Chapter 3

Methodology

3.1 Introduction

This section explains the mathematical formulations of natural and mixed convection in square and vertical cylindrical porous annulus. The heat transfer in square annulus with outside and inside heating, using thermal equilibrium and thermal non-equilibrium modelling respectively is formulated. Conjugate heat transfer with single and double wall inserted between the porous materials is considered. A solitary case of heat and mass transfer in vertical cylinder is included with appropriate boundary conditions followed by the segmental heating for different size and location of heater is discussed. Mixed convection in vertical cylinder with thermal non- equilibrium modelling is described. The corresponding boundary conditions with schematic representation and meshed diagram for separate cases of the each model are explained.

The governing partial differential equations can be solved either analytically or numerically in general. The nonlinear partial differential equations are difficult to be solved analytically, within a specified time, thus, the numerical solution becomes inevitable. The governing equations involved in the problem, with its non-dimensional derivation and conversion to algebraic equations by employing Finite Element Method is presented in subsequent sub sections. The solution methodology and the detail procedure for evaluating physical parameters are described.

3.2 Investigation of the heat transfer in square porous annulus.

Heat transfer characteristics in a square porous annulus are analyzed with specific emphasis on effect of width ratio of cavity, on heat and fluid flow, inside the porous medium. This particular problem is considered for two separate boundary conditions such as internal and external heating in square annulus.

The schematic of problem under consideration and coordinate system is shown in Figure 3.2 where *L* represents the length or breadth of the annulus with *D* indicating the portion unoccupied by porous medium.

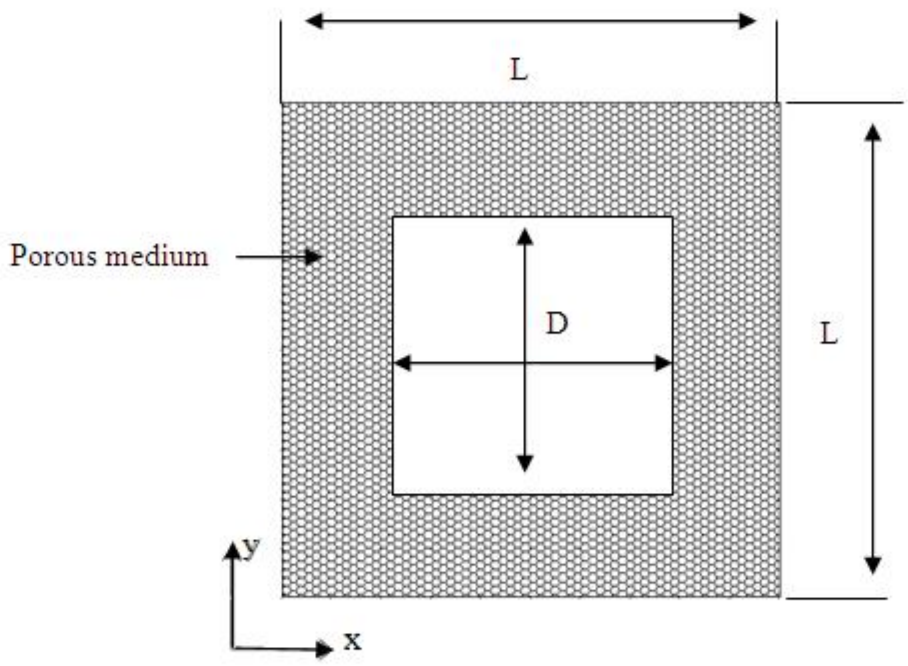


Figure 3.2.1: Schematic of square annulus

3.2.1 Governing equations for heat transfer with equilibrium modelling

The present study deals with the heat transfer analysis in the square porous annulus and the governing equations are dealt in the Cartesian co-ordinates. The governing heat transfer equations can be given as:

Continuity equation

$$\frac{\partial \mathrm{u}}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0 \qquad (3.2.1.1)$$

Where the velocity in horizontal ‘x’ direction;

$$\mathrm{u}=\frac{-K}{\mu}\frac{\partial p}{\partial x} \qquad (3.2.1.2)$$

Velocity in vertical ‘y’ direction;

$$\mathrm{v}=\frac{-K}{\mu}\left(\frac{\partial p}{\partial y}+\rho g\right) \qquad (3.2.1.3)$$

‘Whereas’ K ’ is the permeability of porous medium suggested by Bejan (2004):

$$K=\frac{D_p{}^2\varphi^3}{180(1-\varphi)^2} \qquad (3.2.1.4)$$

Density variation can be incorporated be Boussinesq approximation as:

$$\rho=\rho_\infty\left[1-\beta_T\left(T-T_\infty\right)\right] \qquad (3.2.1.5)$$

In order to accomplish the required parametrical solution, the pressure terms in the equations (3.2.1.2) and (3.2.1.3) are simplified by means of suitable mathematical simplifications. Differentiating equation (3.2.1.2) with respect to *y* and (3.2.1.3) with the respect to x lead to:

$$\frac{\partial u}{\partial y} = \frac{-K}{\mu} \frac{\partial^2 p}{\partial x \partial y} \tag{3.2.1.6}$$

$$\frac{\partial \mathrm{v}}{\partial x} = \frac{-K}{\mu}\left(\frac{\partial^2 p}{\partial y \partial x} - \rho_\infty \beta_T g \frac{\partial T}{\partial x}\right) \tag{3.2.1.7}$$

After eliminating the pressure terms from equations (3.2.1.6) and (3.2.1.7) we are left with momentum equation as:

$$\frac{\partial \mathrm{v}}{\partial x} - \frac{\partial \mathrm{u}}{\partial y} = \frac{gK\beta}{\nu} \frac{\partial T}{\partial x} \tag{3.2.1.8}$$

The energy equation is given by:

$$\mathrm{u}\frac{\partial T}{\partial x} + \mathrm{v}\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{\mu}{K(\rho c)_f}\left(u^2 + v^2\right) - \frac{1}{\rho C_p}\frac{\partial q_r}{\partial x} \tag{3.2.1.9}$$

The left hand side of above equation (3.2.1.9) highlights the convection of energy due to velocity *u* and *v*. the first term in bracket on right hand side of (3.2.1.9) indicates the conduction of heat in porous medium, the second term reflects the viscous dissipation and last term highlights the radiation effect. The radiation can be approximated by Rosseland hypothesis as:

$$q_r = -\frac{4n^2\sigma}{3\beta_R} \frac{\partial T^4}{\partial x} \tag{3.2.1.10}$$

Incorporating (3.2.1.10) into (3.2.1.9) results into:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{\mu}{K\left(\rho c\right)_f}\left(u^2+v^2\right)+\frac{1}{\rho C_p}\frac{4n^2\sigma}{3\beta_R}\frac{\partial^2 T^4}{\partial x^2} \qquad (3.2.1.11)$$

Velocity can be expressed in terms of stream functions ψ as:

$$u=\frac{\partial\psi}{\partial y} \qquad (3.2.1.12a)$$

$$v=-\frac{\partial\psi}{\partial x} \qquad (3.2.1.12b)$$

The non-dimensionalisation of the governing equations (3.2.1.8) and (3.2.1.11) have been accomplished by introducing the suitable non-dimensional parameters as.

Non-dimensional width $\bar{x}=\frac{x}{L}$ (3.2.1.13a)

Non-dimensional Height $\bar{y}=\frac{y}{L}$ (3.2.1.13b)

Non-dimensional Stream function $\bar{\psi}=\frac{\psi}{\alpha}$ (3.2.1.13c)

Non-dimensional Temperature $\bar{T}=\frac{\left(T-T_\infty\right)}{\left(T_w-T_\infty\right)}$ (3.2.1.13d)

Rayleigh Number $Ra=\frac{g\beta_T\Delta TKL}{\nu\alpha}$ (3.2.1.13e)

Radiation parameter $R_d=\frac{4\sigma n^2\,{T_\infty}^3}{\beta_R k_s}$ (3.2.1.13f)

Viscous dissipation parameter $\varepsilon=\frac{\alpha\ \mu}{\Delta TK\rho c}$ (3.2.1.14g)

Expanding the term T^4 in the equation (3.2.1.10) with the help of Taylor series results into (Raptis 1998):

$$T^4 \approx 4TT_\infty^{\;3} - 3T_\infty^{\;4} \qquad (3.2.1.14)$$

Substitution of equations (3.2.1.12 –3.2.1.14) into equations (3.2.1.8) and (3.2.1.11) gives rise to following non-dimensional equations:

$$\frac{\partial^2 \overline{\psi}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{\psi}}{\partial \overline{y}^2} = -Ra\frac{\partial \overline{T}}{\partial \overline{x}} \qquad (3.2.1.15)$$

$$\left[\frac{\partial \overline{\psi}}{\partial \overline{y}}\frac{\partial \overline{T}}{\partial \overline{x}} - \frac{\partial \overline{\psi}}{\partial \overline{x}}\frac{\partial \overline{T}}{\partial \overline{y}}\right] = \left(\left(1+\frac{4R_d}{3}\right)\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2}\right) + \varepsilon\left[\left(\frac{\partial \overline{\psi}}{\partial \overline{y}}\right)^2 + \left(\frac{\partial \overline{\psi}}{\partial \overline{x}}\right)^2\right] \qquad (3.2.1.16)$$

Equations (3.2.1.15) and (3.2.1.16) are two coupled partial differential equations which are interlinked together thus a change of variable in one equation affects the other equation.

3.2.2 Governing equations for heat transfer with thermal non-equilibrium modelling

There are two basic approaches adopted for the analysis in heat transfer through porous medium, thermal equilibrium and thermal non-equilibrium. Thermal equilibrium model assumes that the solid porous matrix and the fluid are in same temperature, thus thermal equilibrium condition prevails whereby single energy equation is required to represent the entire porous domain. In most of the real problems the temperature of the porous matrix is not same with that of the fluid, thus requires separate energy equations, each for solid and fluid phases. However, the momentum equation remains the same with thermal energy carried by fluid phase but the energy equation will be split into 2 separate equations representing fluid and solid phases of porous medium as given below,

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} = 0 \qquad (3.2.2.1)$$

Momentum equation

$$\frac{\partial u}{\partial y}-\frac{\partial \mathrm{v}}{\partial x}=-\frac{g\beta K}{\nu}\frac{\partial T_f}{\partial x} \qquad (3.2.2.2)$$

Energy equation for fluid

$$u\frac{\partial T_f}{\partial x}+v\frac{\partial T_f}{\partial y}=\varphi\alpha_f\left(\frac{\partial^2 T_f}{\partial x^2}+\frac{\partial^2 T_f}{\partial y^2}\right)+\frac{h}{(\rho c)_f}\left(T_s-T_f\right)+\frac{\mu}{K(\rho c)_f}\left(u^2+v^2\right)-\frac{\varphi}{(\rho c)_f}\frac{\partial q_r}{\partial x}$$

(3.2.2.3a)

Energy equation for solid

$$(1-\varphi)k_s\left(\frac{\partial^2 T_s}{\partial x^2}+\frac{\partial^2 T_s}{\partial y^2}\right)=h\left(T_s-T_f\right) \qquad (3.2.2.3b)$$

The following non-dimensional parameters are utilized:

Non-dimensional width $\bar{x}=\frac{x}{L}$ (3.2.2.4a)

Non-dimensional Height $\bar{y}=\frac{y}{L}$ (3.2.2.4b)

Non-dimensional Stream function $\bar{\psi}=\frac{\psi}{\varphi\alpha_f}$ (3.2.2.4c)

Non-dimensional Temperature $\bar{T}=\frac{(T-T_o)}{(T_h-T_c)}$ (3.2.2.4d.i)

Where $T_o=\frac{(T_h+T_c)}{2}$ (3.2.2.4d.ii)

Radiation parameter $R_d=\frac{4\sigma n^2 T_c^{\ 3}}{\beta_R k_s}$ (3.2.2.4e)

Rayleigh Number $Ra=\frac{g\beta_T \Delta TKL}{\phi\nu\alpha_f}$ (3.2.2.4f)

Inter-phase heat transfer coefficient $H=\frac{hL^2}{\varphi k_f}$ (3.2.2.4g)

Modified conductivity ratio $$Kr=\frac{\phi k_f}{(1-\phi)k_s}$$ (3.2.2.4h)

Viscous dissipation parameter $$\varepsilon=\frac{\varphi\alpha_f\mu}{\Delta TK\rho c}$$ (3.2.2.4g)

Utilisation of above parameters leads to following non-dimensional equations as:

Momentum equation

$$\frac{\partial^2\overline{\psi}}{\partial\overline{x}^2}+\frac{\partial^2\overline{\psi}}{\partial\overline{y}^2}=-Ra\frac{\partial\overline{T_f}}{\partial\overline{x}} \quad (3.2.2.5)$$

Energy equation for fluid

$$\left[\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial\overline{T_f}}{\partial\overline{x}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial\overline{T_f}}{\partial\overline{y}}\right]=\left(\left(1+\frac{4}{3}R_d\right)\frac{\partial^2\overline{T}_f}{\partial\overline{x}^2}+\frac{\partial^2\overline{T_f}}{\partial\overline{y}^2}\right)+\varepsilon\left[\left(\frac{\partial\overline{\psi}}{\partial\overline{y}}\right)^2+\left(\frac{\partial\overline{\psi}}{\partial\overline{x}}\right)^2\right]+H\left(\overline{T}_s-\overline{T}_f\right)$$

(3.2.2.6a)

Energy equation for solid

$$\frac{\partial^2\overline{T}_s}{\partial\overline{x}^2}+\frac{\partial^2\overline{T}_s}{\partial\overline{y}^2}=HKr\left(\overline{T}_s-\overline{T}_f\right) \quad (3.2.2.6b)$$

3.2.3 Boundary conditions for heat transfer when outside walls of square annulus is heated to isothermal temperature T_h

The schematic of the problem under consideration and coordinate system is shown in Figure 3.2.1. The outer boundaries of the duct are exposed to isothermal temperature T_h and the inner walls are maintained isothermally at cooler temperature T_c. *L* indicates the length or breadth of the annulus with *D* highlighting the portion unoccupied by porous medium.

The heating of outside walls sets the convection of heat into the porous medium due to temperature difference between outside and inside walls. The following boundary conditions are applied

$$x = 0, \quad u = 0, \quad \mathrm{v} = 0, \quad T = T_h \tag{3.2.3.1a}$$

$$x = L, \quad u = 0, \quad \mathrm{v} = 0, \quad T = T_h \tag{3.2.3.1b}$$

$$y = 0, \quad u = 0, \quad \mathrm{v} = 0, \quad T = T_h \tag{3.2.3.1c}$$

$$y = L, \quad u = 0, \quad \mathrm{v} = 0, \quad T = T_h \tag{3.2.3.1d}$$

$$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L-D}{2}, \quad u = 0, \quad \mathrm{v} = 0, \quad T = T_c \tag{3.2.3.2a}$$

$$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L+D}{2}, \quad u = 0, \quad \mathrm{v} = 0, \quad T = T_c \tag{3.2.3.2b}$$

$$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L-D}{2}, \quad u = 0, \quad \mathrm{v} = 0, T = T_c \tag{3.2.3.2c}$$

$$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L+D}{2}, \quad u = 0, \quad \mathrm{v} = 0, \quad T = T_c \tag{3.2.3.2d}$$

The final non-dimensional boundary conditions are:

$$\bar{x} = 0, \quad \bar{\psi} = 0, \quad \bar{T} = 1 \tag{3.2.3.3a}$$

$$\bar{x} = 1, \quad \bar{\psi} = 0, \quad \bar{T} = 1 \tag{3.2.3.3b}$$

$$\bar{y} = 0, \quad \bar{\psi} = 0, \quad \bar{T} = 1 \tag{3.2.3.3c}$$

$$\bar{y} = 1, \quad \bar{\psi} = 0, \quad \bar{T} = 1 \tag{3.2.3.3d}$$

$$\frac{1-W}{2} \le \bar{x} \le \frac{1+W}{2}, \bar{y} = \frac{1-W}{2}, \qquad \bar{\psi} = 0 \qquad \bar{T} = 0 \qquad (3.2.3.4a)$$

$$\frac{1-W}{2} \le \bar{x} \le \frac{1+W}{2}, \bar{y} = \frac{1+W}{2}, \qquad \bar{\psi} = 0 \qquad \bar{T} = 0 \qquad (3.2.3.4b$$

$$\frac{1-W}{2} \le \bar{y} \le \frac{1+W}{2}, \bar{x} = \frac{1-W}{2}, \qquad \bar{\psi} = 0 \qquad \bar{T} = 0 \qquad (3.2.3.4c)$$

$$\frac{1-W}{2} \le \bar{y} \le \frac{1+W}{2}, \bar{x} = \frac{1+W}{2}, \qquad \bar{\psi} = 0 \qquad \bar{T} = 0 \qquad (3.2.3.4d)$$

The heat transfer rate at wall surface q_w is given by:

$$q_w = -\left[\{k + \frac{16\,\sigma\, T_c^{\,3}}{3\beta_r}) \frac{\partial T}{\partial x} \right] \qquad (3.2.3.5)$$

The Nusselt number is expressed as

At vertical surfaces

$$Nu = -\left[\left(1 + \frac{4Rd}{3}\right) \frac{\partial \bar{T}}{\partial \bar{x}} \right]_{\substack{\bar{x}=0, \\ \bar{x}=1}} \qquad (3.2.3.6a)$$

At horizontal surfaces

$$Nu = -\left[\left(1 + \frac{4Rd}{3}\right) \frac{\partial \bar{T}}{\partial \bar{y}} \right]_{\substack{\bar{y}=0 \\ \bar{y}=1}} \qquad (3.2.3.6b)$$

The term $R_d=0$ reduces the problem under investigation to pure natural convection without radiation. As per the convenience, the radiation term can be defined as R_d or the reciprocal of R_d which is generally denoted as N or N_R.

