

ENHANCING COOPERATION IN NON-LINEAR SYSTEMS IN  
GAME THEORY

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FACULTY OF SCIENCE  
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ENHANCING COOPERATION IN NON-LINEAR SYSTEMS  
IN GAME THEORY

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## ABSTRACT

Game theory is a branch of mathematics involving the study of cooperation and conflicts in the society. Given the importance of cooperation in the fight for common goods, the emergence of cooperation amongst selfish individuals is a fundamental and important issue in the economics and behavioural sciences. The aim of this thesis is to further our understanding of the roles of various factors such as incentives and network on the enhancement of cooperation in different economic models involving non-linear systems. In particular, two models, one involving social dilemma with N-players and the other involving economic behaviours with two players are studied. These two models are used to develop a third model which retains the main features of the second model, but modified to include N-players with an evolutionary trait as in the first model. This is to give some insights on the effects of the various features in the first model on the frequency of cooperation and magnitude of incentives, i.e. technological leapfrogging in the second model. Punishing strategy, which can be regarded as a form of direct or indirect reciprocity, is another important mechanism in promoting cooperation. In a recent model of N-player Snowdrift game with evolutionary trait incorporating a costly punishing strategy, the role of punishment and the effects of a structured population connected through a square lattice in promoting cooperation are investigated. One of the main challenges in the studies of evolutionary games is bridging the gap between theoretical and empirical research. Different problems have been studied in the hope of applying the findings to implement game theory to a practical scenario where the market is dominated by only a few players. Therefore, the role of punishment is studied in a Cournot duopoly. In the industry, the role of the punisher in Snowdrift game can be taken up by the patent system as the latter punishes the free-riders by giving intellectual

property rights to the innovators (cooperators), thereby causing technological lagging in the free-riders (defectors). Therefore, punishment in this case is given in the form of incentive-denial. The effect of patenting on cooperation and defection is thus studied in a long-term research and development (R&D) Cournot duopoly differential game, as well as to determine the sustainability of R&D incentives in an environment where technological innovation is almost a public good. Finally, the R&D Cournot duopoly differential game model is simplified to an extent which allows the study of the model in an evolutionary well-mixed N-player setting to identify precisely the factors directly affecting the firms' investment rate and technological leapfrogging. With the introduction of an evolutionary feature to the simplified R&D Cournot duopoly model, the latter allows the study of the effect of an N-player evolutionary game on the various factors in the original R&D Cournot duopoly model. The R&D Cournot duopoly model is modified with the view that it can be readily generalized to incorporate other interesting and practical features such as real-life networking effects not present in the original Cournot duopoly game.

## ABSTRAK

Teori permainan adalah salah satu cabang dalam bidang matematik yang mengkajiselidik isu kerjasama dan konflik dalam masyarakat. Kerjasama adalah asas dan isu penting dalam kajiselidik ekonomi dan perangai manusia kerana kepentingan kerjasama adalah untuk kebaikan semua. Objektif tesis ini adalah untuk mendalami kefahaman tentang faktor-faktor yang akan menggalakkan kerjasama dalam model ekonomi yang berlainan dan melibatkan sistem tak-linear seperti insentif dan struktur rangkaian. Khususnya, model sosial dilema yang melibatkan  $N$  pemain dan model tingkah-laku ekonomi yang melibatkan dua pemain dikaji. Kedua-dua model ini digunakan untuk membina model ketiga yang megekalkan ciri-ciri model kedua dan diubahsuai untuk mengandungi  $N$  pemain dan ciri evolusi seperti dalam model pertama. Ini adalah untuk mengkaji kesan ciri-ciri dalam model pertama terhadap kekerapan kerjasama dan pintasan teknologi dalam model kedua. Strategi denda adalah mekanisme penting dalam menggalakkan kerjasama. Kami mengkaji watak dan kesan strategi denda dalam menggalakkan kerjasama dengan menggunakan model evolusi permainan "Snowdrift" yang melibatkan  $N$  pemain dan strategi denda yang terkini dalam populasi yang dihubungkan dalam struktur "lattice". Dalam kajian evolusi permainan, mendekatkan jurang antara teori dan kajian empirikal adalah salah satu cabaran besar. Pelbagai kajian telah dilakukan dalam pencarian kaedah untuk mengimplikasi teori permainan dalam senario dunia nyata dimana pasaran dikuasai oleh beberapa pemain sahaja. Justeru itu, watak denda dikaji dengan menggunakan "Cournot duopoly" yang lebih praktikal. Dalam industri, watak denda dalam permainan "Snowdrift" boleh digantikan dengan sistem paten kerana sistem paten menghukum pihak yang menyalin dengan memberikan pihak inovator hak monopoli dalam penggunaan sesuatu inovasi yang boleh menjauhkan jurang kepakaran teknologi penyalin. Dengan itu, keuntungan

pihak yang berkerjasama boleh dilindungi melalui paten. Dalam kes ini, denda adalah dalam bentuk insentif-penafian. Kami mengkaji kesan paten terhadap strategi kerjasama dan strategi pembelotan dengan menggunakan model pembezaan, penyelidikan dan pembangunan jangka panjang "Cournot duopoly" yang lebih praktikal. Kami juga mengkaji sama ada insentif penyelidikan dan pembangunan boleh dikekalkan dalam situasi dimana inovasi teknologi adalah milik umum. Akhir sekali, kami meringkaskan model pembezaan "Cournot duopoly" agar model ini boleh dikaji dengan tetapan evolusi "well-mixed" yang melibatkan N pemain dalam mengenal pasti secara tepat faktor-faktor penting yang mempengaruhi kadar pelaburan firma-firma dan pintasan teknologi. Kami memasukkan elemen evolusi dalam model yang telah dimodifikasi ini agar model yang telah dimodifikasi ini boleh digunakan untuk mengkaji kesan permainan yang melibatkan N pemain ke atas factor-factor penting yang mempengaruhi strategi pemain dalam model asal. Kami memodifikasikan model asal penyelidikan dan pembangunan "Cournot duopoly" dengan tujuan model ini siap sedia untuk pengubahsuaian atau penggabungan unsur-unsur lain yang menarik dan praktikal yang tidak hadir dalam model asal "Cournot duopoly" seperti kesan rangkain dunia nyata.

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## CHAPTER 1: INTRODUCTION TO GAME THEORY

Game theory is a branch of applied mathematics frequently applied to economics to study strategic situations involving several stakeholders, each with different goals, whose actions can affect one another. In game theory, a game is any situation where multiple players can affect the outcome, a player is a stakeholder, a move or option is an action a player can take and, at the end of the game, the payoff for each player is the outcome. In general, the value of game theory lies in understanding the interactions between players and the likely outcome which is dependent on the actions of players with potentially conflicting motives. Game theory is a well-developed field of study that has attracted some of the world's greatest mathematicians, with two Nobel Prizes won in the field.

### 1.1 History of Game Theory

The origins of game theory go far back in time. Recent work suggests that the division of an inheritance described in the Talmud (in the early years of the first millennium) predicts the modern theory of cooperative games and, in 1713, James Waldegrave developed a strategy for a card game (Bellhouse, 2007) that provided the first known solution to a two player game.

The earliest example of a formal game-theoretic analysis is the study of a duopoly by Antoine Cournot in his book of 1838, titled "Researches into the Mathematical Principles of the Theory of Wealth" (Cournot, 1838) where he attempted to explain the underlying rules governing the behaviour of duopolists.

In 1883, J. Bertrand presented a model of price competition in a duopoly market (Bertrand, 1883). Emile Borel suggested a formal theory of games in 1921 (Borel, 1921), which was furthered by John von Neumann in 1928 in his paper "On the Theory of Parlor Games" (von Neumann, 1928). The works of A. Cournot were continued by Heinrich Von

Stackelberg, who describe the quantity leadership model in his 1934 book titled “Market Structure and Equilibrium” (von Stackelberg, 1934).

As the science of industrial organization gained its popularity, mathematical methods, especially game theory, proved to be the mainstream in analyzing the economic markets. John von Neumann and Oskar Morgenstern are considered to be the pioneers of game theory, releasing their monograph, named “Theory of Games and Economic Behaviour”, a 600 pages mathematical treatment that laid the groundwork of the field, in 1944 (von Neumann & Morgenstern, 1944). Their work provided much of the basic terminologies and problem setup that are still in use today and is usually credited as the origin of the formal study of game theory. This work focused on finding unique strategies that allowed players to minimize their maximum losses (minimax solution) by considering all the possible responses of other players, for every possible strategy of their own. Building upon their 1928 work on two player games where the winnings of one player are equal and contrary to the losses of his opponent (zero-sum) and where each player knows the strategies available to all players and their consequences (perfect information), von Neumann and Morgenstern (1944) extended the minimax theorem to include games involving imperfect information and games with more than two players.

The golden age of game theory occurred in the 1950s and 1960s when researchers focused on finding sets of strategies, known as equilibria, to “solve” a game if all players behaved rationally. The most famous of these is the Nash equilibrium proposed by John Nash (Nash, 1950, 1951), later made famous in the film “A Beautiful Mind” starring Russell Crowe. Nash equilibrium exists if no player can unilaterally move to improve their own outcome. In other words, they have no incentive to change, since their strategy is the best they can do given the actions of the other players. Nash also made significant contributions to bargaining theory and examined cooperative games where threats and

promises are fully binding and enforceable (Drechsel, 2010).

In 1965, Reinhard Selten introduced the concept of subgame perfect equilibria (Selten, 1965), which describes strategies that deliver Nash equilibrium across every sequential subgame of the original game. Such subgame perfect equilibria may be found by first determining optimal action of the player who makes the last move of the game. Then, the optimal action of the next to last moving player is determined assuming the last player's action as given. The process, known as backward induction, continues until all players' actions have been determined. In 1967 and 1968, John Harsanyi formalized Nash's work and developed incomplete information games (Harsanyi, 1967, 1968a, 1968b). He, along with John Nash and Reinhard Selten, won the Nobel Prize for Economics in 1994. Another important contribution to game theory during the 1950s and 1960s was Luce and Raiffa's book, "Games and Decisions" (Luce & Raiffa, 1957). The Prisoner's Dilemma, introduced by the RAND Corporation (Poundstone, 1992) is also a product of this period.

Further adding to the acclaim of game theory, another Nobel Prize was awarded to game theorists, Robert Aumann and Thomas Schelling, in 2005. Schelling used game theory in his 1960 book, "The Strategy of Conflict" (Schelling, 1960) to explain why credible threats of nuclear annihilation from the U.S. and the former Soviet Union were counterbalancing through mutually assured destruction and therefore were not likely to be used. He also argued that the ability to retaliate was more useful than the ability to withstand an attack. Aumann's work (e.g. Aumann et al., 1995) was mathematical and focused on whether cooperation increases if games are continually repeated rather than played out in a single encounter. He showed that cooperation is less likely when there are many participants, when interactions are infrequent, when the time horizon is short or when others' actions cannot be clearly observed.



Throughout the years, game theory has been applied to many different fields of study. Biologists have used it to learn species behaviour (Hofbauer & Sigmund, 1998). Algorithmic game theory is an example of application in computer science (Roughgarden, 2010). In mathematics, there is a complete branch that studies decision making process (Mazalov, 2014). It also has its influences in business where it can model interaction of stakeholder, dynamics in interest rates etc. (Geckil & Anderson, 2009).

## **1.2 Definitions of Games**

The object of study in game theory is the game. What is a Game? Every child understands what games are. When someone overreacts, we sometimes say “it’s just a game”. Games are often not serious. Mathematical games are different. From its beginning, it was the purpose of game theory to be applied to serious situations in economics, politics, business, and other areas. Even wars can be analyzed by mathematical game theory.

