolation of Iterative Solution	4.6768
Our Method SHWCM	4.66185392

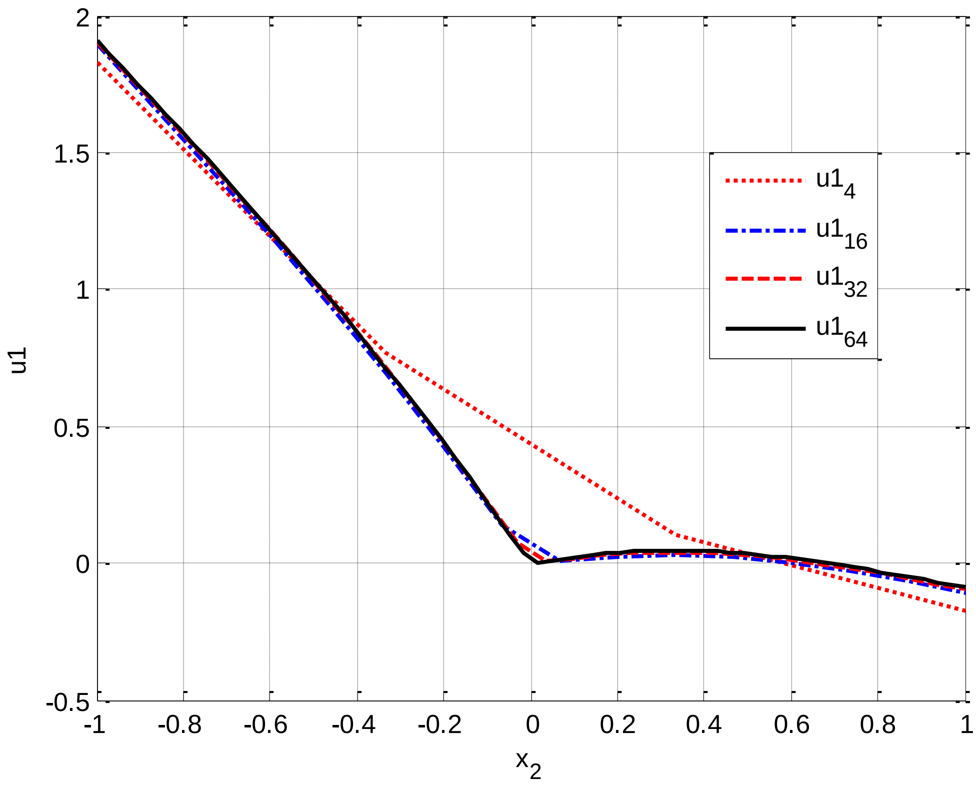


Figure 4.16: First optimal feedback control for Example 5 via the SHWCM with

$m = 4, 16, 32, 64$ **and** $x_1 = -\frac{1}{3}, -\frac{1}{15}, -\frac{1}{31}, -\frac{1}{63}$**, respectively**

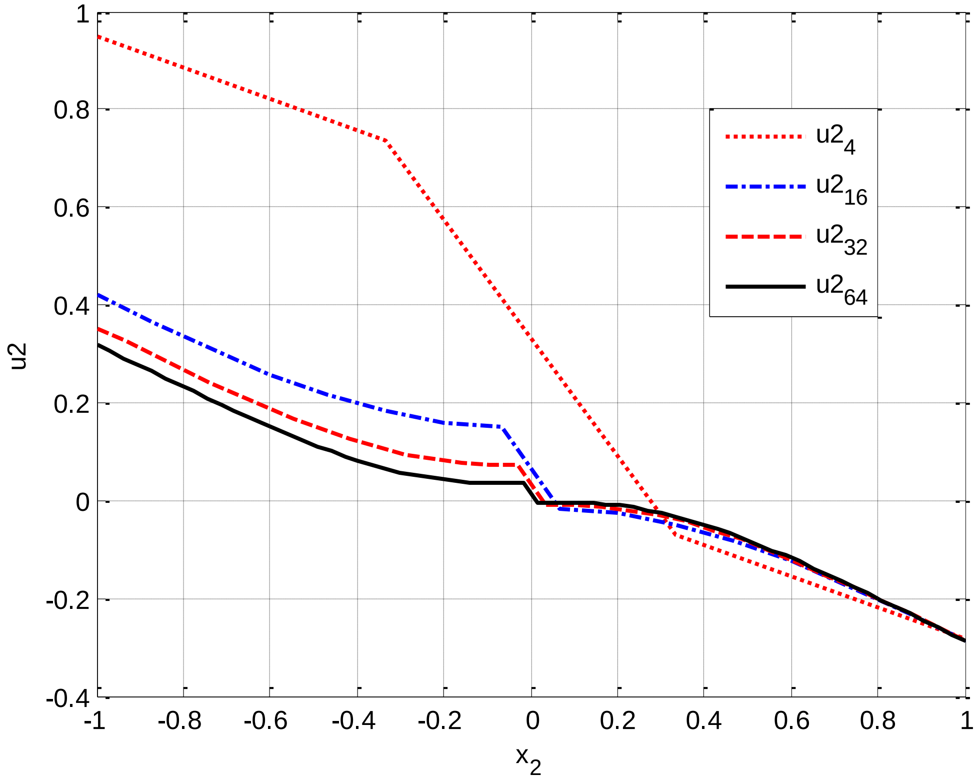


Figure 4.17: Second optimal feedback control for Example 5 via the SHWCM with $m = 4, 16, 32, 64$ and $x_1 = -\frac{1}{3}, -\frac{1}{15}, -\frac{1}{31}, -\frac{1}{63}$, respectively

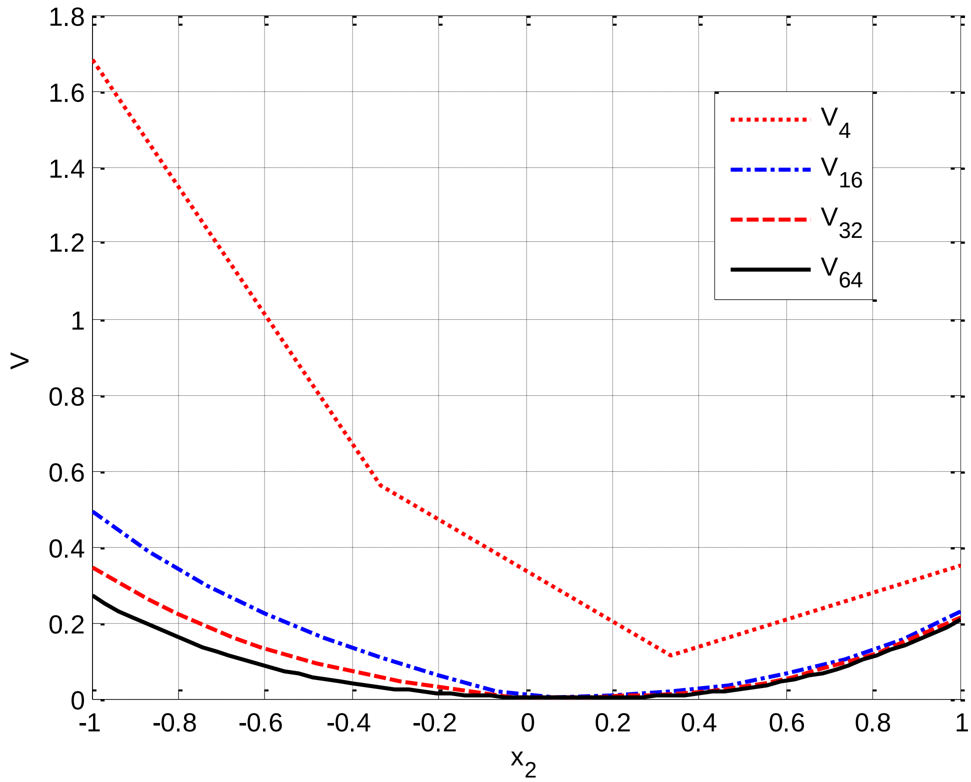


Figure 4.18: Value for cost function for Example 5 via the SHWCM with $m = 4, 16, 32, 64$ and $x_1 = -\frac{1}{3}, -\frac{1}{15}, -\frac{1}{31}, -\frac{1}{63}$, respectively

Figure 4.19 illustrates the results obtained by the proposed SHWCM for cost function, whereas Figures 4.20 and 4.21 demonstrate the first and second optimal feedback control of the nonlinear optimal control problem with $m = 64$, respectively. The numerical results are obtained within 31 successive controls iterations and a criteria error of $\varepsilon = 0.01$.

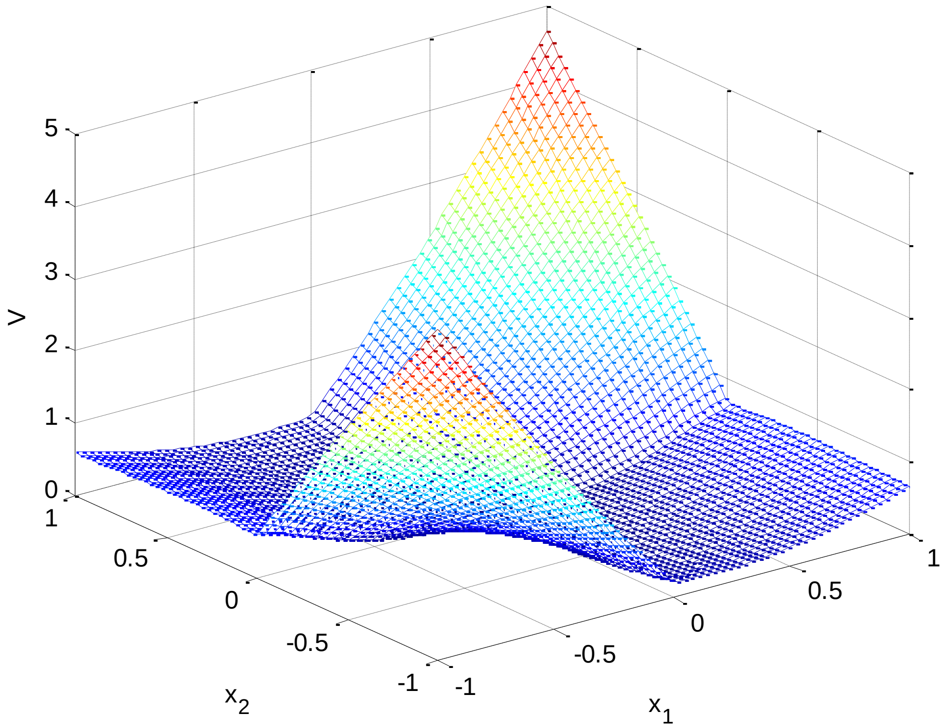


Figure 4.19: Approximate solution for cost function with $m = 64$, $\varepsilon = 0.01$ and 31 iterations for Example 5

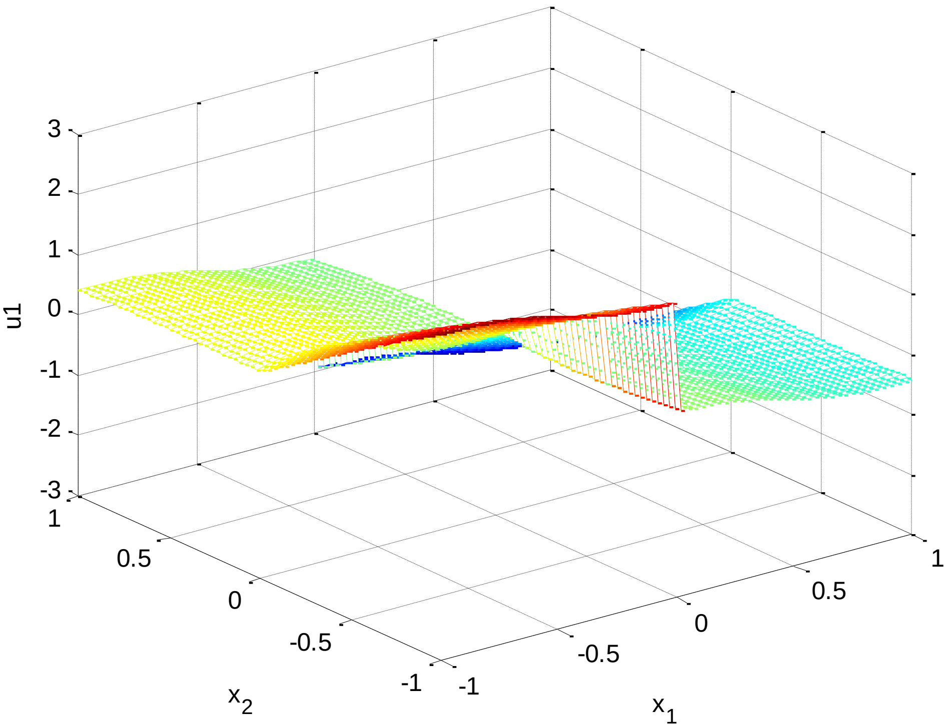


Figure 4.20: Approximate solution for first optimal feedback control with $m = 64$, $\varepsilon = 0.01$ and 31 iterations for Example 5

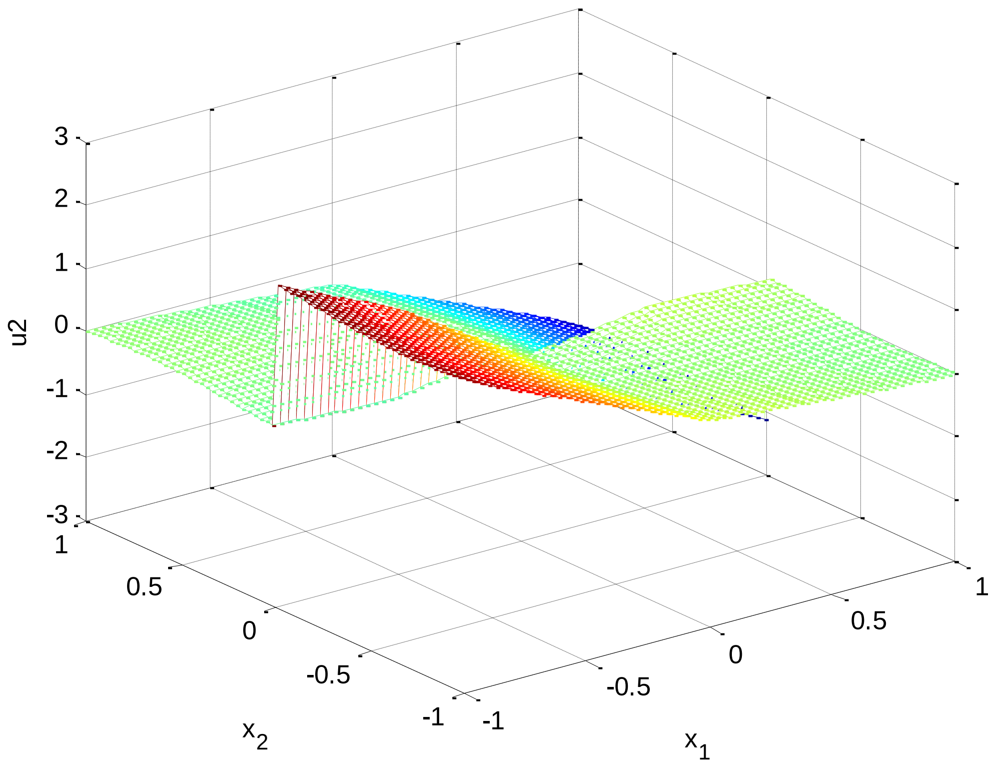


Figure 4.21: Approximate solution for second optimal feedback control with $m = 64$, $\varepsilon = 0.01$ and 31 iterations for Example 5

4.8 Numerical Discussions

The results of all examples are depicted in figures. Each figure is plotted with the solution obtained from the proposed numerical method (SHWCM) with various Haar wavelet resolutions. The results in figure form provide a better visualization regarding of the agreement between numerical and exact solutions, if available in some examples.

By using the SHWCM, which involves Haar wavelet operational matrices to solve the GHJB equation, the problem is reduced to a matrix computation that is much simpler than a method that requires the computation of multidimensional integrals. The proposed method can be easily coded.

This advantage suggests that the Haar wavelet has a great potential as a numerical tool. Additionally, other benefits of this tool include faster computation and attractiveness.

All of figures clearly indicate that the convergence of approximate solutions for optimal feedback control requires lower resolution than that required by the convergence of approximate solutions for the cost function. For instance, see Figures 4.13 and Figure 4.14. However, in both cases, more accurate results can be obtained by increasing the resolution of the Haar wavelet.

The proposed numerical method presents encouraging results even for a small value of Haar wavelet resolution of $m = 8$. The accuracy of the solution in the numerical results is improved as larger values of m are used. An example of this phenomenon is the nonlinear optimal control problem in Example 3, which is depicted in Figure 4.7 and Figure 4.8 with $m = 16$ and $m = 32$. In addition, the proposed numerical method agrees well with the exact solution, as shown in Examples 1, 2, and 3. The simulation results indicate that the accuracy of the control and cost can be improved by increasing the Haar wavelet resolution.

This work will serve as foundation for finding the solution to the Hamilton-Jacobi-Bellman equation in view of the sparse matrices that appeared during the calculation, which contribute to a faster computational analysis.

4.9 Conclusion

In this chapter, we used the Haar wavelets operational matrices to approximate the solution to the GHJB equation in the interval $[-\tau,\tau)\times[-\tau,\tau)$. When Haar wavelets operational matrix methods are used to approximate the GHJB equation, and the result is plugged into the successive GHJB equation, we obtain algorithm 4.4.1 (Figure 4.2), which improves the closed-loop performance of $u^{(0)}(x)$. The GHJB equation is the key to finding the results in this chapter because it answers three fundamental questions that are presented in Section 2 of Chapter 1. First, its solution provides a compact representation of the performance of any admissible control. Second, its solution allows users to find a control law that improves the performance of the original control. Third, by iterating the process, we converge uniformly to the solution of HJB equations. The advantages of the proposed method are as follows:

- All of the computations are performed off-line.
- The resulting controls are in feedback form, and they stabilize the closed-loop system.
- The algorithm converges uniformly to the optimal control.
- By increasing the complexity of the approximating control, it can be made arbitrarily close to the optimal.
- More accurate results can be obtained by increasing the resolution of the Haar wavelet, and the approximate controls are guaranteed to stabilize on Ω.

Finally, the proposed approach is simple and stable, and it has been tested on linear and nonlinear infinite time horizon optimal control problems in one or two-dimensional state space with one or two controllers.

CHAPTER 5

DIRECT METHOD

CONSTRAINED OPTIMAL CONTROL PROBLEMS

5.1 Introduction

Optimal control problems without constraints can be successfully solved with the use of the majority of direct and indirect techniques. However, analytical and computational difficulties often arise because of inequality constraints. Thus, researchers aim to solve constrained optimal control problems with numerical methods. Direct methods are widely used to solve nonlinear optimal control problems. Direct methods obtain an optimal solution by directly minimizing the constrained performance index. Furthermore, this type of method utilizes either discretization or parameterization technique to convert the optimal control problem into a mathematical programming problem, which is typically solved by an optimization code. Therefore, the application of direct methods does not require the use of first-order necessary conditions for optimality that arise from the use of the minimum principle of Pontryagin on optimal control problems. Moreover, integrating the system of the adjoint equations is not needed.

In this chapter, we propose a new numerical method for solving the linear and nonlinear constraints of finite time horizon optimal control problems. This approach uses quasilinearization technique, and the state and control variables are parameterized using Haar wavelet functions and the Haar wavelet operational matrix to convert the nonlinear optimal control problem into a quadratic programming problem. The linear inequality constraints for trajectory variables are converted to quadratic programming constraints by using Haar wavelet collocation method. The terminal state constraints are converted using Haar wavelet functions and adjoined to the system dynamics constraints. Then, the quadratic programming problem with linear inequality constraints is solved by using the MATLAB command `quadprog()`.

The advantages of the proposed method are summarized as follows:

1. This method facilitates easy approximation.
2. This method can be applied on constrained optimal control problems with unequal number of state variables and control variables.
3. This method can handle inequality and equality constraints.

Numerical examples, results, and discussions are shown at the end of this chapter. These numerical examples are computed and compared with others existing methods. The accuracy of the state and control variables, as well as the cost, can be improved by increasing the Haar wavelet resolution.

5.2 Problem Statement

In this chapter, we consider the following optimal control problem:

The system to be controlled is given by nonlinear differential equations of the form

$$\dot{x} = f(x(t), u(t), t), \qquad x(0) = \mathbf{x}_0, \qquad 0 \le t \le t_f \;, \tag{5.1}$$

where $x(t) \in \Re^{n_1}$ is the state vector, $u(t) \in \Re^{n_2}$ is the control vector, f is continuously differentiable with respect to all its arguments, $\mathbf{x}_0$ is the initial condition vector, and 0, t_f are a known initial and finite time, respectively, subject to the following constraints:

1. Terminal state constraints:

$$\Psi(x(t_f), t_f) = 0. \tag{5.2}$$

2. Saturation state and control constraints:

$$x(t) \le \mathbf{x}_{\mathbf{max}} \ , \quad x(t) \ge \mathbf{x}_{\mathbf{min}} \ , \tag{5.3}$$

$$u(t) \le \mathbf{u}_{\mathbf{max}} \ , \quad u(t) \ge \mathbf{u}_{\mathbf{min}} \, . \tag{5.4}$$

The vector inequalities such as $x(t) \le \mathbf{x}_{\mathbf{max}}$ means $x_i(t) \le \mathbf{x}_{\mathbf{max},i}$ for all $i = 1, 2, \ldots, n_1$.

The problem is finding the optimal control $u^*(t)$ that minimizes the following performance index:

$$J = x^{\mathbf{T}}(t_f)\ \mathbf{S}\ x(t_f) + \int_0^{t_f} (x^{\mathbf{T}}\mathbf{Q}\, x + u^{\mathbf{T}}\mathbf{R}\, u)\, dt \ , \tag{5.5}$$

where $\mathbf{Q} \in \Re^{n_1 \times n_1}$ is a positive semi-definite matrix, $\mathbf{R} \in \Re^{n_2 \times n_2}$ is a positive definite matrix and the terminal cost given by the scrap function $x^{\mathbf{T}}(t_f)\ \mathbf{S}\ x(t_f)$ and $\mathbf{S}$ is a symmetric and positive definite (or semi definite) matrix.

5.3 Proposed Method

The proposed method for solving the stated optimal control problem consists mainly of three steps:

1. Using quasilinearization technique to replace the constrained nonlinear optimal control problem by a sequence of constrained linear optimal control problems.
2. Using the Haar wavelet operational matrix and approximation functions to convert the optimal control problem into a quadratic programming problem. The linear inequality constraints for trajectory variables are transformed into

quadratic programming constraints by using the Haar wavelet collocation method.

3. Improving the solving of the quadratic programming problem by utilizing the obtained trajectories as the new nominal trajectories and control until the stopping criteria is satisfied.

5.4 Numerical Solution to the Nonlinear Optimal Control Problem

We propose the following numerical solution to a nonlinear optimal control problem with inequality constraints and terminal state constraints: At each step of this algorithm, we identify an approximate solution to the optimal control problems Eqns. (5.1) to (5.5). The orthogonal Haar wavelet is used as a basis to approximate state $x(t)$ and control $u(t)$.

5.4.1 Quasilinearization Technique

Applying the quasilinearization method proposed by Bellman and Kalaba, (1965), we can replace the optimal control problem in Eqns. (5.1) - (5.5) with the following sequence of constrained linear-quadratic optimal control problems:

Minimizes

$$J^{[k]} = x^{[k]\mathbf{T}}(t_f)\ \mathbf{S}\ x^{[k]}(t_f) + \int_0^{t_f} (x^{[k]T}\mathbf{Q}\, x^{[k]} + u^{[k]T}\mathbf{R}\, u^{[k]})dt\ , \qquad (5.6)$$

with is subject to the linearized time varying state equations:

$$\frac{dx^{[k]}(t)}{dt} = \mathbf{A}^{[k-1]}(t)x^{[k]} + \mathbf{B}^{[k-1]}(t)u^{[k]},\ x^{[\mathbf{k}]}(\mathbf{0}) = \mathbf{x}_0 \quad , \quad k \geq 1 \qquad (5.7)$$

where

$$\mathbf{A}^{[k]}(t) = \left. \frac{\partial f(x,u,t)}{\partial x} \right|_{x^k, u^k}, \tag{5.8}$$

$$\mathbf{B}^{[k]}(t) = \left. \frac{\partial f(x,u,t)}{\partial u} \right|_{x^k, u^k}, \tag{5.9}$$

are the $n_1 \times n_1$ and $n_1 \times n_2$ matrix, respectively, and the terminal state and the inequality constraints are expressed as follows:

$$\Psi(x(t_f), t_f) = 0 \ , \tag{5.10}$$

$$x^{[k]}(t) \le \mathbf{x_{max}} \ , \ \ x^{[k]}(t) \ge \mathbf{x_{min}} \ , \tag{5.11}$$

$$u^{[k]}(t) \le \mathbf{u_{max}} \ , \ \ u^{[k]}(t) \ge \mathbf{u_{min}} \ . \tag{5.12}$$

The initial matrices $\mathbf{A}^0(t)$ and $\mathbf{B}^0(t)$ are determined using an approximately accurate initial assumption of $x^0(t)$ and $u^0(t)$ that does not cause the algorithm to diverge. We suggest starting from the initial condition vector $\mathbf{x}_0$.

5.4.2 Optimal Control Problem using Haar Wavelet Method

Haar wavelet operational matrix and Haar wavelet functions are used to approximate the optimal control problem in terms of unknown coefficients of state and control variables.