The average Nusselt number is evaluated as;

Left wall $\overline{N}u_L = \frac{1}{L}\int_{\bar{y}=0}^{\bar{y}=1} Nu \quad _{@\bar{x}=0}$ (3.2.3.7a)

Right wall $\overline{N}u_R = \frac{1}{L}\int_{\bar{y}=0}^{\bar{y}=1} Nu \quad _{@\bar{x}=1}$ (3.2.3.7b)

Bottom wall $\overline{N}u_B = \frac{1}{L}\int_{\bar{x}=0}^{\bar{x}=1} Nu \quad _{@\bar{y}=0}$ 3.2.3.7c)

Top wall $\overline{N}u_T = \frac{1}{L}\int_{\bar{x}=0}^{\bar{x}=1} Nu \quad _{@\bar{y}=1}$ (3.2.3.7d)

Total Average Nusselt number

$$\overline{N}u_{Tot} = \frac{1}{4}\left(\overline{N}u_L + \overline{N}u_R + \overline{N}u_B + \overline{N}u_T\right) \tag{3.2.3.8}$$

3.2.4 Boundary conditions for square annulus subjected to internal wall heating to isothermal temperature T_h

The schematic of problem under consideration along with its boundary condition is depicted in Figure 3.2.4a. The all 4 inner walls are heated to temperature T_h and all outside walls are maintained at cooler temperature T_c.

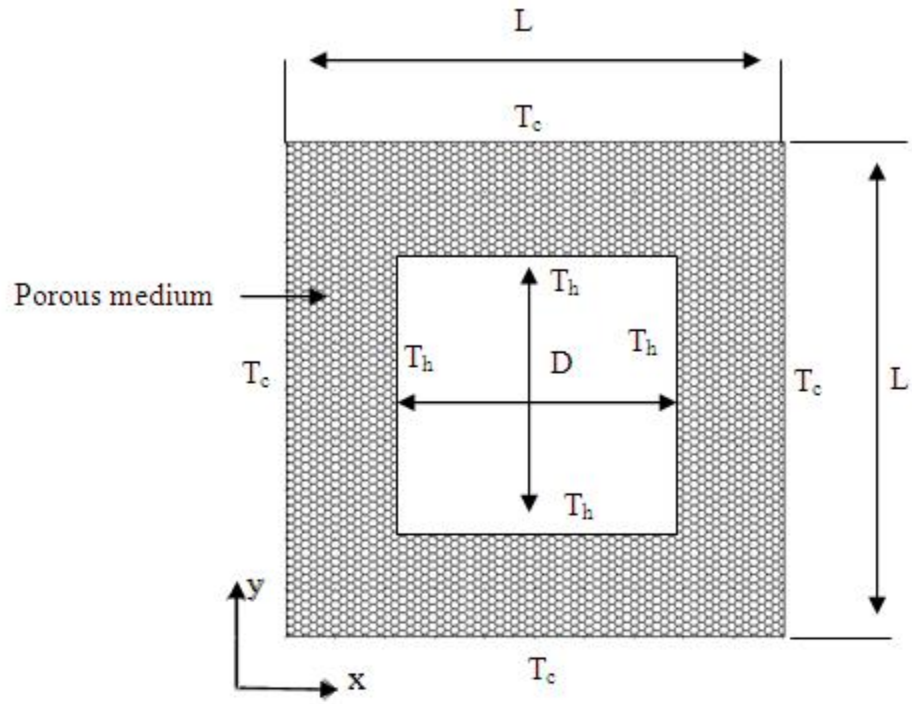


Figure 3.2.4a: Schematic of square annulus with inner heating

(a) Thermal Equilibrium

$$x = 0 \qquad u = 0 \qquad \mathrm{v} = 0 \qquad T = T_c \tag{3.2.4.1 a}$$

$$x = L \qquad u = 0 \qquad \mathrm{v} = 0 \qquad T = T_c \tag{3.2.4.1 b}$$

$$y = 0 \qquad u = 0 \qquad \mathrm{v} = 0 \qquad T = T_c \tag{3.2.4.1 c}$$

$$y = L \qquad u = 0 \qquad \mathrm{v} = 0 \qquad T = T_c \tag{3.2.4.1 d}$$

$$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L-D}{2} \qquad u = 0 \qquad \mathrm{v} = 0 \qquad T = T_h \tag{3.2.4.2 a}$$

$$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L+D}{2} \qquad u = 0 \qquad \mathrm{v} = 0 \qquad T = T_h \tag{3.2.4.2 b}$$

$$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L-D}{2} \qquad u = 0 \qquad \mathrm{v} = 0 \qquad T = T_h \tag{3.2.4.2 c}$$

$$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L+D}{2} \qquad u = 0 \qquad \mathrm{v} = 0 \qquad T = T_h \tag{3.2.4.2d}$$

$$\varepsilon = \frac{\alpha\mu}{\Delta TK\rho c} \tag{3.2.4.3}$$

The non-dimensionalisation of boundary conditions leads to:

$\bar{x} = 0$	$\bar{\psi} = 0$	$\bar{T} = 0$	(3.2.4.4a)
$\bar{x} = 1$	$\bar{\psi} = 0$	$\bar{T} = 0$	(3.2.4.4b)
$y = 0$	$\bar{\psi} = 0$	$\bar{T} = 0$	(3.2.4.4c)
$y = L$	$\bar{\psi} = 0$	$\bar{T} = 0$	(3.2.4.4d)
$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L-D}{2}$	$\bar{\psi} = 0$	$\bar{T} = 1$	(3.2.4.5a)
$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L+D}{2}$	$\bar{\psi} = 0$	$\bar{T} = 1$	(3.2.4.5b)
$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L-D}{2}$	$\bar{\psi} = 0$	$\bar{T} = 1$	(3.2.4.5c)
$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L+D}{2}$	$\bar{\psi} = 0$	$\bar{T} = 1$	(3.2.4.5d)

(b) Thermal Non Equilibrium

$x = 0$	$u = 0$	$\mathrm{v} = 0$	$T_f = T_s = T_c$	(3.2.4.6a)
$x = L$	$u = 0$	$\mathrm{v} = 0$	$T_f = T_s = T_c$	(3.2.4.6b)
$y = 0$	$u = 0$	$\mathrm{v} = 0$	$T_f = T_s = T_c$	(3.2.4.6c)
$y = L$	$u = 0$	$\mathrm{v} = 0$	$T_f = T_s = T_c$	(3.2.4.6d)
$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L-D}{2}$	$u = 0$	$\mathrm{v} = 0$	$T_f = T_s = T_h$	(3.2.4.7a)
$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L+D}{2}$	$u = 0$	$\mathrm{v} = 0$	$T_f = T_s = T_h$	(3.2.4.7b)
$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L-D}{2}$	$u = 0$	$\mathrm{v} = 0$	$T_f = T_s = T_h$	(3.2.4.7c)
$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L+D}{2}$	$u = 0$	$\mathrm{v} = 0$	$T_f = T_s = T_h$	(3.2.4.7d)

$$\varepsilon = \frac{\varphi\alpha_f\mu}{\Delta TK\rho c} \tag{3.2.4.8}$$

Non-dimensional form of above equations as:

$$\bar{x} = 0 \qquad \bar{\psi} = 0 \qquad \bar{T}_f = \bar{T}_s = -\frac{1}{2} \qquad (3.2.4.9a)$$

$$\bar{x} = 1 \qquad \bar{\psi} = 0 \qquad \bar{T}_f = \bar{T}_s = -\frac{1}{2} \qquad (3.2.4.9b)$$

$$y = 0 \qquad \bar{\psi} = 0 \qquad \bar{T}_f = \bar{T}_s = -\frac{1}{2} \qquad (3.2.4.9c)$$

$$y = L \qquad \bar{\psi} = 0 \qquad \bar{T}_f = \bar{T}_s = -\frac{1}{2} \qquad (3.2.4.9d)$$

$$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L-D}{2} \qquad \bar{\psi} = 0 \qquad \bar{T}_f = \bar{T}_s = \frac{1}{2} \qquad (3.2.4.10a)$$

$$\frac{L-D}{2} \le x \le \frac{L+D}{2}, y = \frac{L+D}{2} \qquad \bar{\psi} = 0 \qquad \bar{T}_f = \bar{T}_s = \frac{1}{2} \qquad (3.2.4.10b)$$

$$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L-D}{2} \qquad \bar{\psi} = 0 \qquad \bar{T}_f = \bar{T}_s = \frac{1}{2} \qquad (3.2.4.10c)$$

$$\frac{L-D}{2} \le y \le \frac{L+D}{2}, x = \frac{L+D}{2} \qquad \bar{\psi} = 0 \qquad \bar{T}_f = \bar{T}_s = \frac{1}{2} \qquad (3.2.4.10d)$$

The Nusselt number for thermal equilibrium and thermal non-equilibrium cases can be calculated as follows.

(a) Thermal Equilibrium

horizontal hot walls

$$Nu = -\left(\frac{\partial \bar{T}}{\partial \bar{y}}\right)_{T=T_h} \qquad (3.2.4.11a)$$

vertical hot walls

$$Nu = -\left[\left(1+\frac{4}{3}R_d\right)\frac{\partial \bar{T}}{\partial \bar{x}}\right]_{T=T_h} \qquad (3.2.4.11b)$$

Average Nu at Left hot surface

$$\overline{Nu}_B = \frac{1}{L}\int_{\frac{1-W}{2},}^{\frac{1+W}{2}} Nu \quad @\,\bar{x}=\frac{1-W}{2} \qquad (3.2.4.11c)$$

Average Nu at Right hot surface

$$\overline{Nu}_T = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu \quad @\,\bar{x}=\frac{1+W}{2} \qquad (3.2.4.11d)$$

Average *Nu* at bottom hot surface

$$\overline{N}u_L = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu_L \Big|_{@\,\bar{y}=\frac{1-W}{2}}, \qquad (3.2.4.11e)$$

Average *Nu* at Top hot surface

$$\overline{N}u_R = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu \Big|_{@\,\bar{y}=\frac{1+W}{2}} \qquad (3.2.4.11f)$$

Total Average *Nu*

$$\overline{N}u_{Tot} = \frac{1}{4}\left(\overline{N}u_L + \overline{N}u_R + \overline{N}u_B + \overline{N}u_T\right) \qquad (3.2.4.11g)$$

(b) Thermal Non-Equilibrium

Horizontal hot walls

$$Nu_f = -\left(\frac{\partial \overline{T}_f}{\partial \bar{y}}\right)_{T_f=T_h} \quad Nu_s = -\left(\frac{\partial \overline{T}_s}{\partial \bar{y}}\right)_{T_s=T_h} \qquad (3.2.4.12a)$$

vertical hot walls

$$Nu_f = -\left(\left(1+\frac{4}{3}R_d\right)\frac{\partial \overline{T}_f}{\partial \bar{x}}\right)_{T_f=T_h} Nu_s = -\left(\frac{\partial \overline{T}_s}{\partial \bar{x}}\right)_{T_s=T_h} \qquad (3.2.4.12b)$$

Average *Nu* at Left hot surface

$$\overline{N}u_B = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu_f \Big|_{@\,\bar{x}=\frac{1-W}{2}}, \quad \overline{N}u_B = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu_s \Big|_{@\,\bar{x}=\frac{1-W}{2}} \qquad (3.2.4.12c)$$

Average *Nu* at Right hot surface

$$\overline{N}u_T = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu_f \Big|_{@\,\bar{x}=\frac{1+W}{2}} \quad \overline{N}u_T = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu_s \Big|_{@\,\bar{x}=\frac{1+W}{2}} \qquad (3.2.4.12d)$$

Average *Nu* at bottom hot surface

$$\overline{N}u_L = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu_f \Big|_{@\,\bar{y}=\frac{1-W}{2}}, \quad \overline{N}u_L = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu_s \Big|_{@\,\bar{y}=\frac{1-W}{2}}, \qquad (3.2.4.12e)$$

Average *Nu* at Top hot surface

$$\overline{N}u_R = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu_f \Big|_{@\,\bar{y}=\frac{1+W}{2}} \quad \overline{N}u_R = \frac{1}{L}\int_{\frac{1-W}{2}}^{\frac{1+W}{2}} Nu_s \Big|_{@\,\bar{y}=\frac{1+W}{2}} \qquad (3.2.4.12f)$$

Total Average *Nu*

$$\overline{N}u_t = \frac{-1}{1+Kr}\left[\int\left\{Kr\left(Nu_f\right)+Nu_s\right\}d\overline{y}\right] or\ d\overline{x} \qquad (3.2.4.12g)$$

Average *Nu* of 4 walls

$$\overline{N}u_{Tot} = \frac{1}{4}\left(\overline{N}u_{tL} + \overline{N}u_{tR} + \overline{N}u_{tB} + \overline{N}u_{tT}\right) \qquad (3.2.4.12f)$$

3.3 Governing equations for conjugate heat transfer in vertical cylinder

The physical model of conjugate heat transfer in an annular porous annulus along with the coordinate system is depicted in Figure 3.3.1. The coordinate system is chosen in such a way that the *r and z* axis points towards the radial and vertical direction of the annulus. The model includes a solid wall having a finite thickness at the inner radius of the annulus which is followed by the porous region until the outer radius of annulus. The solid wall thickness is defined as a fraction of the total thickness of the annulus between inner and outer radii. The inner surface of the annulus is heated to constant temperature T_h and the outer surface is maintained at constant temperature T_∞ such that $T_h > T_\infty$.