A game is any situation with three aspects, consisting of the players, the strategies, and the payoffs. There is a set of participants, whom we call the players. Each player has a set of options for how to behave; we will refer to these as the player’s possible strategies. For each choice of strategy, each player receives a payoff that can depend on the strategies selected by the others. The payoffs are generally represented by numbers-the higher the number, the more favourable the outcome is for the player.

Despite that, mathematical games have strict rules that specify exactly what the players are allowed to do. Though many real-world games allow for discovering new moves or ways to act, games that can be analyzed mathematically have a rigid set of possible moves, usually all known in advance.

Mathematical games may have many possible outcomes, each producing payoffs for the players. The payoffs may be monetary, or they may express satisfaction. A mathematical game is “thrilling” in that its outcome cannot be predicted in advance. Since

its rules are fixed, this implies that a game must either contain some random elements or have more than one players. Mathematical games also involve decision making, and can therefore at least partly be analyzed via game theory.

In real-life, game cheating is possible. Cheating means players consciously choose to not play by the rules. Game theory doesn't even acknowledge the existence of cheating. It simply determines how to win without cheating. Rational behaviour is usually assumed for all players. That is, players have preferences, beliefs about the world (including the other players), and try to optimize their individual payoffs while being aware that other players are trying to do the same.

Equilibrium is the point in a game where both players have made their decisions and an outcome is reached. A Nash equilibrium, named after the Nobel winning economist, John Nash, is a solution to a game involving two or more players who want the best outcome for themselves and must take the actions of others into account. When a Nash equilibrium is reached, players cannot improve their payoffs by independently changing their strategy. This means that it is the best strategy assuming the other has chosen a strategy and will not change it. For example, in the Prisoner's Dilemma game, confessing is a Nash equilibrium because it is the best outcome, taking into account the likely actions of others.

### **1.3 Types of Games**

In game theory, different types of games help in the analysis of different types of problems. The different types of games are formed on the basis of the number of players involved in a game, symmetry of the game, cooperation amongst players, etc. Cooperative games are the one in which the players are convinced to adopt a particular strategy through negotiations and agreements between players (Curiel, 1997). Non-cooperative games are the games in which the players decide on their own strategies to maximize their profits

(Nash, 1951). Of the two types of games, non-cooperative games are able to model situations to the finest details, producing accurate results.

A symmetric game (see Shapley, 1964) is a game where the payoffs for playing a particular strategy depend only on the other strategies employed, not on who is playing them. If the identities of the players can be changed without changing the payoff to the strategies, then a game is symmetric. Many of the commonly studied  $2 \times 2$  games are symmetric. Examples of symmetric games are the chicken game (Rapoport & Chammah, 1966), the Prisoner's Dilemma, and the Stag Hunt game (Skyrms, 2004).

On the other hand, asymmetric games (see Samuelson & Zhang, 1992) are games where there are no identical strategy sets for both players. The strategy that provides benefit to one player may not be equally beneficial for the other player. For instance, the ultimatum game and similarly the dictator game have different strategies for each player.

Constant sum games (see Straffin, 1993) are games in which the sum of outcomes of all the players remains constant no matter which strategy is chosen by each player. Zero sum game (see Straffin, 1993) is a type of constant sum game in which the sum of outcomes of all players is zero. In a zero sum game, the strategies of different players cannot affect the available resources. Moreover, in a zero sum game, the gain of one player is always equal to the loss of the other player. Examples of zero sum games are chess and gambling.

On the other hand, non-zero sum games (see Straffin, 1993) are the games in which sum of the outcomes of all players is not zero. A non-zero sum game can be transformed to a zero sum game by adding a dummy player. The losses of dummy player are overridden by the net earnings of players.

Simultaneous games (see Prisner, 2014) are games where both players move simultaneously, or if they do not move simultaneously, the later players do not have knowledge

about the move of other players. On the contrary, sequential games (see Haurie & Krawczyk, 2000) are games in which each player knows the moves of the players who have already adopted a strategy. However, in sequential games, the players do not have deep knowledge about the strategies of the other players. For example, a player may know that an earlier player did not perform one particular action, while he does not know which of the other available actions the first player actually performed. Often, simultaneous games are represented in normal form while sequential games are represented in extensive form.

Normal (or strategic) form games (see Ozdaglar, 2015) refer to description of game in the form of matrix where the payoff and the strategies of a game are represented in a tabular form. Normal form games help in identifying the dominated strategies and Nash equilibrium. A classical example of a normal form game is the Prisoner's Dilemma.

On the other hand, the extensive form, also called a game tree (see Straffin, 1993), is more detailed than the normal form of a game. It is a complete description of how the game is played over time. This includes the order in which players take actions, the information that players have at the time they must take those actions, and the times at which any uncertainty in the situation is resolved. A game in extensive form may be analyzed directly, or can be converted into an equivalent normal form.

In game theory, an extensive-form game (see Luce & Raiffa, 1957) has perfect information if each player, when making any decision, is perfectly informed of all the events that have already occurred. Chess is an example of a game with perfect information as each player can see all of the pieces on the board at all times. Other examples of perfect games include Tic-tac-toe, Irensei, and Go. Card games where each player's cards are hidden from other players, as in contract bridge and poker, are examples of games with imperfect information.

Combinatorial games (see Beck, 2008) are two-person games with perfect information and no chance moves, and with a win-or lose outcome. Such a game is determined by a set of positions, including an initial position, and the player whose turn it is to move. Player moves from one position to another, with the players usually alternating moves, until a terminal position is reached. A terminal position is one from which no moves are possible. Then one of the players is declared the winner and the other the loser.

Much of game theory is concerned with finite, discrete games (see Bajari et al., 2010), that have a finite number of players, moves, events, outcomes, etc. Continuous games (see Webb, 2007) extends the notion of a discrete game, where the players choose from a finite set of pure strategies. The continuous game concepts allow games to include more general sets of pure strategies, which may be uncountable infinite.

Differential games (see Isaacs, 1999; Tanimoto, 2015) such as the continuous pursuit and evasion game are continuous games where the evolution of the players' state variables are governed by differential equations. The problem of finding an optimal strategy in a differential game is closely related to the one in optimal control theory (Kirk, 2012). Typical cases of differential games are the games with a random time horizon. In such games, the terminal time is a random variable with a given probability distribution function.

Games with an arbitrary, but finite, number of players are often called N-person games (see Straffin, 1993). An important difference between two person games and N-person games, apart from the complexity of the mathematics involved, is the fact that coalitions may form between players of N-person games affecting the game dynamics. Since we are concerned with opposing interests, a player in this perspective need not be a single person but can be a nation, a football team or a pair of Bridge partners.

Some of the best-known games used to illustrate social dilemma are the Prisoner's Dilemma, Stag Hunt game, Chicken, Hawk-dove (see Binmore, 2007), and Snowdrift

game (see Doebeli & Hauert, 2005). And some of the best known games used to model the economic behaviours are Cournot game, Stackelberg game and Bertrand game. They will be discussed in the following sections.

### **1.3.1 Prisoner's Dilemma**

As a branch of applied mathematics, game theory has been used to study a wide variety of human and animal behaviours. One of the most famous classic examples in the development of game theory is the Prisoner's Dilemma. Originally framed by Merrill Flood and Melvin Dresher working at RAND organization in 1950, it is the standard model of social dilemma which has been studied extensively. Albert W. Tucker formalized the game with prison sentence rewards and named it "Prisoner's Dilemma" (Poundstone, 1992).

The story behind the name "Prisoner's Dilemma" is that of two suspects who have been apprehended by the police and are being interrogated in separate rooms. The police strongly suspect that these two individuals are responsible for a robbery, but there is not enough evidence to charge either of them for the crime. However, if they both chose to resist arrest, they could be charged with a lesser crime, carrying a one-year sentence. Each of the suspects is told the following story. "If you confess, and your partner doesn't confess, then you will be released and your partner will be charged with the crime. Your confession will be sufficient to convict him of the robbery and he will be sent to prison for 10 years. If you both confess, then we don't need either of you to testify against the other, and you will both be convicted of the robbery. (Although in this case your sentence will be less—4 years only—because of your guilty plea.) Finally, if neither of you confess, and then we can't convict either of you of the robbery, so we will charge each of you with resisting arrest. Your partner is being offered the same deal. Do you want to confess?"

To formalize this story as a game, we need to identify the players, the possible strategies, and the payoffs. The two suspects are the players, and each has to choose between two possible strategies: Confess (defect) or Not-Confess (cooperate). The situation is best illustrated in what is called a “payoff matrix” as follows:

	Cooperator	Defector
Cooperator	$R$	$S$
Defector	$T$	$P$

If both players cooperate, they both receive the reward  $R$  for cooperating. If both players defect, they both receive the punishment payoff  $P$ . If first player defects while the opponent cooperates, then the first player receives the temptation payoff  $T$ , while the opponent receives the “sucker’s” payoff  $S$ . Similarly, if the first player cooperates while the opponent defects, then the first player receives the sucker’s payoff  $S$ , while the opponent receives the temptation payoff  $T$ . To be a Prisoner’s Dilemma game in the strong sense, the following condition must hold for the payoffs:  $T > R > P > S$ .

The payoff relationship  $R > P$  implies that mutual cooperation is superior to mutual defection, while the payoff relationships  $T > R$  and  $P > S$  imply that defection is the dominant strategy for both agents. A state where none of the participants can improve its payoff by unilaterally changing its strategy is called a Nash equilibrium (Nash, 1951), and in the case of this game, mutual defection is the only strong Nash equilibrium in the game. The dilemma then is that mutual cooperation yields a better outcome than mutual defection but it is not the rational outcome because from a self-interested perspective, the choice to cooperate, at the individual level, is irrational.

The Prisoner’s Dilemma is a standard example of a game analysed in game theory that shows why two completely “rational” individuals might not cooperate, even if it appears that it is in their best interests to do so. Prisoner’s Dilemma games are models of

societal and organizational conflict situations. The intriguing nature of the dilemmas and the potential importance of the situations they model have led to the study of Prisoner's Dilemma "problems" in several disciplines. Economists have used Prisoner's Dilemma games as a model of oligopolistic price setting (Shubik, 1955). Political scientists have studied them as models of the free-rider problem associated with public goods (Hardin, 1971). Psychologists have studied behaviour in different forms of the Prisoner's Dilemma game (Rapaport & Chammah, 1965) and have often used them as a vehicle for studying personality differences (Terhune, 1968). This intense scrutiny has focused on techniques for increasing the amount of cooperation in Prisoner's Dilemma games (Dawes et al., 1977).

The Prisoner's Dilemma is meant to study short term decision-making where the actors do not have any specific expectations about future interactions or collaborations. There is a substantial relationship between the Stag Hunt game and the Prisoner's Dilemma. In biology, many circumstances that have been described as Prisoner's Dilemma might also be interpreted as a Stag Hunt game, depending on how fitness is calculated. It is also the case that some human interactions that seem like Prisoner's Dilemmas may in fact be Stag Hunts.

### **1.3.2 Stag Hunt**

In game theory, the Stag Hunt (see Skyrms, 2004) is a game which describes a conflict between safety and social cooperation. Other names for it or its variants include "assurance game", "coordination game", and "trust dilemma". The Stag Hunt is a story that became a game. The story is briefly told by Rousseau, in "A Discourse on Inequality" in 1755 (Rousseau, 1992). Jean-Jacques Rousseau described a situation in which two individuals go out on a hunt. Each can individually choose to hunt a stag or hunt a hare. Each player must choose an action without knowing the choice of the other. If an individual hunts a



stag, he must have the cooperation of his partner in order to succeed. An individual can get a hare by himself, but a hare is worth less than a stag. This is taken to be an important analogy for social cooperation. If we illustrate the situation in a payoff matrix as in the Prisoner's Dilemma,

	Cooperator	Defector
Cooperator	$R$	$S$
Defector	$T$	$P$

the following condition must hold for the payoffs:  $R > T > P > S$ . So you can see in the payoff matrix that if both players cooperate, i.e. both player 1 and player 2 choose a stag, then they will both get the largest payoff of  $R$ . But if both players do not cooperate, i.e. player 1 chooses a hare and player 2 chooses a stag, then player 2 will get the smallest payoff of  $S$  and player 1 will get a payoff of  $T$ , and vice versa. If both players choose to hunt the hare, then both players will get a payoff of  $P$ .