5.4.2.1 Parameterization using Haar Wavelet Functions

To formulate the optimal control problem in Eqns. (5-6)-(5-12) into a quadratic programming problem, the proposed method, which is based on parameterizing the state and control variables using Haar wavelet functions, is applied. At first, the state vector $\dot{x}(t)$ and control vector $u(t)$ are expanded in terms of Haar wavelet basis by using Eqn. (3.10) as follow:

$$\dot{x}_k(t) = \sum_{i=0}^{m-1} c_{ki} h_i(t), \qquad k = 1, 2, \cdots, n_1, \tag{5.13}$$

$$u_l(t) = \sum_{i=0}^{m-1} d_{li} h_i(t) , \qquad l = 1, 2, 3, \cdots, n_2 , \tag{5.14}$$

where $c_{k0}, c_{k1}, c_{k2}, \cdots, c_{km-1}$, $k = 1, 2, 3, \cdots, n_1$ are unknown parameters for the state variables and $d_{l0}, d_{l1}, d_{l2}, \cdots, d_{lm-1}$ for $l = 1, 2, 3, \cdots, n_2$, are unknown parameters for the control variables

Eqns. (5.13) and (5.14) can be written in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n_1} \end{bmatrix} = \begin{bmatrix} c_{10} & c_{11} & \cdots & c_{1m-1} \\ c_{20} & c_{21} & \cdots & c_{2m-1} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n_1 0} & c_{n_1 1} & \cdots & c_{n_1 m-1} \end{bmatrix} \begin{bmatrix} h_0(t) \\ h_1(t) \\ \vdots \\ h_{m-1}(t) \end{bmatrix} \tag{5.15}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n_2} \end{bmatrix} = \begin{bmatrix} d_{10} & d_{11} & \cdots & d_{1m-1} \\ d_{20} & d_{21} & \cdots & d_{2m-1} \\ \vdots & \vdots & \cdots & \vdots \\ d_{n_2 0} & d_{n_2 1} & \cdots & d_{n_2 m-1} \end{bmatrix} \begin{bmatrix} h_0(t) \\ h_1(t) \\ \vdots \\ h_{m-1}(t) \end{bmatrix} \tag{5.16}$$

These equations can be rewritten in compact form as follows:

$$\dot{x}(t) = \mathbf{c}^{\mathbf{T}} \mathbf{h}(t) \tag{5.17}$$

$$u(t) = \mathbf{d}^{\mathbf{T}} \mathbf{h}(t) \tag{5.18}$$

where $\mathbf{c}^{\mathbf{T}}$ and $\mathbf{d}^{\mathbf{T}}$ are now an $n_1 \times m$ and $n_2 \times m$ unknown coefficient matrices for Haar wavelet functions, respectively; and $\mathbf{h}(t)$ is the vector of known Haar wavelet function with dimension of $m \times 1$, where $\mathbf{h}(t) = [h_0(t) \;\; h_1(t) \;\; \cdots \;\; h_{m-1}(t)]^{\mathbf{T}}$ and $\mathbf{T}$ is the transpose.

By integrating Eqn. (5.17) with respect to t and applying Eqn. (3.33), we obtain $x(t)$, which is expressed in terms of Haar wavelet functions and the Haar operational matrix as

$$x(t) = \int_0^t \mathbf{c}^{\mathbf{T}} \mathbf{h}(t)\, dt \; + \mathbf{x}_0 \tag{5.19}$$

Thus

$$x(t) = \mathbf{c}^{\mathbf{T}} \mathbf{P}\, \mathbf{h}(t) + \mathbf{x}_0\, \theta^{\mathbf{T}}\, \mathbf{h}(t), \tag{5.20}$$

where $\mathbf{x}_0$ is the $n_1 \times 1$ column vector of the initial conditions that is $\mathbf{x}_0 = \left[x_{01} \;\; x_{02} \;\; x_{03} \ldots x_{0n_1} \right]^{\mathbf{T}}$, and $\theta = \left[1, \; 0, \; 0, \; \cdots \; , \; 0 \right]^{\mathbf{T}}$ is an $m \times 1$ vector.

Eqns. (5.17), (5.18), and (5.20) can then be rewritten in compact form by using the properties of the operation vec, where $vec\,\mathbf{(ABC)} = \mathbf{(A \otimes C^T)}\, vec\,\mathbf{(B)}$ [see A.15 (Brewer, 1978)], as follows:

$$\dot{x}(t) = (\mathbf{I}_{n_1} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{c}) \tag{5.21}$$

$$u(t) = (\mathbf{I}_{n_2} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{d}) \tag{5.22}$$

$$x(t) = (\mathbf{I}_{n_1} \otimes \mathbf{h}^{\mathbf{T}}(t)\, \mathbf{P}^{\mathbf{T}})\, vec(\mathbf{c}) + (\mathbf{I}_{n_1} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{x}_0\, \theta^{\mathbf{T}}) \tag{5.23}$$

where $\mathbf{I}_{n_1}$ and $\mathbf{I}_{n_2}$ denote $n_1 \times n_1$ and $n_2 \times n_2$ identity matrices, respectively. In addition $vec(\mathbf{c}) = \left[c_{10} \;\; c_{20} \;\; \cdots \;\; c_{n_1 0} \;\; c_{11} \;\; c_{21} \cdots c_{n_1 1} \;\; \cdots \;\; c_{1m-1} \;\; c_{2m-1} \cdots c_{n_1 m-1} \right]^{\mathbf{T}}$ is the vector of unknown Haar wavelet coefficients with dimension $n_1 m \times 1$, $vec(\mathbf{d}) = \left[d_{10} \;\; d_{20} \;\; \cdots \;\; d_{n_1 0} \;\; d_{11} \;\; d_{21} \cdots d_{n_1 1} \;\; \cdots \;\; d_{1m-1} \;\; d_{2m-1} \cdots d_{n_1 m-1} \right]^{\mathbf{T}}$ is an $n_2 m \times 1$ vector of unknown Haar wavelet coefficients, and $vec(\mathbf{x}_0\, \theta^{\mathbf{T}})$ is an $n_1 m \times 1$ vector of known coefficients that can be framed as $vec(\mathbf{x}_0\ \theta^{\mathbf{T}}) = [\ x_{01}\ 0\ 0\ 0\ \cdots\ \ x_{02}\ 0\ 0\ 0\ \ \cdots\ x_{03}\ 0\ 0\ 0\ \ \cdots\ ]^{\mathbf{T}}$.

5.4.2.2 Approximation of the Performance Index

To approximate the performance index of the optimal control problem described in Eqn. (5.6), the performance index is formalized by using Haar wavelet functions. We first expand the second term of the performance index by substituting Eqns. (5.22) and (5.23) into Eqn. (5.6). Thus, we obtain

$$\begin{aligned} J_1 \ = \int_0^{t_f} \Big\{ & \big[(\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t)\ \ \mathbf{P}^{\mathbf{T}})\, vec(\mathbf{c}) + (\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{x}_0\ \theta^{\mathbf{T}}) \big]^{\mathbf{T}} \\ & \mathbf{Q} \big[(\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t)\ \ \mathbf{P}^{\mathbf{T}})\, vec(\mathbf{c}) + (\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{x}_0\ \theta^{\mathbf{T}}) \big] \\ & + \big[(\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{d}) \big]^{\mathbf{T}} \mathbf{R}\ \big[(\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{d}) \big] \Big\} dt \end{aligned} \tag{5.24}$$

Rearranging and simplifying Eqn. (5.24) yields

$$\begin{aligned} J_1 \ = \int_0^{t_f} \Big\{ & vec^{\mathbf{T}}(\mathbf{c})(\mathbf{I} \otimes \mathbf{P}\,\mathbf{h}(t))\, \mathbf{Q}\, (\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t)\, \mathbf{P}^{\mathbf{T}})\, vec(\mathbf{c}) \\ & + vec^{\mathbf{T}}(\mathbf{c})(\mathbf{I} \otimes \mathbf{P}\,\mathbf{h}(t))\ \ \mathbf{Q}\, (\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{x}_0\ \theta^{\mathbf{T}}) \\ & + vec^{\mathbf{T}}(\mathbf{x}_0\ \theta^{\mathbf{T}})(\mathbf{I} \otimes\ \mathbf{h}(t))\, \mathbf{Q}\ \ (\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t) \mathbf{P}^{\mathbf{T}})\, vec(\mathbf{c}) \\ & + vec^{\mathbf{T}}(\mathbf{x}_0\ \theta^{\mathbf{T}})(\mathbf{I} \otimes \mathbf{P}\,\mathbf{h}(t))\, \mathbf{Q}\ \ (\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{x}_0\ \theta^{\mathbf{T}}) \\ & + vec^{\mathbf{T}}(\mathbf{d})\, (\mathbf{I} \otimes \mathbf{h}(t))\ \ \mathbf{R}\ \ (\mathbf{I} \otimes \mathbf{h}^{\mathbf{T}}(t))\, vec(\mathbf{d})\ \Big\} dt \end{aligned} \tag{5.25}$$

According to the Kronecker product properties (Brewer, 1978), if matrices A, B, C, D and E are appropriate dimensions matrices, then $(A \otimes B)(C \otimes D) = AC \otimes BD$ and $A(\mathbf{I} \otimes E) = A \otimes E$ [see A.4 (Lancaster, 1969)]. Therefore, Eqn. (5.25) can be rewritten as

$$J_1 = \int_0^{t_f} \{ vec^{\mathbf{T}}(\mathbf{c})(\mathbf{Q}\otimes \mathbf{P}\,\mathbf{h}(t)\mathbf{h}^{\mathbf{T}}(t)\,\mathbf{P}^{\mathbf{T}})\,vec(\mathbf{c})$$

$$+ vec^{\mathbf{T}}(\mathbf{c})(\mathbf{Q}\otimes \mathbf{P}\,\mathbf{h}(t)\,\mathbf{h}^{\mathbf{T}}(t))\,vec(\mathbf{x}_0\,\theta^{\mathbf{T}})$$

$$+ vec^{\mathbf{T}}(\mathbf{x}_0\,\theta^{\mathbf{T}})(\mathbf{Q}\otimes\ \mathbf{h}(t)\,\mathbf{h}^{\mathbf{T}}(t)\mathbf{P}^{\mathbf{T}})\,vec(\mathbf{c})$$

$$+ vec^{\mathbf{T}}(\mathbf{x}_0\,\theta^{\mathbf{T}})(\mathbf{Q}\otimes \mathbf{P}\,\mathbf{h}(t)\,\mathbf{h}^{\mathbf{T}}(t))\,vec(\mathbf{x}_0\,\theta^{\mathbf{T}})$$

$$+ vec^{\mathbf{T}}(\mathbf{d})\,(\mathbf{R}\otimes \mathbf{h}(t)\,\mathbf{h}^{\mathbf{T}}(t)\,)\,vec(\mathbf{d})\ \}dt \tag{5.26}$$

The integration of the product of two Haar wavelet function vectors has been discussed in Section 3.7. By applying Eqn. (3.53) on Eqn. (5.26), we obtain

$$J_1 = vec^{\mathbf{T}}(\mathbf{c})(\mathbf{Q}\otimes \mathbf{P}\,\mathbf{E}\,\mathbf{P}^{\mathbf{T}})vec(\mathbf{c}) + vec^{\mathbf{T}}(\mathbf{c})(\mathbf{Q}\otimes \mathbf{P}\,\mathbf{E})vec(\mathbf{x}_0\,\theta^{\mathbf{T}})$$

$$+ vec^{\mathbf{T}}(\mathbf{x}_0\,\theta^{\mathbf{T}})(\mathbf{Q}\otimes \mathbf{E}\,\mathbf{P}^{\mathbf{T}})vec(\mathbf{c}) + vec^{\mathbf{T}}(\mathbf{x}_0\,\theta^{\mathbf{T}})(\mathbf{Q}\otimes \mathbf{E})vec(\mathbf{x}_0\,\theta^{\mathbf{T}})$$

$$+ vec^{\mathbf{T}}(\mathbf{d})(\mathbf{R}\otimes \mathbf{E})vec(\mathbf{d}) \tag{5.27}$$

where

$$vec^{\mathbf{T}}(\mathbf{x}_0\,\theta^{\mathbf{T}})(\mathbf{Q}\otimes \mathbf{E}\,\mathbf{P}^{\mathbf{T}})vec(\mathbf{c}) = vec(\mathbf{c})(\mathbf{Q}\otimes \mathbf{P}\,\mathbf{E})vec(\mathbf{x}_0\,\theta^{\mathbf{T}}) \text{ and } \mathbf{E} = \mathbf{E}^{\mathbf{T}}.$$

Finally, the performance index in Eqn. (5.27) can be written in quadratic form as follows:

$$J_1 = \frac{1}{2}\mathbf{Z}^T\,\mathbf{H}_{\mathbf{ess}}\,\mathbf{Z} + \mathbf{F}^{\mathbf{T}}\mathbf{Z} + e\ , \tag{5.28}$$

where

$$\mathbf{Z} = \left[vec^{\mathbf{T}}(\mathbf{c}) \qquad vec^{\mathbf{T}}(\mathbf{d}) \right]^{\mathbf{T}}, \tag{5.29}$$

$$\mathbf{H}_{\mathbf{ess}} = \begin{bmatrix} \mathbf{Q}\otimes \mathbf{P}\,\mathbf{E}\,\mathbf{P}^{\mathbf{T}} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}\otimes \mathbf{E} \end{bmatrix} \tag{5.30}$$

$$\mathbf{F} = \left[2vec^{\mathbf{T}}(\mathbf{x}_0\ \theta^{\mathbf{T}})(\mathbf{Q} \otimes \mathbf{E}\,\mathbf{P}^{\mathbf{T}}) \qquad \mathbf{O} \right] \tag{5.31}$$

$$e = \left[vec^{\mathbf{T}}(\mathbf{x}_0\ \theta^{T})(\mathbf{Q} \otimes \mathbf{E}) vec(\mathbf{x}_0\ \theta^{\mathbf{T}}) \right]$$

(5.32)

are $m \times (n_1 + n_2) \times 1$, $m(n_1 + n_2) \times m(n_1 + n_2)$, $m \times (n_1 + n_2) \times 1$ and 1×1 matrices, respectively.

Further, we need to expand the first term of performance index in Eqn. (5.6) by converting it to a quadratic programming problem with the use of the scrap function, which is define as follows:

$$J_2 = x^{\mathbf{T}}(t_f)\,\mathbf{S}\,x(t_f) \tag{5.33}$$

where $x(t_f)$ is vector of final conditions.

First, we assume that

$$\dot{x}(t_f) = \mathbf{c}^{\mathbf{T}}\mathbf{h}(t_f) \tag{5.34}$$

By applying Eqn. (3.52), we obtain

$$\mathbf{P}\,\mathbf{h}(t_f) = \int_0^{t_f} \mathbf{h}(t)dt$$

$$= t_f\ \theta \tag{5.35}$$

where $\theta = \left[1,\ 0,\ 0,\ \cdots\ ,\ 0 \right]^{\mathbf{T}}$ is $m \times 1$ vector.

Integrating Eqn. (5.34) and utilizing Eqns. (3.33) and (3.52), we obtain

$$x(t_f) = t_f\ \mathbf{c}^{\mathbf{T}}\theta + \mathbf{x}_0\ \theta^{\mathbf{T}}\theta\ , \tag{5.36}$$

The term $\theta^{\mathbf{T}}\theta$ can be evaluated as follows:

$$\theta^{\mathbf{T}}\theta = \begin{bmatrix} 1, & 0, & 0, & \cdots, & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \mathbf{1} \tag{5.37}$$

Finally, Eqn. (5.36) can be written as

$$x(t_f) = t_f \; \mathbf{c}^{\mathbf{T}}\theta + \mathbf{x}_0 \tag{5.38}$$

Rewriting Eqn. (5.38) in compact form by using the properties of the operation vec (see A.16), we have

$$vec\,(x(t_f)) = t_f \; vec(\mathbf{c}^{\mathbf{T}}\theta) + vec(\mathbf{x}_0) \tag{5.39}$$

$$= t_f \, (\mathbf{I}_{n_1} \otimes \theta^{\mathbf{T}}) \, vec(\mathbf{c}) + vec(\mathbf{x}_0)$$

where $vec\,(\mathbf{x_0})$ is $n_1 \times 1$ vector.

When Eqn. (5.39) is substituted into Eqn. (5.33), we obtain

$$J_2 = [t_f \, (\mathbf{I}_{n_1} \otimes \theta^{\mathbf{T}}) vec(\mathbf{c}) + vec(\mathbf{x}_0)]^{\mathbf{T}} \mathbf{S} \, [t_f \, (\mathbf{I}_{n_1} \otimes \theta^{\mathbf{T}}) vec(\mathbf{c}) + vec(\mathbf{x}_0)] \tag{5.40}$$

Simplifying and rearranging Eqn. (5.40), we obtain

$$J_2 = t_f^2 \; vec^{\mathbf{T}}(\mathbf{c})(\mathbf{I}_{n_1} \otimes \theta) \, \mathbf{S} \, (\mathbf{I}_{n_1} \otimes \theta^{\mathbf{T}}) vec(\mathbf{c}) + vec^{\mathbf{T}}(\mathbf{x}_0) \, \mathbf{S} \, vec(\mathbf{x}_0)$$

$$+ t_f \, vec^{\mathbf{T}}(\mathbf{c})(\mathbf{I}_{n_1} \otimes \theta) \, \mathbf{S} \, vec(\mathbf{x}_0) + t_f \, vec^{\mathbf{T}}(\mathbf{x}_0) \, \mathbf{S} \, (\mathbf{I}_{n_1} \otimes \theta^{\mathbf{T}}) vec(\mathbf{c}), \tag{5.41}$$

Utilizing Kronecker product properties (Brewer, 1978), we have

$$J_2 = t_f^2 \; vec^{\mathbf{T}}(\mathbf{c})(\mathbf{S} \otimes \theta\theta^{\mathbf{T}}) vec(\mathbf{c}) + vec^{\mathbf{T}}(\mathbf{x}_0) \, \mathbf{S} \, vec(\mathbf{x}_0)$$

$$+ \; t_f \, vec^{\mathbf{T}}(\mathbf{c})(\mathbf{S} \otimes \theta) vec(\mathbf{x}_0) + t_f \, vec^{\mathbf{T}}(\mathbf{x}_0)(\mathbf{S} \otimes \theta^{\mathbf{T}}) vec(\mathbf{c}). \tag{5.42}$$

We should take note of the term

$$t_f\, vec^{\mathbf{T}}(\mathbf{c})(\mathbf{S}\otimes\theta)\, vec(\mathbf{x}_0) = t_f\, vec^{\mathbf{T}}(\mathbf{x}_0)(\mathbf{S}\otimes\theta^{\mathbf{T}})\, vec(\mathbf{c}) \tag{5.43}$$

Therefore, we have

$$J_2 = t_f^2\ vec^{\mathbf{T}}(\mathbf{c})(\mathbf{S}\otimes\theta\theta^{\mathbf{T}})vec(\mathbf{c}) + 2\,t_f\, vec^{\mathbf{T}}(\mathbf{x}_0)(\mathbf{S}\otimes\theta^{\mathbf{T}})vec(\mathbf{c})$$

$$+\, vec^{\mathbf{T}}(\mathbf{x}_0)\,\mathbf{S}\, vec(\mathbf{x}_0)\,. \tag{5.44}$$

The performance index for the scrap function can be written in quadratic form as follows:

$$J_2 = \frac{1}{2}\mathbf{Z}^T\ \mathbf{H}_{\mathbf{ess}}\ \mathbf{Z} + \mathbf{F_2}^{\mathbf{T}}\mathbf{Z} + e_{\mathbf{2}}\ , \tag{5.45}$$

where

$$\mathbf{Z} = \left[vec^{\mathbf{T}}(\mathbf{c}) \qquad vec^{\mathbf{T}}(\mathbf{d}) \right]^{\mathbf{T}}, \tag{5.46}$$

$$\mathbf{H}_{\mathbf{ess}} = t_f^2 \begin{bmatrix} (\mathbf{S}\otimes\theta\theta^{\mathbf{T}}) & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}, \tag{5.47}$$

$$\mathbf{F_2} = t_f\ \left[\, 2vec^{\mathbf{T}}(\mathbf{x}_0\,)(\mathbf{I}_{n_1}\otimes\theta^{\mathbf{T}}) \qquad \mathbf{O}\,\right]^{\mathbf{T}}, \tag{5.48}$$

$$e_2 = \left[vec^{\mathbf{T}}(\mathbf{x}_0)\,\mathbf{S}\, vec\,(\mathbf{x}_0) \right], \tag{5.49}$$

are $m\times(n_1+n_2)\times 1$, $m(n_1+n_2)\times m(n_1+n_2)$, $m\times(n_1+n_2)\times 1$, and 1×1 matrices, respectively.

Finally, the performance index for both parts can be written as

$$J = J_1 + J_2\,. \tag{5.50}$$

5.4.2.3 Approximations of System Dynamics

State equations are approximated in terms of the unknown coefficients of the state and control variables by substituting Eqns. (5.21), (5.22) and (5.23) into Eqn. (5.7). Once these equations are simplified, the time varying matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ should be expressed in terms of the Haar wavelets.

The function of the $(i,\, j)^{th}$ element of $\mathbf{A}(t)$ can be approximated using Eqn. (3.22) as

$$[\mathbf{A}(t)]_{i,j} = G_{i,j}^{\mathbf{T}}\mathbf{h}(t) \tag{5.51}$$

where $G_{ij}^{\mathbf{T}} = \mathbf{[}\ g_0^{ij}\ \ g_1^{ij}\ \ g_2^{ij}\ \dots\ \ g_{m-1}^{ij}\ \mathbf{]}$ is the m row vector of the known coefficients of the Haar wavelet function for each $i = 1,2,\dots,n_1$, and $j = 1,2,\dots,n_1$, can be calculated using Eqn. (3.31) as follows:

$$G_{i,j}^{\mathbf{T}} = [\mathbf{A}(t)]_{i,j}\ \mathbf{H^{-1}} \tag{5.52}$$

where $\mathbf{H^{-1}}$ is the inverse of the Haar wavelet matrix at collocation points.

Similarly, the elements of $\mathbf{B}(t)$ can be expanded using the Haar wavelet function:

$$[\mathbf{B}(t)]_{i,j} = L_{i,j}^{\mathbf{T}}\mathbf{h}(t) \tag{5.53}$$

where $L_{i\,j}^{\mathbf{T}} = \mathbf{[}\ L_0^{ij}\ \ L_1^{ij}\ \ L_2^{ij}\ \dots\ \ L_{m-1}^{ij}\ \mathbf{]}$ is the constant $1\times m$ row coefficients of Haar wavelet functions for each $i = 1,2,\dots,n_1$ and $j = 1,2,\dots,n_2$.

Then Eqns. (5.51) and (5.53) can be rewritten in compact form by using Kronecker product properties [see A.6 (Brewer, 1978)]:

$$\mathbf{A}(t) = \mathbf{G}^{\mathbf{T}}(\mathbf{I}_{n_1} \otimes \mathbf{h}(t)) \tag{5.54}$$

$$\mathbf{B}(t) = \mathbf{L}^{\mathbf{T}}(\mathbf{I}_{n_2} \otimes \mathbf{h}(t)) \tag{5.55}$$

where the block matrices $\mathbf{G}^{\mathbf{T}} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1n_1} \\ G_{21} & G_{22} & \cdots & G_{2n_1} \\ \vdots & \vdots & \cdots & \vdots \\ G_{n_1 1} & G_{n_1 2} & \cdots & G_{n_1 n_1} \end{bmatrix}$ and $\mathbf{L}^{\mathbf{T}} = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n_2} \\ L_{21} & L_{22} & \cdots & L_{2n_2} \\ \vdots & \vdots & \cdots & \vdots \\ L_{n_1 1} & L_{n_1 2} & \cdots & L_{n_1 n_2} \end{bmatrix}$

are of size $n_1 \times n_1 m$ and $n_1 \times n_2 m$, respectively.

Given the notation above, the transpose of Eqns. (5.21), (5.22), (5.23) with Eqns. (5.54) and (5.55) are substituted into Eqn. (5.7) to obtain

$$vec^{\mathbf{T}}(\mathbf{c})\,(\mathbf{I}_{n_1} \otimes \mathbf{h}(t)) = \mathbf{G}^{\mathbf{T}}(\mathbf{I}_{n_1} \otimes \mathbf{h}(t))\,\left\{vec^{\mathbf{T}}(\mathbf{c})(\mathbf{I}_{n_1} \otimes \mathbf{P}\,\mathbf{h}(t)) + vec^{\mathbf{T}}(\mathbf{x}_0\theta^{\mathbf{T}})(\mathbf{I}_{n_1} \otimes \mathbf{h}(t))\right\}$$

$$+\mathbf{L}^{\mathbf{T}}(\mathbf{I}_{n_1} \otimes \mathbf{h}(t))\,vec\,(\mathbf{d})\,(\mathbf{I}_{n_2} \otimes \mathbf{h}(t)) \tag{5.56}$$

Simplifying Eqn. (5.56) by utilizing Kronecker product properties, we have

$$\begin{aligned} vec^{\mathbf{T}}(\mathbf{c})(\mathbf{I}_{n_1} \otimes \mathbf{h}(t)) \;&=\; \mathbf{G}^{\mathbf{T}}(\mathbf{I}_{n_1} \otimes \mathbf{h}(t))\,vec^{\mathbf{T}}(\mathbf{c})\,(\mathbf{I}_{n_1} \otimes \mathbf{P})(\mathbf{I}_{n_1} \otimes \mathbf{h}(t)) \\ &\quad + \mathbf{G}^{\mathbf{T}}(\mathbf{I}_{n_1} \otimes \mathbf{h}(t))\,vec^{\mathbf{T}}(\mathbf{x}_0\,\theta^{\mathbf{T}})\,(\mathbf{I}_{n_1} \otimes \mathbf{h}(t)) \\ &\quad + \mathbf{L}^{\mathbf{T}}(\mathbf{I}_{n_1} \otimes \mathbf{h}(t))\,vec^{\mathbf{T}}(\mathbf{d})(\mathbf{I}_{n_2} \otimes \mathbf{h}(t)) \end{aligned} \tag{5.57}$$

Then

$$\begin{aligned} vec^{\mathbf{T}}(\mathbf{c})(\mathbf{I}_{n_1} \otimes \mathbf{h}(t)) \;&=\; vec^{\mathbf{T}}(\mathbf{c})\,(\mathbf{I}_{n_1} \otimes \mathbf{P})(\mathbf{I}_{n_1} \otimes \mathbf{h}(t))(\mathbf{I}_{n_1} \otimes \mathbf{h}^{\mathbf{T}}(t))\mathbf{G} \\ &\quad + vec^{\mathbf{T}}(\mathbf{x}_0\,\theta^{\mathbf{T}})\,(\mathbf{I}_{n_1} \otimes \mathbf{h}(t))(\mathbf{I}_{n_1} \otimes \mathbf{h}^{\mathbf{T}}(t))\mathbf{G} \\ &\quad + vec^{\mathbf{T}}(\mathbf{d})(\mathbf{I}_{n_2} \otimes \mathbf{h}(t))(\mathbf{I}_{n_1} \otimes \mathbf{h}^{\mathbf{T}}(t))\mathbf{L} \end{aligned} \tag{5.58}$$

The product of $(\mathbf{I}\otimes\mathbf{h}(t))$ and $(\mathbf{I}\otimes\mathbf{h}^{\mathbf{T}}(t))$ can be expanded into a Haar series with a Haar coefficient block matrix $\hat{\mathbf{M}}$, which is given in Eqn. (3.51) as

$$
\begin{aligned}
vec^{\mathbf{T}}(\mathbf{c})(\mathbf{I}_{n_1}\otimes\mathbf{h}(t)) &= vec^{\mathbf{T}}(\mathbf{c})\,(\mathbf{I}_{n_1}\otimes\mathbf{P})\hat{\mathbf{M}}(\mathbf{G})(\mathbf{I}_{n_1}\otimes\mathbf{h}(t)) \\
&\quad + vec^{\mathbf{T}}(\mathbf{x}_0\,\theta^{\mathbf{T}})\,\hat{\mathbf{M}}(\mathbf{G})(\mathbf{I}_{n_1}\otimes\mathbf{h}(t)) \\
&\quad + vec^{\mathbf{T}}(\mathbf{d})\hat{\mathbf{M}}(\mathbf{L})(\mathbf{I}_{n_2}\otimes\mathbf{h}(t))
\end{aligned}
\qquad (5.59)
$$

where the block matrices $\hat{\mathbf{M}}(\mathbf{G})=\begin{bmatrix} \mathbf{M}(G_{11}) & \mathbf{M}(G_{21}) & \cdots & \mathbf{M}(G_{n_1 1}) \\ \mathbf{M}(G_{12}) & \mathbf{M}(G_{22}) & \cdots & \mathbf{M}(G_{n_1 2}) \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{M}(G_{1n_1}) & \mathbf{M}(G_{2n_1}) & \cdots & \mathbf{M}(G_{n_1 n_1}) \end{bmatrix}$ and

$\hat{\mathbf{M}}(\mathbf{L})=\begin{bmatrix} \mathbf{M}(L_{11}) & \mathbf{M}(L_{21}) & \cdots & \mathbf{M}(L_{n_1 1}) \\ \mathbf{M}(L_{12}) & \mathbf{M}(L_{22}) & \cdots & \mathbf{M}(L_{n_1 2}) \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{M}(L_{1n_2}) & \mathbf{M}(L_{2n_2}) & \cdots & \mathbf{M}(L_{n_1 n_2}) \end{bmatrix}$ are $n_1 m\times n_1 m$ and $n_2 m\times n_1 m$ matrices, respectively.