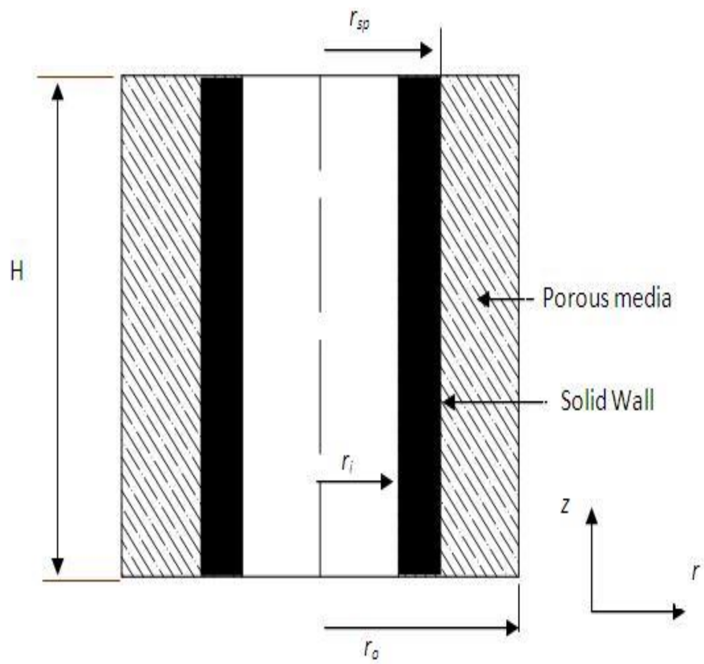


Figure 3.3.1: Schematic of the physical model

The governing equations of heat and fluid flow in the porous solid regions of the domain are given by:

For porous region

$$\frac{\partial (ru)}{\partial r} + \frac{\partial (rw)}{\partial z} = 0 \tag{3.3.1}$$

$$\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z} = \frac{gK\beta}{\nu}\frac{\partial T_p}{\partial r} \tag{3. 3.2}$$

$$u\frac{\partial T_p}{\partial r} + w\frac{\partial T_p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_p}{\partial r}\right) + \frac{\partial^2 T_p}{\partial z^2} \tag{3. 3.3}$$

For solid wall:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right) + \frac{\partial^2 T_s}{\partial z^2} = 0 \tag{3.3.4}$$

Subjected to boundary conditions:

At $r = r_i$ $T_s = T_h$ (3.3.5a)

At $r = r_o$, $T_p = T_\infty$ $u = 0$ (3.3.5b)

Since there is no heat storage in the medium, the following condition at solid-porous interface has to be satisfied, thus at

$$r = r_{sp} \qquad u = 0 \qquad T_s = T_p \qquad k_s \frac{\partial T_s}{\partial r} = k_p \frac{\partial T_p}{\partial r} \tag{3.3.5c}$$

The continuity equation (3.3.1) can be satisfied automatically by introducing the stream function ψ as:

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial z} \tag{3.3.6a}$$

$$w = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{3.3.6b}$$

The following parameters have been used for non-dimensionalisation of the governing equations.

$$\bar{r} = \frac{r}{L}, \quad \bar{z} = \frac{z}{L}, \quad \bar{\psi} = \frac{\psi}{\alpha_f \varphi L}, \quad \bar{T} = \frac{(T - T_\infty)}{(T_h - T_\infty)}$$
$$Ra = \frac{g\beta \Delta T K L}{\nu \alpha_p} \tag{3.3.7}$$

Substitution of equations (3.3.6)-(3.3.7) into equations (3.3.2)-(3.3.4) results into:

$$\frac{\partial^2 \bar{\psi}}{\partial \bar{z}^2} + \bar{r}\frac{\partial}{\partial \bar{r}}\left(\frac{1}{\bar{r}}\frac{\partial \bar{\psi}}{\partial \bar{r}}\right) = \bar{r} Ra \frac{\partial \bar{T_p}}{\partial \bar{r}} \tag{3.3.8}$$

$$\frac{1}{\bar{r}}\left[\frac{\partial\bar{\psi}}{\partial\bar{r}}\frac{\partial\overline{T_p}}{\partial\bar{z}}-\frac{\partial\bar{\psi}}{\partial\bar{z}}\frac{\partial\overline{T_p}}{\partial\bar{r}}\right]=\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\overline{T_p}}{\partial\bar{r}}\right)+\frac{\partial^2\overline{T_p}}{\partial\bar{z}^2} \qquad (3.3.9)$$

$$\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\overline{T_s}}{\partial\bar{r}}\right)+\frac{\partial^2\overline{T_s}}{\partial\bar{z}^2}=0 \qquad (3.3.10)$$

The corresponding boundary conditions take the form as:

At $\bar{r}=r_i$ $\qquad \bar{T}_s=1$ (3.3.11a)

At $\bar{r}=r_o$ $\qquad \bar{\psi}=0 \qquad \bar{T}_p=0$ (3.3.11b)

At $\bar{r}=r_{sp}$ $\qquad \bar{\psi}=0, \qquad \bar{T}_s=\bar{T}_p \qquad Kr\frac{\partial\bar{T}_s}{\partial\bar{r}}=\frac{\partial\bar{T}_p}{\partial\bar{r}}$ (3.3.11c)

At $\bar{z}=0$ *and* $\bar{z}=A_r$ $\qquad \frac{\partial\bar{T}}{\partial\bar{z}}=0$ (3.3.11d)

The Nusselt number can be calculated using following expression:

$$\bar{N}u_p=-\frac{1}{A_r}\int_{\bar{z}=0}^{\bar{z}=A_r}\left(\frac{\partial\bar{T}_p}{\partial\bar{r}}\right)_{\bar{r}=\bar{r}_{sp}}d\bar{z} \qquad (3.3.12)$$

3.4 Conjugate heat transfer in a vertical annulus with porous medium sandwiched between two solids

The schematic representation of the physical model of the conjugate heat transfer in an annular porous annulus with coordinate system is depicted in Figure 3.4.1. The coordinate system is chosen in such a way that the *r and z* axis points towards the radial and vertical direction of the annulus. Since this is a conjugate problem, a solid wall with finite thickness exists at the inner and outer radii of the annulus. The porous medium is sandwiched between these two solid walls. The solid wall thickness is defined as a fraction of the total thickness of the annulus between inner and outer radii. *DL* and *DR* refer to the fraction of solid wall at inner and outer surfaces respectively. The conductivity ratio *Kr* indicates the ratio of thermal conductivity between inner solid to porous medium where as solid conductivity ratio *Krs* is the ratio of inner to outer wall thermal conductivity. The inner surface of the annulus is heated to constant temperature T_h whereas the outer surface is maintained at constant temperature T_∞ such that $T_h > T_\infty$.

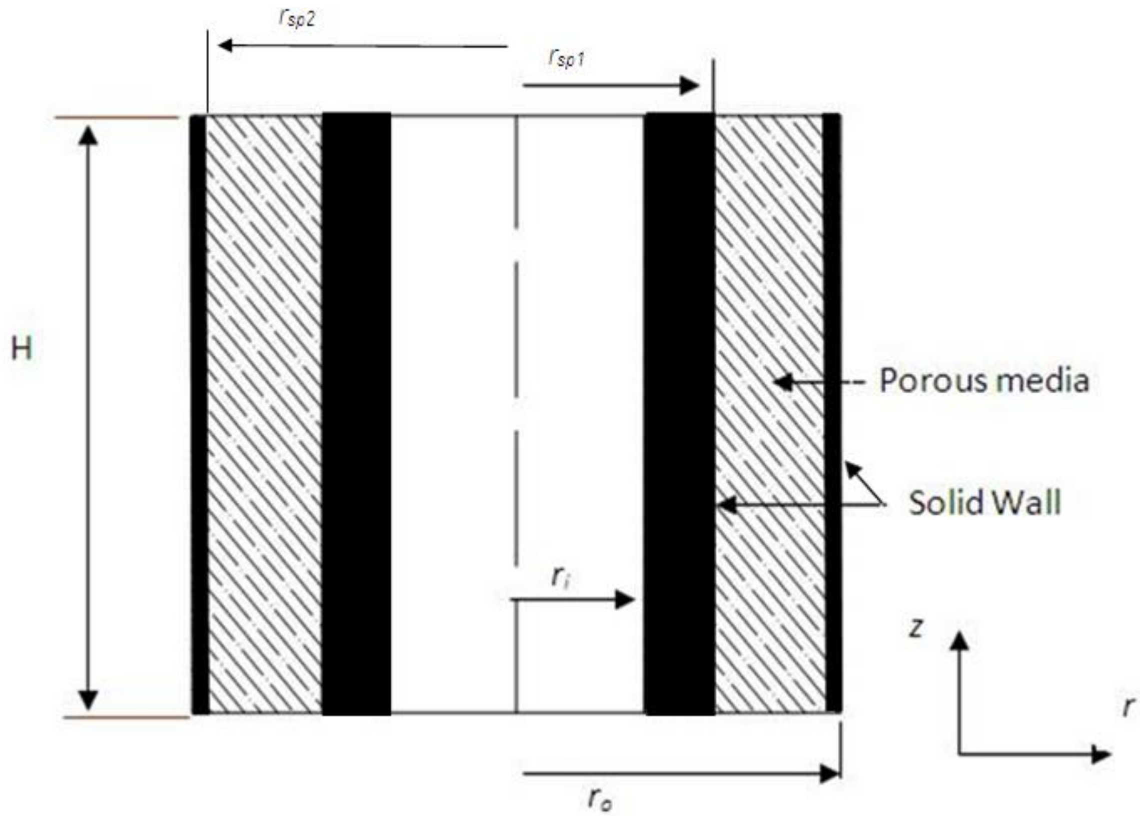


Figure 3.4.1: Schematic of the physical model

The governing equations for this case are same as equations (3.3.8)-(3.3.10) with different boundary conditions due to additional solid wall in the geometry, as given below:

At $\quad r = r_i \qquad T_s = T_h$ (3.4.1a)

At $\quad r = r_o, \qquad T_s = T_\infty$ (3.4.1b)

Since there is no heat storage in the medium, the following condition at solid-porous interface has to be satisfied

At $r = r_{sp1}$ $u = 0$ $k_{si}\dfrac{\partial T_{s1}}{\partial r} = k_p \dfrac{\partial T_p}{\partial r}$ (3.4.1c)

At $r = r_{sp2}$ $u = 0$ $k_p \dfrac{\partial T_p}{\partial r} = k_{s2}\dfrac{\partial T_{s2}}{\partial r}$ (3.4.1d)

Corresponding non-dimensional form of boundary conditions are:

At $\bar{r} = r_i$ $\bar{T}_{s1} = 1$ (3.4.2a)

At $\bar{r} = r_o$ $\bar{T}_{s2} = 0$ (3.4.2b)

At $\bar{r} = r_{sp1}$ $\bar{\psi} = 0,$ $K_1 \dfrac{\partial \bar{T}_{s1}}{\partial \bar{r}} = \dfrac{\partial \bar{T}_p}{\partial \bar{r}}$ (3.4.2c)

At $\bar{r} = r_{sp2}$ $\bar{\psi} = 0,$ $\dfrac{\partial \bar{T}_p}{\partial \bar{r}} = K_2 \dfrac{\partial \bar{T}_{s2}}{\partial \bar{r}}$ (3.4.2d)

Nusselt number is calculated using following expression:

$$\overline{N}u_p = -\frac{1}{\bar{z}}\int_0^{\bar{z}} \left(\frac{\partial \bar{T}_p}{\partial \bar{r}}\right)_{\bar{r} = \bar{r}_{isp1}} d\bar{z} \tag{3.4.3}$$

3.5 Governing equations for heat and mass transfer in vertical cylinder

The combined heat and mass transfer in porous medium is a phenomenon whereby thermosolutal transport occurs due to temperature and concentration gradient. It is also known as double diffusion or thermo-solutal transport.

The governing equations for the double diffusion in a vertical cylinder can be given as:

$$\frac{\partial (ru)}{\partial r} + \frac{\partial (rw)}{\partial z} = 0 \tag{3.5.1}$$

$$u = \frac{-K}{\mu}\frac{\partial p}{\partial r} \tag{3.5.2a}$$

$$\mathrm{w} = \frac{-K}{\mu}\left(\frac{\partial p}{\partial z} + \rho g\right) \tag{3.5.2b}$$

Energy equation

$$u\frac{\partial T_p}{\partial r}+\mathrm{w}\frac{\partial T_p}{\partial z}=\alpha\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_p}{\partial r}\right)+\frac{\partial^2 T_p}{\partial z^2}\right) \quad (3.5.3)$$

Concentration equation

$$u\frac{\partial C}{\partial r}+\mathrm{w}\frac{\partial C}{\partial z}=D\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)+\frac{\partial^2 C}{\partial z^2}\right) \quad (3.5.6)$$

The heat transfer in solid wall is described by:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right)+\frac{\partial^2 T_s}{\partial z^2} = 0 \quad (3.\ 5.7)$$

The continuity equation (3.5.1) can be satisfied by introducing the stream function ψ as:

$$u=-\frac{1}{r}\frac{\partial\psi}{\partial z} \quad (3.\ 5.8a)$$

$$\mathrm{w}=\frac{1}{r}\frac{\partial\psi}{\partial r} \quad (3.\ 5.8b)$$

The density variation can be described by Boussinesq approximation

$$\rho=\rho_\infty\left[1-\beta_T\left(T-T_\infty\right)-\beta_C\left(C-C_\infty\right)\right] \quad (3.5.9)$$

The initial boundary conditions are:

At $r = r_i$, $T = T_w$, $u = 0$, (3.5.10a)

At $r = r_o$, $T = T_\infty$, $C = C_\infty$, $u = 0$, (3.5.10b)

For a steady state flow, following condition at solid-porous interface must be satisfied

At $r = r_{sp}$, $u=0$, $T_s = T_p$, $C = C_h$, $k_s\frac{\partial T_s}{\partial r} = k_p\frac{\partial T_p}{\partial r}$ (3.5.10c)

The following parameters have been used for non-dimensionalisation

$$\bar{r}=\frac{r}{L_t},\quad \bar{z}=\frac{z}{L_t},\quad \bar{\psi}=\frac{\psi}{\alpha L_t},\quad \bar{T}=\frac{\left(T-T_\infty\right)}{\left(T_w-T_\infty\right)},\quad \bar{C}=\frac{\left(C-C_\infty\right)}{\left(C_w-C_\infty\right)} \quad (3.5.11a)$$

$$Ra=\frac{g\beta_T\Delta TKL_t}{\nu\alpha},\quad Le=\frac{\alpha}{D},\quad N=\frac{\beta c\left(C_w-C_\infty\right)}{\beta_T\left(T_w-T_\infty\right)} \quad (3.5.11b)$$

After mathematical simplification the final non-dimensional equations are:

$$\frac{\partial^2 \overline{\psi}}{\partial \overline{z}^2} + \overline{r}\frac{\partial}{\partial \overline{r}}\left(\frac{1}{\overline{r}}\frac{\partial \overline{\psi}}{\partial \overline{r}}\right) = \overline{r}Ra\left[\frac{\partial \overline{T_p}}{\partial \overline{r}} + N\frac{\partial \overline{C}}{\partial \overline{r}}\right] \tag{3.5.12}$$

$$\frac{1}{\overline{r}}\left[\frac{\partial \overline{\psi}}{\partial \overline{r}}\frac{\partial \overline{T}}{\partial \overline{z}} - \frac{\partial \overline{\psi}}{\partial \overline{z}}\frac{\partial \overline{T}}{\partial \overline{r}}\right] = \left(\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\left(\overline{r}\frac{\partial \overline{T_p}}{\partial \overline{r}}\right) + \frac{\partial^2 \overline{T_p}}{\partial \overline{z}^2}\right) \tag{3.5.13}$$

$$\frac{1}{\overline{r}}\left[\frac{\partial \overline{\psi}}{\partial \overline{r}}\frac{\partial \overline{C}}{\partial \overline{z}} - \frac{\partial \overline{\psi}}{\partial \overline{z}}\frac{\partial \overline{C}}{\partial \overline{r}}\right] = \frac{1}{Le}\left(\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\left(\overline{r}\frac{\partial \overline{C}}{\partial \overline{r}}\right) + \frac{\partial^2 \overline{C}}{\partial \overline{z}^2}\right) \tag{3.5.14}$$

$$\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\left(\overline{r}\frac{\partial \overline{T_s}}{\partial \overline{r}}\right) + \frac{\partial^2 \overline{T_s}}{\partial \overline{z}^2} = 0 \tag{3.5.15}$$

Thus the final boundary conditions are

At $\overline{r} = r_i$ $\quad \overline{T}_s = 1$ (3.5.16a)

At $\overline{r} = r_o$ $\quad \overline{\psi} = 0 \quad \overline{C} = 0 \quad \overline{T}_p = 0$ (3.5.16b)

At $\overline{r} = r_{sp}$ $\quad \overline{\psi} = 0, \quad \overline{T}_s = \overline{T}_p \quad \overline{C} = 1 \quad Kr\frac{\partial \overline{T}_s}{\partial \overline{r}} = \frac{\partial \overline{T}_p}{\partial \overline{r}}$ (3.5.16c)

The Nusselt number is calculated using following expressions:

$$\overline{N}u_p = -\frac{1}{\overline{z}}\int_{\overline{z}=0}^{\overline{z}=Ar}\left(\frac{\partial \overline{T}_p}{\partial \overline{r}}\right)_{\overline{r}=\overline{r}_{sp}} d\overline{z} \tag{3.5.17}$$

$$\overline{S}h = -\frac{1}{\overline{z}}\int_{\overline{z}=0}^{\overline{z}=Ar}\left(\frac{\partial \overline{C}_p}{\partial \overline{r}}\right)_{\overline{r}=\overline{r}_{sp}} d\overline{z} \tag{3.5.18}$$

3.6 Governing equations for discrete heating in vertical annular cylinder

Investigation of heat transfer in a vertical annular cylinder subjected to discrete heating is of considerable importance because in many practical applications, heating takes place over a portion of one of the vertical walls of the porous enclosure (Sankar et al., 2011). An annulus with inner radius r_i and outer radius r_o having porous medium fixed in between inner and outer radii is considered. The coordinate system is chosen in such a way that the r *and* z axis represents the radial and vertical direction respectively of the annulus. A section of the inner surface of the annulus is heated to constant temperature T_w and the outer surface is maintained at constant temperature T_∞ such that $T_w > T_\infty$. The heated length of annulus is referred as HL henceforth to indicate the % of heater length considered. The schematic of the problem under investigation for different heater length position is shown in Figures 3.6.1a, 3.6.1b, and 3.6.1c, where the 50% heater length is placed at bottom, middle and top sections of the hot surface respectively.