A Stag Hunt is a game with two pure strategy Nash equilibria—one that is risk dominant another that is payoff dominant. The strategy pair (Stag, Stag) is payoff dominant since payoffs are higher for both players compared to the other pure Nash equilibrium, (Hare, Hare). On the other hand, (Hare, Hare) risk dominates (Stag, Stag) strategy since if uncertainty exists with regard to the other player's action, gathering will provide a higher expected payoff. The more uncertainty players have concerning the actions of the other player(s), the more likely they will choose the strategy corresponding to it.

The Stag Hunt differs from the Prisoner's Dilemma in that there are two Nash equilibria: when both players cooperate and both players defect. In the Prisoners Dilemma, the only Nash equilibrium is when both players choose to defect. Stag Hunt is in many ways like Prisoner's Dilemma except that in Stag Hunt, mutual cooperation is individually preferable to unilateral defection. Therefore, Stag Hunts are more likely to foster cooper-

ation than Prisoner's Dilemma. Moreover, a Prisoner's Dilemma can be changed into a Stag Hunt in situations where the benefits of an opponent's defection can be mitigated.

### 1.3.3 Chicken Game

Chicken game (see Rapoport & Chammah, 1966) is an influential model of conflict for two players in game theory. The name "chicken" has its origins in a game in which two drivers drive towards each other on a collision course: one must swerve, or both may die in the crash, but if one driver swerves and the other does not, the one who swerved will be called a "chicken", meaning a coward. This terminology is most prevalent in political science and economics. It is presumed that the best thing for each driver is to stay straight while the other swerves (since the other is the "chicken" while a crash is avoided). Additionally, a crash is presumed to be the worst outcome for both players. This yields a situation where each player, in attempting to secure his best outcome, risks the worst. The pure strategy equilibria are the two situations wherein one player swerves while the other does not. If we illustrate the situation in a payoff matrix as in the Prisoner's Dilemma,

	Cooperator	Defector
Cooperator	$R$	$S$
Defector	$T$	$P$

the order of preference is  $T > R > S > P$ .

Chicken game is like Prisoner's Dilemma except that unrequited cooperation is preferable to mutual defection. Defection is therefore less likely than Prisoner's Dilemma because its potential costs are greater. Unlike Stag Hunt game and Prisoner's Dilemma, iteration may reduce the likelihood for cooperation because a reputation for cooperation reduces an opponent's timidity towards defection.

### 1.3.4 Hawk-Dove Game

Hawk-Dove game (see Binmore, 2007) is motivated by the following story. Suppose two animals are engaged in a contest to decide how a piece of food will be divided between them. Each animal can choose to behave aggressively (the Hawk strategy) or passively (the Dove strategy). If the two animals both behave passively, they divide the food evenly, and each get a payoff of  $R$ . If one behaves aggressively while the other behaves passively, then the aggressor gets most of the food, obtaining a payoff of  $T$ , while the passive one only gets a payoff of  $S$ . But if both animals behave aggressively, then they destroy the food (and possibly injure each other), each getting a payoff of  $P$ . Thus we have the payoff matrix

$$\begin{array}{cc} & \begin{array}{cc} \text{Cooperator} & \text{Defector} \end{array} \\ \begin{array}{c} \text{Cooperator} \\ \text{Defector} \end{array} & \left( \begin{array}{cc} R & S \\ T & P \end{array} \right) \end{array}$$

with the order of preference  $T > R > S > P$ . This game has two Nash equilibria: (Dove, Hawk) and (Hawk, Dove). Without knowing more about the animals, we cannot predict which of these equilibria will be played. The Hawk-Dove game is identical to the Chicken game from a game-theoretic point of view.

### 1.3.5 Snowdrift Game

The study of game theory has slowly evolved over the years, from coordination games and domination games like Stag Hunt and Prisoner's Dilemma respectively to anti-coordination games like Snowdrift game (see Doebeli & Hauert, 2005). The Snowdrift game is a theoretical model of cooperation within the context of game theory. A mixed strategy of cooperation and defection can emerge under a Snowdrift game system of payoffs, which makes it very different from the Prisoner's Dilemma. The problem of cooperation is easily

illustrated in the famous Prisoner's Dilemma where two players have the opportunity to either cooperate or defect, with cooperation resulting in a benefit to the opposing player but entailing a cost to the cooperator. In this situation, an individual player in a one-shot interaction is always better off when defecting, independent of what the other player does.

However, the Prisoner's Dilemma does not represent the frequent situation where individuals obtain immediate direct benefits from the cooperative acts they perform and costs of cooperation are shared between cooperators. Such a situation is encapsulated in the Snowdrift game, which derives its name from the following situation. There are two drivers driving in opposite direction on a road blocked by a snowdrift. Both the drivers share the same interest: both want the snowdrift to be removed. But who's going to get out and shovel? It might seem fair just to get out and shovel the snow together—in other words, to cooperate. But what if the other driver just sits there and refuses to help? If the cost of shoveling is low compared to the benefit of getting out of the drift, it will be in the driver interest to shovel. Indeed, the other driver is a freeloader who shares the benefit undeservedly. If the cost of shoveling was too high to bear, the driver would have refused to do it, letting both of them freeze there. That would be the Prisoner's Dilemma. But if the cost of shoveling is lower than the costs of doing nothing, then a mixed strategy will be optimal. As long as freeloaders aren't too common, that strategy will pay off. So a population engaged in the Snowdrift game will come to a mixed proportion of shovelers and freeloaders. The situation can be illustrated in a payoff matrix below:

	Cooperator	Defector
Cooperator	$R$	$S$
Defector	$T$	$P$

In this game, cooperation yields a benefit  $b$  that is accessible to both players (i.e. free passage to go home), whereas the cost  $c$  (i.e. removing the snowdrift) is shared between

cooperators (Doebeli & Hauert, 2005). If both players cooperate, they both receive the reward  $R = b - c/2$  for cooperating. If both players defect, they both receive the punishment payoff  $P = 0$ . If first player defects while the opponent cooperates, then the first player receives the temptation payoff  $T = b$ , while the opponent receives the “sucker’s” payoff  $S = b - c$ . Similarly, if the first player cooperates while the opponent defects, then the first player receives the sucker’s payoff  $S = b - c$ , while the opponent receives the temptation payoff  $T = b$ . In Snowdrift game, the benefit of getting out of the drift is higher than the cost of shovelling,  $b > c > 0$ . Thus, the following condition must hold for the payoffs:  $T > R > S > P$ .

Situations similar to the Snowdrift game are ubiquitous in human working life. For example, two scientists accomplishing a research project would each benefit if the other invests more time than oneself in the writing of the paper reporting the collaborative work. But if one of the collaborators does not contribute at all, the best option probably remains to do all the work on one’s own. The Snowdrift game has the same characteristics as the game of chicken or the Hawk-Dove game, but these games are usually framed in terms of competitive interactions.

### **1.3.6 Cournot Duopoly Game**

As a branch of applied mathematics, game theory was initially developed in economics to understand a large collection of economic behaviors, including behaviors of firms, markets, and consumers. The first use of game-theoretic analysis was by a well-known mathematician, philosopher and economist Antoine Augustin Cournot in 1838 with his solution of the Cournot duopoly in his book titled “Researches into the Mathematical Principles of the Theory of Wealth” (Cournot, 1838). He applied mathematical models for analysing market demand and production costs, provided profit maximization conditions for different types of market structures and presented the classic duopoly model, named

in his honour.

Duopoly is a form of oligopoly market having two participants only: producers or sellers. Oligopoly is a market structure in which a small number of firms have the large majority of market share (i.e. cell phone services market, food product markets, internet services market and etc. in Malaysia). An oligopoly is similar to a monopoly, except that rather than one firm, two or more firms dominate the market. There is no precise upper limit to the number of firms in an oligopoly, but the number must be low enough that the actions of one firm significantly impact and influence the others.

The Cournot Duopoly is a classic oligopolistic market in which there are two enterprises producing the same commodity and selling it in the same market. As the number of competitors is limited to just two, their interaction becomes even more important. This is in contrast to the N-player games where the decision of individuals have less impact on the dynamics of the system. In a classic Cournot duopoly, every producer, before making decisions on prices and quantities, has to take into account not only the current strategy of the competitor, but his forthcoming responsive actions as well.

In a Cournot game, the players are the firms, the actions of each firm are the set of possible outputs (any nonnegative amount) and the payoff of each firm is its profit. This game models a situation in which each player chooses quantities (outputs) of homogeneous products simultaneously without communication and the market determines the price at which they are sold. Research has shown that decision-makers operating in the same market over an extended period of time tend to have similar views of market demand and good knowledge of one another's cost structure. Hence, the Cournot duopoly game assumes that the two players have the same view of market demand, have good knowledge of each other's cost functions, and choose their profit-maximizing output with the belief that their rival chooses the same way.

Specifically, if firm 1 produces the output  $q_1$  and firm 2 produces the output  $q_2$ , then the price at which each unit of output is sold is  $P(q_1 + q_2)$ , where  $P$  is the inverse demand function. Let us denote firm 1's total cost function by  $TC_1(q_1)$  and firm 2's by  $TC_2(q_2)$ . Then firm 1's total revenue when the pair of outputs chosen by the firms is  $(q_1, q_2)$  is  $TR_1 = P(q_1 + q_2)q_1$ , so that its profit is  $V_1 = P(q_1 + q_2)q_1 - TC_1(q_1)$ ; firm 2's total revenue is  $TR_2 = P(q_1 + q_2)q_2$ , and hence its profit is  $V_2 = P(q_1 + q_2)q_2 - TC_2(q_2)$ . Firm 1's marginal revenue,  $MR_1$ , is determined by taking the derivative of total revenue  $TR_1$  with respect to  $q_1$ , while treating  $q_2$  as a constant because firm 1 can't change the quantity of output produced by firm 2. Firm 1's marginal cost,  $MC_1$ , is determined by taking the derivative of total cost  $TC_1$  with respect to  $q_1$ . Similar functions are constructed for firm 2.

Now, how do we determine the firms' outputs so that they are in Nash equilibrium? First of all, we need to have a specific cost function and demand function to determine the profit function,  $V$ . Then, we find the firms' best reaction functions by taking the derivatives of firms' profit functions with respect to firms' outputs, and setting these derivatives equal to zero. Since

$$\begin{aligned}\frac{\partial V_i}{\partial q_i} &= \frac{\partial}{\partial q_i} [P(q_i + q_j)q_i - TC_i(q_i)] = 0, \\ P'(q_i + q_j)q_i + P(q_i + q_j) - \frac{\partial [TC_i(q_i)]}{\partial q_i} &= 0, \\ P'(q_i + q_j)q_i + P(q_i + q_j) - MC_i(q_i) &= 0,\end{aligned}$$

for  $i, j = 1, 2$  and  $i \neq j$ , we have

$$P'(q_1 + q_2)q_1 + P(q_1 + q_2) - MC_1(q_1) = 0, \quad (1.1)$$

$$P'(q_1 + q_2)q_2 + P(q_1 + q_2) - MC_2(q_2) = 0. \quad (1.2)$$

By solving Eq.(1.1) and Eq.(1.2) simultaneously, we can find a pair of firms' outputs,

$(q_1, q_2)$ , that maximizes the firms' profits. In the Cournot duopoly model, both firms determine the profit-maximizing quantities simultaneously.