At collocation points, we obtain

$$vec^{\mathbf{T}}(\mathbf{c})\text{ - }vec^{\mathbf{T}}(\mathbf{c})\,(\mathbf{I}_{n_1}\otimes\mathbf{P})\hat{\mathbf{M}}(\mathbf{G})-vec^{\mathbf{T}}(\mathbf{d})\hat{\mathbf{M}}(\mathbf{L}) = vec^{\mathbf{T}}(\mathbf{x}_0\theta^{\mathbf{T}})\hat{\mathbf{M}}(\mathbf{G}) \qquad (5.60)$$

Transforming Eqn. (5.60) into a standard system of linear equations. we obtain

$$\begin{bmatrix}\mathbf{I}_{n_1 m}-\hat{\mathbf{M}}^{\mathbf{T}}(\mathbf{G})(\mathbf{I}_{n_1}\otimes\mathbf{P}^{\mathbf{T}}) & -\hat{\mathbf{M}}^{\mathbf{T}}(\mathbf{L})\end{bmatrix}\begin{bmatrix} vec(\mathbf{c}) \\ vec(\mathbf{d})\end{bmatrix}=\begin{bmatrix}\hat{\mathbf{M}}^{\mathbf{T}}(\mathbf{G})vec\,(\mathbf{x}_0\theta^{\mathbf{T}})\end{bmatrix} \qquad (5.61)$$

In this equation, all the multiplications must be performed blockwise (Lancaster and Tismenetsky, 1985).

5.4.2.4 Approximations of Equality Constraints

The equality constraint is of the form

$$x(t_f) = \mathbf{x}_{t_f} \tag{5.62}$$

where $x(t_f)$ is the terminal state variable and $\mathbf{x}_{t_f}$ is an $n_1 \times 1$ of the known finite time condition vector.

This constraint can be treated as the system dynamics. We substitute $x(t_f)$ from Eqn. (5.39) in the given constraint (5.62), and expand it in the Haar wavelet. The resulting constraint in $vec(\mathbf{c})$ and $vec(\mathbf{d})$ are then adjoined to the other constraints.

Substituting Eqn. (5.62) into Eqn. (5.39), we obtain

$$vec(\mathbf{x}_{t_f}) = t_f\,(\mathbf{I}_{n_1} \otimes \theta^{\mathbf{T}})\,vec(\mathbf{c}) + vec(\mathbf{x}_0) \tag{5.63}$$

Moving $vec(\mathbf{x}_0)$ to the other side, we have

$$t_f\,(\mathbf{I}_{n_1} \otimes \theta^{\mathbf{T}})\,vec(\mathbf{c}) = vec(\mathbf{x}_{t_f}) - vec(\mathbf{x}_0) \tag{5.64}$$

Eqn. (5.64) is rewriting by adding zero coefficients for the missing variable $vec(\mathbf{d})$ as bellows:

$$t_f\begin{bmatrix}(\mathbf{I}_{n_1} \otimes \theta^{\mathbf{T}}) & \mathbf{O}\end{bmatrix}\begin{bmatrix} vec(\mathbf{c}) \\ vec(\mathbf{d})\end{bmatrix} = \begin{bmatrix} vec(\mathbf{x}_{t_f}) - vec(\mathbf{x}_0)\end{bmatrix} \tag{5.65}$$

The resulting constraints in $vec(\mathbf{c})$ and $vec(\mathbf{d})$ are then adjoined to the other constraints in Eqn. (5.61) to form

$$\begin{bmatrix} \mathbf{I}_{n_1 m} - \hat{\mathbf{M}}^{\mathbf{T}}(\mathbf{G})(I_{n_1} \otimes \mathbf{P}^{\mathbf{T}}) & -\hat{\mathbf{M}}^{\mathbf{T}}(\mathbf{L}) \\ t_f\,(\mathbf{I}_{n_1} \otimes \theta^{\mathbf{T}}) & \mathbf{O} \end{bmatrix}\begin{bmatrix} vec(\mathbf{c}) \\ vec(\mathbf{d})\end{bmatrix} = \begin{bmatrix} \hat{\mathbf{M}}^{\mathbf{T}}(\mathbf{G})vec(\mathbf{x}_0\theta^{\mathbf{T}}) \\ vec(\mathbf{x}_{t_f}) - vec(\mathbf{x}_0)\end{bmatrix} \tag{5.66}$$

5.4.2.5 Approximations of Inequality Constraints

For inequality constraints the linear inequalities for the state and control variables in this study should also be determined. Haar wavelet collocation method is used to convert these constraints into quadratic programming constraints. Substituting Eqns. (5.22) and (5.23) into Eqns. (5.3) and (5.4) at collocation points, respectively, we form:

$$(\mathbf{I}_{n_1} \otimes \mathbf{H}^{\mathbf{T}}(t)\,\mathbf{P}^{\mathbf{T}})vec(\mathbf{c}) + vec(\mathbf{x}_0\theta^{\mathbf{T}}\mathbf{H}(t)) \le vec(\mathbf{x}_{\max}\theta^{\mathbf{T}}\mathbf{H}) \ , \tag{5.67}$$

$$(\mathbf{I}_{n_1} \otimes \mathbf{H}^{\mathbf{T}}(t)\,\mathbf{P}^{\mathbf{T}})vec(\mathbf{c}) + vec(\mathbf{x}_0\theta^{\mathbf{T}}\mathbf{H}(t)) \ge vec(\mathbf{x}_{\min}\theta^{\mathbf{T}}\mathbf{H}) \ , \tag{5.68}$$

$$(\mathbf{I}_{n_2} \otimes \mathbf{H}^{\mathbf{T}}(t)\,)vec(\mathbf{d}) \le vec(\mathbf{u}_{\max}\theta^{\mathbf{T}}\mathbf{H}) \ , \tag{5.69}$$

$$(\mathbf{I}_{n_2} \otimes \mathbf{H}^{\mathbf{T}}(t)\,)vec(\mathbf{d}) \ge vec(\mathbf{u}_{\min}\theta^{\mathbf{T}}\mathbf{H}) \, . \tag{5.70}$$

By moving the constant vector of Eqns. (5.67) and (5.68) to the other side and by changing the signs of Eqns. (5.68) and (5.70) to match the command of quadprog() at MATLAB, we generate:

$$(\mathbf{I}_{n_1} \otimes \mathbf{H}^{\mathbf{T}}\ \mathbf{P}^{\mathbf{T}})vec(\mathbf{c}) \le vec(\mathbf{x}_{\max}\ \theta^{\mathbf{T}}\mathbf{H}) - vec(\mathbf{x}_0\ \theta^{\mathbf{T}}\ \mathbf{H}) \ , \tag{5.71}$$

$$-(\mathbf{I}_{n_1} \otimes \mathbf{H}^{\mathbf{T}}\ \mathbf{P}^{\mathbf{T}})vec(\mathbf{c}) \le vec(\mathbf{x}_0\ \theta^{\mathbf{T}}\ \mathbf{H}) - vec(\mathbf{x}_{\min}\theta^{\mathbf{T}}\mathbf{H}) \ , \tag{5.72}$$

$$(\mathbf{I}_{n_2} \otimes \mathbf{H}^{\mathbf{T}}\ )vec(\mathbf{d}) \le vec(\mathbf{u}_{\max}\theta^{\mathbf{T}}\mathbf{H}) \ , \tag{5.73}$$

$$-(\mathbf{I}_{n_2} \otimes \mathbf{H}^{\mathbf{T}}\ )vec(\mathbf{d}) \le -vec(\mathbf{u}_{\min}\theta^{\mathbf{T}}\mathbf{H}) \, . \tag{5.74}$$

Combining Eqns. (5.71)-(5.74) after the zeros of the missing variables are added in the above equations, we obtain the following form of inequality constraints:

$$\begin{bmatrix} (\mathbf{I}_{n_1} \otimes \mathbf{H}^{\mathbf{T}}\ \mathbf{P}^{\mathbf{T}}) & \mathbf{O}_{n_1 m \times n_2 m} \\ -(\mathbf{I}_{n_1} \otimes \mathbf{H}\ \ \mathbf{P}^{\mathbf{T}}) & \mathbf{O}_{n_1 m \times n_2 m} \\ \mathbf{O}_{n_2 m \times n_1 m} & (\mathbf{I}_{n_2} \otimes \mathbf{H}^{\mathbf{T}}) \\ \mathbf{O}_{n_2 m \times n_1 m} & -(\mathbf{I}_{n_2} \otimes \mathbf{H}^{\mathbf{T}}) \end{bmatrix} \begin{bmatrix} vec\,(\mathbf{c}) \\ vec\,(\mathbf{d}) \end{bmatrix} \leq \begin{bmatrix} vec(\mathbf{x}_{\max}\ \theta^T \mathbf{H}) - vec(\mathbf{x}_0\ \theta^T \mathbf{H}) \\ vec(\mathbf{x}_0\ \theta^T \mathbf{H}) - vec(\mathbf{x}_{\min}\ \theta^T \mathbf{H}) \\ vec(\mathbf{u}_{\max}\ \theta^T \mathbf{H}) \\ -vec(\mathbf{x}_{\min}\ \theta^T \mathbf{H}) \end{bmatrix} \tag{5.75}$$

On the basis of the previous reformulation, the optimal control problems in Eqns. (5.6)–(5.12) can be approximated by the following quadratic programming problem:

$$\min_{Z}\ \frac{1}{2}\mathbf{Z}^{\mathbf{T}}\mathbf{H}_{\mathbf{ess}}\ \mathbf{Z} + \mathbf{F}^{\mathbf{T}}\mathbf{Z} + e \tag{5.76}$$

subject to

$$\mathbf{F_1 Z} = \mathbf{b}_1\ , \tag{5.77}$$

$$\mathbf{F_2 Z} \leq \mathbf{b}_2\ , \tag{5.78}$$

where

$$\mathbf{F_1} = \left[\mathbf{I}_{n_1 m} \quad -\hat{\mathbf{M}}^{\mathbf{T}}(\mathbf{G})(\mathbf{I}_{n_1} \otimes \mathbf{P}^{\mathbf{T}}) \qquad -\hat{\mathbf{M}}^{\mathbf{T}}(\mathbf{L})\right], \tag{5.79}$$

$$\mathbf{b}_1 = \begin{bmatrix} \hat{\mathbf{M}}^{\mathbf{T}}(\mathbf{G}) vec(\mathbf{x}_0\ \theta^{\mathbf{T}}) \\ vec(\mathbf{x}_{t_f}) - vex(\mathbf{x}_0) \end{bmatrix} \quad , \tag{5.80}$$

$$\mathbf{F}_2 = \begin{bmatrix} (\mathbf{I}_{n_1} \otimes \mathbf{H}^{\mathbf{T}}\ \mathbf{P}^{\mathbf{T}}) & \mathbf{O}_{n_1 m \times n_2 m} \\ -(\mathbf{I}_{n_1} \otimes \mathbf{H}\ \ \mathbf{P}^{\mathbf{T}}) & \mathbf{O}_{n_1 m \times n_2 m} \\ \mathbf{O}_{n_2 m \times n_1 m} & (\mathbf{I}_{n_2} \otimes \mathbf{H}^{\mathbf{T}}) \\ \mathbf{O}_{n_2 m \times n_1 m} & -(\mathbf{I}_{n_2} \otimes \mathbf{H}^{\mathbf{T}}) \end{bmatrix} \tag{5.81}$$

$$\mathbf{b_2} = \begin{bmatrix} vec(\mathbf{x}_{\max}\ \theta^T \mathbf{H}) - vec(\mathbf{x}_0\ \theta^T \mathbf{H}) \\ vec(\mathbf{x}_0\ \theta^T \mathbf{H}) - vec(\mathbf{x}_{\min}\ \theta^T \mathbf{H})\ \text{-} \\ vec(\mathbf{u}_{\max}\ \theta^T \mathbf{H}) \\ -vec(\mathbf{x}_{\min}\ \theta^T \mathbf{H}) \end{bmatrix} \tag{5.82}$$

The problems denoted by Eqns. (5.76)–(5.82) represent a standard quadratic programming problem that can be solved by using a solver such as quadprog() in MATLAB (Xue and Chen, 2011). Once the optimal solution to the unknown parameters $\mathbf{Z}$ is obtained, we substitute these parameters into Eqns. (5.18) and (5.20) to determine the new nominal states $x^{[k]}(t)$ and controllers $u^{[k]}(t)$ to be used in subsequent iterations. These new nominal trajectories should be substituted into Eqn. (5.7) to drive the next optimal control problem that is constrained linear quadratic. This procedure should be repeated until an acceptable convergence is achieved.

$$|J^{k+1} - J^{k}| < \varepsilon \tag{5.83}$$

The iteration is terminated when the difference between the two cost functions $|J^{k+1} - J^{k}|$ is sufficiently small.

5.5 Numerical Results and Discussions

In this section, a few examples of finite time horizon optimal control problems are solved using the method illustrated above. The proposed method is applied to linear and nonlinear quadratic optimal control problems that may be subject to one or two constraints. Examples in the succeeding subsection include linear optimal control problems with and without constraints. Examples of nonlinear optimal control problems with and without constraints are also illustrated to demonstrate the simplicity, effectiveness, and accuracy of the proposed numerical method.

We use a Haar wavelet algorithm implemented in the MATLAB for all of the examples presented in this section.

5.5.1 Linear Optimal Control Problem

In this subsection, the numerical method developed in this chapter is tested on the unconstrained linear quadratic optimal control problem and the inequality state constraint examples. These two examples will be illustrate and discuss in Example 1 and Example 2, respectively.

5.5.1.1 Example 1

Minimizes,

$$J = \int_0^{10} [x_1^2(t) + x_2^2(t) + u^2(t)]dt \, , \tag{5.84}$$

subject to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix} , \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \end{bmatrix}. \tag{5.85}$$

The optimal control problem in Eqns. (5.84)-(5.85) is linear. Thus, it can be solved by using the method described in the previous section; that is, directly transforming the problem into a quadratic programming problem subject to equality constraints directly without the need to apply quasilinearization method.

With the proposed method, the numerical solution to this problem is obtained by approximating both the state and control variables based on the Haar wavelet series of unknown parameters. The optimal value is found to be $J = 2744.15391937$ for $m = 256$, which is very close to the exact value of $J = 2744.154$.

The minimum value of J^* using Haar wavelet functions approximation with Haar wavelets resolutions of $m = 8, 16, 32, 64, 128$, and 256 are listed in Table 5.1. This example shows that even a coarse Haar wavelet resolution of $m = 8$ already yields an accurate result.

Table 5.1: Results of the performance index for Example 1 with resolution of Haar wavelet $m = 8, 16, 32, 64, 128$, and 256

Haar wavelet resolution m	J^*
8	2744.15466281
16	2744.15429557
32	2744.15403243
64	2744.15394865
128	2744.15392642
256	2744.15392078

Figures (5.1) - (5.3) present the graphical representations of the numerical solution with the optimal trajectories for different resolutions of Haar wavelets approximations functions for $m = 8, 16, 64$, and 256 for state variables and $m = 8, 16$, and 64 for control variable. These figures clearly show that the Haar wavelets approximation functions converges to the correct optimal trajectories as the resolution of the Haar wavelet functions increases.

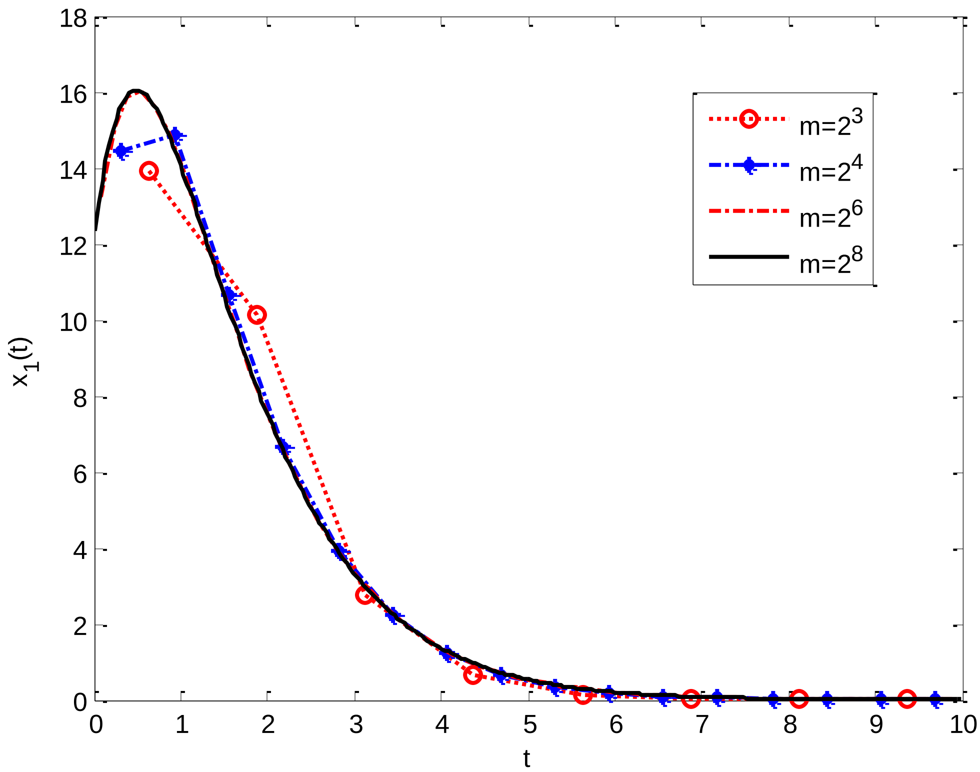


Figure 5.1: State variable $x_1(t)$ for Haar wavelet resolutions $m = 2^3, 2^4, 2^6, 2^8$ and $t_f = 10$ obtained from Example 1

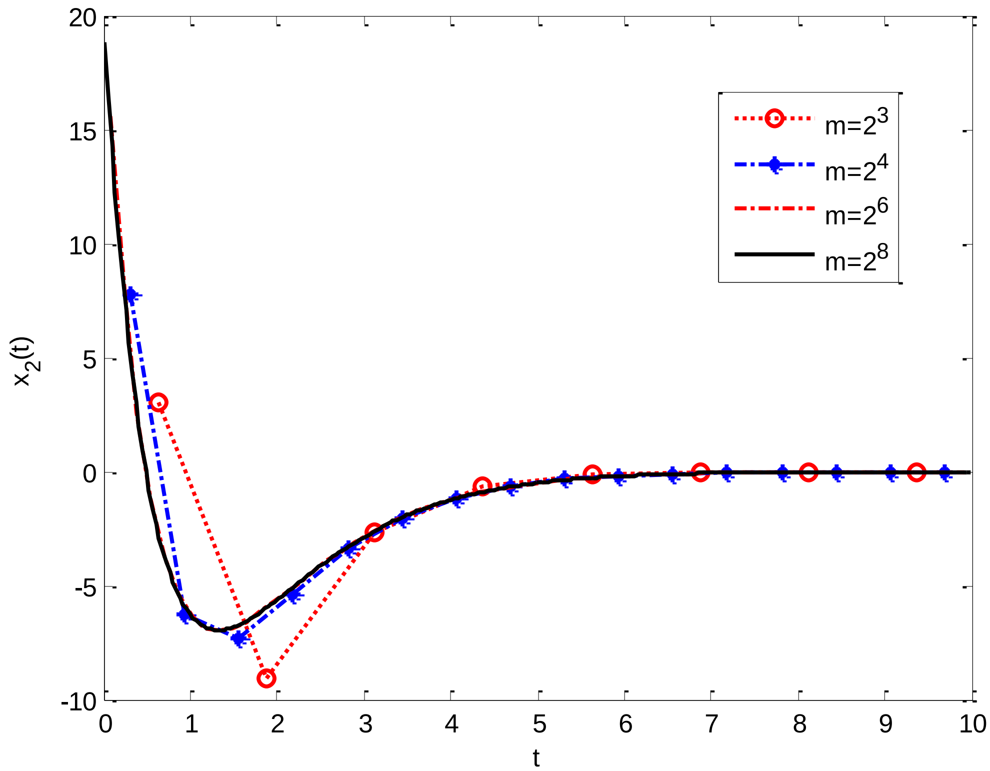


Figure 5.2: State variable $x_2(t)$ with for Haar wavelet resolutions and $m = 2^3, 2^4, 2^6, 2^8$ and $t_f = 10$ obtained from Example 1

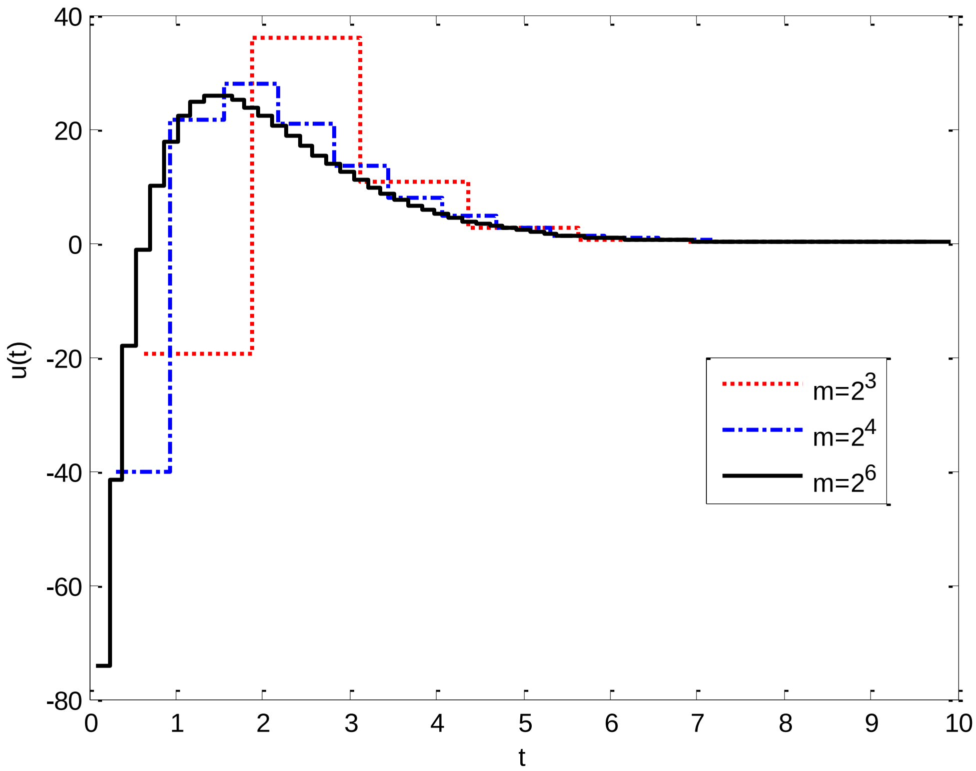


Figure 5.3: Control variable $u(t)$ with Haar wavelet resolutions $m = 2^3, 2^4, 2^6$ and $t_f = 10$ obtained from Example 1

5.5.1.2 Example 2

Consider the following performance index that minimizes, as presented by Kleinman et al. (1968):

$$J = \int_0^1 [x_1^2(t) + x_2^2(t) + 0.005u^2(t)]dt\,, \tag{5.86}$$

subject to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u(t)]\,, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{5.87}$$

with the following state variable inequality constraint

$$x_2(t) \le r(t) \ , \tag{5.88}$$

where $r(t)$ is an arbitrary known function

$$r(t) = 8(t - 0.5)^2 - 0.5 \tag{5.89}$$

Here, we solve the problem with Haar wavelet functions by choosing Haar wavelet resolutions of $m = 8, 16, 32, 64, 128,$ and 256. Expanding $r(t)$ and $x_2(t)$ in terms of Haar wavelet approximation functions, we obtain

$$r(t) = \beta^{\mathbf{T}}\mathbf{H} \tag{5.90}$$

$$x_2(t) = (\mathbf{I} \otimes \mathbf{H}^{\mathbf{T}}\ \mathbf{P}^{\mathbf{T}})\, vec(\mathbf{c}) + (\mathbf{I} \otimes \mathbf{H}^{\mathbf{T}})\, vec(\mathbf{x}_{02}\ \theta^{\mathbf{T}}) \tag{5.91}$$

For the inequality constraint in Eqn. (5.88) a treatment similar to that applied in Eqn. (5.3) is suggested. First, the inequality constraint from the state variable is converted into Haar wavelet collection points by substituting Eqns. (5.90) and (5.91) into Eqn. (5.88). Then, we have

$$\begin{bmatrix} \mathbf{O}_{m\times m} & (\mathbf{I} \otimes \mathbf{H}^{\mathbf{T}}\ \mathbf{P}^{\mathbf{T}}) & \mathbf{O}_{m\times m} \end{bmatrix} \begin{bmatrix} vec\,(\mathbf{c}) \\ vec\,(\mathbf{d}) \end{bmatrix} \le \left[vec(\beta^{\mathbf{T}}\mathbf{H}) - vec(\mathbf{x}_{02}\ \theta^{T}\mathbf{H}) \right] \tag{5.92}$$

where the inequality constraint in Eqn. (5.92) represents the standard form of the quadratic programming problem in Eqn. (5.78).

Table 5.2 shows the cost values for different values of Haar wavelet resolutions of $m = 8$, 16, 32, 64, 128, and 256. These values are obtained from applying the proposed method on Example 2.

Table 5.2: Results of the performance index for Example 2 for Haar wavelet resolutions $m = 8, 16, 32, 64, 128$ and 256

Haar wavelet resolution m	J^*
8	0.17254748
16	0.17109637
32	0.17011179
64	0.16989690
128	0.16983953
256	0.16983337

The computational result for $r(t)$ together with $x_1(t)$ and $x_2(t)$ and $u(t)$ using the present method for $m = 128$ and $t_f = 1$ are given in Figures 5.4 and 5.5, respectively.