As explained in section 3.2.2 the thermal non-equilibrium approach require two separate energy equarions to be solved. The following equations govern the thermal non equilibrium heat transfer in a vertical cylinder.

Momentum equation

$$\frac{\partial \mathrm{w}}{\partial r} - \frac{\partial \mathrm{u}}{\partial \mathrm{z}} = \frac{gK\beta}{\nu}\frac{\partial T}{\partial r} \tag{3.6.1}$$

Energy equation for fluid

$$\left(\rho c_p\right)_f \left(\mathrm{u}\frac{\partial T_f}{\partial r} + \mathrm{w}\frac{\partial T_f}{\partial z} \right) = \varphi k_f \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_f}{\partial r} \right) + \frac{\partial^2 T_f}{\partial z^2} \right) + h\left(T_s - T_f\right) \tag{3.6.2}$$

Energy equation for solid

$$(1-\varphi)k_s \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r} \right) + \frac{\partial^2 T_s}{\partial z^2} \right) = h\left(T_s - T_f\right) + (1-\varphi)\frac{1}{r}\frac{\partial}{\partial r}(rq_r) \tag{3.6.3}$$

In this case the non-dimensional stream function can be defined as

Stream function $$\overline{\psi} = \frac{\psi}{\alpha\,\phi L_t} \tag{3.6.4}$$

Substitution of dimensionless parameters, described in section 3.2.1, gives the following non-dimensional coupled partial differential equations:

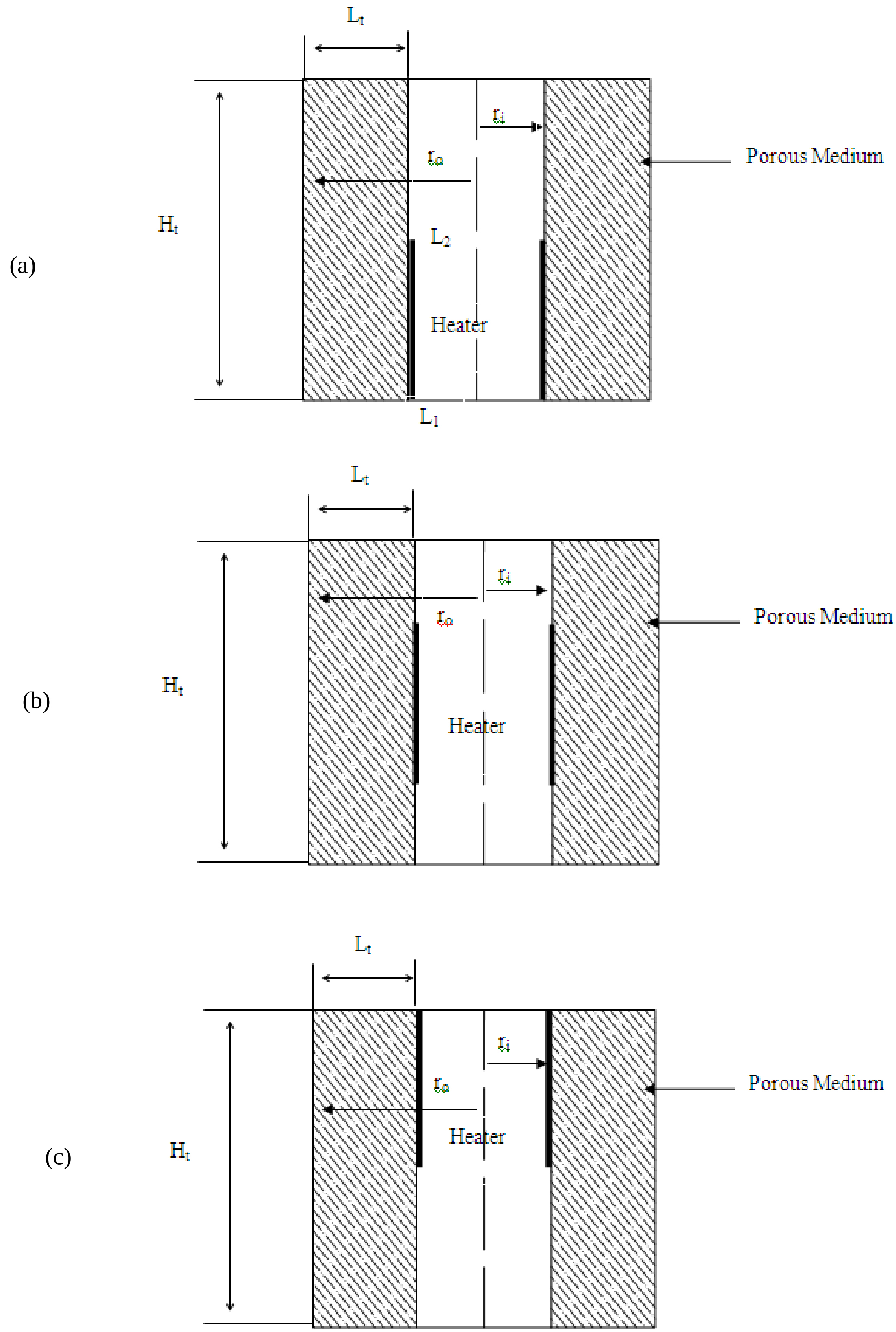


Figure 3.6.1: Schematic of the annulus with 50% HL heater a) bottom b) mid-and c) top sections of the annulus

There are two separate sets of boundary conditions are considered for this problem. CASE I represents the temperature of fluid and solid part of the porous domain are equal to hot wall at inner surface whereas CASE II represents that only solid material of the porous domain temperature is equal to hot wall and fluid temperature is unknown at inner surface.

3.6.1 Boundary conditions

Case I

At $r = r_i \; and \; L_1 \le z \le L_2$, $T_f = T_s = T_w$ $\quad u = 0$ (3.6.5a)

At $r = r_o$, $\quad T_f = T_s = T_w$ $\quad u = 0$ (3.6.5b)

Case II

At $r = r_i \; and \; L_1 \le z \le L_2$, $T_s = T_w$ $\quad u = 0$ (3.6.6a)

At $r = r_o$, $\quad T_f = T_s = T_w$ $\quad u = 0$ (3.6.6b)

Following dimensionless parameters are used to non-dimensionalise the governing equations

$$\bar{r} = \frac{r}{L_t}, \quad \bar{z} = \frac{z}{L_t}, \quad \bar{T} = \frac{(T - T_0)}{(T_w - T_\infty)} \quad \text{where} \quad T_0 = \frac{(T_w - T_\infty)}{2},$$

$$R_d = \frac{4\sigma n^2 T_\infty{}^3}{\beta_R K_s}, \quad \bar{\psi} = \frac{\psi}{\alpha \phi L_t}, \quad H = \frac{h L_t}{\phi k_f}$$

$$Kr = \frac{\phi k_f}{(1-\phi) k_s}, \quad Ra = \frac{g \beta K \Delta T L_t}{\phi \nu \alpha_f} \qquad (3.6.7)$$

Momentum equation

$$\frac{\partial^2 \overline{\psi}}{\partial \bar{z}^2} + \frac{1}{\bar{r}} \frac{\partial^2}{\partial \bar{r}} \left(\bar{r} \frac{\partial \overline{\psi}}{\partial \bar{r}} \right) = \bar{r} Ra \frac{\partial \overline{T_f}}{\partial \bar{r}} \tag{3.6.8}$$

Energy equation for fluid

$$\frac{1}{\bar{r}} \left[\frac{\partial \overline{\psi}}{\partial \bar{r}} \frac{\partial \overline{T_f}}{\partial \bar{z}} - \frac{\partial \overline{\psi}}{\partial \bar{z}} \frac{\partial \overline{T_f}}{\partial \bar{r}} \right] = \left(\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\left(1 + \frac{4R_d}{3} \right) \bar{r} \frac{\partial \overline{T_f}}{\partial \bar{r}} \right) + \frac{\partial^2 \overline{T_f}}{\partial \bar{z}^2} \right) + H \left(\overline{T}_s - \overline{T}_f \right) \tag{3.6.9}$$

Energy equation for solid

$$\left(\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\left(1 + \frac{4R_d}{3} \right) \bar{r} \frac{\partial \overline{T_s}}{\partial \bar{r}} \right) + \frac{\partial^2 \overline{T_s}}{\partial \bar{z}^2} \right) = HKr \left(\overline{T}_s - \overline{T}_f \right) \tag{3.6.10}$$

Corresponding boundary conditions as

Case I

At $\quad \bar{r} = r_i \ and\ \overline{L}_1 \geq \bar{z} \geq \overline{L}_2, \quad \overline{\psi} = 0, \quad \overline{T}_f = \overline{T}_s = \frac{1}{2}$ (3.6.11a)

At $\quad \bar{r} = r_o, \quad \overline{\psi} = 0 \quad \overline{T}_f = \overline{T}_s = -\frac{1}{2}$ (3.6.11b)

Case II

At $\quad \bar{r} = r_i, \ and\ \overline{L}_1 \geq \bar{z} \geq \overline{L}_2, \quad \overline{\psi} = 0, \quad \overline{T}_s = \frac{1}{2}$ (3.6.12a)

At $\quad \bar{r} = r_o, \quad \overline{\psi} = 0 \quad \overline{T}_f = \overline{T}_s = -\frac{1}{2}$ (3.6.12b)

The Nusselt number is calculated using following expressions:

For fluid

$$\overline{N}u_f = -\frac{1}{\left(\overline{L}_2 - \overline{L}_1 \right)} \int_{\overline{L}_1}^{\overline{L}_2} \left(\frac{\partial \overline{T}_f}{\partial \bar{r}} \right)_{\bar{r} = \bar{r}_i} d\bar{z} \tag{3.6.13}$$

For solid

$$\bar{N}u_s = -\frac{1}{\left(\bar{L}_2 - \bar{L}_1\right)} \int_{\bar{L}_1}^{\bar{L}_2} \left(\left(1 + \frac{4}{3} R_d \right) \frac{\partial \bar{T}_s}{\partial \bar{r}} \right)_{\bar{r} = \bar{r}_i} d\bar{Z} \tag{3.6.14}$$

The total heat transfer rate for the present problem can be expressed as:

$$q_t = \left\{ \varphi k_f \left(\frac{\partial T_f}{\partial r} \right)_{r = r_i, r_o} + (1 - \varphi) k_s \left(1 + \frac{4}{3} R_d \right) \left(\frac{\partial T_s}{\partial r} \right)_{r = r_i} \right\} \tag{3.6.15}$$

Using equation (3.6.14) it can be shown that the average total Nusselt number is:

$$\bar{N}u_t = \left(\frac{-1}{Kr + 1} \right) \frac{1}{\left(\bar{L}_2 - \bar{L}_1\right)} \int_{\bar{L}_1}^{\bar{L}_2} \left\{ Kr \left(\frac{\partial \bar{T}_f}{\partial \bar{r}} \right)_{\bar{r} = \bar{r}_o} + \left(1 + \frac{4}{3} R_d \right) \left(\frac{\partial \bar{T}_s}{\partial \bar{r}} \right)_{\bar{r} = \bar{r}_i} \right\} d\bar{z} \tag{3.6.16}$$

3.7 Governing equations for mixed convection in an annular cylinder, thermal non-equilibrium modelling.

This section describes the heat transfer characteristics of mixed convection in a porous medium for aiding and opposing flow. The aiding and opposed flow is governed by the direction of applied velocity. It must be noted that the aiding flow refers to a condition when the applied velocity and the buoyancy force act in same direction, assisting each other. However, for opposed flow, the buoyancy force and applied velocity act in opposite direction. The annulus is subjected to discrete heating of 20%, 35% and 50% at bottom, mid and top sections of the annulus. One of the cases, 20% heating at bottom section for aiding and opposing flow is shown in Figure 3.7.

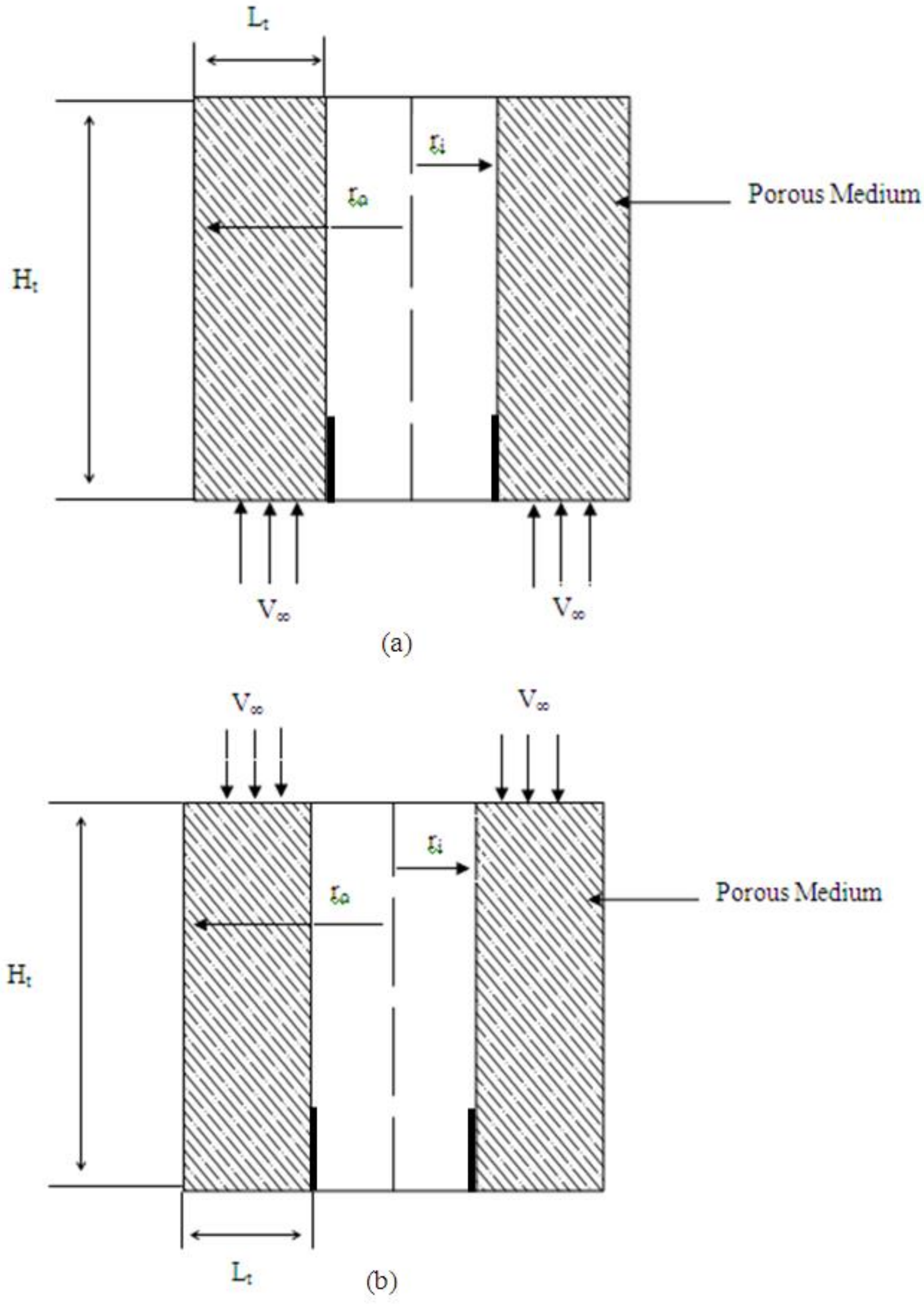


Figure 3.7.1: Schematic physical model of a) Aiding Flow b) Opposing flow

The governing equations for mixed convection in an annular cylinder are given by:

Momentum equation

$$\frac{\partial w}{\partial r}-\frac{\partial u}{\partial z}=\frac{gK\beta}{\nu}\frac{\partial T}{\partial r} \tag{3.7.1}$$