### 1.3.7 Stackelberg Game

In a Cournot game both producers choose their outputs simultaneously. If one of them could outrun his competitor and become the first to credibly announce the planned output, the game would become a Stackelberg game. It was developed in 1934 by Heinrich Von Stackelberg in his "Market Structure and Equilibrium" (von Stackelberg, 1934). The game would be of two stages: first, a Stackelberg leader chooses his output, and then a Stackelberg follower, having all the information on the leader's choice at his disposal, makes the decision on his output (a perfect information game).

In a Stackelberg model of duopoly, one firm serves as the industry leader. As the industry leader, the firm is able to implement its decision before its rivals. Thus, if firm 1 makes its decision first, firm 1 is the industry leader and firm 2 reacts to or follows firm 1's decision. However, in making its decision, firm 1 must anticipate how firm 2 reacts to that decision. An example of such leadership may be Microsoft's dominance in software markets. Although Microsoft can make decisions first, other smaller companies react to Microsoft's actions when making their own decisions. The actions of these followers, in turn, affect Microsoft. The primary difference between the Cournot and Stackelberg duopoly models is that firms choose simultaneously in the Cournot model and sequentially in the Stackelberg model.

To find the Nash equilibrium of a Stackelberg game, we need to use backward induction, as in any sequential game. That is, we start by analyzing the decision of the follower. Because firm 2 reacts to firm 1's output decision, we begin by deriving firm 2's reaction function. Firm 2 produces the output  $q_2$ , so that the price at which each unit of output is sold is  $P(q_1 + q_2)$ , where  $P$  is the inverse demand function. Denote firm 2's total



cost function by  $TC_2(q_2)$ . Then firm 2's total revenue is  $TR_2 = P(q_1 + q_2)q_2$ , and hence its profit is  $V_2 = P(q_1 + q_2)q_2 - TC_2(q_2)$ . As usual, we need to have a specific cost function and demand function to determine the profit function  $V_2$ . Firm 2's marginal cost  $MC_2$  is determined by taking the derivative of total cost  $TC_2$  with respect to  $q_2$ , while treating  $q_1$  as a constant because firm 2 is unable to change the quantity of output produced by firm 1. Then, we find firm 2's best reaction function by taking the derivative of firm 2's profit function with respect to firm 2's output, and setting the derivative equal to zero. We have

$$P'(q_1 + q_2)q_2 + P(q_1 + q_2) - MC_2(q_2) = 0. \quad (1.3)$$

Note that firm 2 has exactly the same reaction function as in the Cournot duopoly. We solve Eq.(1.3) for  $q_2$  to get firm 2's reaction function.

At this point, substitute firm 2's reaction function into firm 1's demand curve. This is the critical difference from the Cournot duopoly. By substituting firm 2's reaction function in its decision-making process, firm 1 is anticipating firm 2's reaction to its output decision. Now, if firm 1 produces the output  $q_1$ , then the price at which each unit of output is sold is  $P(q_1)$ , where  $P$  is the inverse demand function of  $q_1$  (by substituting firm 2's reaction function,  $q_2(q_1)$ ). Denoting firm 1's total cost function by  $TC_1(q_1)$ , the total revenue of firm 1 is given by  $TR_1 = P(q_1)q_1$ , so that its profit is  $V_1 = P(q_1)q_1 - TC_1(q_1)$ . Firm 1's marginal revenue  $MR_1$  is determined by taking the derivative of total revenue  $TR_1$  with respect to  $q_1$ . Firm 1's marginal cost  $MC_1$  is determined by taking the derivative of total cost  $TC_1$  with respect to  $q_1$ . Then, we find firm 1's best output by taking the derivative of firm 1's profit  $V_1$  with respect to  $q_1$ , setting the derivative equal to zero, and solving for  $q_1$ . We have

$$\begin{aligned} \frac{dV_1}{dq_1} &= \frac{d}{dq_1}[P(q_1)q_1 - TC_1(q_1)] = 0, \\ P'(q_1)q_1 + P(q_1) - MC_1(q_1) &= 0. \end{aligned} \quad (1.4)$$

Finally, substitute the value  $q_1$  obtained into firm 2's reaction function to determine  $q_2$ . In the Cournot model, firm 1 simply notes that the market demand is satisfied by the output produced by it and firm 2 and both firms make simultaneous decisions. On the other hand, the Stackelberg model reflects sequential decisions where firm 1 substitutes an equation to represent how firm 2 reacts to its production decision. The simultaneous decision-making associated with the Cournot model leads to different outcomes from the outcomes associated with sequential decisions of the Stackelberg model. The Stackelberg leadership model results in a higher market quantity and lower price for the good as compared to the Cournot model.

### 1.3.8 Bertrand Game

The Cournot and Stackelberg duopoly theories in managerial economics focus on firms competing through the quantity of output they produce. In 1883, J. Bertrand proposed a different model of competition between two duopolists (Bertrand, 1883), based on allowing the firms to set prices rather than to fix production quantities. The Bertrand duopoly model examines price competition amongst firms that produce differentiated but highly substitutable products. Each firm's quantity demanded is a function of not only the price it charges but also the price charged by its rival. Coca-Cola and Pepsi are examples of Bertrand duopolists.

With the Bertrand model, we focus on what price is selected to maximize the profits. The quantity demanded for firm 1 and firm 2 is a function of both the price the firm establishes and the price established by their rival because the goods are highly substitutable. Thus, each firm has the demand function  $q_i(P_1, P_2)$ , relating quantity demanded to its price and its rival's price.

To find the Nash equilibrium of the game, we first need to find the Bertrand reaction function for each firm. In the Bertrand model, firms compete with the price. Therefore,

reaction functions are expressed in terms of price, not quantities. Each firm's total revenue equals price times quantity, so  $TR_1 = P_1 q_1(P_1, P_2)$  and  $TR_2 = P_2 q_2(P_1, P_2)$ . To simplify the analysis, assume that both firms have zero marginal costs for their products. Profit maximization then requires each firm to choose a price that maximizes its total revenue. Hence, we find the firms' best reaction functions by taking the derivatives of the firms' total revenue function, with respect to the firms' prices, and setting the derivatives equal to zero. Since

$$\begin{aligned} \frac{dTR_i}{dP_i} &= \frac{d}{dP_i} [P_i q_i(P_i, P_j)] = 0, \\ P_i \frac{dq_i(P_i, P_j)}{dP_i} + q_i(P_i, P_j) &= 0, \end{aligned}$$

for  $i, j = 1, 2$  and  $i \neq j$ , we have

$$P_1 \frac{dq_1(P_1, P_2)}{dP_1} + q_1(P_1, P_2) = 0, \quad (1.5)$$

$$P_2 \frac{dq_2(P_1, P_2)}{dP_2} + q_2(P_1, P_2) = 0. \quad (1.6)$$

By solving Eq.(1.5) and Eq.(1.6) simultaneously, we can find a pair of firms' price  $(P_1, P_2)$  that maximizes the firms' profits. In the Bertrand duopoly model, both firms determine the profit-maximizing price simultaneously.

#### 1.4 Evolutionary Games

In the above sections, we discussed the basic ideas of game theory, in which individual players make decisions, and the payoff to each player depends on the decisions made by all. In this section, we explore the notion of evolutionary game theory (see Sigmund & Nowak, 1999; Tanimoto, 2015), which shows that the basic ideas of game theory can be applied even to situations in which no individual is overtly reasoning, or even making explicit decisions. Rather, game-theoretic analysis will be applied to settings in which individuals can exhibit different forms of behaviour (including those that may not be the

result of conscious choices), and determine which forms of behaviour have the ability to persist in the population. This shares some similarities with the herding models (Rodgers & Yap, 2002) where the agents select the larger groups to join due to the higher return associated with the latter.

As its name suggests, this approach has been applied most widely in the area of evolutionary biology, the domain in which the idea was first articulated by John Maynard Smith in “Evolution and the Theory of Games”, Cambridge University Press, 1982. Evolutionary biology is based on the idea that an organism’s genes largely determine its observable characteristics, and hence its fitness in a given environment. Organisms that are more fit will tend to produce more offspring, causing genes that provide greater fitness to increase their representation in the population. In this way, fitter genes tend to win over time, because they provide higher rates of reproduction.

The key insight of evolutionary game theory is that much behaviour involves the interaction of multiple organisms in a population, and the success of any one of these organisms depends on how its behaviour interacts with that of others. So the fitness of an individual organism can’t be measured in isolation; rather it has to be evaluated in the context of the full population in which it lives. This opens the door to a natural game-theoretic analogy: an organism’s genetically-determined characteristics and behaviours resemble its strategy in a game, its fitness resembles its payoff, and this payoff depends on the strategies (characteristics) of the organisms with which it interacts.

Evolutionary game theory considers games involving a population of decision makers, where the frequency with which a particular decision is made can change over time in response to the decisions made by all individuals in the population. In biology, this is intended to model (biological) evolution, where genetically programmed organisms pass along some of their strategy programming to their offspring. In economics, the same

theory is intended to capture population changes because people play the game many times within their lifetime, and consciously (and perhaps rationally) switch strategies.

In the evolutionary interpretation, there is a large population of individuals, each of which can adopt one of the strategies. The game describes the payoffs that result when two of these individuals meet. The dynamics of this game are based on the assumption that each strategy is played by a certain fraction of individuals. Consequently, given this distribution of strategies, individuals with better average payoff will be more successful than others, so that their proportion in the population increases over time. This, in turn, may affect which strategies are better than others. In many cases, in particular in symmetric games with only two possible strategies, the dynamic process will move to equilibrium.

Although evolutionary game theory has provided numerous insights to particular evolutionary questions, a growing number of social scientists have become interested in evolutionary game theory in hopes that it will provide tools for addressing a number of deficiencies in the traditional theory of games like the equilibrium selection problem, the problem of hyper-rational agents and the lack of a dynamical theory in the traditional theory of games.

The concept of Nash equilibrium has been the most used solution concept in game theory since its introduction by John Nash in 1950. A selection of strategies by a group of agents is said to be in Nash equilibrium if each agent's strategy is a best-response to the strategies chosen by the other players. By best-response, we mean that no individual can improve her payoff by switching strategies unless at least one other individual switches strategies as well. This need not mean that the payoffs to each individual are optimal in a Nash equilibrium: indeed, one of the disturbing facts of the Prisoner's Dilemma is that the only Nash equilibrium of the game, when both agents defect is suboptimal. Yet a difficulty arises with the use of Nash equilibrium as a solution concept for games: if we

restrict players to using pure strategies, not every game has a Nash equilibrium.

A more significant problem with invoking the Nash equilibrium as the appropriate solution concept arises because games with multiple Nash equilibria exist. When there are several different Nash equilibria, how is a rational agent to decide which of the several equilibria is the “right one” to settle upon? Attempts to resolve this problem have produced a number of possible refinements to the concept of a Nash equilibrium, each refinement having some intuitive purchases. Unfortunately, so many refinements of the notion of a Nash equilibrium have been developed that, in many games which have multiple Nash equilibria, each equilibrium could be justified by some refinement present in the literature. The problem has thus shifted from choosing amongst multiple Nash equilibria to choosing amongst the various refinements.

In evolutionary game theory, we no longer think of individuals as choosing strategies as they move from one game to another. This is because our interests are different. We’re now concerned less with finding the equilibria of single games than with discovering which equilibria are stable, and how they will change over time. So we now model “the strategies themselves” as playing against each other. One strategy is ‘better’ than another if it is likely to leave more copies of itself in the next generation, when the game will be played again. We study the changes in distribution of strategies in the population as the sequence of games unfolds.

For evolutionary game theory, a new equilibrium concept is introduced by Smith (1982). In this new concept, a set of strategies is considered to be evolutionary stable strategy (ESS) if, when adopted, serves to stop the spreading of an initially rare strategy.