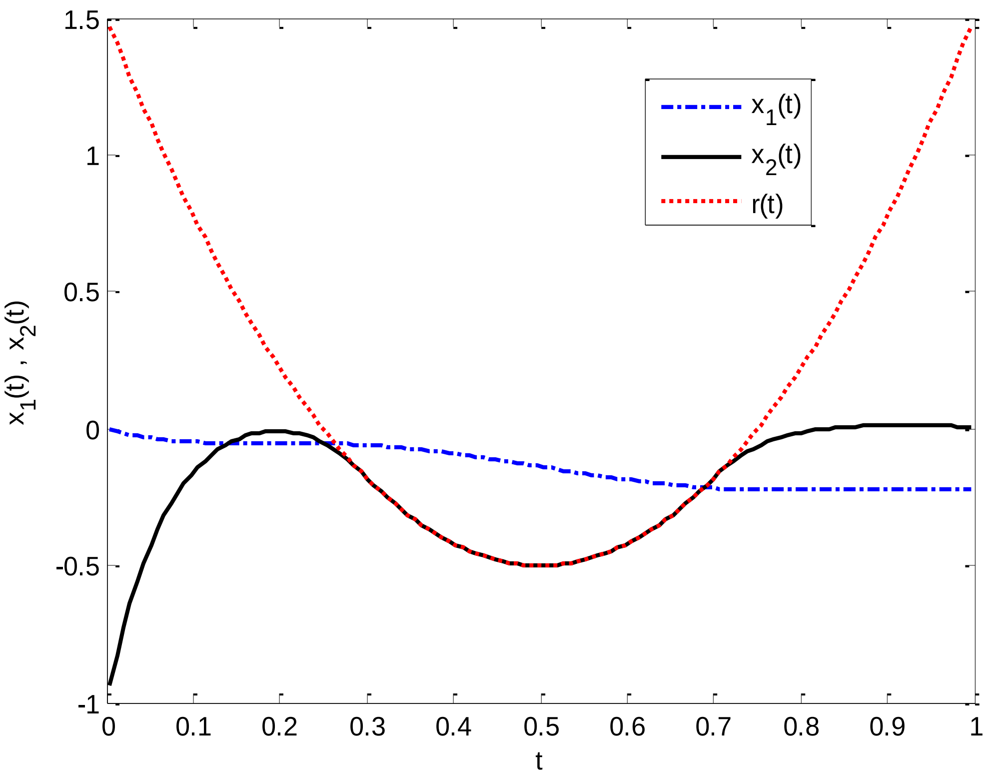


Figure 5.4: State variables $x_1(t)$, $x_2(t)$ and inequality constraint $r(t)$ obtained in Example 2 for $m = 128$ and $t_f = 1$

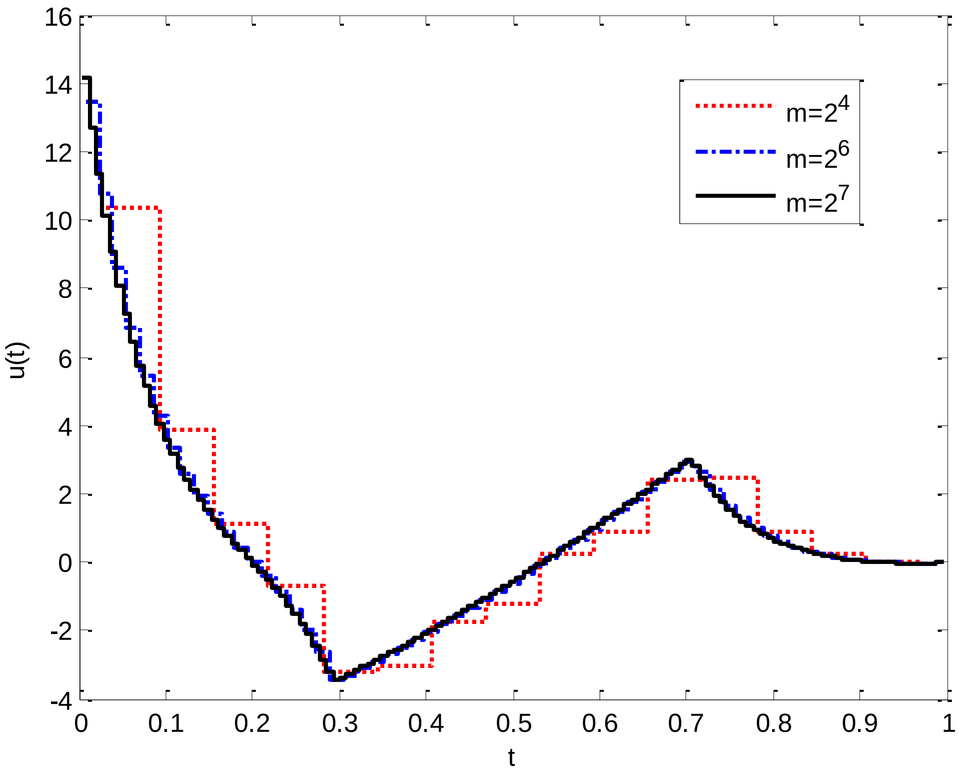


Figure 5.5: Control variable $u(t)$ obtained in Example 2 for $m=128$ and $t_f=1$

This example has been solved by using generalized gradient method (Jacobson and Lele, 1969; Mehra and Davis, 1972); classical Chebyshev polynomial (Neuman and Sen, 1973; Vlassenbroeck, 1988); Chebyshev spectral method (Jaddu, 2002); hybrid functions approximations (Marzban and Razzaghi, 2003); rationalized Haar functions (Marzban and Razzaghi, 2010); Triangular orthogonal Function (Han et al., 2012); Bézier control points (Ghomanjani et al., 2012). The performance index can be compared to the findings of other researchers in Table 5.3, which indicates that Jacobson and Lele, (1969) offer the lowest performance index. Our result for optimal values is also shown in Table 5.3 for comparison.

Based on Table 5.3, we can be concluded that the proposed technique exhibits competitive performance, as demonstrated in Example 2.

Table 5.3: Simulation results of the performance index for Example 2 (Vlassenbroeck, 1988)

Source	J^*
Jacobson and Lele (1969)	0.164
Mehra and Davis (1972)	0.178
Neuman and Sen (1973)	
N=9	0.16946
Vlassenbroeck (1988)	
$m=13$, $K=28$	0.17185
Jaddu(2002)	0.17078488
Marzban and Razzaghi (2003)	
M=4, N=4	0.17013640
Marzban and Razzaghi (2010)	0.170103
Han *et al.* (2012)	0.170835
Ghomanjani *et al.* (2012)	0.17289045
Present result $m=256$	0.16983337

5.5.2 Nonlinear Optimal Control Problem

In this subsection, we consider Van der Pol oscillator problem which is adapted from Jaddu (1998). We consider two cases: (1) unconstrained problem; and (2) terminal states and control constrained problem. These two cases will be discussed in Examples 3 and 4, respectively.

5.5.2.1 Example 3

Consider the following nonlinear system state equations, as presented by Jaddu (1998):

$$\dot{x}_1(t) = x_2(t) \tag{5.93}$$

$$\dot{x}_2(t) = -x_1(t) + (1 - x_1^2(t))\, x_2(t) + u(t) \tag{5.94}$$

The cost function to be minimized, starting from the initial states $x_1(0)=1$ and $x_2(0)=0$, is

$$J=\int_0^5\left(x_1^2(t)+x_2^2(t)+u^2(t)\right)dt \tag{5.95}$$

To solve this example using the proposed method, the system Eqns. (5.93) and (5.94) as well as the performance index (5.95) are expanded up to the first order around nominal trajectories of $x_1^{[k]}(t)$ and $x_2^{[k]}(t)$ by using quasilinearization technique.

The expanded performance index is

$$J^{[k]}=\int_0^5\left((x_1^{[k]})^2+(x_2^{[k]})^2+(u^{[k]})^2\right)dt \tag{5.96}$$

and the linearized state equations are

$$\begin{bmatrix}\dfrac{dx_1^{[k]}(t)}{dt}\\ \dfrac{dx_2^{[k]}(t)}{dt}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1-2\,x_1^{[k-1]}(t)x_2^{[k-1]}(t) & 1-\ (x_1^{[k-1]}(t))^2\end{bmatrix}\begin{bmatrix}x_1^{[k]}\\ x_2^{[k]}\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u^{[k]}\ ,\quad k\geq 1 \tag{5.97}$$

The state and control variables are approximated by using Haar wavelet approximation functions. Then, the linear quadratic optimal control problems in Eqns. (5.96)-(5.97) are converted into a quadratic programming problem, which is then solved successively until the difference between the two cost functions satisfy $\left|\,J^{k+1}-J^{k}\,\right|<10^{-4}$. This difference is achieved in five quasilinearization iterations for a Haar wavelet resolution of $m=64$. The approximate optimal value for the cost function and the difference $\left|\,J^{k+1}-J^{k}\,\right|$ of these five quasilinearization iterations, starting from initial nominal trajectories with $m=64$, are summarized in Table 5.4.

Table 5.4: Optimal value of the performance index and convergence error for Example 3 with Haar wavelet resolution $m = 64$

Iteration (k)	$J^*_{\min}$	Convergence error $\lvert J^{k+1} - J^k \rvert$
1	0.95333975	-
2	1.44082952	0.48748977
3	1.43778685	0.00304267
4	1.43744694	0.00033991
5	1.43747817	0.00003123

The optimal trajectories for five quasilinearization iterations with $m = 64$ are shown in Figures 5.6 - 5.8, which clearly demonstrate that the trajectories nearly converge to the optimal ones, after the second quasilinearization step.

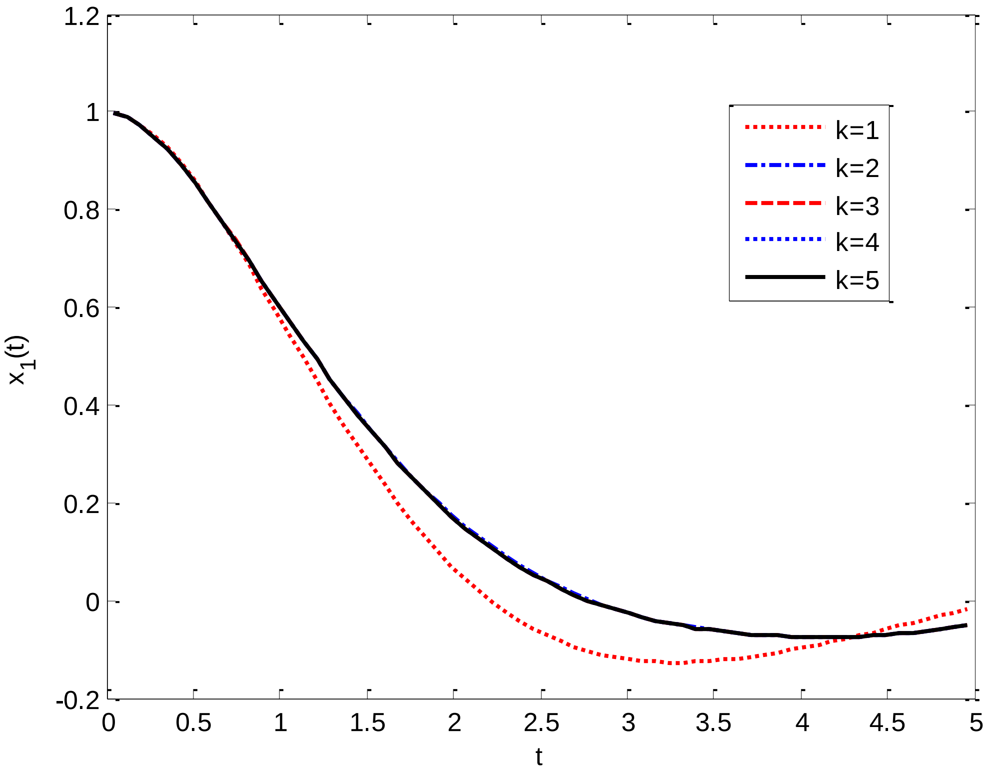


Figure 5.6: State variable $x_1(t)$ of Example 3 for 5 quasilinearization iterations with $m = 64$ and $t_f = 5$

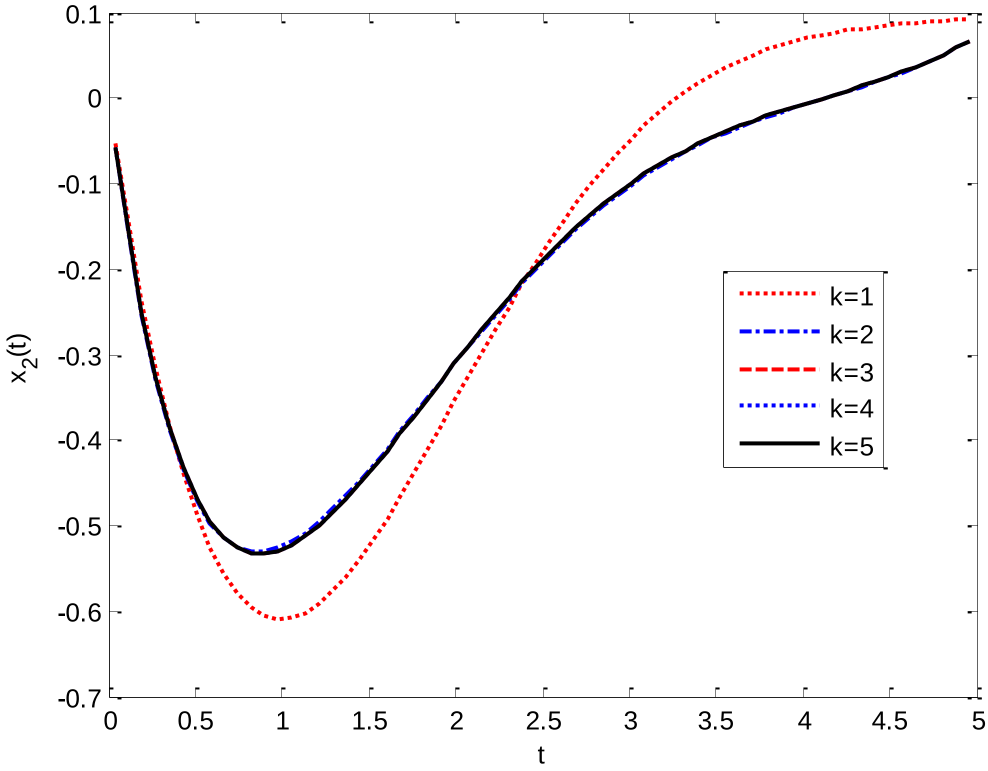


Figure 5.7: State variable $x_2(t)$ of Example 3 for 5 quasilinearization iterations with $m = 64$ and $t_f = 5$

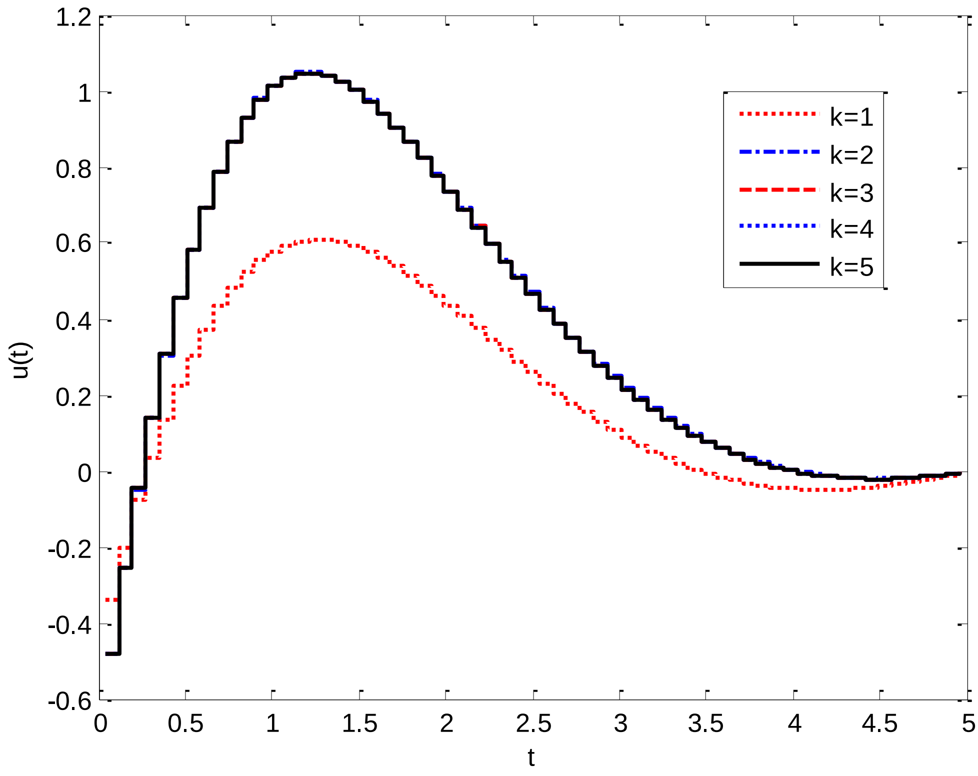


Figure 5.8: Control variable $u(t)$ of Example 3 for 5 quasilinearization iterations with $m = 64$ and $t_f = 5$

The optimal values with different values of Haar wavelet resolutions of $m = 8, 16, 32, 64, 128$, and 256 that are obtained from Example 3 are shown in Table 5.5. Figures 5.9-5.11 illustrate the optimal trajectories with Haar wavelet resolutions of $m = 8, 16, 32$, and 64 and $t_f = 5$.

Table 5.5: Results of the performance index for Example 3 with Haar wavelet resolutions $m = 8, 16, 32, 64, 128$ and 256

Haar wavelet resolution m	J^*
8	1.45505319
16	1.44169372
32	1.43831978
64	1.43747817
128	1.43726793
256	1.43721538

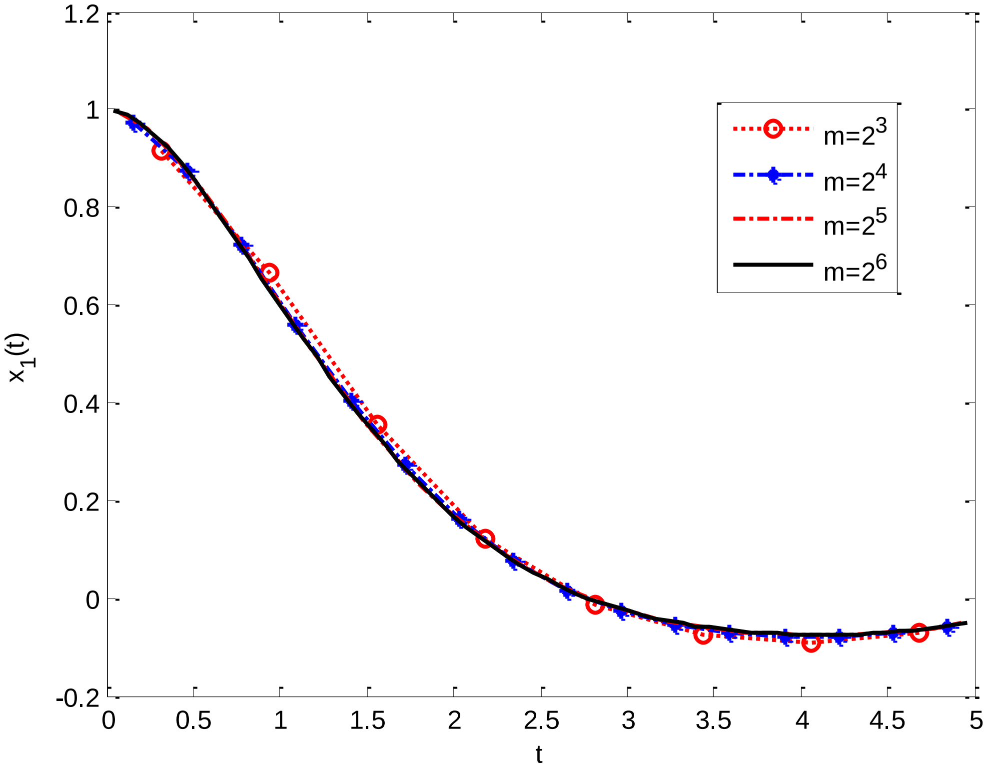


Figure 5.9: State variable $x_1(t)$ obtained in Example 3 for and $m = 2^3, 2^4, 2^5$ and 2^6 with $t_f = 5$

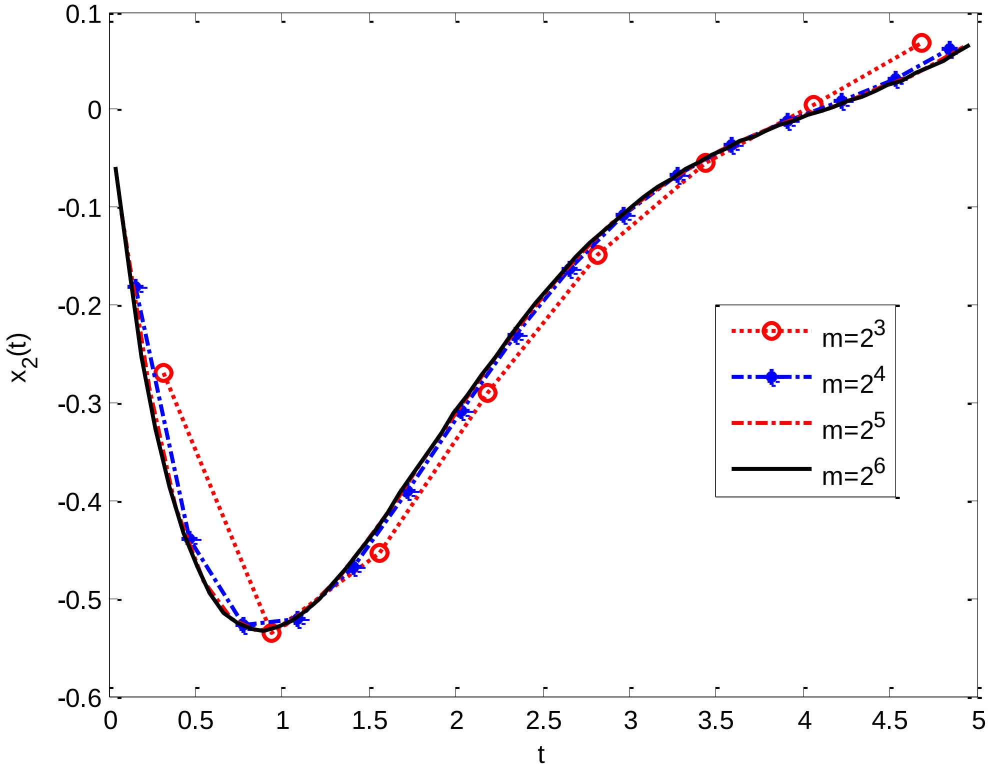


Figure 5.10: State variable $x_2(t)$ obtained in Example 3 for and $m = \mathbf{2}^3, \mathbf{2}^4, \mathbf{2}^5$ and $\mathbf{2}^6$ with $t_f = 5$

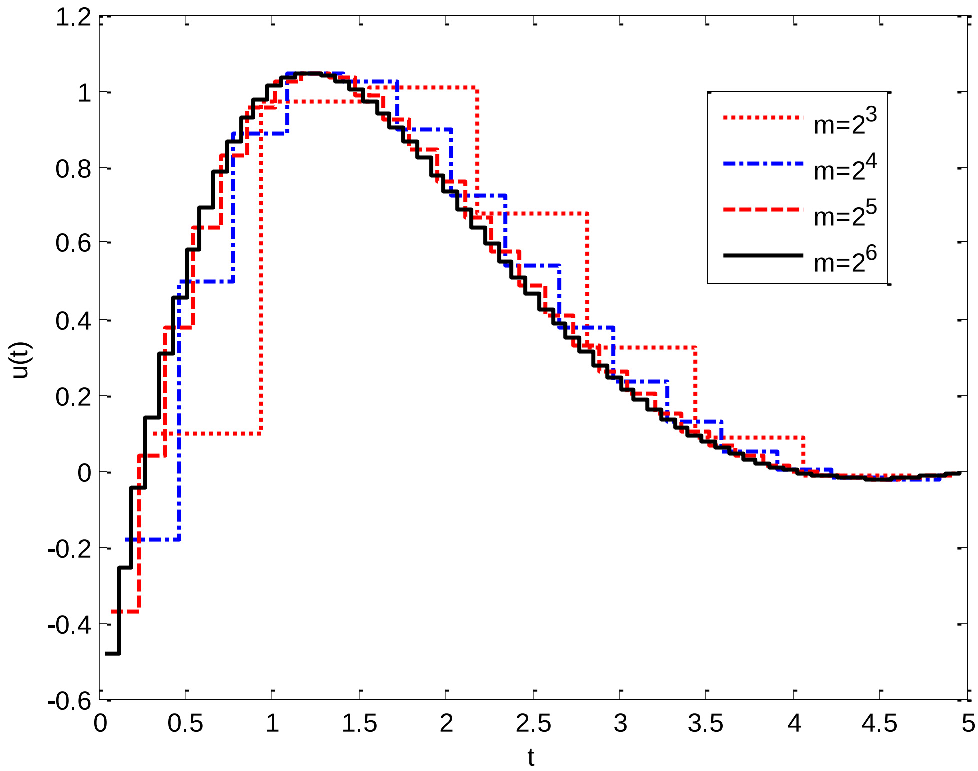


Figure 5.11: Control variable $u(t)$ obtained in Example 3 for $m = 2^3, 2^4, 2^5$ and 2^6 with $t_f = 5$

This problem has been solved by using second variation method (Bullock and Franklin, 1967) and quasilinearization and discretization method (Bashein and Enns, 1972). Jaddu (1998) used Chebyshev polynomials with parameterize the state variables to study this example. In a recent study, Jaddu and Majdalawi (2014) solved this proplem by using Legendre polynomials iterative technique. Table 5.6 shows the optimal values obtained from these methods for comparison.