Energy equation for fluid

$$\left(\rho c_p\right)_f\left(u\frac{\partial T_f}{\partial r}+w\frac{\partial T_f}{\partial z}\right)=\varphi k_f\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_f}{\partial r}\right)+\frac{\partial^2 T_f}{\partial z^2}\right)+h\left(T_s-T_f\right) \tag{3.7.2}$$

Energy equation for solid

$$(1-\varphi)k_s\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right)+\frac{\partial^2 T_s}{\partial z^2}\right) = h\left(T_s-T_f\right)+(1-\varphi)\frac{1}{r}\frac{\partial}{\partial r}\left(rq_r\right) \tag{3.7.3}$$

Subjected to the boundary conditions

3.7.1 Boundary conditions for aiding flow

At $r=r_i \; and \; L_1\le z\le L_2,\; T_f=T_s=T_w \quad u=0$ (3.7.4a)

At $r=r_o,\quad T_f=T_s=T_w \quad u=0$ (3.7.4b)

At $z=0,\quad v=V_\infty,\; \frac{\partial T}{\partial z}=0 \quad u=0$ (3.7.4c)

At $z=H,\quad \frac{\partial T}{\partial z}=0 \quad u=0$ (3.7.4d)

3.7.2 Boundary conditions for opposing flow

At $r=r_i \; and \; L_1\le z\le L_2,\; T_f=T_s=T_w \quad u=0$ (3.7.5a)

At $r=r_o,\quad T_f=T_s=T_w \quad u=0$ (3.7.5b)

At $z = 0$, $\frac{\partial T}{\partial z} = 0$ $u = 0$ (3.7.5c)

At $z = H$, $v = -V_{\infty}$, $\frac{\partial T}{\partial z} = 0$ $u = 0$ (3.7.5d)

Following non-dimensional parameters are utilized

$$\bar{r} = \frac{r}{L_t}, \quad \bar{z} = \frac{z}{L_t}, \quad \bar{T} = \frac{(T - T_0)}{(T_w - T_\infty)} \quad \text{where} \quad T_0 = \frac{(T_w - T_\infty)}{2},$$

$$R_d = \frac{4\sigma n^2 T_\infty^{\ 3}}{\beta_R K_s}, \quad \bar{\psi} = \frac{\psi}{\alpha \phi L_t}, \quad H = \frac{h L_t}{\phi k_f}$$

$$Kr = \frac{\phi k_f}{(1-\phi) k_s}, \quad Pe = \frac{V_\infty L_t}{\varphi \alpha}, \quad Ra = \frac{g \beta K \Delta T L_t}{\phi \nu \alpha_f} \tag{3.7.6}$$

Substitution of dimensionless parameters gives the following non-dimensional coupled partial differential equations:

Momentum equation

$$\frac{\partial^2 \bar{\psi}}{\partial \bar{z}^2} + \frac{1}{\bar{r}} \frac{\partial^2}{\partial \bar{r}} \left(\bar{r} \frac{\partial \bar{\psi}}{\partial \bar{r}} \right) = \frac{\bar{r} Ra}{Pe} \frac{\partial \bar{T_f}}{\partial \bar{r}} \tag{3.7.7}$$

Energy equation for fluid

$$Pe \left[\frac{\partial \bar{\psi}}{\partial \bar{r}} \frac{\partial \bar{T_f}}{\partial \bar{z}} - \frac{\partial \bar{\psi}}{\partial \bar{z}} \frac{\partial \bar{T_f}}{\partial \bar{r}} \right] = \left(\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\left(1 + \frac{4R_d}{3} \right) \bar{r} \frac{\partial \bar{T_f}}{\partial \bar{r}} \right) + \frac{\partial^2 \bar{T_f}}{\partial \bar{z}^2} \right) + H \left(\bar{T_s} - \bar{T_f} \right) \tag{3.7.8}$$

Energy equation for solid

$$\left(\frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left(\left(1 + \frac{4R_d}{3} \right) \bar{r} \frac{\partial \bar{T_s}}{\partial \bar{r}} \right) + \frac{\partial^2 \bar{T_s}}{\partial \bar{z}^2} \right) = HKr \left(\bar{T_s} - \bar{T_f} \right) \tag{3.7.9}$$

With corresponding boundary conditions the non-dimensional boundary conditions for aiding flow:

At $\bar{r} = r_i \; and \; \bar{L}_1 \geq \bar{z} \geq \bar{L}_2$, $\bar{\psi} = 0$, $\bar{T}_f = \bar{T}_s = \frac{1}{2}$ (3.7.10a)

At $\bar{r} = r_o$, $\bar{\psi} = 0$ $\bar{T}_f = \bar{T}_s = -\frac{1}{2}$ (3.7.10b)

At $\bar{z} = 0$, $\bar{\psi} = 1,$ $\frac{\partial \bar{T}}{\partial \bar{z}} = 0$ (3.7.10c)

With corresponding boundary conditions the non-dimensional boundary conditions for opposing flow:

At $\bar{r} = r_i \; and \; \bar{L}_1 \geq \bar{z} \geq \bar{L}_2$, $\bar{\psi} = 0$, $\bar{T}_f = \bar{T}_s = \frac{1}{2}$ (3.7.11a)

At $\bar{r} = r_o$, $\bar{\psi} = 0$ $\bar{T}_f = \bar{T}_s = -\frac{1}{2}$ (3.7.11b)

At $\bar{z} = H$, $\bar{\psi} = -1,$ $\frac{\partial \bar{T}}{\partial \bar{z}} = 0$ (3.7.11c)

At $\bar{z} = 0$, $\bar{\psi} = 0,$ $\frac{\partial \bar{T}}{\partial \bar{z}} = 0$ (3.7.11d)

3.8 Solution methodology

As stated in the section 3.1 the governing partial differential equations can be solved either analytically or numerically. The analytical solution is limited to the problems which are simple in nature i.e. with simple geometries and boundary conditions. But with complex geometries and boundary conditions, the exact solution to the nonlinear partial differential equations is rather difficult, thus numerical methods are preferred for the solution of nonlinear partial differential equations with acceptable accuracy in recent years. Finite element method (FEM) is extensively used in the research due to compatibility with the solution procedures and accuracy. FEM is useful for solving even the problems with the irregular or contour boundaries thus has greater recognition in research community, a powerful tool for solving the linear and non-linear partial differential equations. The versatility of FEM is well understood due to the fact that, application of the FEM for one engineering area can be extended to solve the other engineering disciplines. The computer code written for heat transfer problem can be used to analyze the other field problems such as solid mechanics, aerodynamics, and hydraulics with minimal changes.

3.8.1 Finite element formulation

Basically the FEM consist of five steps which are:

- Specifying the approximate equation,
- Discretizing the region,
- Developing the algebraic system of equations,
- Solving the system of equations,
- Calculating the quantities of interests.

In this study, three- nodded linear triangular element is used to discretize the region, which is often recommended due to its reasonable representation of curved boundaries. The accuracy can increased with the use large number of smaller elements at the region of higher temperature variation.

The schematic diagram of the three-node triangular element is shown in the Figure 3.8.1.

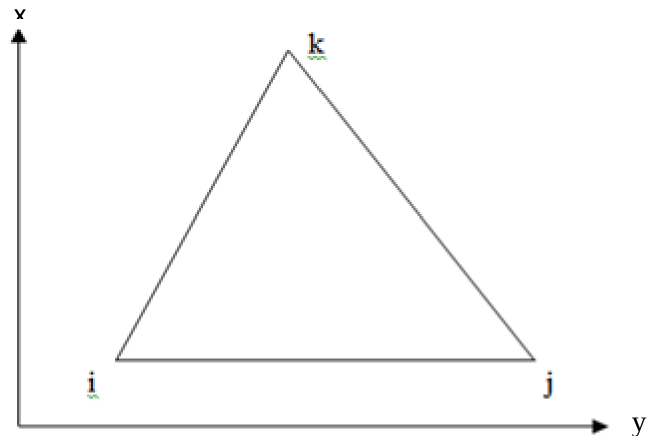


Figure 3.8.1: Typical triangular element

For a triangular element as shown above figure 3.8.1, the temperature variation is represented with the help of polynomial as:

$$T = \alpha_1 + \alpha_2 x + \alpha_3 y \tag{3.8.1}$$

$T_i, T_j \& T_k$ are the values of '*T*' at i, j and k nodes with x_i, x_j, x_k and y_i, y_j, y_k representing the *r* and *z* co-ordinates respectively. The values of α_1, α_2 and α_3 can be evaluated by substituting the values of *x* and *y* at nodes i, j and k in the equation

$$\alpha_1 = \frac{1}{2A}\left[\left(x_j y_k - x_k y_j\right)T_i + \left(x_k y_i - x_i y_k\right)T_j + \left(x_i y_k - x_j y_i\right)T_k\right] \tag{3.8.2a}$$

$$\alpha_2 = \frac{1}{2A}\left[\left(y_j - y_k\right)T_i + \left(y_k - y_i\right)T_j + \left(y_i - y_j\right)T_k\right] \tag{3.8.2b}$$

$$\alpha_3 = \frac{1}{2A}\left[\left(x_k - x_j\right)T_i + \left(x_i - x_k\right)T_j + \left(x_j - x_i\right)T_k\right] \tag{3.8.2c}$$

A Indicates the area of triangle and is given by,

$$A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} \tag{3.8.2d}$$

On substitution of the values $\alpha_1, \alpha_2, \alpha_3$ into (3.8.1) and mathematical rearrangement, we derive the following equation,

$$T = N_i T_i + N_j T_j + N_k T_k \tag{3.8.3}$$

Where N_i, N_j, N_k called the shape functions and is given by,

$$N_p = \frac{a_p + b_p x + c_p y}{2A}, \quad p = i, j, k \tag{3.8.4}$$

The constants a_p, b_p, c_p can be expressed in terms of coordinate system,

$$\begin{aligned} a_i &= x_j y_k - x_k y_j \\ b_i &= y_j - y_k \\ c_i &= x_k - x_j \end{aligned} \tag{3.8.5a}$$

$$\begin{aligned} a_j &= x_k y_i - x_i y_k \\ b_j &= y_k - y_i \\ c_j &= x_i - x_k \end{aligned} \tag{3.8.5b}$$

$$\begin{aligned} a_k &= x_i y_j - x_j y_i \\ b_k &= y_i - y_j \\ c_k &= x_j - x_i \end{aligned} \tag{3.8.5c}$$

The momentum equation is descritised into algebraic form of equations with the help of Galerkin’s method by integrating the product of shape function and the momentum equation over the elemental area as given by

$$\left\{R^e\right\} = -\int_A N^T \left(\frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} + Ra \frac{\partial \bar{T}}{\partial \bar{x}} \right) dA \qquad (3.8.6)$$

Where R^e is the residue left over that has to be minimised. Considering the individual terms and converting the higher order differential term into its first order simpler form yields,

$$\int_A N^T \frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} dA = \int_A \frac{\partial}{\partial \bar{x}} \left([N]^T \frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} \right) dA - \int_A \frac{\partial [N]^T}{\partial \bar{x}} \frac{\partial \bar{\psi}}{\partial \bar{x}} \qquad (3.8.7)$$

By applying Green theorem

$$\int_A N^T \frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} dA = -\int_A \frac{\partial N^T}{\partial \bar{x}} \frac{\partial N}{\partial \bar{x}} \begin{Bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{Bmatrix} dA \qquad (3.8.8)$$

By substituting (3.8.3) into (3.8.8)

,

$$\int_A N^T \frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} dA = -\frac{1}{(2A)^2} \int_A \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{Bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{Bmatrix} dA$$
$$= \frac{1}{4A^2} \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} \begin{Bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{Bmatrix} \qquad (3.8.9)$$

The differential with respect to x axis resulted into above equation. The second term of momentum equation is similar to first term but with the difference that it is differential in y direction. Integrating the second terms of momentum results into

$$\int_A N^T \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} dA = -\frac{1}{4A^2} \begin{bmatrix} c_1^2 & c_1 c_2 & c_1 c_3 \\ c_1 c_2 & c_2^2 & c_2 c_3 \\ c_1 c_3 & c_2 c_3 & c_3^2 \end{bmatrix} \begin{Bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{Bmatrix} \qquad (3.8.10)$$

The last term of momentum equation, is (3.8.8) subjected to similar integration

$$\int_A N^T Ra \frac{\partial \overline{T}}{\partial \overline{x}} dA = Ra \int_A N^T \frac{\partial \overline{T}}{\partial \overline{x}} dA \tag{3.8.11}$$

Consider that the triangular element is divided into three sub triangles as shown below.

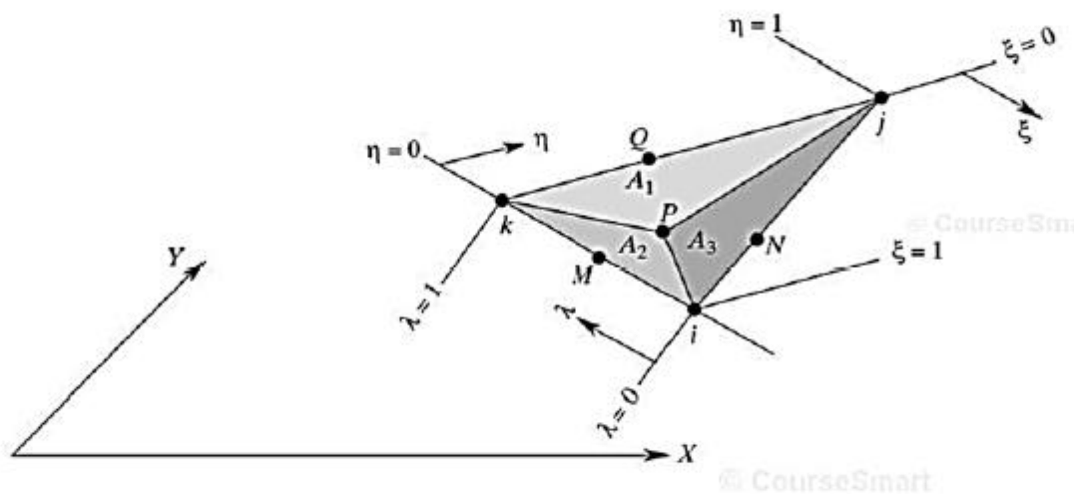


Figure 3.8.2: Triangular element divided into three sub triangles (Moaveni 2010)

Where

$$\zeta = \frac{Area\ pij}{Area\ ijk}$$
$$\eta = \frac{Area\ pjk}{Area\ ijk} \tag{3.8.12}$$
$$\lambda = \frac{Area\ pki}{Area\ ijk}$$

It can be shown (Moaveni 2010) that,

$$\zeta = N_1$$
$$\eta = N_2 \tag{3.8.13}$$
$$\lambda = N_3$$

Now by replacing the shape functions in above equation

$$\int_A N^T Ra \frac{\partial \overline{T}}{\partial \overline{x}} dA = Ra \int_A \begin{bmatrix} \zeta \\ \eta \\ \lambda \end{bmatrix} \frac{\partial [N]}{\partial \overline{x}} \begin{bmatrix} \overline{T}_1 \\ \overline{T}_2 \\ \overline{T}_3 \end{bmatrix} dA \tag{3.8.14}$$

By using a simple relation(Moaveni 2010) the area integration can be evaluated as:

$$\int_A L_1^d L_2^e L_3^f dA = \frac{d!e!f!}{(d+e+f+2)!} 2A \tag{3.8.15}$$

Substituting (3.8.15) into (3.8.14) we get,

$$\int_A N^T Ra \frac{\partial \overline{T}}{\partial \overline{x}} dA = Ra \frac{A}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{2A} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \overline{T}_1 \\ \overline{T}_2 \\ \overline{T}_3 \end{bmatrix}$$

$$= \frac{Ra}{6} \begin{Bmatrix} b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \\ b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \\ b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \end{Bmatrix} \tag{3.8.16}$$

Now we are ready with the FE formulation of all terms of momentum equation which can be written as

$$\frac{1}{4A} \begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^2 \end{bmatrix} \begin{Bmatrix} \overline{\psi}_1 \\ \overline{\psi}_2 \\ \overline{\psi}_3 \end{Bmatrix} + \frac{1}{4A} \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{bmatrix} \begin{Bmatrix} \overline{\psi}_1 \\ \overline{\psi}_2 \\ \overline{\psi}_3 \end{Bmatrix} = \frac{Ra}{6} \begin{bmatrix} b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \\ b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \\ b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \end{bmatrix} \tag{3.8.17}$$