The traditional theory of games imposes a very high rationality requirement upon agents (see Luce & Raiffa, 1957). This requirement originates in the development of the theory of utility which provides the underpinnings of game theory. Numerous results

from experimental economics have shown that these strong rationality assumptions do not describe the behaviour of real human subjects. Humans are rarely (if ever) the hyper-rational agents described by traditional game theory. The evolutionary, population-dynamic view of games is useful because it does not require the assumption that all players are sophisticated and think the others are also rational, which is often unrealistic. Instead, the notion of rationality is replaced with the much weaker concept of reproductive success: strategies that are successful on average will be used more frequently and thus prevail in the end.

The theory of evolution is a dynamical theory. The approach to evolutionary game theory involves the construction of an explicit model of the process by which the frequency of strategies changes in the population, and the properties of the evolutionary dynamics within that model are studied. This approach explicitly models the dynamics present in interactions amongst individuals in the population. Since the traditional theory of games lacks an explicit treatment of the dynamics of rational deliberation, evolutionary game theory can be seen, in part, as filling an important lacuna of traditional game theory.

For much of the history of game theory, a great deal of research in evolutionary game theory has focused on the properties and applications of the replicator equation. The replicator equation has become an essential tool over the past 40 years in applying evolutionary game theory to behavioral models in the biological and social sciences. The replicator equation was introduced in 1978 by Taylor and Jonker (Taylor & Jonker, 1978) and describes the evolution of the frequencies of population types, taking into account their mutual influence on their fitness. This important property allows the replicator equation to capture the essence of selection and, amongst other key results; it provides a connection between the biological concepts of evolutionarily stable strategies with the economical concept of Nash equilibrium. The replicator equation, due to Taylor and Jonker, was the

first and most successful proposal of an evolutionary game dynamics.

The replicator equation is derived in a specific framework that involves a number of assumptions, beginning with that of an infinite, well mixed population with no mutations. By well mixed population it is understood that every individual either interacts with every other one or at least has the same probability to interact with any other individual in the population. This hypothesis implies that any individual effectively interacts with a player which uses the average strategy within the population, an approach that has been traditionally used in physics under the name of mean-field approximation (Hauert & Szabo, 2005). Deviations from the well mixed population scenario affect strongly and non-trivially the outcome of the evolution, in a way which is difficult to apprehend in principle. Such deviations can arise when one considers, for instance, finite size populations, alternative learning/reproduction dynamics, or some kind of structure (spatial or temporal) in the interactions between individuals.

Within the population dynamics framework, the state of the population, i.e. the distribution of strategy frequencies, is given by  $\mathbf{x}$ . A first key point is that we assume that  $x_i$  are differentiable functions of time  $t$ . This requires in turn assuming that the population is infinitely large (or that  $x_i$  are expected values for an ensemble of populations). Within this hypothesis, we can now postulate a law of motion for  $\mathbf{x}(t)$ . Assuming further that individuals meet randomly, engaging in a game with payoff matrix  $\mathbf{W}$ , then  $f_i = (\mathbf{W}\mathbf{x})_i$  is the expected payoff for an individual using strategy  $s_i$ , and  $\bar{f} = \mathbf{x}^T \mathbf{W} \mathbf{x}$  is the average population payoff in the population state  $\mathbf{x}$ . If we, consistently with our interpretation of payoff as fitness, postulate that the per capita rate of growth of the subpopulation using strategy  $s_i$  is proportional to its payoff, we arrive at the replicator equation

$$\dot{x}_i = x_i (f_i - \bar{f}),$$



where  $f_i$  and  $\bar{f}$  are the fitness for an individual using strategy  $s_i$  and average population fitness (or instantaneous average fitness), respectively. The term  $\bar{f}$  arises to ensure the constraint  $\sum_i x_i = 1$ , and  $\dot{x}_i$  denotes the time derivative of  $x_i$ . In mathematical terms, this equation translates into the elementary principle of natural selection: individuals with strategies that enable efficient reproduction will replace those employing strategies with smaller fitness.

In the next section, two famous evolutionary games, the evolutionary Prisoner's Dilemma game and the evolutionary Snowdrift game, will be discussed in detail.

### 1.4.1 Evolutionary Prisoner's Dilemma

The evolutionary Prisoner's Dilemma games were introduced by Robert Axelrod in "The Evolution of Cooperation" in 1984 (Axelrod, 1984) to study the emergence of cooperation rather than exploitation amongst selfish individuals. Since the pioneering work of Axelrod, this approach has become a fruitful tool in the area of political and behaviour sciences, biology, and economics.

In evolutionary biology, the Prisoner's Dilemma is usually framed in terms of fitness costs and benefits. Cooperators provide a benefit  $b$  to their co-player at a cost  $c$  to themselves ( $b > c$ ) and defectors neither provide benefits nor pay costs. The payoffs for the joint behaviour of two interacting individuals are usually written in the form of a payoff matrix:

$$\begin{array}{cc}
 & \begin{array}{cc} \text{Cooperator} & \text{Defector} \end{array} \\
 \begin{array}{c} \text{Cooperator} \\ \text{Defector} \end{array} & \left( \begin{array}{cc} R & S \\ T & P \end{array} \right).
 \end{array}$$

Mutual cooperation pays  $R = b - c$  whereas mutual defection pays nothing. However if only one player defects and the other cooperates, then the defectors gets the benefit  $T = b$

without having to pay the costs and the cooperators faces the costs  $c$  without receiving any benefit, hence  $S = -c$ . Thus, just as in the case of the prisoners, it is always better to defect irrespective of the other players behaviour, but if both players follow this reasoning, they end up with nothing instead of  $b - c$ . The elements of the payoff matrix satisfy the following conditions:  $T > R > P > S$  and  $2R > T + S$ . In what follows, it will be assumed that the payoffs for the Prisoner's Dilemma are the same for everyone in the population.

Evolutionary dynamics is about populations and in this case about the change in frequencies of cooperators and defectors. In an infinite population with a fraction of co-operators  $x_C$ , a fraction of defectors  $x_D = 1 - x_C$  and randomly interacting individuals, the evolutionary fate of cooperators is given by the replicator equation:

$$\dot{x}_i = x_i[f_i - \bar{f}],$$

$$\begin{bmatrix} \dot{x}_C \\ \dot{x}_D \end{bmatrix} = \begin{bmatrix} x_C \\ x_D \end{bmatrix} \cdot \begin{bmatrix} f_C - \bar{f} \\ f_D - \bar{f} \end{bmatrix}$$

where  $f_C$  and  $f_D$  represent the average payoffs (fitness) of cooperators and defectors respectively, and  $\bar{f} = x_C f_C + x_D f_D$  denotes the average population payoff (fitness). The average payoff (fitness) of cooperators is simply  $f_C = x_C(b - c) + x_D(-c)$ . Similarly, the average payoff (fitness) of defectors is  $f_D = x_C b + x_D(0)$ . We can rewrite them as  $f_C = x_C b - c$  and  $f_D = x_C b$  respectively, where  $x_D = 1 - x_C$ . Thus, cooperators are always worse off ( $f_D > f_C$ ) and irrespective of their initial frequency, they will dwindle and eventually disappear.

The only stable equilibrium under the replicator dynamics occurs when everyone in the population follows the only evolutionary stable strategy. For the evolutionary Prisoner's Dilemma, the state where everybody defects ( $x_D = 1, x_C = 0$ ) is the only evolutionary stable strategy. This nicely illustrates the fact that evolutionary dynamics

represents a myopic optimization process: even though fitter individuals are selected in every time step, the overall fitness of the population decreases.

The Prisoner's Dilemma is the "standard model" of a social dilemma because if everybody defects, the mean group pay-off is lower than if everybody had cooperated. In the Prisoner's Dilemma, cooperators are doomed if interactions occur randomly. In structured populations, individuals interact only with their neighbours and cooperators may thrive by aggregating in clusters and thereby reducing exploitation by defectors. In finite populations, a surprisingly simple rule determines whether evolution favours cooperation:  $b > ck$ , that is, if the benefits  $b$  exceed  $k$ -times the costs  $c$  of cooperation, where  $k$  is the (average) number of neighbours. The spatial Prisoner's Dilemma has led to the general belief that spatial structure is beneficial for cooperation. Interestingly, however, this no longer holds when relaxing the social dilemma and considering the Snowdrift game.

In the continuous Prisoner's Dilemma, cooperative investments gradually decrease and defection dominates just as in the traditional Prisoner's Dilemma. In the Prisoner's Dilemma, defection always generates a higher payoff than cooperation, regardless of the opponent's strategy. Consequently, stable cooperation can only evolve under a Prisoner's Dilemma system of payoffs if some kind of information transfer is possible. One example is the iterated Prisoner's Dilemma, in which two players encounter each other repeatedly. In this circumstance, one player can punish defection, leading to conditional strategies — the most famous of which is "tit for tat" which yield a positive payoff for cooperation.

#### **1.4.2 Evolutionary Snowdrift Game**

The Snowdrift game derives its name from a situation in which two drivers are trapped on either side of a snowdrift and have the options of staying in the car or removing the snowdrift. Letting the opponent do all the work is the best option, but if the other player

stays in the car it is better to shovel, lest one never gets home (Sugden, 1986). Similar situations occur whenever the act of cooperation provides a common good that can be exploited by others, but that also provides some benefits to the cooperator itself.

In the Snowdrift game, cooperation yields a benefit  $b$  to the cooperator as well as to the opposing player, and incurs a cost  $c$  if the opponent defects, but only a cost  $c/2$  if the opponent cooperates. If both players cooperate, they both receive the reward  $R = b - c/2$  for cooperating. If both players defect, they both receive the punishment payoff  $P = 0$ . If first player defects while the opponent cooperates, then the first player receives the temptation payoff  $T = b$ , while the opponent receives the “sucker’s” payoff  $S = b - c$ . Similarly, if the first player cooperates while the opponent defects, then the first player receives the sucker’s payoff  $S = b - c$ , while the opponent receives the temptation payoff  $T = b$ .

In contrast to the Prisoner’s Dilemma, the best strategy now depends on the co-player’s decision: if the other driver shovels it is best to be lazy but when facing a lazy bum it is better to start shoveling instead of remaining stuck in the snow. The payoff matrix of the Snowdrift game is given by:

	Cooperator	Defector
Cooperator	$R$	$S$
Defector	$T$	$P$

The benefit of getting out of the drift is higher than the cost of shovelling ( $b > c > 0$ ). Thus, the following condition must hold for the payoffs:  $T > R > S > P$ . In what follows, it will be assumed that the payoffs for the Snowdrift game are the same for everyone in the population. Note that for  $2b > c > b$ , the Snowdrift game turns into the Prisoner’s Dilemma. For even higher cost  $c > 2b$ , mutual defection becomes the mutually preferred outcome.

In the evolutionary Snowdrift game, the players wish to achieve a higher reward by changing their strategy through referencing to the strategy of better performing opponent in multiple rounds of the game. How will a population of individuals that repeatedly plays the Snowdrift game evolve? In infinite populations with a fraction of cooperators  $x_C$ , and a fraction of defectors  $x_D = 1 - x_C$ , the evolutionary dynamics is again determined by the replicator equation:

$$\dot{x}_i = x_i[f_i - \bar{f}],$$

$$\begin{bmatrix} \dot{x}_C \\ \dot{x}_D \end{bmatrix} = \begin{bmatrix} x_C \\ x_D \end{bmatrix} \cdot \begin{bmatrix} f_C - \bar{f} \\ f_D - \bar{f} \end{bmatrix}$$

with  $f_C = x_C(b - c/2) + x_D(b - c)$  and  $f_D = x_C b + x_D(0)$ . We can rewrite them as  $f_C = b - c(1 - x_C/2)$  and  $f_D = x_C b$ , respectively, where  $x_D = 1 - x_C$ .