Table 5.6: Simulation results of the performance index for Example 3 (Vlassenbroeck, 1988)

Source	J^*
Bullock and Franklin (1967)	1.433508
Bashein and Enns (1972)	1.438097
Jaddu (1998)	1.433487
Jaddu and Majdalawi (2014)	1.449396
Present result, $m = 256$	1.43721538

5.5.2.2 Example 4

Consider the following performance index that minimizes (Jaddu, 1998):

$$J = \int_0^5 \left(x_1^2(t) + x_2^2(t) + u^2(t)\right)dt \tag{5.98}$$

subject to

$$\dot{x}_1(t) = x_2(t) \tag{5.99}$$

$$\dot{x}_2(t) = -x_1(t) + (1 - x_1^2(t))\, x_2(t) + u(t) \tag{5.100}$$

$$|u(t)| \le 0.75 \tag{5.101}$$

$$x_1(0) = 1 \text{ , } x_2(0) = 0 \tag{5.102}$$

$$x_1(t_f) = -1 \text{ , } x_2(t_f) = 0 \tag{5.103}$$

To find the solution numerically using the proposed method, we need to express the equality and inequality constraints on the state and control variables from Eqns. (5.101) and (5.103) in a Haar wavelet series and apply the Haar wavelet collection points. These equations can be treated like Eqns. (5.2) and (5.4) after simplifying Eqn. (5.101). We substitute $x(t_f)$ and $u(t)$ from Eqns. (5.64) and (5.22) into the following equations.

$$u(t) \le 0.75 \tag{5.104}$$

$$-u(t) \le 0.75 \tag{5.105}$$

$$x(t_f) = \mathbf{x}_{t_f} \tag{5.106}$$

where $\mathbf{x}_{t_f}$ is the finite time vector and $t_f = 5$ is the finite time.

Finally, the resulting constraints in $vec(\mathbf{c})$ and $vec(\mathbf{d})$ are then adjoined to the other constraints, which are then explanted in Sections 5.6.4 and 5.6.5, respectively. Then, the problem is solved with the use of for using Haar wavelet resolutions of $m = 8, 16, 32, 64, 128$ and 256. For each m, convergence is achieved in six quasilinearization iterations except for $m = 8$ and 16, which are achieved in eight and seven quasilinearization iterations, respectively. The iteration is terminated when the difference between two cost functions $|J^{k+1} - J^k|$ is less than $\varepsilon = 0.0001$.

Table 5.7 shows the optimal cost function for different values of Haar wavelet resolutions that are obtained from Example 4.

Table 5.7: Results of the performance index for Example 4 with Haar wavelet resolution for $m = 8, 16, 32, 64, 128, 256$

Haar wavelet resolution m	J^*
8	2.1448152143
16	2.1397055318
32	2.1423550823
64	2.1429211308
128	2.1430566067
256	2.1430830369

Table 5.8 shows that the optimal value after six quasilinearization iterations is $J_{\min} = 2.1430566067$ for a Haar wavelet resolution of $m = 128$. The optimal values of each iteration with the convergence errors are also shown in this table.

Table 5.8: Optimal value of the performance index and convergence error for Example 4 with resolution of Haar wavelet $m = 128$

Iteration (k)	$J_{\min}$	**Convergence error** $\lvert J^{k+1} - J^k \rvert$
1	1.8540226638	-
2	2.1812152841	0.32719262
3	2.1390510574	0.04216423
4	2.1433925447	0.00434149
5	2.1430318903	0.00036065
6	2.1430566067	0.00002472

The computational result for $x_1(t)$, $x_2(t)$ and $u(t)$ using the proposed method for $m = 128$ are given in Figures 5.12 and 5.13, respectively.

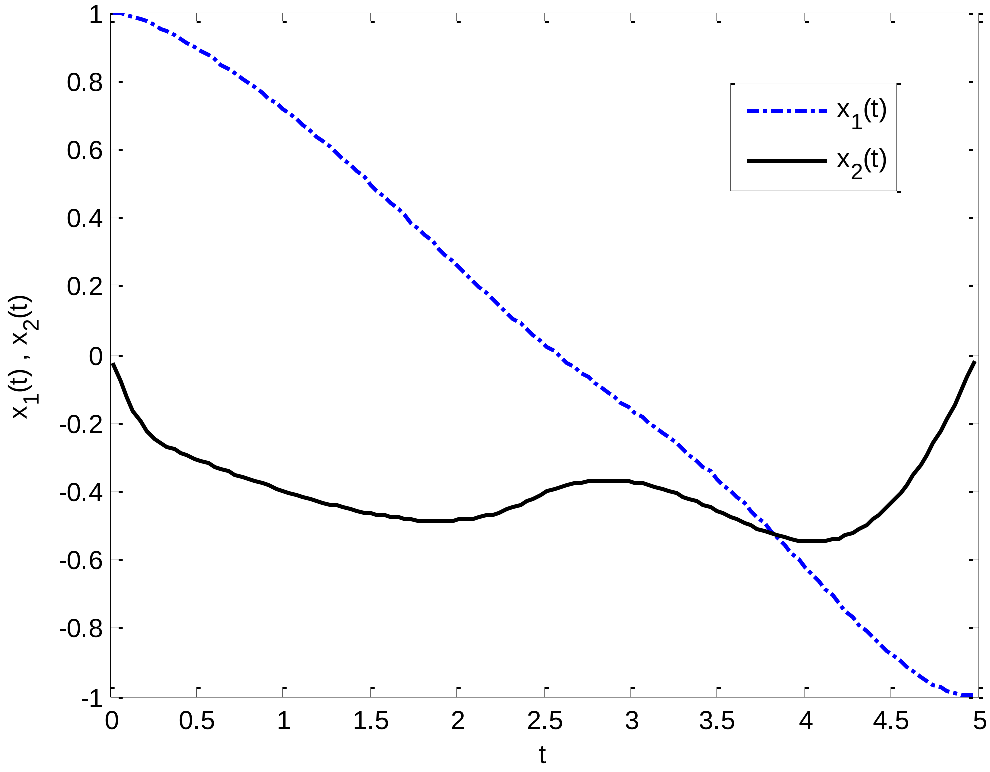


Figure 5.12: States variables $x_1(t)$ and $x_2(t)$ obtained in Example 4 for $m = 128$ and $t_f = 5$

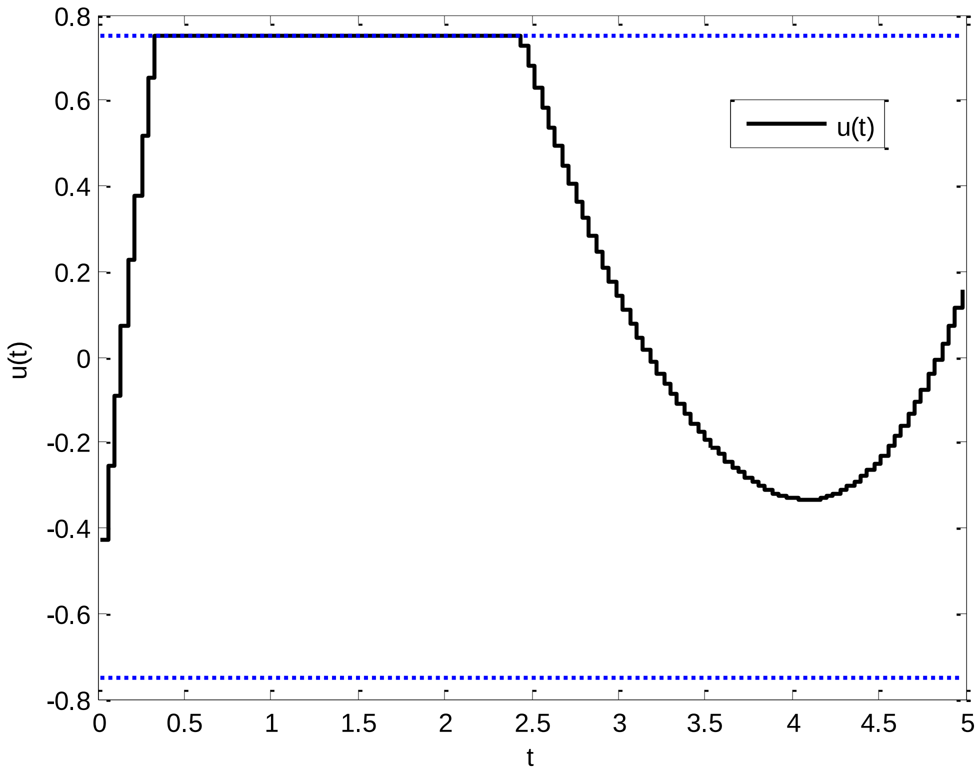


Figure 5.13: Control variable $u(t)$ obtained in Example 4 for $m = 128$ and $t_f = 5$

As shown in Table 5.9, example 4 has been investigated using Chebyshev polynomials with parameterize the state variables (Jaddu, 2002); rationalized Haar functions (Han and Li, 2011) and Triangular orthogonal Function (Han et al., 2012).

Table 5.9: Simulation results of the performance index for Example 4

Source	J^*
Jaddu (2002) N=12	2.14141
Han and Li (2011) k=16	2.14959
Han et al. (2012) M=64	2.14056
Present result, $m = 256$	2.1430830369

5.6 Conclusion

In this chapter, we proposed a new numerical method for solving finite time horizon nonlinear optimal control problems with state and control inequality constraints. The proposed approach employs quasilinearization method and parameterization of state and control variables using Haar wavelet functions and the Haar wavelet operational matrix to convert the nonlinear optimal control problem into a sequence of constrained time-varying linear quadratic programming problem. The linear inequality constraints for trajectory variables are converted to quadratic programming constraints by using Haar wavelet collocation. Further, the terminal state constraints are converted using Haar wavelet functions and are adjoined to the system dynamics constraints. The proposed method is simple and it has been tested for a constrained nonlinear quadratic optimal control problem in two-dimensional state space with one controller. The accuracy of the state and control variables, as well as the cost, can be improved by increasing the Haar wavelet resolution.

In contrast to the HJB equation solution to infinite time horizon optimal control problem presented in Chapter 4, the direct solution is characterized as follows:

Open-loop: The resulting optimal trajectory is explicitly solved as a function of time $u(t)$ and not as a feedback control law.

Local: The resulting solution is only valid for the specified initial condition $x(0)$. When a new initial condition is specified, the problem must be solved again.

However, this method provides an alternative way of finding the solution for constrained optimal control problems in a fastest time. Furthermore, the sparse matrices that appeared during the calculation contribute to a faster computational analysis. Numerical results demonstrate the good performance of the proposed method used in term of accuracy and competitiveness compared with existing approaches. The proposed method is very convenient, as it requires only simple computing systems and low computer memory with small m.

CHAPTER 6

APPLICATION OF HAAR WAVELET METHOD TO PRDOUCTION-INVENTORY MODEL

6.1 Introduction

Application of optimization methods to production and inventory problems date back to the classical economic order quantity (EOQ) model or the lot size formula by Harris (1913). The EOQ is a static model wherein the demand is constant and only a stationary solution is sought. An important dynamic production-planning model was developed by Holt et al. (1960), where both production costs and inventory holding costs over time were considered. They used various calculus techniques to solve the continuous-time version of their model. Furthermore, Dobos (1999) studied the effect of constraints on the production and inventory model. Dobos (1999) modified the Holt et al. (1960) model and used optimal control theory to derive the optimal production rate.

Most inventory models deal with a single-item (Balkhi and Benkherouf, 2004). However, such models are seldom applied in the real world. Hence, multi-item inventory models are more realistic than single-item models are. In multi-item models, the second item in an inventory favors the demand for the first item, and vice-versa (Sethi and Thompson, 2006). This is why retailers deal with several items and stock them in their show rooms (Bhattacharya, 2005).

In this chapter, the direct method proposed in the previous chapter is applied to optimize the control of the two-item production-inventory model with stock-dependent deterioration rates and deterioration due to self-contact and the presence of the other stock. Four different types of demand rates are used, namely, constant, linear, logistic and periodic demand rates. The solution to the model is discussed numerically and presented graphically. By enhancing the resolution of the Haar wavelet, we can improve the accuracy of the states, controls, and cost. Simulation results are also compared with the work of other researchers.

6.2 Optimal Control of Two-Item Production-Inventory Model

This section is devoted to the mathematical formulations and model assumptions for the optimal control of the two-item inventory model with deteriorating items of different deterioration types. In two-item models, the second item in an inventory favors the demand for the first item, and vice-versa. We consider a factory that produces two items and has a finished goods warehouse. The objective function includes the sum of inventory holding costs, the holding costs of one item as a result of the presence of other items, and production costs. The problem is considered an optimal control problem with two state and two control variables, which are inventory levels y_i and production rates v_i, respectively. The following variables and parameters are used:

$y_i(t)$	:	the inventory levels at time t
$u_i(t)$	:	the production rates at time t
t_f	:	the length of the planning horizon
$\hat{y}_i$	:	the inventory goal levels
$\hat{v}_i$	:	the production goal rates
y_{io}	:	the initial inventory levels
c_{ii}	:	the production cost coefficients
h_{ii}	:	the inventory holding cost coefficients
h_{12}	:	the inventory holding cost coefficient of y_1 due to the presence of unit of y_2, or vice- versa

$D_i(y_1, y_2, t)$: the demand rates at the instantaneous levels of the inventory $(y_1(t), y_2(t))$ and time t

a_{ii} : the deterioration coefficient due to self-contact of y_i

a_{ij} : the demand coefficient of y_i due to presence of unit of y_j , $i \neq j$

θ_i : the natural deterioration rate of y_i

The optimal control problem is defined to determine the production rate, which minimizes the total cost,

$$J = \min \int_0^{t_f} \left\{ \sum_{i=1}^{2} (h_{ii}(y_i - \hat{y}_i)^2 + c_{ii}(v_i - \hat{v}_i)^2) + 2h_{12}(y_1 - \hat{y}_1)(y_1 - \hat{y}_1) \right\} dt \tag{6.1}$$

where $t \in [0, t_f]$, $h_{11}h_{22} > h_{12}^2$, $h_{ii} > 0$, $c_{ii} > 0$

subject to

$$\dot{y}_1 = -y_1(t)(\theta_1 + a_{12}y_2(t) + a_{11}y_1(t)) - D_1(y_1, y_2, t) + v_1(t), \tag{6.2}$$

$$\dot{y}_2 = -y_2(t)(\theta_2 + a_{21}y_1(t) + a_{22}y_2(t)) - D_2(y_1, y_2, t) + v_2(t), \tag{6.3}$$

with constraints,

$$y_i(t) \geq 0 \quad , \tag{6.4}$$

$$v_i(t) \geq 0 \, . \tag{6.5}$$

This system is nonlinear and is difficult to solve analytically. Therefore, we address the system numerically and display the results graphically. The objective function Eqn. (6.1) can be economically interpreted as an effort to keep the inventory levels $(y_1(t), y_2(t))$ and production rates ($(v_1(t), v_2(t))$ as close as possible to the target levels ($\hat{y}_1, \hat{y}_2$) and rates ($\hat{v}_1, \hat{v}_2$), respectively. The system dynamics in Eqns. (6.2) and (6.3) can be used to describe the time evolution of inventory levels and production rates (El-Gohary and Elsayed, 2008).

6.3 Reformulation of the Optimal Control of the Two-Item Production-Inventory Model

In this section, we reformulate the two-item production-inventory model as a nonlinear quadratic problem with the following substitution:

$$x_i(t) = y_i(t) - \hat{y}_i\,, \tag{6.6}$$

$$u_i(t) = v_i(t) - \hat{v}_i\,. \tag{6.7}$$

In particular, we have $x_{0i} = y_{0i} - \hat{y}_i$ and $\dot{x}_i(t) = \dot{y}_i(t)$.

The problem in Eqns. (6.1) and (5.5) can be reformulated as:

$$J = \min \int_0^{t_f} \mathbf{(}x^{\mathbf{T}}(t)\,\mathbf{Q}\,x(t) + u^{\mathbf{T}}(t)\,\mathbf{R}\,u(t)\mathbf{)}\,dt\,, \tag{6.8}$$

subject to

$$\dot{x}_1(t) = -(x_1(t)+\hat{y}_1)(\theta_1 + a_{12}(x_2(t)+\hat{y}_2) + a_{11}(x_1(t)+\hat{y}_1) - D_1(x_1, x_2, t) + u_1(t) + \hat{v}_1 \tag{6.9}$$

$$\dot{x}_2(t) = -(x_2(t)+\hat{y}_2)(\theta_2 + a_{21}(x_1(t)+\hat{y}_1) + a_{22}(x_2(t)+\hat{y}_2) - D_2(x_1, x_2, t) + u_2(t) + \hat{v}_2 \tag{6.10}$$

with constraints

$$x_i(t) + \hat{y}_i \geq 0 \tag{6.11}$$

$$u_i(t) + \hat{v}_i \geq 0 \tag{6.12}$$

where $\mathbf{Q} = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix}$.

The numerical solution to this problem as obtained using the proposed method is discussed for the following four different types of demand rates:

1. Constants: $D_i(y_1, y_2, t) = \alpha_i$

2. Linear functions of inventory levels: $D_i(y_1, y_2, t) = d_i y_i + \alpha_i$

3. Logistic functions of inventory levels: $D_i(y_1, y_2, t) = y_i(g_i - y_i)$

4. Periodic function of time: $D_i(y_1, y_2, t) = 1 + k_i \sin(t)$

where α_i, d_i, g_i, and k_i are positive constants for $i = 1, 2$.

6.4 Numerical Solution

In this section, we solve the optimal control problem of the two-item production-inventory model by using four different types of demand rates, namely, constant, linear, logistic and periodic demand rates. The solution to the model is discussed numerically and presented graphically.

6.4.1 Constant Demand Rates

In this subsection, we present the numerical solution in the case of constant demand rates. We substitute the controlled system Eqns. (6.9) to (6.10) for $D_i = \alpha_i$, where $\alpha_1 = 0.6$ and $\alpha_2 = 0.8$, using the parameter values and initial states presented in Table 6.1.

Table 6.1: Values of system parameters and initial states (El-Gohary and Elsayed, 2008)

Parameter	h_{11}	c_{11}	θ_1	a_{11}	a_{12}	$\hat{v}_1$	y_{01}	$\hat{y}_1$	h_{12}
Value	4	6	0.02	0.04	0.7	9	2	4	-4
Parameter	h_{22}	c_{22}	θ_2	a_{22}	a_{21}	$\hat{v}_2$	y_{20}	$\hat{y}_2$	t_f
Value	5	5	0.03	0.05	0.6	8	1	3	5

The numerical solution in this application is obtained by using the algorithm presented in Chapter 5, Section 6. Each state and control variable is first approximated using Haar wavelet functions at m^{th} resolution and the Haar wavelet operational matrix. Then, the objective function Eqn. (6.8) subject to nonlinear dynamic system Eqns. (6.9) and (6.10) is converted into a sequence quadratic programming problem through the quasilinearization iterative technique. The inequality constraints for state and control variables Eqns. (6.11) and (6.12) are transformed into quadratic programming constraints by using the Haar wavelet collocation method. The quadratic programming problem with linear inequality constraints is then solved using a standard QP solver. The optimal control problem, which is subject to constraints Eqns. (6.11) and (6.12), is solved beginning with nominal trajectories $x_1^0 = -2$ and $x_2^0 = -2$ for $m = 8, 16, 32, 64, 128,$ and 256. For each m, convergence is achieved in three quasilinearization iterations. The iteration is terminated when the difference between two cost functions $| J^{k+1} - J^k |$ is less than $\varepsilon = 0.0001$.

Table 6.2 summarizes the results obtained from these six cases of Haar wavelet resolution, including the simulated optimal values of inventory levels and production rates, as well as the total cost at the end of the planning horizon period. Figures 6.1 to 6.4 show the optimal values of the inventory levels and the production rates for $m = 256$ and its successive quasilinearization iteration. Table 6.2 indicates that the approximated cost function converges with the true cost function as we increase the resolution of the Haar wavelet. Figures 6.1 and 6.2 also suggest that the optimal inventory levels increase over time. Figures 6.3 and 6.4 show how production rates were optimized and tended to their goal rates at the end of the planning horizon period.

Table 6.2: Simulation results of the application of the direct method using constant demand rates for m = 8, 16, 32, 64, 128, and 256

m	$y_1^*(t_f)$	$y_2^*(t_f)$	$v_1^*(t_f)$	$v_2^*(t_f)$	J^*
8	3.83137254	2.89765654	9.01076105	7.98865953	0.38994675
16	3.82833273	2.90028336	9.00648218	7.99359316	0.40195069
32	3.82633976	2.90198127	9.00363305	7.99661281	0.40686114
64	3.82518677	2.90295078	9.00193897	7.99826898	0.40828178
128	3.82453701	2.90345359	9.00100455	7.99912747	0.40872216
256	3.82421337	2.90372111	9.00051167	7.99956248	0.40881472

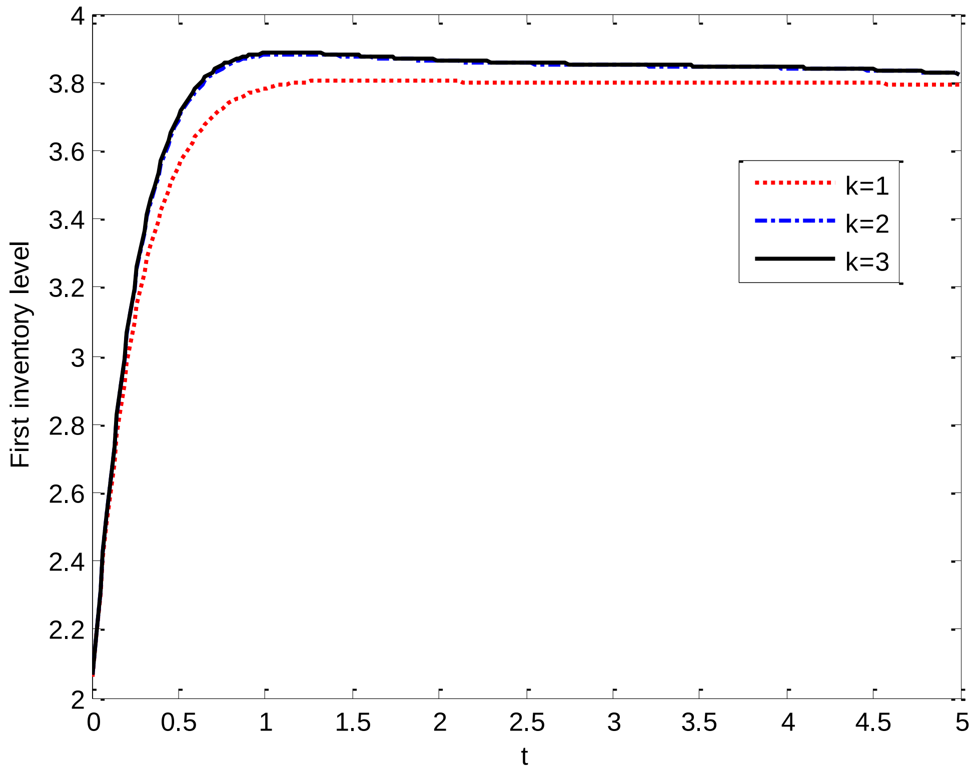


Figure 6.1: First inventory level against time $t_f = 5$ and m = 256 using constant demand rates for k = 1, 2, 3 quasilinearization iterations

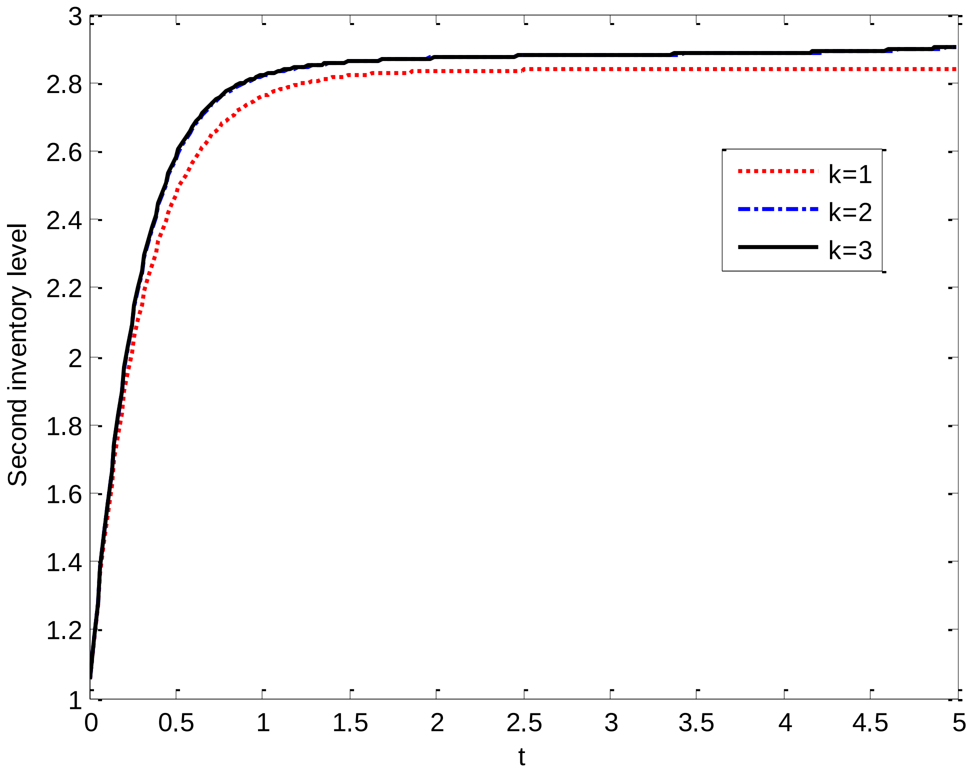


Figure 6.2: Second inventory level against time $t_f = 5$ and m = 256 using constant demand rates for k = 1, 2, 3 quasilinearization iterations

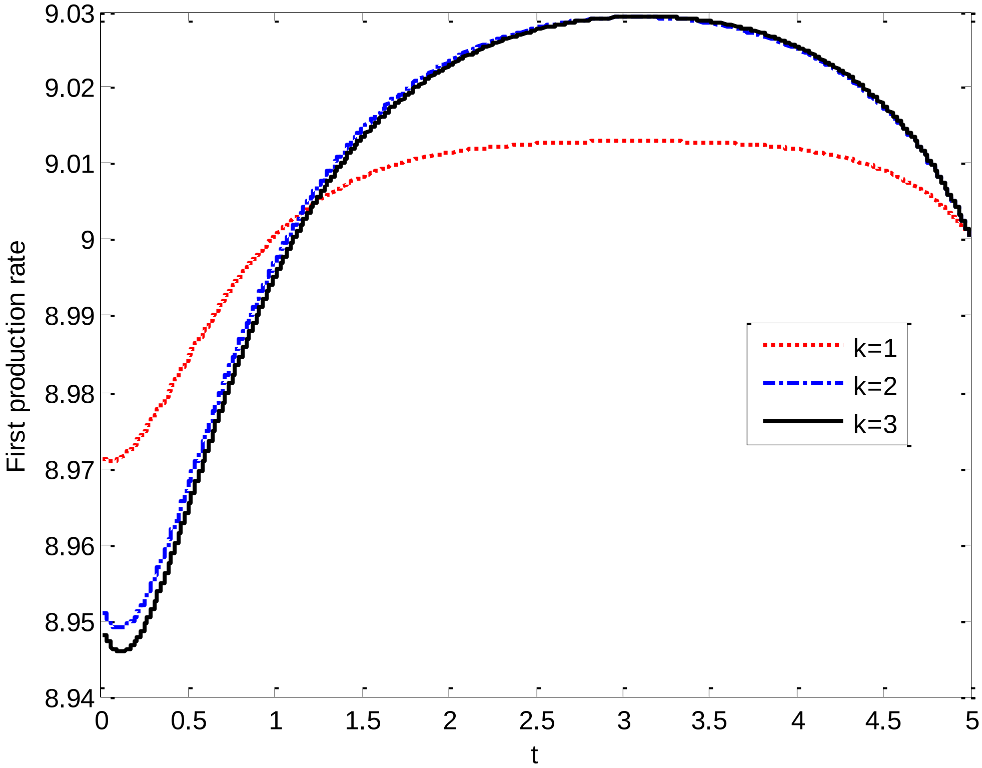


Figure 6.3: First production rate against time $t_f = 5$ and m = 256 using constant demand rates for k = 1, 2, 3 quasilinearization iterations