Consider the energy equation in cartesian coordinate

$$\left[\frac{\partial \overline{\psi}}{\partial \overline{y}} \frac{\partial \overline{T}}{\partial \overline{x}} - \frac{\partial \overline{\psi}}{\partial \overline{x}} \frac{\partial \overline{T}}{\partial \overline{y}} \right] = \left(\left(1 + \frac{4R_d}{3}\right) \frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} \right) \tag{3.8.18}$$

Applying the Galerkin method to equation (3.8.18)

$$\{R^e\} = -\int_A [N]^T \left\{ \left[\frac{\partial \overline{\psi}}{\partial \overline{y}} \frac{\partial \overline{T}}{\partial \overline{x}} - \frac{\partial \overline{\psi}}{\partial \overline{x}} \frac{\partial \overline{T}}{\partial \overline{y}} \right] - \left(1 + \frac{4R_d}{3}\right) \left[\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} \right] \right\} dA \tag{3.8.19}$$

On separate term evaluation we get,

$$\int_A [N]^T \frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial\bar{T}}{\partial\bar{x}}dA = \int_A \begin{bmatrix}\zeta\\ \eta\\ \lambda\end{bmatrix}\frac{\partial[N]}{\partial\bar{y}}\{\bar{\psi}\}\frac{\partial[N]}{\partial\bar{x}}\{\bar{T}\}dA$$

$$= \int_A \begin{bmatrix}\zeta\\ \eta\\ \lambda\end{bmatrix}dA\left\{\frac{1}{4A^2}\left[c_1\bar{\psi}_1 + c_2\bar{\psi}_2 + c_3\bar{\psi}_3\right]\left[b_1 \quad b_2 \quad b_3\right]\begin{bmatrix}\bar{T}_1\\ \bar{T}_2\\ \bar{T}_3\end{bmatrix}\right\} \tag{3.8.20}$$

$$= \frac{1}{12A}\begin{Bmatrix}c_1\bar{\psi}_1 + c_2\bar{\psi}_2 + c_3\bar{\psi}_3\\ c_1\bar{\psi}_1 + c_2\bar{\psi}_2 + c_3\bar{\psi}_3\\ c_1\bar{\psi}_1 + c_2\bar{\psi}_2 + c_3\bar{\psi}_3\end{Bmatrix}\left[b_1 \quad b_2 \quad b_3\right]\begin{bmatrix}\bar{T}_1\\ \bar{T}_2\\ \bar{T}_3\end{bmatrix}$$

Similraly the term in y direction yields :

$$\int_A [N]^T \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial\bar{T}}{\partial\bar{y}}dA = \frac{1}{12A}\begin{Bmatrix}b_1\bar{\psi}_1 + b_2\bar{\psi}_2 + b_3\bar{\psi}_3\\ b_1\bar{\psi}_1 + b_2\bar{\psi}_2 + b_3\bar{\psi}_3\\ b_1\bar{\psi}_1 + b_2\bar{\psi}_2 + b_3\bar{\psi}_3\end{Bmatrix}\left[c_1 \quad c_2 \quad c_3\right]\begin{bmatrix}\bar{T}_1\\ \bar{T}_2\\ \bar{T}_3\end{bmatrix} \tag{3.8.21}$$

The second order differential terms gives similar algebraic equation in matrix form as discussed with respect to momentum equation

$$\int_A N^T\left(1+\frac{4}{3}R_d\right)\frac{\partial^2\bar{T}}{\partial\bar{x}^2}dA = -\frac{1}{4A}\left(1+\frac{4}{3}R_d\right)\begin{bmatrix}b_1^2 & b_1b_2 & b_1b_3\\ b_1b_2 & b_2^2 & b_2b_3\\ b_1b_3 & b_2b_3 & b_3^2\end{bmatrix}\begin{bmatrix}\bar{T}_1\\ \bar{T}_2\\ \bar{T}_3\end{bmatrix} \tag{3.8.22}$$

$$\int_A N^T\left(1+\frac{4}{3}R_d\right)\frac{\partial^2\bar{T}}{\partial\bar{y}^2}dA = -\frac{1}{4A}\left(1+\frac{4}{3}R_d\right)\begin{bmatrix}c_1^2 & c_1c_2 & c_1c_3\\ c_1c_2 & c_2^2 & c_2c_3\\ c_1c_3 & c_2c_3 & c_3^2\end{bmatrix}\begin{bmatrix}\bar{T}_1\\ \bar{T}_2\\ \bar{T}_3\end{bmatrix} \tag{3.8.23}$$

Thus the matrix form of energy equation after applying the Galerkin method results into.

$$\frac{1}{12A}\begin{pmatrix} c_1\bar{\psi}_1 + c_2\bar{\psi}_2 + c_3\bar{\psi}_3 \\ c_1\bar{\psi}_1 + c_2\bar{\psi}_2 + c_3\bar{\psi}_3 \\ c_1\bar{\psi}_1 + c_2\bar{\psi}_2 + c_3\bar{\psi}_3 \end{pmatrix}\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}\begin{Bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \end{Bmatrix}$$

$$-\frac{1}{12A}\begin{pmatrix} b_1\bar{\psi}_1 + b_2\bar{\psi}_2 + b_3\bar{\psi}_3 \\ b_1\bar{\psi}_1 + b_2\bar{\psi}_2 + b_3\bar{\psi}_3 \\ b_1\bar{\psi}_1 + b_2\bar{\psi}_2 + b_3\bar{\psi}_3 \end{pmatrix}\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}\begin{Bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \end{Bmatrix}$$

$$+\frac{1}{4A}\left(1+\frac{4}{3}R_d\right)\left\{\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^2 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{bmatrix}\right\}\begin{Bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \end{Bmatrix} = 0$$

(3.8.24)

3.8.2 Finite element equation with viscous dissipation

The residue of energy equation taking into account the viscous dissipation is

$$\left\{R^e\right\} = -\int_A [N]^T \left\{\left[\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial\bar{T}}{\partial\bar{x}} - \frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial\bar{T}}{\partial\bar{y}}\right] - \left(\left(1+\frac{4R_d}{3}\right)\frac{\partial^2\bar{T}}{\partial\bar{x}^2} + \frac{\partial^2\bar{T}}{\partial\bar{y}^2}\right) - \varepsilon\left[\left(\frac{\partial\bar{\psi}}{\partial\bar{y}}\right)^2 + \left(\frac{\partial\bar{\psi}}{\partial\bar{x}}\right)^2\right]\right\} dA$$

(3.8.25)

Considering the last terms of above equation,

$$\int_A N^T \varepsilon\left(\frac{\partial\bar{\psi}}{\partial\bar{x}}\right)^2 dA = \varepsilon\int_A \begin{bmatrix} \zeta \\ \eta \\ \lambda \end{bmatrix}\left(\left(\frac{\partial N}{\partial\bar{x}}\right)\{\bar{\psi}\}\right)^2 dA$$

$$= \frac{\varepsilon}{12A}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\left[b_1\bar{\psi}_1 + b_2\bar{\psi}_2 + b_3\bar{\psi}_3\right]^2 \qquad (3.8.26)$$

similarly for y-direction,

$$\int_A N^T \varepsilon\left(\frac{\partial\bar{\psi}}{\partial\bar{y}}\right)^2 dA = \varepsilon\int_A \begin{bmatrix} \zeta \\ \eta \\ \lambda \end{bmatrix}\left(\left(\frac{\partial N}{\partial\bar{y}}\right)\{\bar{\psi}\}\right)^2 dA$$

(3.8.27)

$$\int_A N^T \varepsilon \left(\frac{\partial \overline{\psi}}{\partial \overline{y}}\right)^2 dA = \frac{\varepsilon}{12A}\begin{bmatrix}1\\1\\1\end{bmatrix}\left[c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3\right]^2 \tag{3.8.28}$$

The complete energy equation with viscous dissipation is

$$\begin{aligned}
&\frac{1}{12A}\begin{Bmatrix} c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3 \\ c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3 \\ c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3 \end{Bmatrix}\left[b_1\ b_2\ b_3\right] - \frac{1}{12A}\begin{Bmatrix} b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3 \\ b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3 \\ b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3 \end{Bmatrix}\left[c_1\ c_2\ c_3\right] \\
&\frac{1}{4A}\left[\left(1+\frac{4}{3}R_d\right)\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_2 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^2 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{bmatrix}\right]\begin{Bmatrix}\overline{T}_1\\ \overline{T}_2\\ \overline{T}_3\end{Bmatrix} \\
&+\frac{\varepsilon}{12A}\left[\begin{bmatrix}1\\1\\1\end{bmatrix}\left[b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3\right]^2 + \begin{bmatrix}1\\1\\1\end{bmatrix}\left[c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3\right]^2\right] = 0
\end{aligned} \tag{3.8.29}$$

3.8.3 FE formulation of non-equilibrium model in cartesian coordinates

As stated the non-equlibruim model contains two equations for energy transport for solid and fluid phase. The momentum equation is

$$\frac{\partial^2 \overline{\psi}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{\psi}}{\partial \overline{y}^2} = -Ra\,\frac{\partial \overline{T_f}}{\partial \overline{x}} \tag{3.8.30}$$

Applying the Galerkin method to above equation results into:

$$\left\{R^e\right\} = -\int_A N^T \left(\frac{\partial^2 \overline{\psi}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{\psi}}{\partial \overline{y}^2} + Ra\,\frac{\partial \overline{T_f}}{\partial \overline{x}}\right) dA \tag{3.8.31}$$

The matrix form of above equation is:

$$\frac{1}{4A}\left\{\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_2 & b_2b_3 & b_3^2 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^2 \end{bmatrix}\right\}\begin{Bmatrix}\overline{\psi}_1\\ \overline{\psi}2\\ \overline{\psi}3\end{Bmatrix} = \frac{Ra}{6}\begin{Bmatrix} b_1\overline{T}_{f1} + b_2\overline{T}_{f2} + b_3\overline{T}_{f3} \\ b_1\overline{T}_{f1} + b_2\overline{T}_{f2} + b_3\overline{T}_{f3} \\ b_1\overline{T}_{f1} + b_2\overline{T}_{f2} + b_3\overline{T}_{f3} \end{Bmatrix} \tag{3.8.32}$$

Energy equation for fluid

$$\left[\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial\overline{T_f}}{\partial\overline{x}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial\overline{T_f}}{\partial\overline{y}}\right]=\left(\frac{\partial^2\overline{T}_f}{\partial\overline{x}^2}+\frac{\partial^2\overline{T_f}}{\partial\overline{y}^2}\right)+\varepsilon\left[\left(\frac{\partial\overline{\psi}}{\partial\overline{y}}\right)^2+\left(\frac{\partial\overline{\psi}}{\partial\overline{x}}\right)^2\right]+H\left(\overline{T}_s-\overline{T}_f\right) \tag{3.8.33}$$

Applying Galerkin method

$$\left\{R^e\right\}=-\int N^T\left[\left[\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial\overline{T_f}}{\partial\overline{x}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial\overline{T_f}}{\partial\overline{y}}\right]-\left(\frac{\partial^2\overline{T}_f}{\partial\overline{x}^2}+\frac{\partial^2\overline{T_f}}{\partial\overline{y}^2}\right)-\varepsilon\left[\left(\frac{\partial\overline{\psi}}{\partial\overline{y}}\right)^2+\left(\frac{\partial\overline{\psi}}{\partial\overline{x}}\right)^2\right]-H\left(\overline{T}_s-\overline{T}_f\right)\right] \tag{3.8.34}$$

Considering last term of above equation

$$\int_A N^T H\left(\overline{T}_{\overline{s}}-\overline{T}_f\right)dA=H\int_A\begin{bmatrix}\zeta\\ \eta\\ \lambda\end{bmatrix}\left[\begin{bmatrix}\zeta & \eta & \lambda\end{bmatrix}\begin{bmatrix}\overline{T}_{s1}\\ \overline{T}_{s2}\\ \overline{T}_{s3}\end{bmatrix}-\begin{bmatrix}\zeta & \eta & \lambda\end{bmatrix}\begin{bmatrix}\overline{T}_{f1}\\ \overline{T}_{f2}\\ \overline{T}_{f3}\end{bmatrix}\right] \tag{3.8.35}$$

$$=\frac{HA}{12}\left[\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{Bmatrix}\overline{T}_{s1}\\ \overline{T}_{s2}\\ \overline{T}_{s3}\end{Bmatrix}-\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{Bmatrix}\overline{T}_{f1}\\ \overline{T}_{f2}\\ \overline{T}_{f3}\end{Bmatrix}\right] \tag{3.8.36}$$

The matrix form of above equation (3.8.33) is:

$$\left[\frac{1}{12A}\begin{Bmatrix}c_1\overline{\psi}_1+c_2\overline{\psi}_2+c_3\overline{\psi}_3\\ c_1\overline{\psi}_1+c_2\overline{\psi}_2+c_3\overline{\psi}_3\\ c_1\overline{\psi}_1+c_2\overline{\psi}_2+c_3\overline{\psi}_3\end{Bmatrix}\begin{bmatrix}b_1 & b_2 & b_3\end{bmatrix}-\frac{1}{12A}\begin{Bmatrix}b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3\\ b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3\\ b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3\end{Bmatrix}\begin{bmatrix}c_1 & c_2 & c_3\end{bmatrix}\right]\begin{Bmatrix}\overline{T}_{f1}\\ \overline{T}_{f2}\\ \overline{T}_{f3}\end{Bmatrix}$$

$$+\frac{1}{4A}\left[\begin{bmatrix}b_1^2 & b_1b_2 & b_1b_2\\ b_1b_2 & b_2^2 & b_2b_3\\ b_1b_3 & b_2b_3 & b_3^2\end{bmatrix}+\begin{bmatrix}c_1^2 & c_1c_2 & c_1c_3\\ c_1c_2 & c_2^2 & c_2c_3\\ c_1c_2 & c_2c_3 & c_3^2\end{bmatrix}+\frac{HA^2}{3}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\right]\begin{Bmatrix}\overline{T}_{f1}\\ \overline{T}_{f2}\\ \overline{T}_{f3}\end{Bmatrix}$$

$$+\frac{\varepsilon}{12A}\left[\begin{bmatrix}1\\1\\1\end{bmatrix}\left[b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3\right]^2+\begin{bmatrix}1\\1\\1\end{bmatrix}\left[c_1\overline{\psi}_1+c_2\overline{\psi}_2+c_3\overline{\psi}_3\right]^2\right]-\frac{HA}{12}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{Bmatrix}\overline{T}_{s1}\\ \overline{T}_{s2}\\ \overline{T}_{s3}\end{Bmatrix}=0 \tag{3.8.37}$$

Similarly energy equation for solid phase can be transformed to matrix form as:

$$\left\{R^e\right\} = -\int_A N^T \left[\frac{\partial^2 \overline{T}_s}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}_s}{\partial \overline{y}^2} + \frac{4R_d}{3}\frac{\partial^2 \overline{T}_s}{\partial \overline{x}^2} - HKr\left(\overline{T}_s - \overline{T}_f\right) \right] dA \tag{3.8.38}$$

$$\frac{1}{4A}\left\{\left(1+\frac{4}{3}R_d\right)\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_2 & b_2b_3 & b_3^2 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^2 \end{bmatrix} + \frac{HA^2Kr}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\right\}\begin{Bmatrix} \overline{T}_{s1} \\ \overline{T}_{s2} \\ \overline{T}_{s3} \end{Bmatrix}$$

$$-\frac{HAKr}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \overline{T}_{f1} \\ \overline{T}_{f2} \\ \overline{T}_{f3} \end{Bmatrix} = 0 \tag{3.8.39}$$

Energy equation of solid

$$\frac{\partial^2 \overline{T}_s}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}_s}{\partial \overline{y}^2} = HKr\left(\overline{T}_s - \overline{T}_f\right) - \frac{4R_d}{3}\frac{\partial^2 \overline{T}_s}{\partial \overline{x}^2} \tag{3.8.40}$$

The application of Galerkin method yields:

$$\left\{R^e\right\} = -\int_A N^T \left[\frac{\partial^2 \overline{T}_s}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}_s}{\partial \overline{y}^2} + \frac{4R_d}{3}\frac{\partial^2 \overline{T}_s}{\partial \overline{x}^2} - HKr\left(\overline{T}_s - \overline{T}_f\right) \right] dA \tag{3.8.41}$$

$$\frac{1}{4A}\left\{\left(1+\frac{4}{3}R_d\right)\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_2 & b_2b_3 & b_3^2 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^2 \end{bmatrix} + \frac{HA^2Kr}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\right\}\begin{Bmatrix} \overline{T}_{s1} \\ \overline{T}_{s2} \\ \overline{T}_{s3} \end{Bmatrix}$$

$$-\frac{HAKr}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \overline{T}_{f1} \\ \overline{T}_{f2} \\ \overline{T}_{f3} \end{Bmatrix} = 0 \tag{3.8.42}$$