In contrast to the Prisoner's Dilemma, the stable equilibrium  $x_D = 1$  and  $x_C = 0$  is now unstable and an interior fixed point exists with  $f_C = f_D$  for  $x_C = 1 - r$  and  $r = c/(2b - c)$ . Thus, in the Snowdrift game cooperators and defectors can co-exist at some equilibrium frequency, which is determined by the costs and benefits of the game. This originates in the fact that in the Snowdrift game it is always better to adopt a strategy that differs from the co-player. As a consequence,  $f_C > f_D$  holds if cooperators are rare  $x_C \rightarrow 0$ , but  $f_D > f_C$  if cooperators are abundant and defectors are rare  $x_D \rightarrow 0$ . Note that at the equilibrium, the population as a whole is worse off than if everybody would cooperate,  $\bar{f} = (1 - r^2)(b - c/2) < b - c/2$  — this is the hallmark of social dilemmas and is another instance of myopic optimization.

The Prisoner's Dilemma does not represent the frequent situation where individuals obtain immediate direct benefits from the cooperative acts they perform and costs of cooperation are shared between cooperators. Such a situation is encapsulated in the

Snowdrift game. Doebeli et al. (2004) considered the Snowdrift game as a model for the evolution of cooperation. A mixed strategy of cooperation and defection can emerge under a Snowdrift game system of payoffs, which makes it very different from the Prisoner's Dilemma.

In contrast to the continuous Prisoner's Dilemma, the continuous Snowdrift game exhibits rich dynamics but most importantly provides an intriguing natural explanation for phenotypic diversification and the evolutionary origin of cooperators and defectors. Thus, selection may not always favour equal contributions but rather promote states where two distinct types co-exist — those that fully cooperate and those that exploit. In the context of human societies and cultural evolution this could be termed the Tragedy of the Commune because differences in contributions to a communal enterprise have significant potential for escalating conflicts on the basis of accepted notions of fairness.

Building on the foundation of iterated Prisoner's Dilemma with conditional strategies, the cumulative payoffs can be used to approximate the payoffs of the Snowdrift game, and this transfer of information changes one payoff structure into another. However, more importantly the Snowdrift game is still a social dilemma because defection is favoured when the other player cooperates, which occurs at the cost of the overall group pay-off.

In one-shot interactions, the predicted proportion of cooperative acts is zero for the Prisoner's Dilemma, while the Snowdrift game results in a mixed evolutionary stable state with the proportion of cooperative acts being  $1 - c/(2b - c)$ . The assumption of one-shot interactions is, however, not always realistic because repeated interactions amongst the same individuals often occur with iteration having been shown to favour cooperation in the iterated Prisoner's Dilemma.

Despite its potential importance for explaining cooperation amongst non-relatives, both the Snowdrift game and the iterated Snowdrift game have received little attention.

This is surprising because similar social dilemmas such as the Hawk–Dove game or the Chicken game, which have the same pay-off ranking but a different matrix structure, have been successfully used in behavioural ecology to study cooperation and conflicts in animals, as well as being employed in politics, economy and sociology to study the effects of various factors on human cooperation. However, despite the difficulty in accurately determining the payoff values, the Snowdrift game is regarded as an important alternative to the Prisoner’s Dilemma.

### **1.5 Research Objectives**

From the ancient tribal villages where humans worked together to hunt for food and fight off predators, to the modern companies where marketing teams carry out brainstorming to come up with ways to improve the sales of products, humans have been cooperating for survival and mutual benefits for millennia. Hence, one of the most important problems to which evolutionary game theory is being applied is the understanding of the emergence of cooperation in human (albeit non-exclusively) societies. This is an evolutionary puzzle that can be accurately expressed within the formalism of game theory.

As we have seen, rational players should unavoidably defect and never cooperate, thus leading to a very bad outcome for both players. On the other hand, it is evident that if both players had cooperated they would have been much better off. This is a prototypical example of a social dilemma which is, in fact, partially solved in societies. Indeed, the very existence of human society, with its highly specialized labour division, is a proof that cooperation is possible.

This thesis serves to study the dynamics of cooperation. In particular, it focuses on studying the factors promoting and prohibiting cooperation in the society.

In reality, some individuals are willing to pay a cost so as to punish others who do not behave well, e.g. non-cooperative opponents. Such a punishing strategy, which can

be regarded as a form of direct or indirect reciprocity, is another important mechanism in promoting cooperation.

Therefore, the first objective of this research is to study the role of punishment and the effects of a structured population in promoting cooperation. As a following on a recent model of Snowdrift game incorporating a costly punishing strategy (Xu et al., 2011), we study the effects of a population connected through a square lattice (N. W. H. Chan et al., 2013). As far as we are aware, such a model has never been studied before, as previous work focused on well-mixed population (Xu et al., 2011).

In the studies of evolutionary games, bridging the gap between theoretical and empirical research is one of the main challenges for the studies of cooperation. Different problems have been studied in the hope of applying the findings to implement the game theory to a practical scenario. An example is the well-known patent war between Apple and Samsung. It is possible to protect the profit of a cooperator or innovator via patenting. In the industry, the role of the punisher in a Snowdrift game can be taken up by the patent system as a patent grants an innovator monopoly rights over the use of an innovation for a given period of time.

Hence, the second objective of this research is to study the role of punishment in a more practical Cournot duopoly economic model (Yap et al., 2014) for an oligopoly. We study the effect of patenting on cooperation and defection in the sustainment of long-term R&D incentives. The patent system seeks to provide incentives to innovate as well as to disseminate the innovation. Therefore, we also investigate whether patenting is effective in giving enough incentives to firms to innovate. The punishment to the free-riders in this model will be in the form of incentive-denial, rather than actual deduction of payoff as in the case of the N-player evolutionary game in Chapter 2.



The third objective of this study is to simplify the above R&D Cournot duopoly model to an extent which allows us to identify precisely the factors that directly affect the strategy of the players and study the effect of patenting by using an evolutionary well-mixed N-player setting to see whether the results in the former can be reproduced. This modified model extends the model to a non-oligopolistic market and serves to study the effect of an N-player evolutionary game on the various factors previously studied.

Thus, three economic models involving non-linear systems are presented in this thesis.

## **1.6 Overview of the Thesis**

This thesis is organized as follows. In Chapter 1 (this chapter), we have introduction of game theory, history of game theory, definitions and types of games, and the objectives of the research. In Chapter 2, our model (N. W. H. Chan et al., 2013) of N-player evolutionary Snowdrift game with the punishing strategy in a structured population with agents connected by a square lattice is presented. In Chapter 3, the effect of patenting on cooperation and defection (Yap et al., 2014) by using a Cournot duopoly model is investigated and discussed. In Chapter 4, the R&D Cournot duopoly model in Chapter 3 is simplified to an extent which allows the study of the effect of patenting by using an evolutionary well-mixed N-player setting and identify precisely the factors that directly affect the strategy of the players. Finally, in Chapter 5, the research is summarized and some avenues for future research are proposed.

## CHAPTER 2: EVOLUTIONARY SNOWDRIFT GAME INCORPORATING COSTLY PUNISHMENT IN STRUCTURED POPULATIONS

### 2.1 Introduction

As discussed in Chapter 1, the understanding of emergence of cooperation in human societies is one of the most important problems to which evolutionary game is being applied. This chapter aims accurately express the evolutionary puzzle within the formalism of game theory.

The role of punishment and the effects of a structured population in promoting cooperation are important issues. Therefore, within a recent model of Snowdrift game incorporating a costly punishing strategy (P) in a well-mixed population (Xu et al., 2011), we study the effects of a population connected through a square lattice (N. W. H. Chan et al., 2013). The punishers, who carry basically a cooperative (C) character, are willing to pay a cost  $\alpha$  so as to punish a non-cooperative (D) opponent by  $\beta$ . Depending on  $\alpha$  and  $\beta$ , the cost to-benefit ratio  $r$  in Snowdrift game, and the initial conditions, the system evolves into different phases that could be homogeneous or inhomogeneous. The spatial structure imposes geometrical constraint on how one agent is affected by neighbouring agents.

In contrast to Snowdrift game incorporating a costly punishing strategy in a well-mixed population (Xu et al., 2011), where punishers are suppressed due to the cost of punishment, the altruistic punishing strategy can flourish and prevail for appropriate values of the parameters, implying an enhancement in cooperation by imposing punishments in a structured population. The system could evolve to a phase corresponding to the coexistence of C, D and P strategies at some particular payoff parameters, and such a phase is absent in a well-mixed population.

We used pair approximation as our analytic approach and we extended it from a two-strategy system to a three-strategy system. We show that the pair approximation

can, at best, capture the numerical results only qualitatively. Due to the improper way of including spatial correlation imposed by the lattice structure, the approximation does not give the frequencies of C, D, and P accurately and fails to give the homogeneous AllD and AllP phases.

The plan of this chapter is as follows. Section 2.2 is the literature review. Section 2.3 introduces the model and the strategy updating process. Results of extensive numerical simulations, both for the steady state and the dynamics, are presented and discussed in Section 2.4. The discussion is supported by results on the time evolution of the system. The different phases for different combinations of the payoff parameters are revealed by extensive simulations, both for the asynchronized and synchronized strategy updating schemes. The isolated phases are identified as special local structures of strategies that are stable due to the lattice structure. In Section 2.5, the analytical results of the pair approximation extended to three strategies are presented. The inadequacy of the pair approximation is discussed. Results are summarized in Section 2.6.

## **2.2 Literature Review**

The emergence of cooperation amongst selfish individuals is a fundamental and important issue in the research on the behaviour in populations (Nowak & Highfield, 2011). It has attracted the attention of researchers across different fields, including ecologists, physicists and applied mathematicians. Game theoretical models give a powerful tool for studying the emergence of cooperative behaviour (Smith & Price, 1973; Smith & Szathmary, 1995). The Prisoner's Dilemma (Rapaport & Chammah, 1965) is the "standard model" of a social dilemma and which has been studied extensively. However, due to the difficulty in accurately determining the payoff values, the Snowdrift game is regarded as an important alternative to Prisoner's Dilemma. In recent years, the Snowdrift game has been applied to study the emergence of cooperation in a competing population (Zheng et al., 2007;

Zhong et al., 2006; Wang et al., 2006; Santos & Pacheco, 2005; Hauert & Doebeli, 2004).

The players in these models take on one of two strategies, cooperative and non-cooperative, and could evolve their strategy by assessing the performance when they play the game repeatedly. It was found that for Snowdrift game in a well-mixed population (Xu et al., 2011) under an imitation mechanism for strategy updates, cooperation could emerge but a phase consisting of entirely cooperative players (i.e., an AllC phase) does not exist (Hauert & Doebeli, 2004). The well-mixed condition is typically not a good representation of how agents are connected in real populations. Hauert & Doebeli (2004) also studied how a structured population with agents connected in the form of regular lattices would affect cooperation. It was found that an AllC phase exists and cooperation could be enhanced or suppressed due to spatial structures (Santos & Pacheco, 2005; Hauert & Doebeli, 2004; Xu et al., 2007). They also applied the pair approximation that gives results in qualitative agreement with numerical results (Hauert & Doebeli, 2004).

In reality, some individuals are willing to pay a cost so as to punish others who do not behave well, e.g., non-cooperative opponents. Such a punishing strategy, which can be regarded as a form of direct or indirect reciprocity (Ohtsuki et al., 2006) is another important mechanism in promoting cooperation. Costly punishment, for example, was found to promote and stabilize cooperative behavior (Yamagishi, 1986; Ostrom et al., 1992; Fehr & Gächter, 2000) in the Public Good game (Kagel & Roth, 1995), which is a generalization of Prisoner's Dilemma to incorporate N-person interactions. However, the role of punishment remains a topic of recent studies, e.g., punishment was found to be not necessarily effective in promoting cooperation (Rand et al., 2009; Ohtsuki et al., 2009; Nowak, 2008; Dreber et al., 2008).