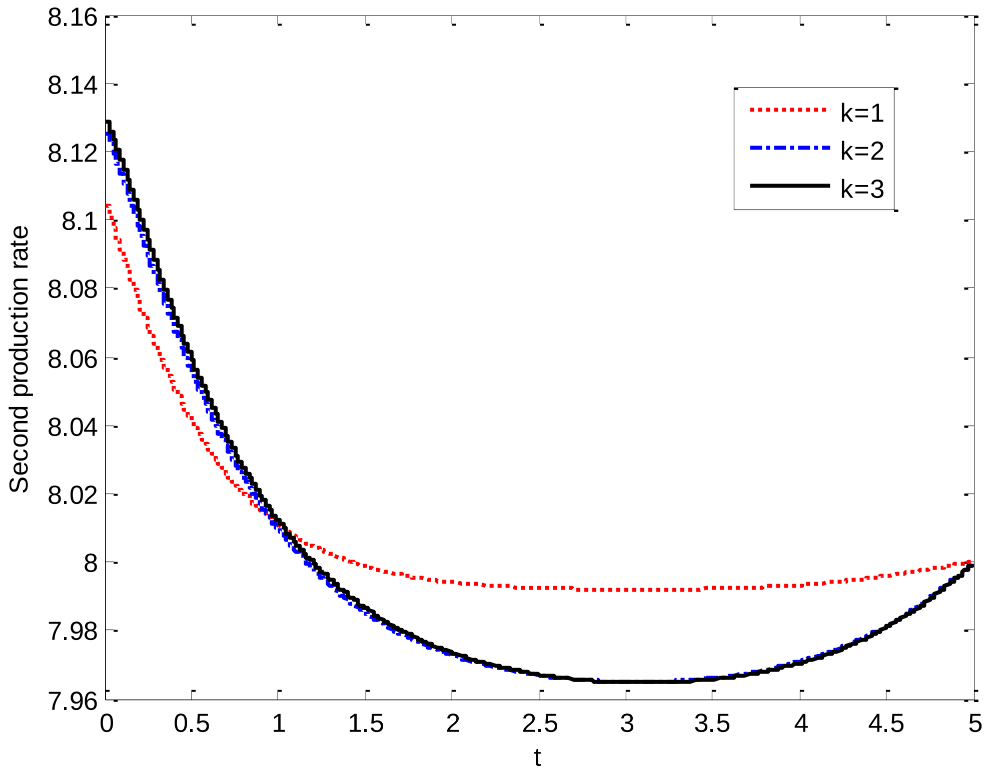


Figure 6.4: Second production rate against time $t_f = 5$ and m = 256 using constant demand rates for k = 1, 2, 3 quasilinearization iterations

6.4.2 Linear Demand Rates

Similar to the numerical solution using constant demand rates, we solve the optimal control problem Eqns. (6.6) to (6.12) by using linear demand rates $D_i(y_1, y_2, t) = d_i y_i(t) + \alpha_i$ and the system parameter values and initial states presented in Table 6.1, with parameters $d_1 = 3$, $d_2 = 4$, $\alpha_1 = 0.6$, and $\alpha_2 = 0.8$. The results from the application of the proposed method for achieving various Haar wavelet resolution m with convergence error less than 10^{-4} are achieved in six quasilinearization iterations for each m, Table 6.3. Meanwhile, Figures 6.5 to 6.8 show the trajectories for successive quasilinearization iterations for m = 256. These illustrations show that in the case of linear demand rates, the iteration converges after six iterations.

Table 6.3: Simulation results of the application of the direct method using linear demand rates for m = 8, 16, 32, 64, 128, and 256

m	$y_1^*(t_f)$	$y_2^*(t_f)$	$v_1^*(t_f)$	$v_2^*(t_f)$	J^*
8	2.08216880	1.34837495	9.02006694	8.00835368	7.59681086
16	2.08158114	1.34834475	9.01471793	8.00698875	7.59683572
32	2.08104995	1.34829464	9.00961866	8.00512824	7.59684504
64	2.08064757	1.34822252	9.00568933	8.00329800	7.59684797
128	2.08038920	1.34815920	9.00313320	8.00191325	7.59684876
256	2.08024061	1.34811638	9.00165070	8.00103800	7.59684896

Additionally, Table 6.4 presents the optimal value J^* for six quasilinearization iterations. The iterations are terminated when the convergence criteria between two cost functions $\left| J^{k+1} - J^k \right| < 10^{-4}$.

Table 6.4: Optimal value of the performance index and convergence error for the application of the direct method using linear demand rates for resolution of Haar wavelet m = 256

Iteration (k)	$J_{\min}$	**Convergence error** $\left\| J^{k+1} - J^k \right\|$
1	7.761526989943635	-
2	7.619255927112216	0.14227106
3	7.599708901525899	0.01954703
4	7.597146554317612	0.00256235
5	7.596864283699386	0.00028227
6	7.596848957719115	0.00001533

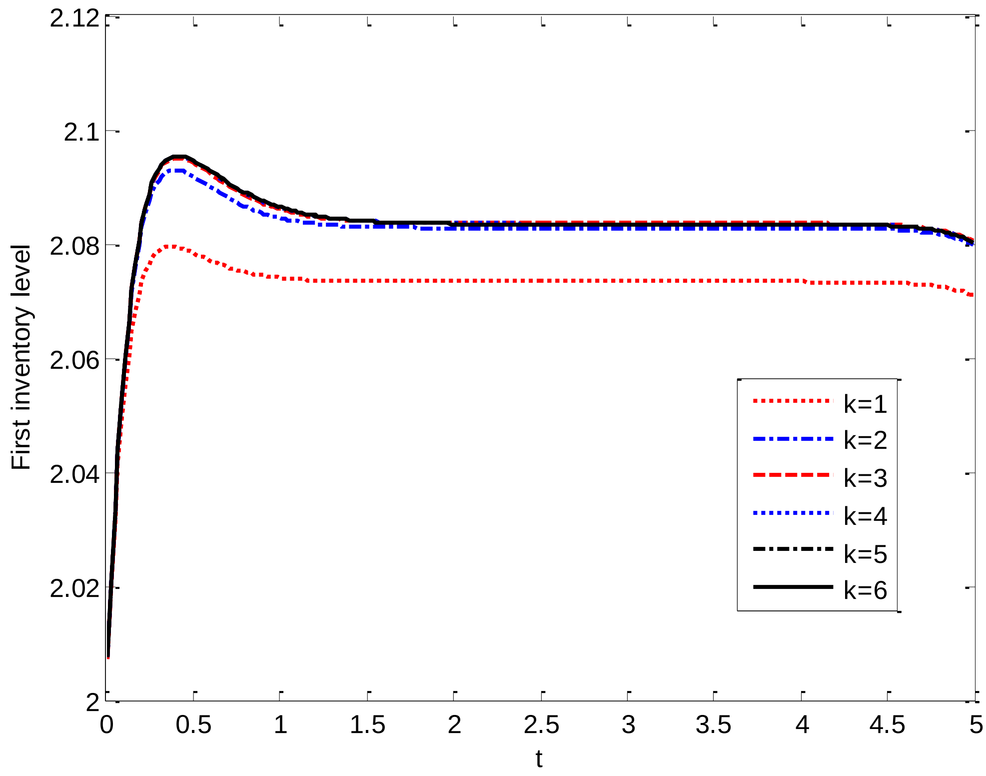


Figure 6.5: First inventory level against time $t_f = 5$ and m = 256 of the application of the direct method using linear demand rates for k = 1, 2, 3, 4, 5, 6 quasilinearization iterations

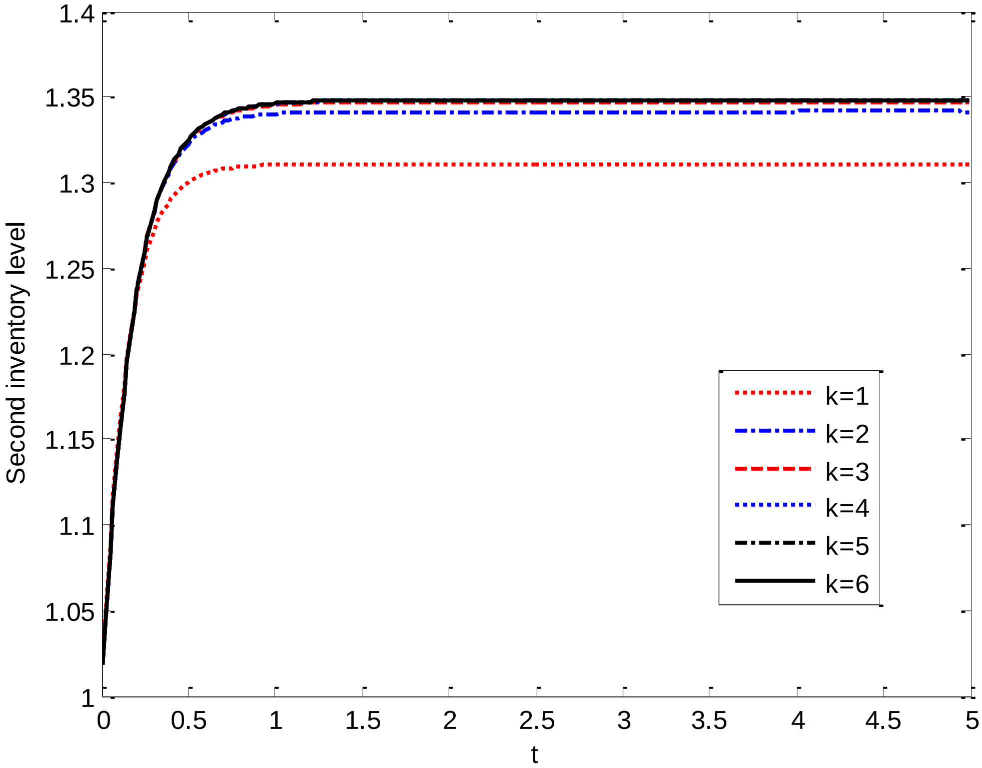


Figure 6.6: Second production rate against time $t_f = 5$ and m = 256 of the application of the direct method using linear demand rates for k = 1, 2, 3, 4, 5, 6 quasilinearization iterations

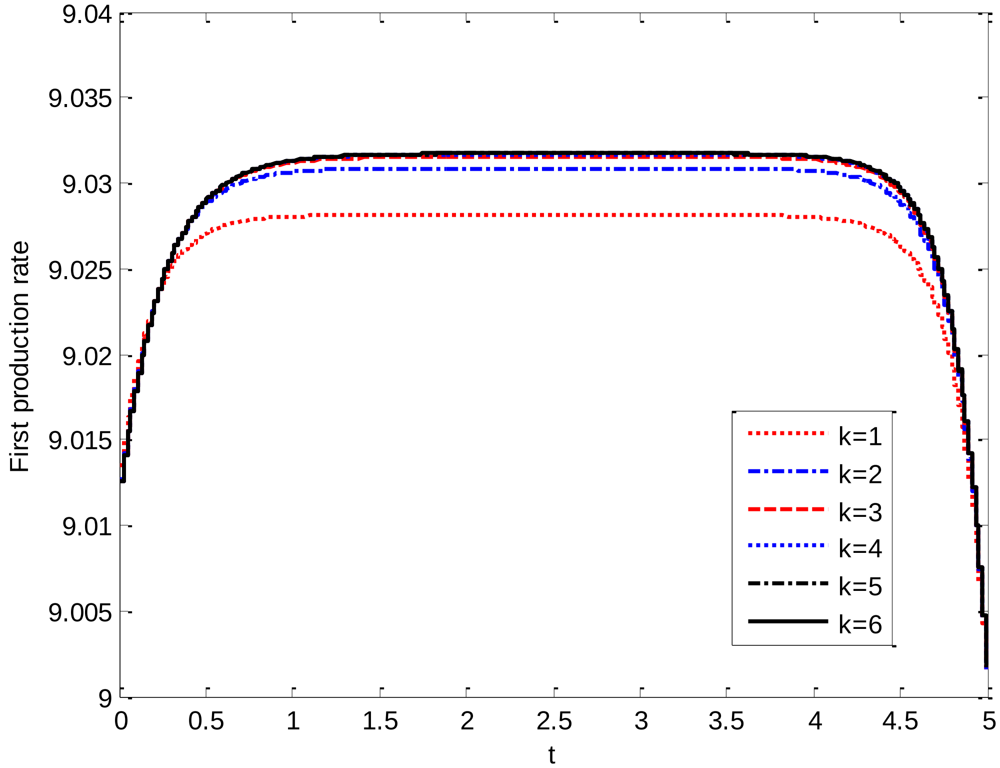


Figure 6.7: First production rate against time $t_f = 5$ and m = 256 of the application of the direct method using linear demand rates for k = 1, 2, 3, 4, 5, 6 quasilinearization iterations

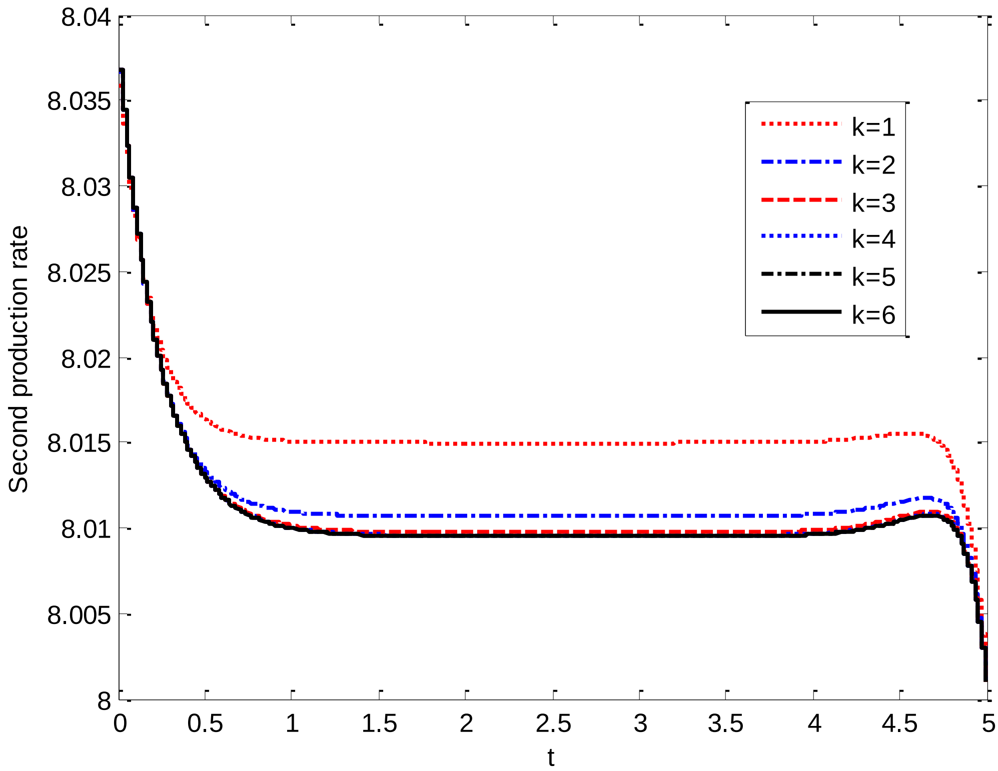


Figure 6.8: Second production rate against time $t_f = 5$ and m = 256 of the application of the direct method using linear demand rates for k = 1, 2, 3, 4, 5, 6 quasilinearization iterations

6.4.3 Logistic Demand Rates

We present the numerical solution when the demand rates are logistic functions, that is, $D_1(y_1, y_2, t) = y_1(t)(g_1 - y_1(t))$ and $D_2(y_1, y_2, t) = y_2(t)(g_2 - y_2(t))$, with system parameters $g_1 = 10$ and $g_2 = 20$. Table 6.5 presents the simulation results obtained from the application of the direct method using logistic demand rates after seven quasilinearization iterations for $m = 8$, 16, 32, 64, 128, and 256. In these six cases, including the simulation results for the optimal values of the inventory levels and the production rates, as well as the total cost at the end of the planning horizon period, the convergence error is less than 10^{-4}.

Table 6.5: Simulation results of the application of the direct method using logistic demand rates for m = 8, 16, 32, 64, 128, and 256

m	$y_1^*(t_f)$	$y_2^*(t_f)$	$v_1^*(t_f)$	$v_2^*(t_f)$	J^*
8	0.96224750	0.38688870	9.02668582	8.00719864	18.66023271
16	0.96341244	0.39575166	9.02086529	8.00617452	18.65963566
32	0.96299653	0.39573929	9.01426343	8.00509685	18.66071869
64	0.96264861	0.39571141	9.00873804	8.00373496	18.66105686
128	0.96240891	0.39568253	9.00492469	8.00241837	18.66115008
256	0.96226488	0.39565973	9.00262992	8.00141464	18.66117449

The approximate optimal controllers and states trajectories for Haar wavelet resolution $m = 256$ are shown in Figures (6.9) to (6.12), whereas the optimal value J^* of each iteration with the difference $\left| J^{k+1} - J^k \right|$ for seven quasilinearization iterations is shown in Table 6.6.

Table 6.6: Optimal value of the performance index and convergence error for the application of the direct method using logistic demand rates for resolution of Haar wavelet $m = 256$

Iteration (k)	$J_{\min}$	Convergence error $\left| J^{k+1} - J^k \right|$
1	20.345714643103790	-
2	18.472739505940940	1.87297514
3	18.688472742265326	0.21573324
4	18.657330877462005	0.03114187
5	18.661705329728910	0.00437445
6	18.661087306230478	0.00061802
7	18.661174489168410	0.00008718

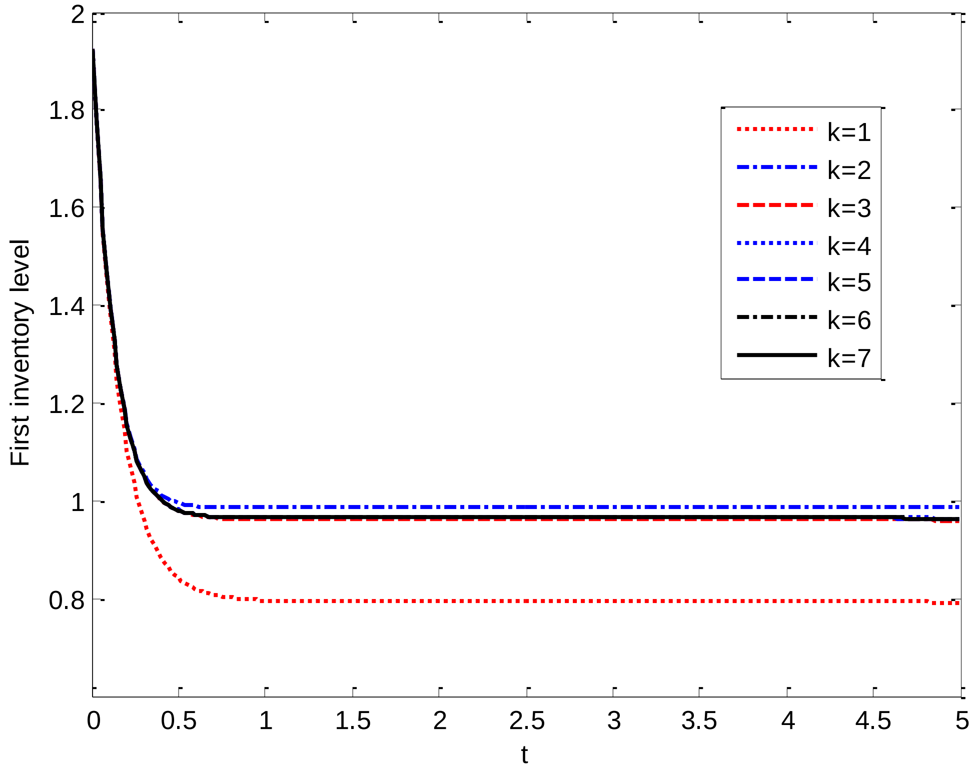


Figure 6.9: First inventory level against time $t_f = 5$ and m = 256 of the application of the direct method using logistic demand rates for k = 1, 2, 3, 4, 5, 6, 7 quasilinearization iterations

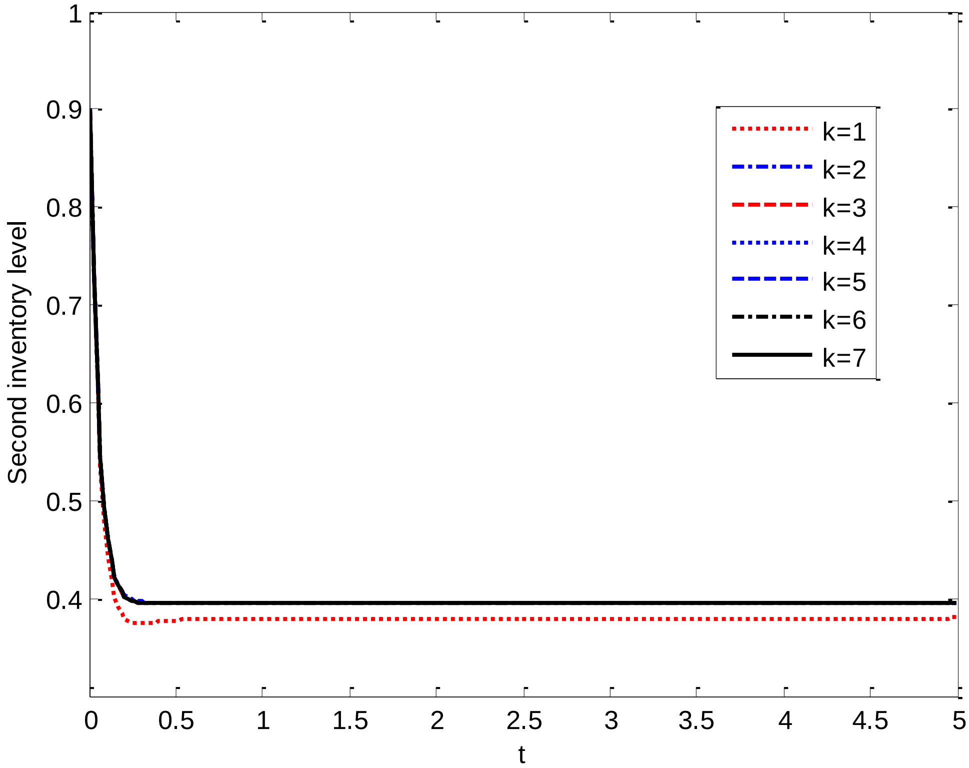


Figure 6.10: Second inventory level against time $t_f = 5$ and m = 256 of the application of the direct method using logistic demand rates for k=1, 2, 3, 4, 5, 6, 7 quasilinearization iterations

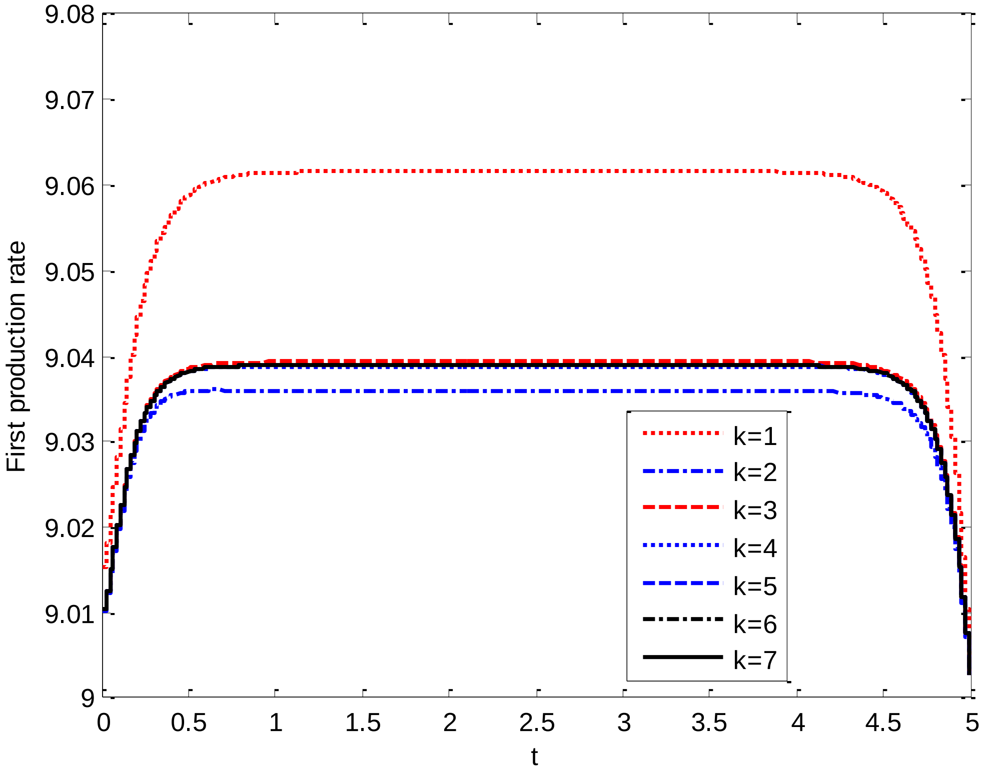


Figure 6.11: First production rate against time $t_f = 5$ and m = 256 of the application of the direct method using logistic demand rates for k = 1, 2, 3, 4, 5, 6, 7 quasilinearization iterations

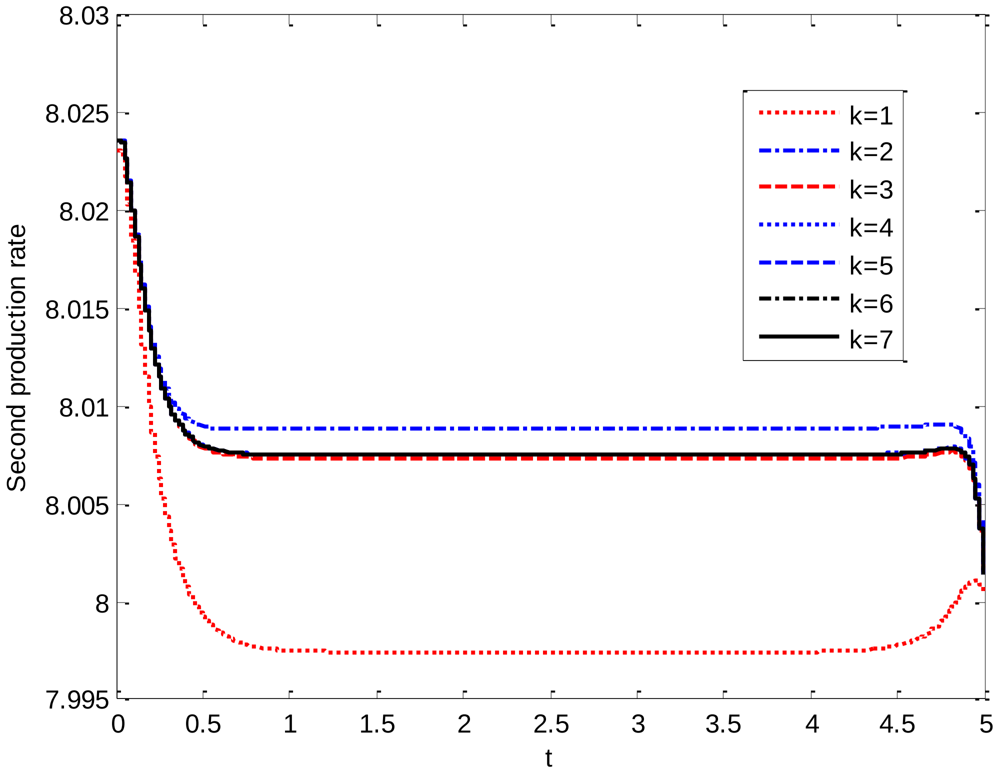


Figure 6.12: Second production rate against time $t_f = 5$ and m = 256 of the application of the direct method using logistic demand rates for k = 1, 2, 3, 4, 5, 6, 7 quasilinearization iterations

Figures (6.13) to (6.16) show the approximate state trajectories and optimal controllers for the various Haar wavelet resolutions $m = 8, 16, 32, 64$, and 128.