3.8.4 FE formulation for cylindrical co-ordinates

Let us consider the linear variation inside the triangular element in r and z direction that can be described the

$$T = \alpha_1 + \alpha_2 r + \alpha_3 z$$

$$\alpha_1 = \frac{1}{2A}\left[\left(r_j z_k - r_k z_j\right)T_i + \left(r_k z_i - r_i z_k\right)T_j + \left(r_i z_k - r_j z_i\right)T_k\right] \tag{3.8.43}$$

$$\alpha_2 = \frac{1}{2A}\left[\left(z_j - z_k\right)T_i + \left(z_k - z_i\right)T_j + \left(z_i - z_j\right)T_k\right] \tag{3.8.44}$$

$$\alpha_3 = \frac{1}{2A}\left[\left(r_k - r_j\right)T_i + \left(r_i - r_k\right)T_j + \left(r_j - r_i\right)T_k\right] \tag{3.8.45}$$

Where A is the area of the triangle and is given by

$$2A = \begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{vmatrix} \tag{3.8.46}$$

Substituting the values of α_1, α_2 and α_3 in the equation (3.8.1) and taking in terms of temperature and the stream functions as

$$T = N_1T_1 + N_2T_2 + N_3T_3 \tag{3.8.47}$$

$$= [N]\ \{T\} \tag{3.8.48}$$

$$\psi = N_1\psi_1 + N_2\psi_2 + N_3\psi_3 \tag{3.8.49}$$

$$= [N]\ \{\psi\} \tag{3.8.50}$$

Where N_1, N_2 and N_3 are the shape function for linear triangle, thus given by:

$$N_1 = \frac{a_i + b_i r + c_i z}{2A} \quad , \qquad \text{i = 1, 2, 3} \tag{3.8.51}$$

Where the variable a, b and c represents:

$$\begin{array}{lll} a_i = r_j z_k - r_k z_j ; & a_j = r_k z_i - r_i z_k ; & a_k = r_i z_j - r_j z_i ; \\ b_i = z_j - z_k ; & b_j = z_k - z_i ; & b_k = z_i - z_j ; \\ c_i = r_k - r_j ; & c_j = r_i - r_k ; & c_k = r_j - r_i ; \end{array} \tag{3.8.52}$$

Applying the Galerkin method and integrating the product of shape function with momentum equation

$$\left\{R^e\right\} = -\int_V N^T \left[\frac{\partial^2 \overline{\psi}}{\partial \overline{z}^2} + \overline{r} \frac{\partial}{\partial \overline{r}} \left(\frac{1}{\overline{r}} \frac{\partial \overline{\psi}}{\partial \overline{r}} \right) - \overline{r} Ra \frac{\partial \overline{T}}{\partial \overline{r}} \right] dV \tag{3.8.53}$$

$$\left\{R^e\right\} = -\int_A N^T \left[\frac{\partial^2 \overline{\psi}}{\partial \overline{z}^2} + \overline{r} \frac{\partial}{\partial \overline{r}} \left(\frac{1}{\overline{r}} \frac{\partial \overline{\psi}}{\partial \overline{r}} \right) - \overline{r} Ra \frac{\partial \overline{T}}{\partial \overline{r}} \right] 2\pi \overline{r} dA \tag{3.8.54}$$

On simplification the equations take the form as

$$\frac{2\pi \overline{R}}{4A} \left[\begin{bmatrix} b_1^{\,2} & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^{\,2} & b_2 b_3 \\ b_1 b_2 & b_2 b_3 & b_3^{\,2} \end{bmatrix} + \begin{bmatrix} c_1^{\,2} & c_1 c_2 & c_1 c_3 \\ c_1 c_2 & c_2^{\,2} & c_2 c_3 \\ c_1 c_2 & c_2 c_3 & c_3^{\,2} \end{bmatrix} \right] \begin{Bmatrix} \overline{\psi}_1 \\ \overline{\psi}_2 \\ \overline{\psi}_3 \end{Bmatrix} + 2\pi \overline{R}^2 Ra \begin{Bmatrix} b_1 \overline{T}_1 + b_2 \overline{T}_2 + b_3 \overline{T}_3 \\ b_1 \overline{T}_1 + b_2 \overline{T}_2 + b_3 \overline{T}_3 \\ b_1 \overline{T}_1 + b_2 \overline{T}_2 + b_3 \overline{T}_3 \end{Bmatrix} = 0 \tag{3.8.55}$$

The energy equation in cylindrical coordinates is

$$\frac{1}{\overline{r}} \left[\frac{\partial \overline{\psi}}{\partial \overline{r}} \frac{\partial \overline{T}}{\partial \overline{z}} - \frac{\partial \overline{\psi}}{\partial \overline{z}} \frac{\partial \overline{T}}{\partial \overline{r}} \right] = \left(\frac{1}{\overline{r}} \frac{\partial}{\partial \overline{r}} \left(\left(1 + \frac{4R_d}{3} \right) \overline{r} \frac{\partial \overline{T}}{\partial \overline{r}} \right) + \frac{\partial^2 \overline{T}}{\partial \overline{z}^2} \right) \tag{3.8.56}$$

The radial distance $\overline{R}$ to the centroid of an element is given by relation:

$$\overline{R} = \frac{\bar{r}_1 + \bar{r}_2 + \bar{r}_3}{3} \tag{3.8.57}$$

Similarly applying the Galerkins method to equation (3.8.56) gives:

$$\left[R^e\right] = -\int_A N^T \left[\frac{1}{\bar{r}} \left[\frac{\partial\overline{\psi}}{\partial\bar{r}} \frac{\partial\overline{T}}{\partial\bar{z}} - \frac{\partial\overline{\psi}}{\partial\bar{z}} \frac{\partial\overline{T}}{\partial\bar{r}} \right] - \left(\frac{1}{\bar{r}} \frac{\partial}{\partial\bar{r}} \left(\left(1 + \frac{4R_d}{3}\right) \bar{r} \frac{\partial\overline{T}}{\partial\bar{r}} \right) + \frac{\partial^2\overline{T}}{\partial\bar{z}^2} \right) \right] 2\pi\bar{r}dA$$

(3.8.58)

$$\left[\frac{2\pi}{12A} \begin{Bmatrix} c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3 \\ c_1\psi_1 + c_2\psi_2 + c_3\overline{\psi}_3 \\ c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3 \end{Bmatrix} \left[b_1 \; b_2 \; b_3 \right] - \frac{2\pi}{12A} \begin{Bmatrix} b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3 \\ b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3 \\ b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3 \end{Bmatrix} \left[c_1 \; c_2 \; c_3 \right] \right] \begin{Bmatrix} \overline{T}_1 \\ \overline{T}_2 \\ \overline{T}_3 \end{Bmatrix}$$

$$+ \frac{2\pi\overline{R}}{4A} \left\{ \left(1 + \frac{4}{3}R_d\right) \begin{bmatrix} b_1^2 & b_1b_2 & b_1b_2 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^2 \end{bmatrix} \begin{Bmatrix} \overline{T}_1 \\ \overline{T}_2 \\ \overline{T}_3 \end{Bmatrix} + \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{bmatrix} \right\} \begin{Bmatrix} \overline{T}_1 \\ \overline{T}_2 \\ \overline{T}_3 \end{Bmatrix} = 0$$

(3.8.59)

The conduction in solid wall for the case of conjugate heat transfer is:

$$\frac{1}{\bar{r}} \frac{\partial}{\partial\bar{r}} \left(\bar{r} \frac{\partial\overline{T_s}}{\partial\bar{r}} \right) + \frac{\partial^2\overline{T_s}}{\partial\bar{z}^2} = 0 \tag{3.8.60}$$

Employing Galerkin method:

$$\left\{R^e\right\} = -\int_A N^T \left[\left(\frac{1}{\bar{r}} \frac{\partial}{\partial\bar{r}} \left(\bar{r} \frac{\partial\overline{T_s}}{\partial\bar{r}} \right) + \frac{\partial^2\overline{T_s}}{\partial\bar{z}^2} \right) \right] 2\pi\bar{r}dA \tag{3.8.61}$$

$$\frac{2\pi\overline{R}}{4A} \left\{ \begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_2 & b_2b_3 & b_3^2 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^2 \end{bmatrix} \right\} \begin{Bmatrix} \overline{T}_{s1} \\ \overline{T}_{s2} \\ \overline{T}_{s3} \end{Bmatrix} = 0 \tag{3.8.62}$$

3.8.5 Finite Element equations for double diffusion in vertical cylinder

The momentum equation taking into account the mass diffusion is given by:

$$\frac{\partial^2 \overline{\psi}}{\partial \overline{z}^2} + \overline{r}\frac{\partial}{\partial \overline{r}}\left(\frac{1}{\overline{r}}\frac{\partial \overline{\psi}}{\partial \overline{r}}\right) = \overline{r}Ra\left[\frac{\partial \overline{T}}{\partial \overline{r}} + N\frac{\partial \overline{C}}{\partial \overline{r}}\right] \tag{3.8.63}$$

$$\left\{R^e\right\} = -\int_A N^T\left[\frac{\partial^2 \overline{\psi}}{\partial \overline{z}^2} + \overline{r}\frac{\partial}{\partial \overline{r}}\left(\frac{1}{\overline{r}}\frac{\partial \overline{\psi}}{\partial \overline{r}}\right) - \overline{r}Ra\left[\frac{\partial \overline{T}}{\partial \overline{r}} + N\frac{\partial \overline{C}}{\partial \overline{r}}\right]\right]2\pi\overline{r}dA \tag{3.8.64}$$

The matrix form, (3.8.65)

$$\frac{2\pi\overline{R}}{4A}\left[\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^2 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{bmatrix}\right]\begin{Bmatrix} \overline{\psi}_1 \\ \overline{\psi}_2 \\ \overline{\psi}_3 \end{Bmatrix} +$$

$$2\pi\overline{R}^2 Ra\left[\begin{Bmatrix} b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \\ b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \\ b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \end{Bmatrix} + N\begin{Bmatrix} b_1\overline{C}_1 + b_2\overline{C}_2 + b_3\overline{C}_3 \\ b_1\overline{C}_1 + b_2\overline{C}_2 + b_3\overline{C}_3 \\ b_1\overline{C}_1 + b_2\overline{C}_2 + b_3\overline{C}_3 \end{Bmatrix}\right] = 0$$

The energy equation

$$\frac{1}{\overline{r}}\left[\frac{\partial \overline{\psi}}{\partial \overline{r}}\frac{\partial \overline{T}}{\partial \overline{z}} - \frac{\partial \overline{\psi}}{\partial \overline{z}}\frac{\partial \overline{T}}{\partial \overline{r}}\right] = \left(\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\left(\left(1 + \frac{4R_d}{3}\right)\overline{r}\frac{\partial \overline{T}}{\partial \overline{r}}\right) + \frac{\partial^2 \overline{T}}{\partial \overline{z}^2}\right) \tag{3.8.66}$$

The formulation of energy equation is similar to that mentioned in previous case

Concentration equation

$$\frac{1}{\overline{r}}\left[\frac{\partial \overline{\psi}}{\partial \overline{r}}\frac{\partial \overline{C}}{\partial \overline{z}} - \frac{\partial \overline{\psi}}{\partial \overline{z}}\frac{\partial \overline{C}}{\partial \overline{r}}\right] = \frac{1}{Le}\left(\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\left(\overline{r}\frac{\partial \overline{C}}{\partial \overline{r}}\right) + \frac{\partial^2 \overline{C}}{\partial \overline{z}^2}\right) \tag{3.8.67}$$

Using the Galerkin method:

$$\left\{R^e\right\} = -\int_A N^T\left[\frac{1}{\overline{r}}\left[\frac{\partial \overline{\psi}}{\partial \overline{r}}\frac{\partial \overline{C}}{\partial \overline{z}} - \frac{\partial \overline{\psi}}{\partial \overline{z}}\frac{\partial \overline{C}}{\partial \overline{r}}\right] - \frac{1}{Le}\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\left(\overline{r}\frac{\partial \overline{C}}{\partial \overline{r}}\right) + \frac{\partial^2 \overline{C}}{\partial \overline{z}^2}\right]2\pi\overline{r}dA \tag{3.8.68}$$

Transform into matrix form,

$$\left[\frac{2\pi}{12A}\begin{Bmatrix} c_1\overline{\psi}_1+c_2\overline{\psi}_2+c_3\overline{\psi}_3 \\ c_1\psi_1+c_2\psi_2+c_3\overline{\psi}_3 \\ c_1\overline{\psi}_1+c_2\overline{\psi}_2+c_3\overline{\psi}_3 \end{Bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}-\frac{2\pi}{12A}\begin{Bmatrix} b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3 \\ b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3 \\ b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3 \end{Bmatrix}\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}\right]\begin{Bmatrix} \overline{C_1} \\ \overline{C_2} \\ \overline{C_3} \end{Bmatrix}$$
$$+\frac{2\pi\overline{R}}{4ALe}\left\{\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_2 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^2 \end{bmatrix}\begin{Bmatrix} \overline{C_1} \\ \overline{C_2} \\ \overline{C_3} \end{Bmatrix}+\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{bmatrix}\right\}\begin{Bmatrix} \overline{C_1} \\ \overline{C_2} \\ \overline{C_3} \end{Bmatrix}=0 \tag{3.8.69}$$

The conduction equation in solid wall

$$\frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}\left(\overline{r}\,\frac{\partial\overline{T_s}}{\partial\overline{r}}\right)+\frac{\partial^2\overline{T_s}}{\partial\overline{z}^2}=0 \tag{3.8.70}$$

Employing Galerkin method:

$$\left\{R^e\right\}=-\int_A N^T\left[\left(\frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}\left(\overline{r}\,\frac{\partial\overline{T_s}}{\partial\overline{r}}\right)+\frac{\partial^2\overline{T_s}}{\partial\overline{z}^2}\right)\right]2\pi\overline{r}dA \tag{3.8.71}$$

$$\frac{2\pi\overline{R}}{4A}\left\{\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_2 & b_2b_3 & b_3^2 \end{bmatrix}+\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^2 \end{bmatrix}\right\}\begin{Bmatrix} \overline{T}_{s1} \\ \overline{T}_{s2} \\ \overline{T}_{s3} \end{Bmatrix}=0 \tag{3.8.72}$$

3.8.6 Energy equation in case of thermal non-equilibrium

Momentum equation

$$\frac{\partial^2\overline{\psi}}{\partial\overline{z}^2}+\overline{r}\,\frac{\partial}{\partial\overline{r}}\left(\frac{1}{\overline{r}}\frac{\partial\overline{\psi}}{\partial\overline{r}}\right)=\overline{r}Ra\frac{\partial\overline{T}}{\partial\overline{r}} \tag{3.8.73}$$

Application of Galerkin method

$$\left\{R^e\right\}=-\int_V N^T\left[\frac{\partial^2\overline{\psi}}{\partial\overline{z}^2}+\overline{r}\,\frac{\partial}{\partial\overline{r}}\left(\frac{1}{\overline{r}}\frac{\partial\overline{\psi}}{\partial\overline{r}}\right)-\overline{r}Ra\frac{\partial\overline{T}}{\partial\overline{r}}\right]dV \tag{3.8.74}$$

$$\frac{2\pi \overline{R}}{4A}\left[\begin{bmatrix} b_1^{\,2} & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^{\,2} & b_2b_3 \\ b_1b_2 & b_2b_3 & b_3^{\,2} \end{bmatrix} + \begin{bmatrix} c_1^{\,2} & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^{\,2} & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^{\,2} \end{bmatrix}\right]\begin{Bmatrix} \overline{\psi}_1 \\ \overline{\psi}_2 \\ \overline{\psi}_3 \end{Bmatrix} + 2\pi \overline{R}^2 Ra \begin{Bmatrix} b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \\ b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \\ b_1\overline{T}_1 + b_2\overline{T}_2 + b_3\overline{T}_3 \end{Bmatrix} = 0$$

(3.8.75)