In a recent work, Xu et al. (2011) had incorporated a punishing strategy into Snowdrift game and studied the effect in a well-mixed population via the replicator dynamics. It

was found that, depending on the payoff parameters, inhomogeneous phases consisting of only two strategies are resulted: a cooperative phase with cooperators and punishers that represents an enhancement in cooperation due to punishment, and a mixed phase with cooperators and non-cooperators similar to Snowdrift game.

In this work, the model of Snowdrift game is studied with the punishing strategy in a structured population with agents connected by a square lattice. Results of extensive numerical simulations, both for the steady state and the dynamics, are presented in and contrasted with those in a well-mixed population. Depending on the payoff parameters, the system evolves into different phases that could be homogeneous and inhomogeneous. Due to the lattice structures, there are special local structures of strategies that are stable at particular values of the parameters.

In contrast to a well-mixed population where punishers are suppressed due to the cost of punishment (Xu et al., 2011), the altruistic punishing strategy can flourish and prevail for appropriate values of the payoff parameters, implying an enhancement in cooperation by imposing punishments. The system could evolve to a phase corresponding to the coexistence of the three strategies, while such a phase is absent in a well-mixed population.

The pair approximation, which is a commonly used analytic approach, is extended to the present three-strategy system. The results show that the pair approximation can, at best, capture the numerical results only qualitatively. It does not give the frequencies of the three strategies and fails to give the homogeneous phases with entirely non-cooperators and punishers scenarios observed in simulations.

### **2.3 Model**

The Snowdrift game reflects a kind of social dilemma typically described by the following scenario. There are two drivers hurrying home in opposite directions on a road blocked

by a snowdrift. Each of them has a choice of two possible actions: to shovel the snowdrift and go home or not to shovel and hope that the driver on the other side would shovel the snowdrift so that he could go home without work. Here, to shovel is a cooperative (C) strategy and not to shovel is a defection (D) strategy, where the usage of C and D follows standard practice in games. If both drivers take C, they could go home and hence get a benefit  $b$ . Shovelling is a laborious job with a total cost of  $c$ . Thus, each cooperative driver gets a net reward of  $R = b - c/2$  by sharing the labour. If they take D, they would be stuck, and each gets a payoff of  $P = 0$ , corresponding to the worst scenario. If only one driver takes C and the other takes D, then both of them could go home and the D-strategy gets a temptation payoff of  $T = b$  but the C-strategy gets a sucker payoff of  $S = b - c$ . We require  $b > c > 0$  so as to have  $T > R > S > P$  as in Snowdrift game. For one shot of the Snowdrift game, the better choice is to take D if his opponent takes C and to take C if his opponent takes D.

In evolutionary Snowdrift game where the players wish to achieve a higher reward by changing their strategy through referencing to the strategy of better performing opponents in multiple rounds of the game, cooperation could emerge. In a well-mixed population, for example, the frequency of cooperation decreases with the ratio  $c/b$ . In a structured population such as that for players located at the nodes of a regular lattice, an AllC phase with enhanced cooperation could appear for small  $c/b$  but the frequency of cooperation is suppressed in a large range of  $c/b$ , when compared with a well-mixed population (Hauert & Doebeli, 2004).

In real-life situation, it has been noticed that there exist other strategies. There are punishers who themselves are cooperative but would pay a cost to punish the non-cooperative players. This is to say that some cooperators would prefer to pay for extra cost  $\alpha$  to punish the defective opponents by letting them lose some benefit  $\beta$ . Such punishing

action is also a kind of altruistic behaviour (Fehr & Gächter, 2002). In Xu et al. (2011), a third punishing (P) strategy was introduced into Snowdrift game. The payoff matrix for Snowdrift game with punishers is given by

$$\begin{array}{c}
 \\
 C \\
 D \\
 P
 \end{array}
 \begin{array}{ccc}
 C & D & P \\
 \left( \begin{array}{ccc}
 R & S & R \\
 T & P & T - \beta \\
 R & S - \alpha & R
 \end{array} \right)
 \end{array}
 \quad (2.1)$$

where an element in the matrix in Eq.(2.1) gives the payoff to the player taking the strategy in the left-hand column when the opponent takes the strategy in the upper row. When P-players meet C-players, they behave just like C-players. When they meet D-players, however, the P-players are still rational where they pay a punishment cost of  $\alpha$  so as to lead to a damage  $\beta$  to a D-player with  $\beta > \alpha$ , where the latter inequality makes the punishing action rational. When a C-player encounters a P-player, the P-player does not behave differently to the C-player and they both receive the same payoff  $R$ . In general, we can simplify the payoffs by setting  $R = 1$  and  $P = 0$ . Further simplification is done by introducing a cost-to-benefit parameter  $r = c/(2b - c)$ ,  $0 < r < 1$  so that  $S = 1 - r$  and  $T = 1 + r$ . The payoff matrix can be rewritten as

$$\begin{array}{c}
 \\
 C \\
 D \\
 P
 \end{array}
 \begin{array}{ccc}
 C & D & P \\
 \left( \begin{array}{ccc}
 1 & 1 - r & 1 \\
 1 + r & 0 & 1 + r - \beta \\
 1 & 1 - r - \alpha & 1
 \end{array} \right)
 \end{array}
 \quad (2.2)$$

with  $\beta > \alpha$ .

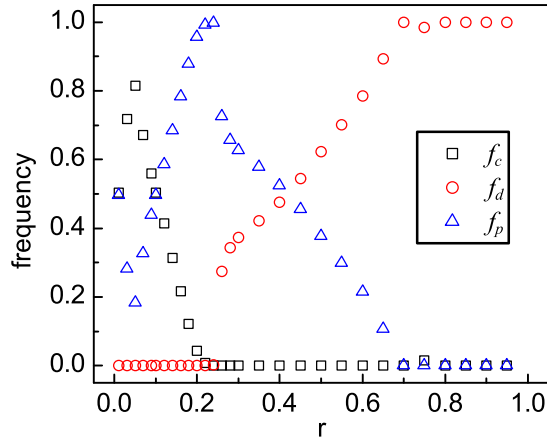
To study the effects of a structured population, we consider a system of  $N$  players occupying the nodes of a two dimensional (2D) square lattice. Every player has  $\bar{k} = 4$

nearest neighbours as his opponents. Therefore, at any instant of time, an average payoff  $\bar{V}_i = V_i/\bar{k}$  can be assigned to every player  $i$  with  $V_i$  calculated by Eq.(2.2) based on the strategies of player  $i$  and its neighbours at that time. The players update their strategies by referencing his opponent's performance. Strategy updates can be carried out either asynchronously or synchronously. In the asynchronous updating scheme, a target player  $i$  is randomly selected amongst all the players for an update at a time step. The targeted player selects a nearest neighbour  $j$  randomly as his reference player. If  $\bar{V}_i \geq \bar{V}_j$ , player  $i$  will keep his strategy. If  $\bar{V}_i < \bar{V}_j$ , player  $i$  will take on the strategy of player  $j$  with the probability  $\omega = (\bar{V}_j - \bar{V}_i)/(1+r)$  (Zhong et al., 2006), where the denominator  $(1+r)$  ensures  $\omega \leq 1$ . An average of  $N$  steps are required for each player to have a chance to update his strategy, constituting a Monte Carlo step (MCS). In the synchronous updating scheme, all the players perform an attempt for strategy update every time step simultaneously.

#### 2.4 Simulation Result and Discussion

We study both the dynamics and the steady states of the model using numerical simulations. To bring out the key features, we focus on the asynchronous updating scheme. Results using the synchronous updating scheme are similar and will be shown later. The initial condition with uniform distribution of the three strategies C, D and P on the nodes of a 2D square lattice is applied and the periodic boundary condition is imposed. For the simulation, one Monte Carlo step (MCS) refers to the evolutionary time scale that every player in the system has, on average, been chosen for a strategy update once. Results are obtained in a  $90 \times 90$  lattice. It has been confirmed that the number of time steps for reaching the long-time behavior depends on the parameters  $r$ ,  $\alpha$  and  $\beta$ . For the steady-state results that follow, sufficient time steps have been used to ensure that the steady state has been reached.



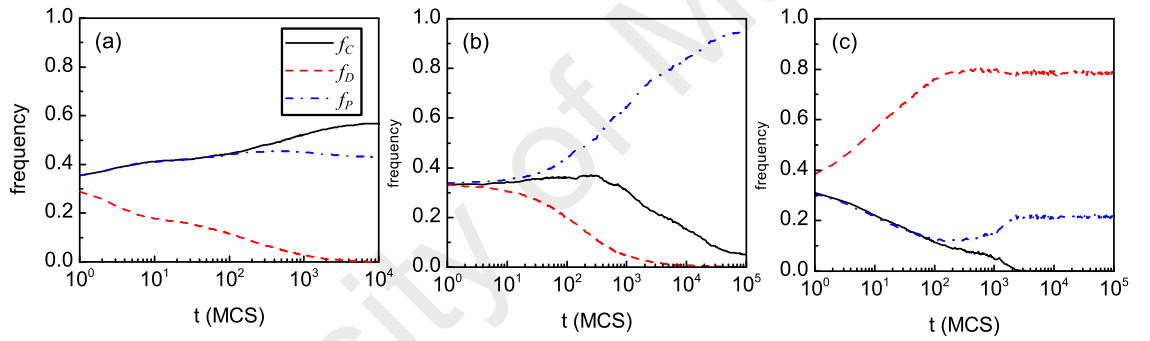


**Figure 2.1:** The frequencies  $f_c$ ,  $f_p$ , and  $f_d$  in the long time limit as functions of  $r$ , as obtained by simulations. The parameters are  $\alpha = 0.01$  and  $\beta = 0.05$ , with a uniform initial distribution of characters and an asynchronous update scheme.

Figure 2.1 shows the steady state frequencies of cooperation  $f_c$ , defection  $f_d$  and punishment  $f_p$ , in the long-time limit, as a function of the cost-to-benefit ratio  $r$  using  $\alpha = 0.01$  and  $\beta = 0.05$  as an example via the asynchronous update. We have checked that even lattices of a smaller size give the same results. Compared with the non-increasing (non-decreasing) dependence of  $f_c$  ( $f_d$ ) against  $r$  in a Snowdrift game without the punishers based on the same strategy updating mechanism (Zhong et al., 2006), introducing the punishing action into Snowdrift game has leads to much richer behavior. In this particular case, there are non-monotonic behavior in the frequencies as  $r$  varies.

For very small  $r$  ( $r \lesssim 0.05$ ), both C and P strategies are effective in suppressing D. As the D-against-D situation is too harmful for small  $r$ , the D strategy almost becomes extinct. The cost of punishment that the P-players pay eventually causes the C strategy to flourish. The non-trivial dynamical evolution of the frequencies in this regime is shown in Figure 2.2(a) for  $r = 0.02$ . In the early stage,  $f_d$  decreases while  $f_c$  and  $f_p$  increase with the same rate, indicating that the dynamics is that of switching from D strategy to C or P strategy with equal probabilities. This is the stage when there are sufficient D-players around for them to be nearest neighbors of each other, resulting in a low payoff. When  $t \gtrsim 100$  MCS,  $f_d$  continues to drop and becomes small, with only  $f_c$  continuing to increase. At this time,

besides the direct switching from D to C, there are also indirect switching to C through the sequence of  $D \rightarrow P \rightarrow C$ . This comes about when  $f_d$  is small so that chances are that a C-player is connected to a P-player. When this C-P pair of neighbors each has a similar competing environment in which there is a same number of D-players, the P-player will get a lower payoff than the C-player in the pair due to the cost of punishment. Thus, when the P-player refers to the C-player in the pair for strategy update, P will switch to C. This mechanism of enhancing the C strategy becomes more apparent after  $t \gtrsim 10^3$  MCS, when  $f_d$  becomes tiny. The presence of D-players is necessary for the dynamical process, since the dynamics stops when the system runs out of D-player by definition. However, under the asynchronous updating scheme there are a few remaining isolated D-players.