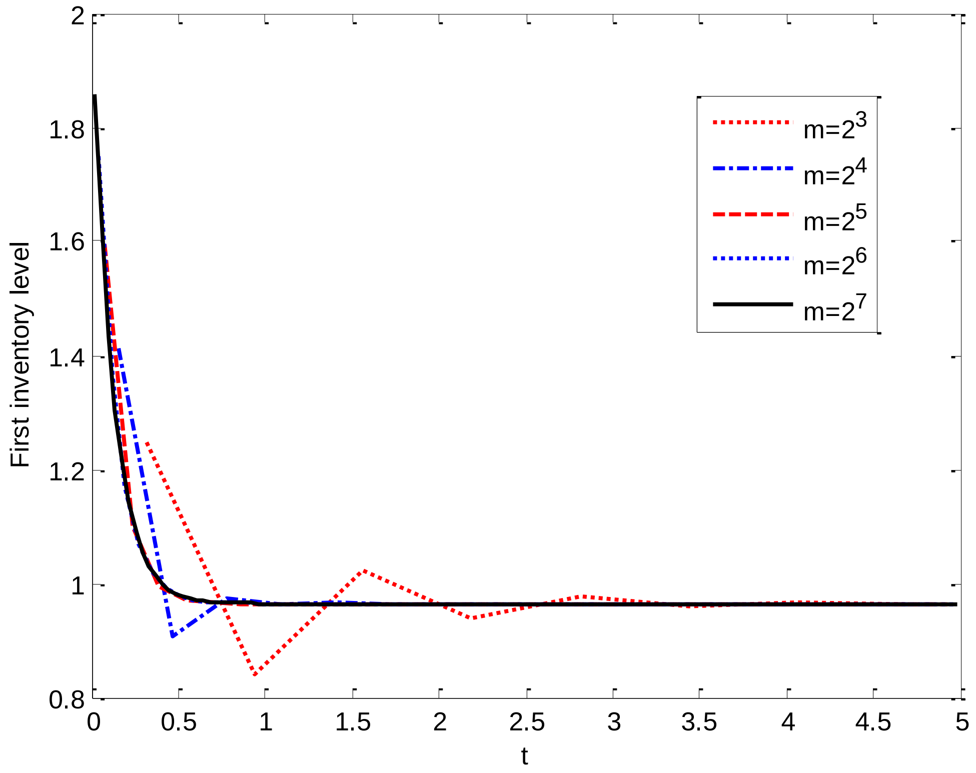


Figure 6.13: First inventory level against time with $t_f = 5$ and $m = 2^3, 2^4, 2^5, 2^6$ and 2^7 for logistic demand rates

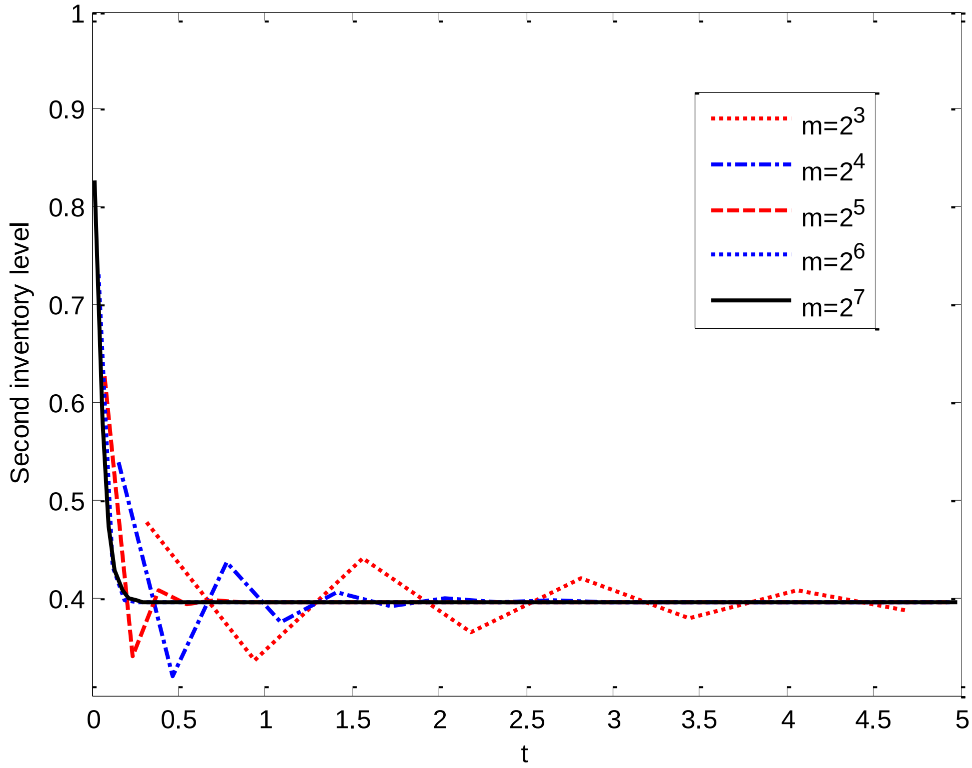


Figure 6.14: Second inventory level against time with $t_f = 5$ and $m = 2^3, 2^4, 2^5, 2^6$ and 2^7 for logistic demand rates

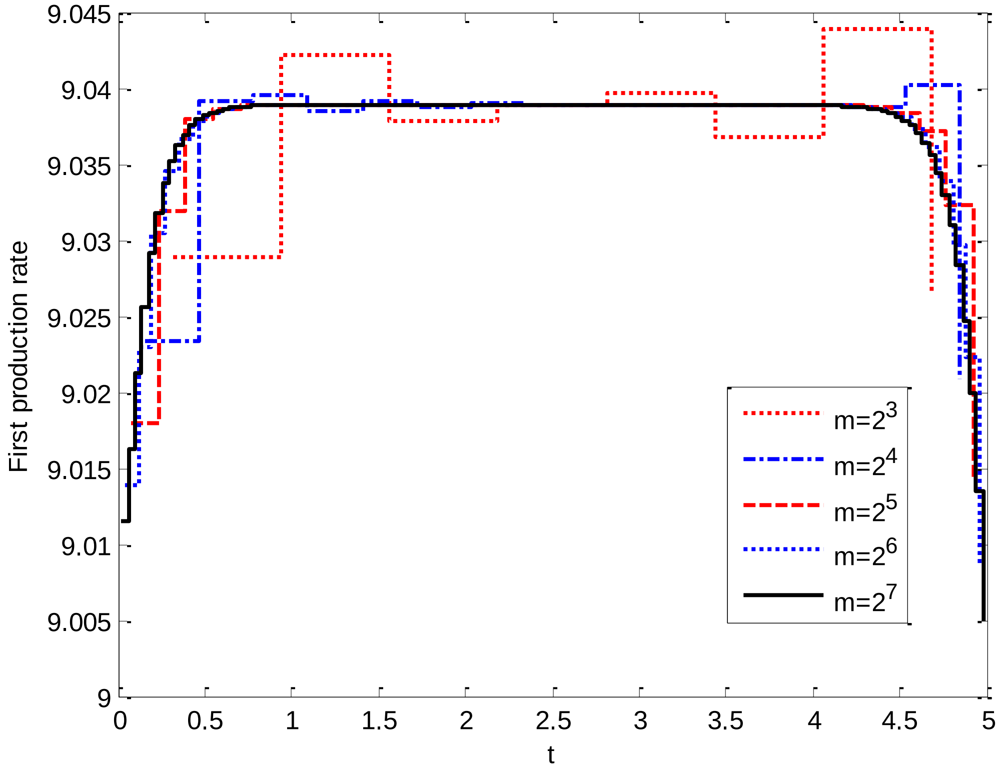


Figure 6.15: First production rate against time with $t_f = 5$ and $m = \mathbf{2}^3, \mathbf{2}^4, \mathbf{2}^5, \mathbf{2}^6$ and $\mathbf{2}^7$ for logistic demand rates

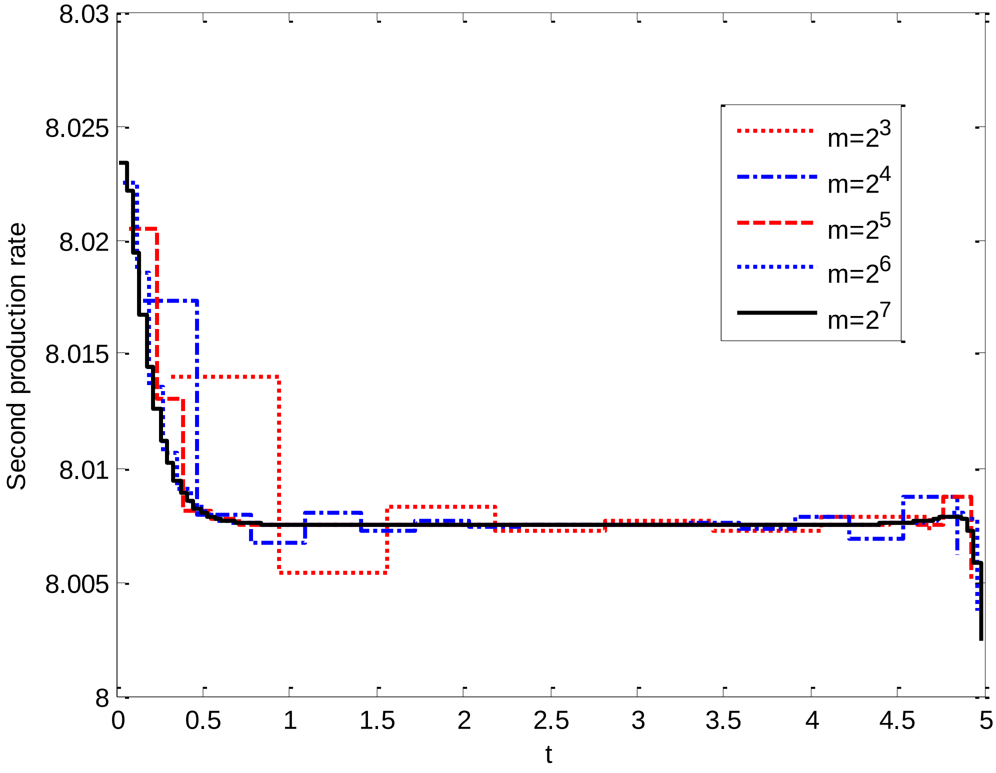


Figure 6.16: Second production rate against time with $t_f = 5$ and $m = 2^3, 2^4, 2^5, 2^6$ and 2^7 for logistic demand rates

6.4.4 Periodic Demand Rates

Finally, we present the numerical solution in the case of periodic demand rates as $D_i(y_1, y_2, t) = 1 + k_i \sin(t)$ with the system parameter values and initial states given in Table 6.1, with parameters $k_1 = 2$ and $k_2 = 1$. Table 6.7 gives the simulation results for $m = 8, 32, 64$, 128 and 256 using periodic demand rates obtained after seven quasilinearization iterations for each m, where the convergence error is less than 10^{-4}.

The optimal trajectories are shown in Figures (6.17) to (6.20), for the various Haar wavelet resolution $m = 8, 16, 32, 64$ and 128. These illustrations show that the trajectories converge with the optimal trajectory at $m = 64$.

Table 6.7: Simulation results of the application of the direct method using periodic demand rates for m = 8, 16, 32, 64, 128, and 256

m	$y_1^*(t_f)$	$y_2^*(t_f)$	$v_1^*(t_f)$	$v_2^*(t_f)$	J^*
8	4.17795557	3.03422206	8.97400136	8.03361326	2.27825152
16	4.21915814	3.01596432	8.98042200	8.02505240	2.30950536
32	4.23718141	3.00357090	8.98834351	8.01465935	2.31888456
64	4.24569862	2.99648260	8.99365695	8.00786091	2.32139612
128	4.24985899	2.99270423	8.99669128	8.00406075	2.32203596
256	4.25191840	2.99075502	8.99831004	8.00206237	2.32219669

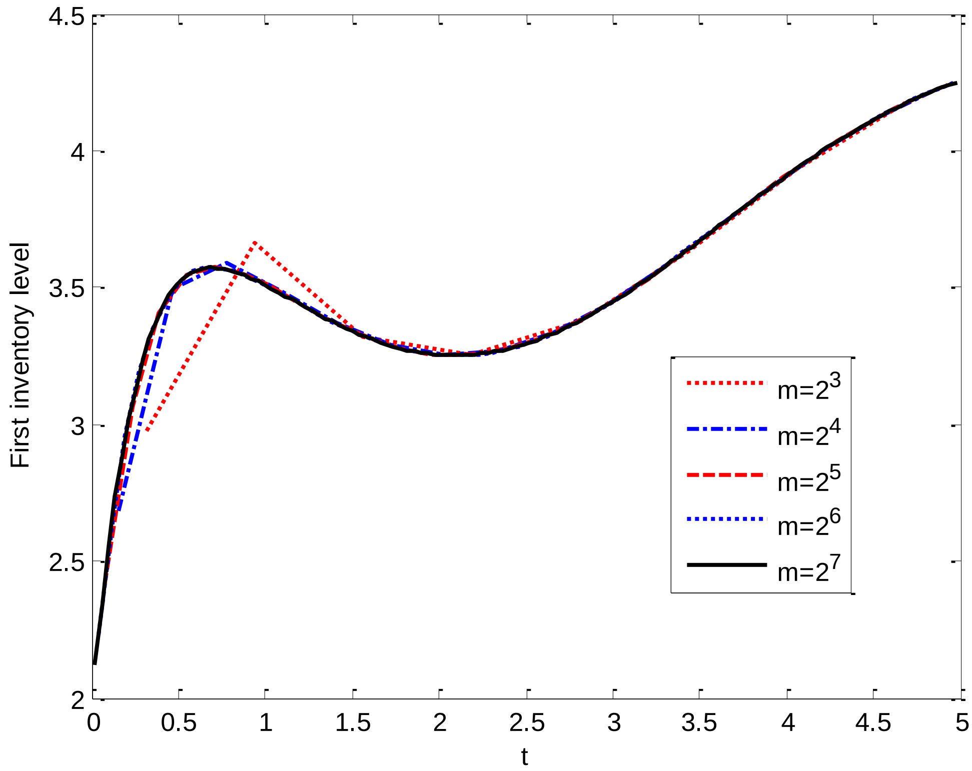


Figure 6.17: First inventory level against time with $t_f = 5$ and $m = 2^3, 2^4, 2^5, 2^6$ and 2^7 using periodic demand rates

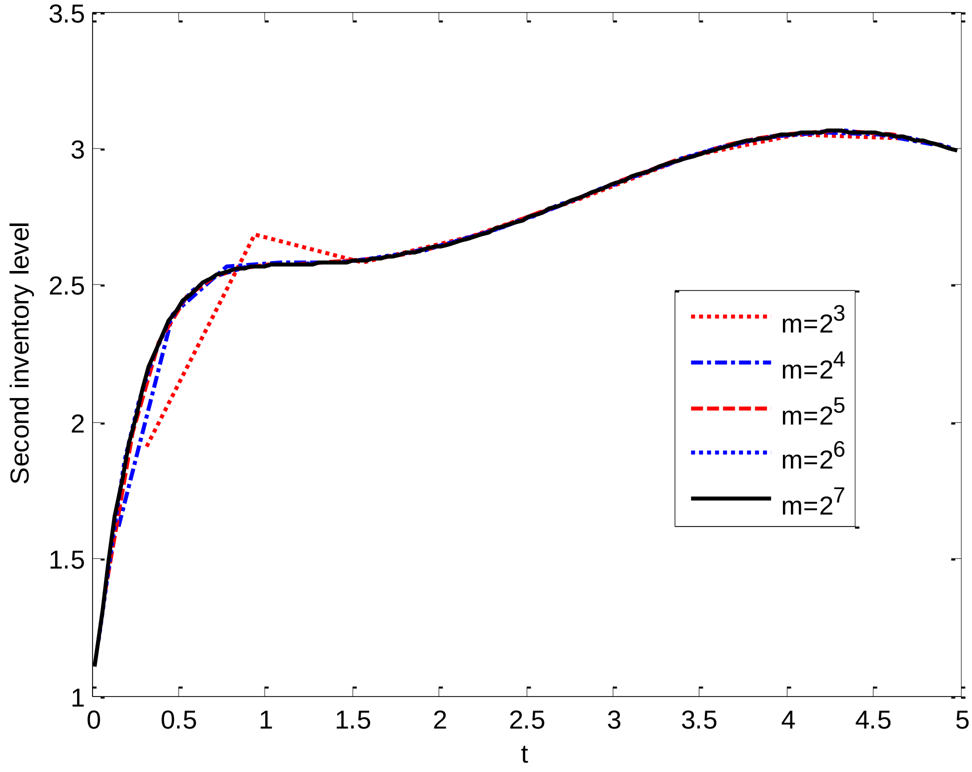


Figure 6.18: Second inventory level against time with $t_f = 5$ and $m = 2^3, 2^4, 2^5, 2^6$ and 2^7 using periodic demand rates

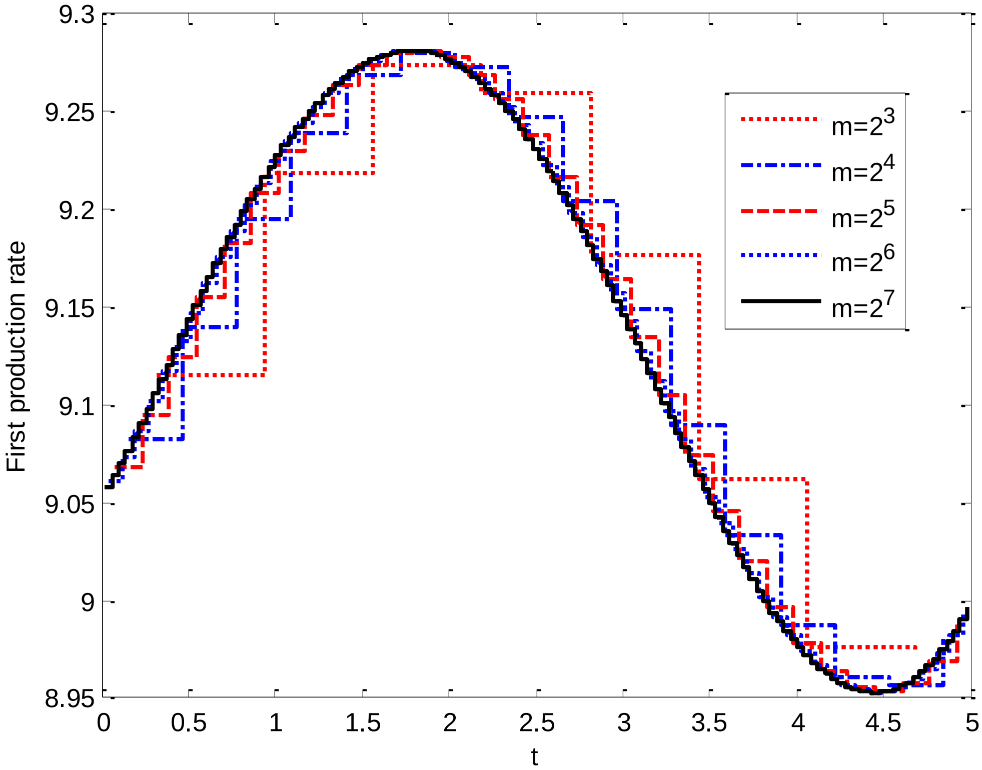


Figure 6.19: First production rate against time with $t_f = 5$ and $m = 2^3, 2^4, 2^5, 2^6$ and 2^7 using periodic demand rates

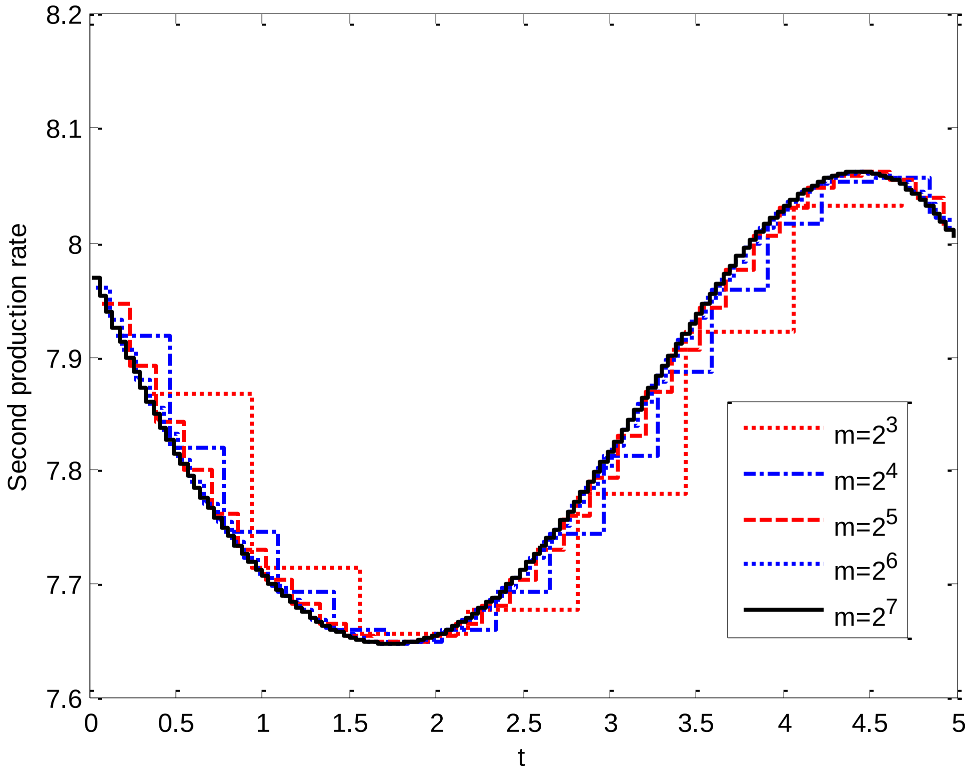


Figure 6.20: Second production rate against time with $t_f = 5$ and $m = 2^3, 2^4, 2^5, 2^6$ and 2^7 using periodic demand rates

6.5 Numerical Discussions and Conclusions

The present numerical method introduce in Chapter 5 solves the application of the direct method for the constrained nonlinear quadratic optimal control problem in two-dimensional state space with two controllers. In particular, we solved the two-item production-inventory model with stock-dependent deterioration rates and deterioration due to self-contact and the presence of the other stock. Four different types of demand rates are used, namely, constant, linear, logistic and periodic demand rates.

The numerical solution for the two-item production-inventory model was obtained by using the new algorithm proposed in Chapter 5, Section 6. We parameterize both the states and the controls by using Haar wavelet functions and operational matrix. The nonlinear optimal control problem is converted into a sequence quadratic programming problem through the quasilinearization iterative technique. Moreover, the inequality constraints for trajectory variables are transformed into quadratic programming constraints by using the Haar wavelet collocation method. The quadratic programming problem with linear inequality constraints is then solved using a standard QP solver.

Additionally, the numerical results of all cases are illustrated in figures and tables. Each figure is plotted according to the solution obtained from the present numerical technique. We conclude that both inventory levels and production rates tend to their real values. Thus both inventory levels and production rates asymptotically tend to their values at the steady state (Alshamrani, 2012). The step functions in Figures 6.3 , 6.4 ,6.7, 6.8, 6.11, 6.12, 6.15, and 6.16 are not visible because the collocation points are too close to each other.

El-Gohary and Elsayed (2008) reduced the same application problem into a system of differential equations according to the Pontryagin principle. To obtain the values in Table 6.8, El-Gohary and Elsayed, (2008) solved this system numerically using the Runge-Kutta method. However, this indirect method has a drawback because the system contains co-state variables, which are not physical entities. Moreover, if the final state is fixed, the indirect method needs to solve a two-point boundary value problem.

Although we consider $m = 256$ in our computation for four types of demand rates, Tables 6.2, 6.3, 6.5, and 6.7 show that the usage of $m = 32$, $m = 16$, $m = 64$ and, $m = 128$, respectively, are enough to approximate with the same accuracy the optimal cost function and trajectory variables obtained as presented in Table 6.8.

Table 6.8: Summary of results obtained from the El-Gohary and Elsayed (2008) method for application with four types of demand rates

Demand rates	$y_1^*(t_f)$	$y_2^*(t_f)$	$v_1^*(t_f)$	$v_2^*(t_f)$	J^*
Constant	3.82	2.9	9	8	0.41
Linear	2.08	1.35	9	8	7.6
Logistic	0.96	0.4	9	8	18.66
Periodic	4.26	2.98	9	8	2.27

In this chapter, present results are compared with the numerical solutions in existing literature. Our method is simple and require fewer collocation points to achieve the same accuracy as the existing numerical solutions. By increasing the Haar wavelet resolution, we can improve the accuracy of the states, controls, and cost.