Energy equation for fluid

$$\frac{1}{\overline{r}}\left[\frac{\partial\overline{\psi}}{\partial\overline{r}}\frac{\partial\overline{T_f}}{\partial\overline{z}} - \frac{\partial\overline{\psi}}{\partial\overline{z}}\frac{\partial\overline{T_f}}{\partial\overline{r}}\right] = \left(\frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}\left(\left(1+\frac{4R_d}{3}\right)\overline{r}\frac{\partial\overline{T_f}}{\partial\overline{r}}\right) + \frac{\partial^2\overline{T_f}}{\partial\overline{z}^2}\right) + H\left(\overline{T}_s - \overline{T}_f\right) \tag{3.8.76}$$

Applying Galerkin method:

$$\left\{R^e\right\} = -\int_A N^T\left[\frac{1}{\overline{r}}\left[\frac{\partial\overline{\psi}}{\partial\overline{r}}\frac{\partial\overline{T_f}}{\partial\overline{z}} - \frac{\partial\overline{\psi}}{\partial\overline{z}}\frac{\partial\overline{T_f}}{\partial\overline{r}}\right] - \left(\frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}\left(\left(1+\frac{4R_d}{3}\right)\overline{r}\frac{\partial\overline{T_f}}{\partial\overline{r}}\right) + \frac{\partial^2\overline{T_f}}{\partial\overline{z}^2}\right) + H\left(\overline{T}_s - \overline{T}_f\right)\right]2\pi\overline{r}dA$$

(3.8.77)

$$\left[\frac{2\pi}{12A}\begin{Bmatrix} c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3 \\ c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3 \\ c_1\overline{\psi}_1 + c_2\overline{\psi}_2 + c_3\overline{\psi}_3 \end{Bmatrix}\left[b_1\ b_2\ b_3\right] - \frac{2\pi}{12A}\begin{Bmatrix} b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3 \\ b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3 \\ b_1\overline{\psi}_1 + b_2\overline{\psi}_2 + b_3\overline{\psi}_3 \end{Bmatrix}\left[c_1\ c_2\ c_3\right]\right]\begin{Bmatrix} \overline{T}_{f1} \\ \overline{T}_{f2} \\ \overline{T}_{f3} \end{Bmatrix}$$

$$+\ \frac{2\pi\overline{R}}{4A}\left[\begin{bmatrix} b_1^{\,2} & b_1b_2 & b_1b_2 \\ b_1b_2 & b_2^{\,2} & b_2b_3 \\ b_1b_3 & b_2b_3 & b_3^{\,2} \end{bmatrix} + \begin{bmatrix} c_1^{\,2} & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^{\,2} & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^{\,2} \end{bmatrix} + \frac{HA^2}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} \overline{T}_{f1} \\ \overline{T}_{f2} \\ \overline{T}_{f3} \end{Bmatrix}$$

$$= \frac{2\pi\overline{R}HA}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \overline{T}_{s1} \\ \overline{T}_{s2} \\ \overline{T}_{s3} \end{Bmatrix} \tag{3.8.78}$$

Energy equation for solid phase of porous medium

$$\left(\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\left(1+\frac{4R_d}{3}\right)\bar{r}\frac{\partial \overline{T_s}}{\partial \bar{r}}\right)+\frac{\partial^2 \overline{T_s}}{\partial \bar{z}^2}\right)=HKr\left(\bar{T}_s-\bar{T}_f\right) \tag{3.8.79}$$

Application of Galerkin method results into:

$$\left\{R^e\right\}=-\int_A N^T\left[\left(\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\left(1+\frac{4R_d}{3}\right)\bar{r}\frac{\partial \overline{T_s}}{\partial \bar{r}}\right)+\frac{\partial^2 \overline{T_s}}{\partial \bar{z}^2}\right)-HKr\left(\bar{T}_s-\bar{T}_f\right)\right]2\pi\bar{r}dA \tag{3.8.80}$$

$$\frac{2\pi\bar{R}}{4A}\left\{\left(1+\frac{4}{3}R_d\right)\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_2 & b_2b_3 & b_3^2 \end{bmatrix}+\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^2 \end{bmatrix}+\frac{HA^2Kr}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\right\}\begin{Bmatrix} \bar{T}_{s1} \\ \bar{T}_{s2} \\ \bar{T}_{s3} \end{Bmatrix}$$

$$=\frac{2\pi\bar{R}HAKr}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \bar{T}_{f1} \\ \bar{T}_{f2} \\ \bar{T}_{f3} \end{Bmatrix} \tag{3.8.81}$$

3.8.7 Finite Element equations for mixed convection

Momentum equation

$$\frac{\partial^2 \overline{\psi}}{\partial \bar{z}^2}+\frac{1}{\bar{r}}\frac{\partial^2}{\partial \bar{r}}\left(\bar{r}\frac{\partial \overline{\psi}}{\partial \bar{r}}\right)=\frac{\bar{r}Ra}{Pe}\frac{\partial \overline{T_f}}{\partial \bar{r}} \tag{3.8.82}$$

Application of Galerkin method yields:

$$\left\{R^e\right\}=-\int_A N^T\left[\frac{\partial^2 \overline{\psi}}{\partial \bar{z}^2}+\bar{r}\frac{\partial}{\partial \bar{r}}\left(\frac{1}{\bar{r}}\frac{\partial \overline{\psi}}{\partial \bar{r}}\right)-\bar{r}\frac{Ra}{Pe}\left[\frac{\partial \overline{T}}{\partial \bar{r}}\right]\right]2\pi\bar{r}dA \tag{3.8.83}$$

$$\frac{2\pi\bar{R}}{4A}\left[\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_2 & b_2b_3 & b_3^2 \end{bmatrix}+\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^2 \end{bmatrix}\right]\begin{Bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{Bmatrix}+2\pi\bar{R}^2\frac{Ra}{Pe}\begin{Bmatrix} b_1\bar{T}_1+b_2\bar{T}_2+b_3\bar{T}_3 \\ b_1\bar{T}_1+b_2\bar{T}_2+b_3\bar{T}_3 \\ b_1\bar{T}_1+b_2\bar{T}_2+b_3\bar{T}_3 \end{Bmatrix}=0$$

(3.8.84)

Energy Equation for fluid

$$Pe\left[\frac{\partial\overline{\psi}}{\partial\overline{r}}\frac{\partial\overline{T_f}}{\partial\overline{z}}-\frac{\partial\overline{\psi}}{\partial\overline{z}}\frac{\partial\overline{T_f}}{\partial\overline{r}}\right]=\left(\frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}\left(\left(1+\frac{4R_d}{3}\right)\overline{r}\frac{\partial\overline{T_f}}{\partial\overline{r}}\right)+\frac{\partial^2\overline{T_f}}{\partial\overline{z}^2}\right)+H\left(\overline{T}_s-\overline{T}_f\right) \tag{3.8.85}$$

$$\left\{R^e\right\}=-\int_A N^T\left[Pe\left[\frac{\partial\overline{\psi}}{\partial\overline{r}}\frac{\partial\overline{T_f}}{\partial\overline{z}}-\frac{\partial\overline{\psi}}{\partial\overline{z}}\frac{\partial\overline{T_f}}{\partial\overline{r}}\right]-\left(\frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}\left(\left(1+\frac{4R_d}{3}\right)\overline{r}\frac{\partial\overline{T_f}}{\partial\overline{r}}\right)+\frac{\partial^2\overline{T_f}}{\partial\overline{z}^2}\right)+H\left(\overline{T}_s-\overline{T}_f\right)\right]2\pi\overline{r}dA$$

(3.8.86)

$$\left[\frac{2\pi Pe}{12A}\begin{Bmatrix}c_1\overline{\psi}_1+c_2\overline{\psi}_2+c_3\overline{\psi}_3\\c_1\overline{\psi}_1+c_2\overline{\psi}_2+c_3\overline{\psi}_3\\c_1\overline{\psi}_1+c_2\overline{\psi}_2+c_3\overline{\psi}_3\end{Bmatrix}\left[b_1\ b_2\ b_3\right]-\frac{2\pi}{12A}\begin{Bmatrix}b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3\\b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3\\b_1\overline{\psi}_1+b_2\overline{\psi}_2+b_3\overline{\psi}_3\end{Bmatrix}\left[c_1\ c_2\ c_3\right]\right]\begin{Bmatrix}\overline{T}_{f1}\\\overline{T}_{f2}\\\overline{T}_{f3}\end{Bmatrix}$$

$$+\frac{2\pi\overline{R}}{4A}\left[\begin{bmatrix}b_1^{\ 2}&b_1b_2&b_1b_2\\b_1b_2&b_2^{\ 2}&b_2b_3\\b_1b_3&b_2b_3&b_3^{\ 2}\end{bmatrix}+\begin{bmatrix}c_1^{\ 2}&c_1c_2&c_1c_3\\c_1c_2&c_2^{\ 2}&c_2c_3\\c_1c_2&c_2c_3&c_3^{\ 2}\end{bmatrix}+\frac{HA^2}{3}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\right]\begin{Bmatrix}\overline{T}_{f1}\\\overline{T}_{f2}\\\overline{T}_{f3}\end{Bmatrix}$$

$$-\frac{2\pi\overline{R}HA}{12}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\begin{Bmatrix}\overline{T}_{s1}\\\overline{T}_{s2}\\\overline{T}_{s3}\end{Bmatrix}=0 \tag{3.8.87}$$

Energy equation for solid pase of porous medium

$$\left(\frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}\left(\left(1+\frac{4R_d}{3}\right)\overline{r}\frac{\partial\overline{T_s}}{\partial\overline{r}}\right)+\frac{\partial^2\overline{T_s}}{\partial\overline{z}^2}\right)=HKr\left(\overline{T}_s-\overline{T}_f\right) \tag{3.8.88}$$

Applying Galerkin method

$$\left\{R^e\right\}=-\int_A N^T\left[\left(\frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}\left(\left(1+\frac{4R_d}{3}\right)\overline{r}\frac{\partial\overline{T_s}}{\partial\overline{r}}\right)+\frac{\partial^2\overline{T_s}}{\partial\overline{z}^2}\right)-HKr\left(\overline{T}_s-\overline{T}_f\right)\right]2\pi\overline{r}dA \tag{3.8.89}$$

The matrix form of equation can be given as:

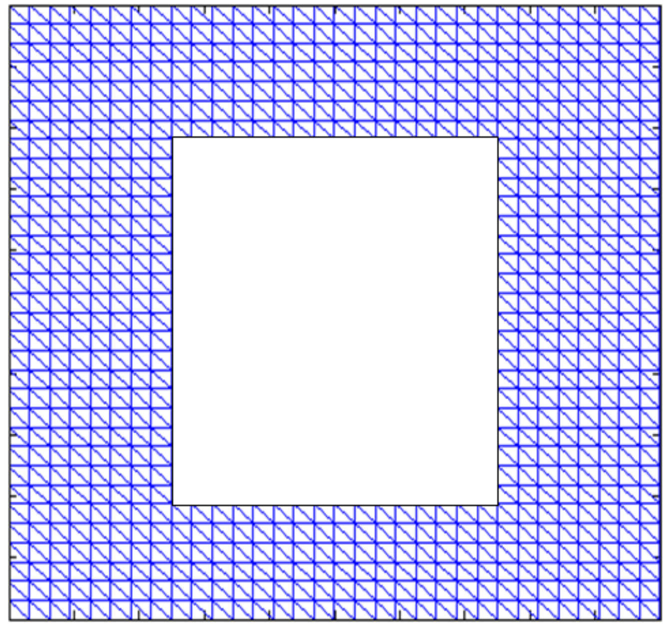
$$\frac{2\pi\bar{R}}{4A}\left\{\left(1+\frac{4}{3}R_d\right)\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ b_1b_2 & b_2b_3 & b_3^2 \end{bmatrix}+\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_2 & c_2c_3 & c_3^2 \end{bmatrix}+\frac{HA^2Kr}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\right\}\begin{Bmatrix} \bar{T}_{s1} \\ \bar{T}_{s2} \\ \bar{T}_{s3} \end{Bmatrix}$$

$$-\frac{2\pi\bar{R}HAKr}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \bar{T}_{f1} \\ \bar{T}_{f2} \\ \bar{T}_{f3} \end{Bmatrix} = 0 \qquad (3.8.90)$$

3.9 Solution Procedure

Mathematical modeling and simulation of a process can be summarized in two steps. The mathematical formulation considering the parameters affecting the process and solving those derived equations to get the values of the unknown solution variables. In this study the heat transfer phenomenon is defined by the mathematical equation as discussed, the finite element method is used to determine the heat transfer characteristics of the flow through porous medium. The discretization of the domain is accomplished by dividing it in to smaller segments known as elements. In present case, the domain of problem under investigation is divided into multiple triangular elements as shown in Figure 3.9.1.

(a)



(b)

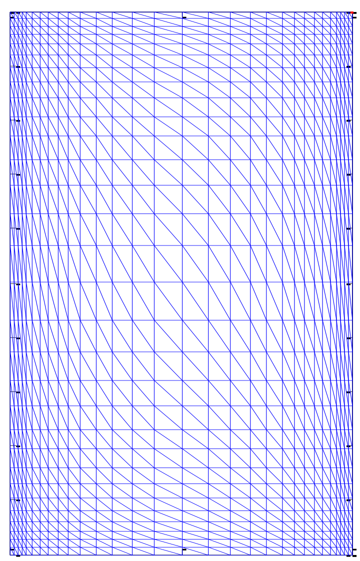


Figure: 3.9.1: Meshed model of a) Square annulus b) vertical annular cylinder

Figures 3.9.1a and 3.9.1b depict the meshed models of square porous annulus and vertical annular cylinder respectively. Due to symmetry, only half the geometry of cylinder is modeled that is sufficient to predict the entire behaviour of cylinder. A computer code is written using the advanced computer software MATLAB to solve the problem under investigation. As there are two or more coupled equations to be solved to get the temperature distribution within porous domain which require an iterative computing method. Two solution variables $\bar{T}$ and $\bar{\psi}$ at various points inside the porous domain is determined by an initial guess value of which are fed in to the computer code. By using the first guess value of $\bar{\psi}$ the new value of $\bar{T}$ is determined and then this new value of $\bar{T}$ is fed back to evaluate the new value of $\bar{\psi}$. The fresh value of $\bar{\psi}$ is then fed to get the improved value of $\bar{T}$ and again this $\bar{T}$ is used to get the fresh value of $\bar{\psi}$. Thus the computer code performs multiple iterations until the difference between previous and current values of both the solution variables is reached the specified tolerance limit. The tolerance for $\bar{T}$ and $\bar{\psi}$ is set as 10^{-5} & 10^{-7} respectively which gives quite accurate results. With the obtained $\bar{T}$ and $\bar{\psi}$ values for the whole domain, the Nusselt number is evaluated to determine the heat transfer rate from the wall to the porous medium. The isothermal lines and streamlines are plotted to analyze the behaviour of heat and fluid flow. It is worth to mention here that the mesh independent study is carried out prior to the selection of a particular mesh size. This ensures that the solution is unaffected due to number of elements selected. Table 3.9.1 shows the mesh independent study carried out for one such case. As evidenced from Table 3.8.1 that the variation in $\overline{Nu}$ and $\overline{Sh}$ is very small when element size is changed from 1800 to 7200. The computational time required to solve the mesh size of 7200 elements is very large compared to 1800 elements. It was found that the time required to solve 7200 elements is approximately 63 times greater than that of 1800 elements. Thus the mesh size of 1800 elements is a better strategy to solve the governing equations for this particular problem since the variation in solution variables is negligible when mesh size is increased from 1800 to 7200. The computations are carried out on a high end computer with intel Xenon (R) processor having 3.1 GHz frequency and 4 GB RAM.

Table 3.9.1 $\overline{Nu}$ variation with mesh size

No of elements	Avg *Nu*	% Change in *Nu*	Avg *Sh*	% Change in *Sh*	Time in seconds	% change in time
1800	21.599	---	3.6999	---	14.23	---
3200	21.800	0.9	3.6583	1.1	78.55	530.5
5000	21.921	1.4	3.6271	1.9	303.00	2456.07
7200	22.001	1.8	3.603	2.6	915.24	6331.76

3.10 Assumptions

The following assumptions are made in the present analyses:

a) The fluid follows Darcy law since the porosity and velocity is low.

b) There is no phase change of fluid in the medium as it operates within lower limits of temperature.

c) The properties of the solid, fluid and those of the porous medium are homogeneous because the porous medium comprises of one material only,

d) Fluid properties are constant except the variation of density with temperature as the operating temperature is very low, variation in other properties are negligible.