CHAPTER 7

CONCLUSION AND FUTURE WORK

7.1 Conclusions

Broadly speaking, solutions to nonlinear optimal control problems with or without constraints often generate both analytical and computational difficulties. Thus, researchers aim to solve these problems by using numerical methods. In general, numerical methods for solving nonlinear optimal control problems fall under two categories: direct and indirect methods. By parameterizing or discretizing the infinite dimensional optimal control problem, into a finite dimensional optimization problem, direct methods reduce the optimal control problem to a nonlinear programming problem. On the other hand, indirect methods solve the Hamilton-Jacobi-Bellman equation or the first-order necessary condition for optimality, which are obtained from the Pontryagin minimum principle. Both the direct and indirect methods are important for solving optimal control problems. The difference between the two methods is that indirect methods are believed to yield more accurate results, whereas direct methods tend to result in better convergence properties.

In this thesis, we proposed direct and indirect numerical methods to solve constrained nonlinear optimal control problems and nonlinear optimal control problems with finite time horizon and infinite time horizon, respectively, by using Haar wavelets operational matrices and Haar wavelets collocation method.

The main contributions of the thesis were presented in Chapters 3, 4, 5 and 6.

Chapter 3 started with an understanding of the mathematical background of Haar wavelet. Many scholars who proposed a numerical method using Haar wavelet basis usually define a Haar wavelet operational matrix within the interval of zero to one. This technique limits the achievement of our ultimate goal because the integration involved in nonlinear optimal control problems does not necessarily cover only the interval between zero to one especially when we have more than one state variables. Therefore, it is more appropriate to derive a Haar wavelet operational matrix in a much more general setting. In Chapter 3, we derived the operational matrix of integration for intervals $[0, \tau)$ and $[-\tau, \tau)$. The new operational matrices for the integration of two-dimension Haar wavelet basis within the interval $[-\tau, \tau)$ were derived using the Haar wavelet basis of two dimensions, the operational matrix for the integration of one dimension within the interval $[0, \tau)$, and Kroneker product properties. Moreover, to simplify the product of two-dimensional functions expressed as Haar series, we derive and prove a new algorithm for the operational matrix of the product of two-dimensional Haar wavelet functions. A general formula in the form of a Haar wavelet matrix with two variables was derived.

In Chapter 4, the solution of the Hamilton-Jacobi-Bellman equation, which appears in the formulation of the nonlinear control system with quadratic cost functional and infinite time horizon, is introduced. The solution was based on the combination of Haar wavelets operational matrices and successive generalized Hamilton-Jacobi-Bellman (GHJB) equation. Although there is no general closed-form solution to this equation, we showed how to approximate the solution of Hamilton-Jacobi-Bellman equation successively. We used the successive GHJB equation to improve the feedback performance of stabilizing controls and reduced the problem of solving the Hamilton-Jacobi-Bellman equation to solving the GHJB equation. When the process of improving the control and solving the GHJB equation is iterated, the solution to the GHJB equation converges uniformly to the solution of the Hamilton-Jacobi-Bellman equation. This result takes the form of the gradient of the Lyapunov function. To determine the Lyapunov function from the resulting solution of the linear system equation, the new method, which depends only on the initial and final states by using variable gradient method, is proposed. In the proposed method, we considered the method of Beard et al. (1997) to successively approximate the solution of the Hamilton-Jacobi-Bellman equation. Instead of using the Galerkin method with polynomial basis, we used collocation method with Haar wavelet basis to solve the GHJB equation. Galerkin method requires the computation of multidimensional integrals which makes the method impractical for higher order systems (Curtis and Beard, 2001). In general, the main advantage of using collocation method, is that the computational burden of solving the GHJB equation is reduced to matrix computation only. A number of numerical examples for linear and nonlinear optimal control problems with one or two state and control variables are given to demonstrate the usefulness, efficiency, and accuracy of the successive Haar wavelet collocation method. To justify our proposed method, the results in the present study are compared with existing or exact results.

In Chapter 5, an efficient new algorithm is proposed to solve constrained nonlinear optimal control problems with finite time horizon under inequality constraints. With this technique, we parameterized both the state and control variables by using Haar wavelet functions and Haar wavelet operational matrix. The nonlinear optimal control problem is converted into a quadratic programming problem through the quasilinearization iterative technique. Moreover, the inequality constraints for trajectory variables were transformed into quadratic programming constraints by using the Haar wavelet collocation method. The quadratic programming problem with linear inequality constraints was then solved using a standard QP solver. The numerical method was tried on several examples, and we found that the proposed method gives results that are better or comparable to results obtained through other established methods. In addition, the proposed method is attractive, stable, convergent, and can be easily coded.

In Chapter 6, the direct method proposed in Chapter 5 was applied to solve a practical optimal control problem: the multi-item production-inventory model with stock-dependent deterioration rates and deterioration due to self-contact and the presence of the other stock. Four different types of demand rates, namely, constant, linear, logistic, and periodic demand rates were used in the method. The solution to the model was discussed numerically and presented graphically. By enhancing the resolution of the Haar wavelet, we improved the accuracy of the states, controls, and cost. Simulation results were also compared with the work of other researchers.

In summary, the study succeeded in achieving all the eight objectives stated in Section 1.4.

7.2 Future Work

Suggestions for future research are summarized below:

1. In the feedback control method for solving nonlinear optimal control problems with infinite time horizon which was reduced to solving a Hamilton-Jacobi-Bellman partial differential equation, we could not handle explicit constraints on the state and control variables. Our method can be extended to a case where explicit constraints are placed on the control, for example, $\|u\| \leq 1$.

2. The Haar wavelet operational matrices with two dimensions defined in the interval $[-\tau, \tau)$ were derived throughout this thesis. For future work, we are also interested in calculating Haar wavelet operational matrices in higher dimensions.

3. The constrained nonlinear optimal control problems with finite time horizon was solved using direct method and open-loop solutions. However, we prefer to use the closed-loop solution because of the advantages it can offer. Therefore, we suggest extending this work to formulate an optimal feedback control solution using this method.

4. In this thesis, we did not conduct an in-depth study to mathematically prove the numerical stability and error analysis of the proposed direct and indirect methods. We suggest to do this in future studies.

5. For our future work, given that Haar wavelet method is relatively easy to implement and computationally inexpensive, we would like to extend the use of this method to partial optimal control problems.

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LIST OF PUBLICATIONS AND SEMINARS

1. Waleeda Swaidan and Amran Hussin, (2013), Feedback control method using Haar wavelet operational matrices for solving optimal control problems, *Published in Abstract and Applied Analysis*, Vol. 2013, Article ID 240352, 8 pages doi:10.1155/2013/240352.

2. Waleeda Swaidan and Amran Hussin, (2014), Haar Wavelet Operational Matrix Method for Solving Constrained Nonlinear Quadratic Optimal Control Problem, Acc*epted in AIP Conference Proceeding Series*.

3. Waleeda Swaidan and Amran Hussin, (2015), Haar Wavelet Method for Constrained Nonlinear Optimal Control Problems with Application to Production-Inventory Model. Accepted in Sains Malaysiana.

4. Present a seminar titled "Haar Operational Matrix Method for Solving Continuous Optimal Control Problems" on 8 February 2013 at Institute of Mathematical Sciences (ISM). University of Malaya, Malaysia.

5. Paper presented titled "Haar Wavelet Operational Matrix Method for Solving Constrained Nonlinear Quadratic Optimal Control Problem" on 24 November 2014 at Simposium Kebangsaan Sains Matematik 22nd .Shah Alam, Malaysia.

APPENDIX A

KRONECKER PRODUCT

The Kronecker product operation ($\otimes$) is a convenient tool for dealing with a large set of matrices. The Kronecker product operation is usually used to formulate the estimation of the parameter vectors for several equations simultaneously. We list some properties of Kronecker the product operation and the *vec* transforms in the Appendix.

A.1 The Kronecker Product of Two Matrices

Let $\mathbf{A} = [a_{ij}]$ be an ($m \times n$) matrix and $\mathbf{B}$ is the ($p \times q$) matrix. The Kronecker product of $\mathbf{A}$, $\mathbf{B}$, and ($\mathbf{A} \otimes \mathbf{B}$) is defined as an ($mp \times nq$) matrix, which can be partitioned as follows:

$$\mathbf{A} \otimes \mathbf{B} \equiv \lfloor a_{ij}\ \mathbf{B} \rfloor, \tag{A.1}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}. \tag{A.2}$$

If $\mathbf{a} = [a_1, a_2, \ldots, a_m]^{\mathbf{T}}$ and $\mathbf{b} = [b_1, b_2, \ldots, b_n]^{\mathbf{T}}$, then the following is obtained:

$$\mathbf{a}\mathbf{b}^{\mathbf{T}} = \mathbf{a} \otimes \mathbf{b}^{\mathbf{T}} = \mathbf{b}^{\mathbf{T}} \otimes \mathbf{a} \tag{A.3}$$

$$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & \cdots & \vdots \\ a_m b_1 & a_m b_2 & \cdots & a_m b_n \end{bmatrix}.$$

A.2 The Kronecker Product of Two Kronecker Products

The matrix product of two Kronecker products of $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{C} \otimes \mathbf{D}$ can be written as a single Kronecker product:

$$\mathbf{(A \otimes B)(C \otimes D)} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \begin{bmatrix} c_{11}\mathbf{D} & \cdots & c_{1s}\mathbf{D} \\ \vdots & \ddots & \vdots \\ c_{n1}\mathbf{D} & \cdots & c_{n1}\mathbf{D} \end{bmatrix},$$

$$= \begin{bmatrix} (\sum_j a_{1j} c_{j1})\,\mathbf{BD} & \cdots & (\sum_j a_{1j} c_{js})\,\mathbf{BD} \\ \vdots & \ddots & \vdots \\ (\sum_j a_{mj} c_{j1})\,\mathbf{BD} & \cdots & (\sum_j a_{mj} c_{js})\,\mathbf{BD} \end{bmatrix},$$

$$= (\mathbf{AC}) \otimes (\mathbf{BD}). \tag{A.4}$$

In general,

$$(\mathbf{A_1} \otimes \mathbf{B_1})(\mathbf{A_2} \otimes \mathbf{B_2})\ldots(\mathbf{A_N} \otimes \mathbf{B_N}) = \mathbf{A_1 A_2} \ldots \mathbf{A_N} \otimes \mathbf{B_1 B_2} \ldots \mathbf{B_N}, \tag{A.5}$$

follows directly for Eqn. (A.4).

Let $\mathbf{A}$ be an $(m \times m)$ matrix and $\mathbf{B}$ is the $(n \times n)$ matrix. Then obtain the following:

$$\mathbf{A} \otimes \mathbf{B} = (\mathbf{A} \otimes \mathbf{I}_n)(\mathbf{I}_m \otimes \mathbf{B}) = (\mathbf{I}_m \otimes \mathbf{B})(\mathbf{A} \otimes \mathbf{I}_n). \tag{A.6}$$

Which indicates that $(\mathbf{I}_m \otimes \mathbf{B})$ and $(\mathbf{A} \otimes \mathbf{I}_n)$ are commutative for square matrices $\mathbf{A}$ and $\mathbf{B}$.

An immediate result is that when $\mathbf{A}$ and $\mathbf{B}$ are square and non-singular, the inverse of their Kronecker product is expressed as follows:

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}, \tag{A.7}$$

$$= \begin{bmatrix} a_{11}^{-1}\mathbf{B}^{-1} & \dots & a_{1n}^{-1}\mathbf{B}^{-1} \\ \vdots & \cdots & \vdots \\ a_{m1}^{-1}\mathbf{B}^{-1} & \dots & a_{mn}^{-1}\mathbf{B}^{-1} \end{bmatrix}.$$

Other results are expressed as follows:

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{A}^{-1}\otimes\mathbf{B}^{-1}) = \mathbf{A}\mathbf{A}^{-1}\otimes\mathbf{B}\mathbf{B}^{-1},$$

$$= \mathbf{I}\otimes\mathbf{I},$$

$$= \mathbf{I}. \tag{A.8}$$

A.3 Other Properties of the Kronecker Product

The transpose, associative, and distributive laws of the Kronecker product are expressed as follows:

$$(\mathbf{A}\otimes\mathbf{B})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}}\otimes\mathbf{B}^{\mathbf{T}}, \tag{A.9}$$

$$\mathbf{A}\otimes(\mathbf{B}+\mathbf{C}) = \mathbf{A}\otimes\mathbf{B}+\mathbf{A}\otimes\mathbf{C}, \tag{A.10}$$

$$(\mathbf{B}+\mathbf{C})\otimes\mathbf{A} = \mathbf{B}\otimes\mathbf{A}+\mathbf{C}\otimes\mathbf{A}, \tag{A.11}$$

$$\mathbf{A}\otimes(\mathbf{B}\otimes\mathbf{C}) = (\mathbf{A}\otimes\mathbf{B})\otimes\mathbf{C}. \tag{A.12}$$

Additionally, another property is expressed as follows:

$$(\mathbf{I}\otimes\mathbf{B}) = diag\,\{\mathbf{B},\ \mathbf{B},\ \dots,\mathbf{B}\}. \tag{A.13}$$

A.4 The *vec* Transform

Let $\mathbf{A} = [a_{ij}]$ be an $(n\times n)$ matrix. The operation $vec(\mathbf{A})$ denotes the vector obtained by transformation of the stacking column of matrix $\mathbf{A}^{\mathbf{T}}$ by putting into one column.

$$vec(\mathbf{A}) \equiv \begin{bmatrix} \mathbf{A}(1,::) & \mathbf{A}(2,:) & \dots & \mathbf{A(n,:)} \end{bmatrix}^{\mathbf{T}} \in \mathbf{C}^{\mathbf{n}^2 \times 1}, \tag{A.14}$$

$$vec\,(\mathbf{BA\,C}) = (\mathbf{B} \otimes \mathbf{C}^{\mathbf{T}})\,vec\,(\mathbf{A})\,, \tag{A.15}$$

$$vec\,(\,\mathbf{A\,B}) = (\mathbf{I} \otimes \mathbf{B}^{\mathbf{T}})\,vec\,(\mathbf{A})\,, \tag{A.16}$$

$$= (\mathbf{A} \otimes \mathbf{I})\,vec\,(\mathbf{B})\,, \tag{A.17}$$

$$= (\mathbf{A} \otimes \mathbf{B}^{\mathbf{T}})\,vec\,(\mathbf{I})\,. \tag{A.18}$$

A.5 Block Matrix Multiplication

A block partitioned matrix product can sometimes be used on equations that involve only algebra on the submatrices of the factors. However, the partitioning of the factors is not arbitrary and requires "conformable partitions" between matrices $\mathbf{A}$ and $\mathbf{B}$, such that all submatrix products that will be used are defined. An $(m \times p)$ matrix $\mathbf{A}$ with q-row partitions and s column partitions is expressed as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} & \cdots & \mathbf{A_{1s}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \cdots & \mathbf{A_{2s}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A_{q1}} & \mathbf{A_{q2}} & \cdots & \mathbf{A_{qs}} \end{bmatrix}. \tag{A.19}$$

A $(p \times n)$ matrix $\mathbf{B}$ with s-row partitions and r-column partitions that are compatible with the partitions of $\mathbf{A}$ is expressed as follows:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B_{11}} & \mathbf{B_{12}} & \cdots & \mathbf{B_{1r}} \\ \mathbf{B_{21}} & \mathbf{B_{22}} & \cdots & \mathbf{B_{2r}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B_{s1}} & \mathbf{B_{s2}} & \cdots & \mathbf{B_{sr}} \end{bmatrix}. \tag{A.20}$$

The matrix product

$$\mathbf{C} = \mathbf{A\,B}, \tag{A.21}$$

can be formed blockwise, yielding **C** as an $(m \times n)$ matrix with q - row partitions and r - column partitions. The submatrices of matrix **C** are calculated by multiplying the following expression (Lancaster, 1969):

$$\mathbf{C}_{qr} = \sum_{p} \mathbf{A}_{qs} \, \mathbf{B}_{sr} \ . \tag{A.22}$$

A.6 Transpose Block Matrix

The transpose block matrix can be defined as follows:

Let $\mathbf{A}$ be a block matrix with q - row partitions and s - column partitions:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} & \cdots & \mathbf{A_{1s}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \cdots & \mathbf{A_{2s}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A_{q1}} & \mathbf{A_{q2}} & \cdots & \mathbf{A_{qs}} \end{bmatrix} \tag{A.23}$$

Thereafter, the transpose block matrix $\mathbf{A^T}$ can be formed as

$$\mathbf{A^T} = \begin{bmatrix} \mathbf{A_{11}^T} & \mathbf{A_{21}^T} & \cdots & \mathbf{A_{1q}^T} \\ \mathbf{A_{12}^T} & \mathbf{A_{22}^T} & \cdots & \mathbf{A_{2q}^T} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A_{s1}^T} & \mathbf{A_{s2}^T} & \cdots & \mathbf{A_{sq}^T} \end{bmatrix}, \tag{A.24}$$

where $\mathbf{A^T}$ is a block matrix with s - row partitions and q - column partitions.

APPENDIX B

MATLAB CODE FOR INDIRECT AND DIRECT METHODS

```
function [P D H] = findp(tau,m)
theta=zeros(m,1);
      theta(1)=1;
% Generate block pulse operational matrix
Q=2*triu(ones(m,m));
for i=1:m
    Q(i,i)=Q(i,i)-1;
end
QB=Q/(2*m);
% Generate Haar matrix
H=ones(m);
nlamb=ones(1,m);
J=log2(m);
%t=1:2:(2*m-1); t=tau*(-1+(t/m))
t=1:2:(2*m-1); t=t/(2*m);
h2=@(t)  (0<=t & t<0.5) - (0.5<=t & t<1);
%h2=@(t)  (-l<=t & t<0.5*l) - (0.5*l<=t & t<l);
for alpha=0:(J-1)
    kk=pow2(alpha)-1;
    for k=0:kk
        i=pow2(alpha)+k+1;
        nlamb(i)=pow2(alpha);
        for j=1:m
            H(i,j)=h2(pow2(alpha)*t(j)-k);
        end
    end
end
lambda=nlamb/m;
D=diag(lambda)
P=2*tau*H*QB*H'*D
end
```

```
function [ A ] = haarbigN(mat,m)
J=log2(m);
R=zeros(m*m,m);
for i=1:m
    b=(mat(:,i))';
    ck=b;
    M=[ck(1) ck(2); ck(2) ck(1)]; NH=[1 1;1 -1];
    for alpha=1:(J-1)
        n=pow2(alpha);
        ca=ck(1:n)'; cb=ck(n+1:2*n)';
        M11=M; M12=NH*diag(cb);
        M21=diag(cb)/NH; M22=diag(ca'*NH);
        M=[M11 M12; M21 M22];
        NH=[kron(NH,[1 1]); kron(eye(n),[1 -1])];
    end
    Mt=M';
    vecMb=Mt(:);
    R(:,i)=vecMb;
end
    R;
for i=1:m*m
    NN=R(i,:);
    ck=NN;
    M=[ck(1) ck(2); ck(2) ck(1)]; NH=[1 1;1 -1];
    for alpha=1:(J-1)
        n=pow2(alpha);
        ca=ck(1:n)'; cb=ck(n+1:2*n)';
        M11=M; M12=NH*diag(cb);
        M21=diag(cb)/NH; M22=diag(ca'*NH);
        M=[M11 M12; M21 M22];
        NH=[kron(NH,[1 1]); kron(eye(n),[1 -1])];
    end
       Mt=M';
       vecM=Mt(:);
       matR(:,:,i)=vec2mat(vecM,m);
end
for i=1:m
    matR(:,:,i);
end
```

```
A=zeros(m*m,m*m);
 e=1;
for i=1:m
    tt=floor(e/m)+1;
    for j=1:m
        e=m*(i-1)+j;
        A(m*(i-1)+1:tt*m,1+(j-1)*m:((j-1)+1)*m)=matR(:,:,e);
    end
  end
end

function [E]= E1(m)
tha=[1;zeros(m-1,1)];
A=[tha';tha';zeros(m-2,m)];
E=sparse(kron(A,eye(m,m)));
end

function [E]= E2(m)
tha=[1;zeros(m-1,1)];
B=[tha';tha';zeros(m-2,m)];
N=eye(m,m);
E=sparse(kron(N,B));
end

% Example 1  for indirect method
% dx/dt=Ax + Bu  by Curtis and Beard
% J=int(x1^2 +u^2) dt
tau=1
%A=[0 1; 0 0];
%B=[0 1]';
%K=[-1  -sqrt(2)];
m=8
[P D H]=findp(tau,m);
t=1:2:(2*m-1); t=tau*(-1+(t/m))
xx=t;
yy=t;
%**********************************%
[Y X]=meshgrid(xx,yy);   % coordinate (x1,x2)
f1=Y;   thata=D*H*f1*H'*D;
```

```
f2=-X-Y;  mu=D*H*f2*H'*D;
%**********intial conditon*********************************%
f5=-0.5*X.^2-0.5*(-X-Y).^2; k=D*H*f5*H'*D;
%****************** uexact=-x1-1.4142*x2 **************
uexact=-X-1.4142*Y;
%***call haarbigN for thata and mu to find Nthata & Nmu******%
mat=thata;
A=haarbigN(mat,m);
Nthata=A;
mat=mu;
A1=haarbigN(mat,m);
Nmu=A1;
that=[1  zeros(m-1,1)];
kl=k';
veck=kl(:);
%******************  new inteval *************************%
Q1=kron(P,eye(m,m));
Q2=kron(eye(m,m),P);
Ea1=E1(m);
Ea2=E2(m);
%*********thata bata and find u1*************************%
M1=((Q2-tau*Ea2)*Nthata+(Q1-tau*Ea1)*Nmu)';
M2=Nthata'*kron(eye(m,m),that);
M3=Nmu'*kron(that,eye(m,m));
     bigW=[M1 M2 M3];
     calfbat=pinv(bigW)*veck;
             C=(calfbat(1:m*m));
             ccdp=vec2mat(C,m);
                  alfa1=(calfbat(m*m+1:m*m+m));
                  alfadp1=[alfa1 zeros(m,m-1)];
                  alfamat=alfadp1';
                  vecalfa=alfamat(:)';
                        bata=(calfbat(m*m+m+1:m*m+2*m));
                       bata1=[bata';zeros(m-1,m)];
                        batamat=bata1';
                        vecbata=batamat(:)';
us1=-0.5*2*(C'*(Q1-tau*Ea1)+vecbata);
u1=us1*kron(H,H);
uaporx1=vec2mat(u1,m)
```

```
%***********************loop for Ui************
    us2=us1;
    f6=-0.5*X.^2; knew=D*H*f6*H'*D;
    knew1=knew';
    vecknew=knew1(:);
    ep=0.0001;
    Uerr=abs(uaporx1);
    u1aporx=vec2mat(us2,m);
    i=1;
while norm(Uerr)>ep
    mat=u1aporx;
    A=haarbigN(mat,m);
    bigu1=A;
    kfinal=vecknew'-0.5*us2*bigu1;
    newf2=u1aporx;
    mat=newf2;
    A1=haarbigN(mat,m);
    Nmu1=A1;
    M1=((Q2-tau*Ea2)*Nthata+(Q1-tau*Ea1)*Nmu1)';
    M2=Nthata'*kron(eye(m,m),that);
    M3=Nmu1'*kron(that,eye(m,m));
     bigW=[M1 M2 M3];
     calfbat=pinv(bigW)*kfinal';
             C=(calfbat(1:m*m));
             ccdp=vec2mat(C,m);
                  alfa1=(calfbat(m*m+1:m*m+m));
                  alfadp1=[alfa1 zeros(m,m-1)];
                  alfamat=alfadp1';
                  vecalfa=alfamat(:)';
                        bata=(calfbat(m*m+m+1:m*m+2*m));
                        bata1=[bata';zeros(m-1,m)];
                        batamat=bata1';
                        vecbata=batamat(:)';
    us1=-0.5*2*(C'*(Q1-tau*Ea1)+vecbata);
    u2=us1*kron(H,H);
    uaporx=vec2mat(u2,m);
        uerr=abs(uaporx-uaporx1);
         us2=us1;
   u1aporx=vec2mat(us1,m);
```

```
   uaporx1=uaporx
   Uerr=uerr;
   i=i+1;
end
%******************find H(x1,0) ****************************
 cofd=C'*(Q2-tau*Ea2)+vecalfa;
    for i=1:m
       cov(i)=cofd((i-1)*m+1:i*m)*H(1:m,(m/2)+1);
   end
   alfanew=cov'*that';
alfanew1=alfanew';
vecalfanew=alfanew1(:)';
Vfinal=vecalfanew*(Q1-tau*Ea1)*kron(H,H)+C'*(Q1-tau*Ea1)*(Q2-
tau*Ea2)*kron(H,H)+vecbata*(Q2-tau*Ea2)*kron(H,H);
Vfinalmat=vec2mat(Vfinal,m)
uaporx
plot(Y(4,:),Vfinalmat(4,:),'-*')

% Example 1 For direct method
% dx/dt=Ax + Bu  by Curtis and Beard
% J=int(x1^2+x2^2+u^2)dt
tau=10
Q=[1 0;0 1];
R=1;
a=[0 1;-1 2];
B=[0 1]';
x0=[12 20];
m=256
[P D H]=findp(tau,m);
P=0.5*P;
t=1:2:(2*m-1); t=tau*t/(2*m);
%**********intial conditon*******************************%
x01=12;
x02=20;
%*****************************************************%
that=[1 zeros(m-1,1)];
A=[eye(2*m,2*m)-kron(a,P')  -kron(B,eye(m,m))];
left=(a*x0'*that');
bleft=left';
```

```
vecb=bleft(:);
for i=1:1:m-1
   J=[floor(log2(i))];
   mm(i)=[1/(2^J)];
end
T=[1 mm];
Equad=10*diag(T);
c1=kron(Q,P*Equad);
c2=zeros(2*m,m);
b1=zeros(m,m);
b2= Equad;
Hquad=2*[kron(Q,P*Equad*P')  zeros(2*m,m);zeros(m,2*m)
kron(R,Equad)];
f1left=x0'*that';f1bleft=f1left';
vecbf1=f1bleft(:)';
fquad=[2*vecbf1*kron(Q,Equad*P') zeros(1,m)];
conquad=(vecbf1*kron(Q,Equad)*vecbf1');
opts = optimset('Algorithm','interior-point-
convex','Display','off');
[x,fval,exitflag,output]=quadprog(Hquad,fquad',[],[],A,vecb,[],[
],[],opts);
min=fval+conquad
xa=x(1:m,1)'*P*H+(x01)*that'*H;
xd=x(m+1:2*m,1)'*P*H+(x02)*that'*H;
xk=x(2*m+1:3*m,1)'*H;
xx=t;
yy=t;
[Y X]=meshgrid(xx,yy);   % coordinate (x1,x2)
uexact=-0.4142*X-4.4142*Y;
plot(t,xa,'-.b',t,xd,'-k','LineWidth',1.5);
xlabel('t'),ylabel('x_1(t) , x_2(t)'),grid off
legend('x_1(t)','x_2(t)')
```