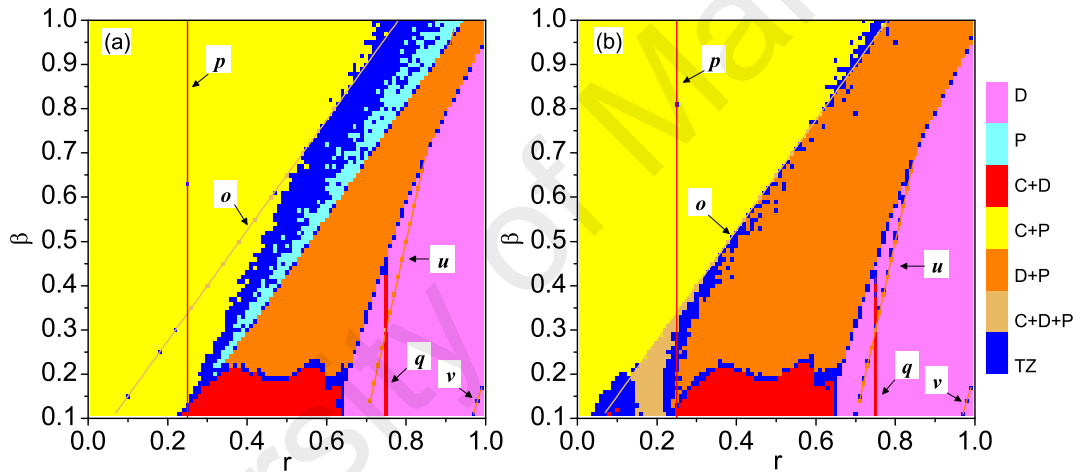
**Figure 2.2:** Time evolution of the frequencies  $f_c(t)$ ,  $f_p(t)$ , and  $f_d(t)$  as obtained by numerical simulations for (a)  $r = 0.02$ , (b)  $r = 0.2$ , and (c)  $r = 0.6$ . The parameters are  $\alpha = 0.01$  and  $\beta = 0.05$  as in Figure 2.1, with a uniform initial distribution of characters and an asynchronous updating scheme.

For  $0.05 < r < 0.24$  (see Figure 2.1), the C strategy loses its edge as the payoff they get becomes smaller when they encounter a D-strategy opponent. In this range of  $r$ , there is a rapid increase in  $f_p$  accompanied by a corresponding rapid drop in  $f_c$ ; and  $f_d$  remains tiny. The increasing trend of  $f_p$  lasts until  $r \approx 0.24$ , where nearly all the players take on the P strategy. Therefore, in this range of  $r$ , the punishing strategy flourishes and suppresses the D strategy. The dynamics typical of this range is shown in Figure 2.2(b) for  $r = 0.2$ . Within a short time,  $f_d$  drops at a more gradual pace when compared to the case

with  $r = 0.02$  because of the better payoff against the C strategy. The initial condition randomly puts the three strategies on to the nodes in the lattice. Thus, the initial switching from D strategy happens at places where a D-player locally has a C or P-neighbor serving as the reference for strategy update and the D-player has one or more D-neighbors to make his payoff low. Thus, they will be switching from the D strategy to either C or P strategy within a short time, as seen in Figure 2.2(b) for  $t < 100$  MCS. However, comparing a D-player referencing to a C-player having a certain local competing environment and to a P-player having the same local competing environment, there is a slightly higher chance for the switching from D to P than the switching from D to C. It is because the switching probability  $\omega$  is related to the difference in average payoffs and thus the positive factor  $(\beta - \alpha)$  gives the switching from D to P an edge. This explains the slightly higher  $f_p$  for  $t < 100$  MCS. This small effect amplifies as the dynamics proceeds, where the D-players remaining in the system tend to have C-players as neighbors and make the payoffs to these C-players low. This is the scenario for  $t > 300$  MCS, when C-players with low payoffs switch to D or P and  $f_c$  drops. At the same time, the switching from D to P continues. For  $t > 1000$  MCS,  $f_d$  stays tiny and the net effect is a switching from C to P directly or through a sequence  $C \rightarrow D \rightarrow P$  indirectly, resulting in a dominating P population. It should be noted that this flourishing of the P strategy at  $r = 0.24$  does not occur in a well-mixed population (Xu et al., 2011) and the local competing environments in a square lattice affect the dynamics.

For  $r > 0.24$ ,  $f_d$  grows at the expense of  $f_p$ , and  $f_c$  becomes extinct at  $r \approx 0.24$  (see Figure 2.1). The increasing  $r$  benefits the D strategy and diminishes the importance of the punishment  $\beta$  of the P strategy. For  $r < 0.67$ , a mixed population with D and P strategies results. Figure 2.2(c) shows the dynamics typical of the mixed population phase at  $r = 0.6$ . Even after 1 MCS, the high value of  $r$  leads to switching from both C and P

strategies to the D strategy. This trend continues up to  $t \sim 100$  MCS. After that time,  $f_d$  levels off. The system now contains many D-players. Recall that D against D leads to the lowest payoff. Too many D-players in the system lower their payoff. They are switching from D strategy to P strategy and at the same time switching from C strategy to D strategy. The net effect is an increase in  $f_p$  beyond  $t = 100$  MCS at the expense of  $f_c$ . Eventually,  $f_c$  vanishes and a finite but small  $f_p$  results. Such a non-trivial dynamics is specific to a structured population. When  $r$  becomes sufficiently large ( $r \gtrsim 0.67$ ), the punishing effect becomes negligible and P strategy also becomes extinct, giving rise to a homogeneous AllD ( $f_d = 1$ ) population.



**Figure 2.3:** Phase diagram showing the nature of the steady state population in the  $r - \beta$  space for the (a) synchronous updating and (b) asynchronous updating schemes, for the case of  $\alpha = 0.1$ . The lines are obtained by considering local stable structures and they explain the presence of the isolated dots.

To explore how the steady state frequencies depend on the interplay between the punishment parameter  $\beta$  and the cost to-benefit ratio  $r$ , detailed simulations are carried out over the range  $0.1 = \alpha < \beta < 1.0$  and  $0 < r < 1$ . The results are displayed in Figure 2.3 in the  $r - \beta$  space for both the synchronous (Figure 2.3(a)) and asynchronous (Figure 2.3(b)) updating schemes. Different colors are used to represent the phases. The results are obtained in square lattices of size  $30 \times 30$ , and it has been confirmed that the results are the same in  $60 \times 60$  lattices. Note that the isolated dots are also simulation results and

the reason of their presence will be discussed.

There are four dominating regions in Figure 2.3. They are the mixed phases of C + P (yellow), C + D (red), D + P (orange) and the homogeneous AllD phase (magenta). For small  $r$  and large  $\beta$ , the D strategy cannot survive and a mixed C + P phase results. Note that AllC phase does not occur. This is because the dynamics stops when the D strategy becomes extinct and does not allow further switching between C and P strategies to arrive at an AllC phase. For intermediate values of  $r$  and relatively large  $\beta$ , the C strategy cannot survive and only the punishers could suppress the D strategy. This gives rise to a region of D + P phase. As  $r$  becomes sufficiently large, the D + P phase eventually loses its stability, leading to the region corresponding to the AllD phase in the right hand side of Figure 2.3.

Interestingly, for a range of small  $\beta$  in an intermediate range of  $r$  (around  $0.25 < r < 0.6$ ), the P strategy becomes extinct as the small punishment cannot suppress the D strategy and the punishment cost  $\alpha$  in turn destabilizes P itself. This gives rise to the regions of C + D phase. The lower left corner in Figure 2.3(b) ( $r < 0.2$ ) is also eye-catching. The 2D square lattice restricts the local competing environment and the P strategy could exist for a long time in the evolutionary process. Thus a small region of the C + D + P phase appears at small  $r$ , and even the results in Figure 2.3(b) were obtained after an extremely long time of  $5 \times 10^6$  time steps. Furthermore, the C strategy may become extinct during the evolutionary process and the system could evolve into a D + P phase. Therefore, there are two narrow phases of C + D + P and D + P around  $r \approx 0.2$  at small  $\beta$  for the asynchronous updating scheme (Figure 2.3(b)). Two points should be noted regarding the C + D + P phase: (i)  $f_d$  is tiny in this phase and (ii) this phase is not found in a well-mixed population (Xu et al., 2011).

Ideally, these phases would be separated by sharp phase boundaries. However, the phases are separated by a narrow transition zone (TZ) in the simulation results. In these

transition zones (blue regions in Figure 2.3), the final phase in different runs could be either one of the phases separated by the transition zone. For synchronous updates (Figure 2.3(a)), there exists a narrow AllP (cyan) phase, which is not found in a well-mixed population, next to the D + P phase; and there is a broader transition zone between the C + P phase and the AllP phase.

The isolated dots in Figure 2.3 are features that come from the spatial structure and can be explained by considering the stability of special local structures of strategies. Here, we use the lines obtained by considering local stable structures to explain the presence of the isolated dots. We consider a local structure with a D-player surrounded by four P-players. Each of the P-neighbors, besides having the D-player as a neighbor, has either C or P players as the other three neighbors. According to Eq.(2.2), the payoffs to the D-player and to the P-neighbors are  $4(1+r-\beta)$  and  $1(1-r-\alpha)+3(1)$  respectively. In this case, the payoffs to the D-player and to the P-neighbors are identical when the condition  $\beta = (5r + \alpha)/4$  is satisfied. This condition gives the line-*o* in Figure 2.3. Under this condition, the local structure of a D-player with four P-neighbors is stable where it will not switch as long as the neighborhoods of its four P-neighbors do not change. Note that the dark yellow dots in Figure 2.3(b) lie exactly on the line-*o*. This explains the presence of the isolated dark yellow dot.

Similarly, we examine the stability of the local structure of a neighboring pair of D strategies surrounded by C players. The payoffs to the D-player and to the C-neighbors according to Eq.(2.2) are  $1(0) + 3(1+r)$  and  $1(1-r) + 3(1)$  respectively. Their payoffs are identical under the condition  $r = 1/4$ . Therefore, we obtained the stability condition,  $r = 1/4$  which gives the line-*p* in Figure 2.3. A final phase with a few D strategy pairs scattered in an otherwise AllC background may thus occur at  $r = 1/4$ . Indeed, the red dots in the C + P region (yellow) correspond to this special case and they lie on the line-*p*.

The local structure of a pair of C strategies surrounded by D players is stable under the condition of  $r = 3/4$  as shown by the line- $q$  in Figure 2.3. According to Eq.(2.2), the payoffs to the C-player and to the D-neighbors are  $1(1) + 3(1 - r)$  and  $1(1 + r) + 3(0)$  respectively. Their payoffs are identical when the condition  $r = 3/4$  is satisfied. This leads to C + D isolated dots within the AllD (magenta) region.

The D + P isolated (orange) dots within the AllD region can correspond to two special local structures and they lie on the line- $u$  and line- $v$  in Figure 2.3. The line- $u$  given by  $\beta = 4r + 3\alpha - 3$  is obtained by examining the stability of the local structure of a neighboring pair of P strategies surrounded by D players. With this local structure, the payoffs to the P-player and to the D-neighbors are  $1(1) + 3(1 - r - \alpha)$  and  $1(1 + r - \beta) + 3(0)$  respectively, according to Eq.(2.2). Their payoffs are identical under the condition  $\beta = 4r + 3\alpha - 3$  and gives the line- $u$ . Another possible structure is that of four P strategies forming a square surrounded by D players. According to Eq.(2.2), the payoffs to the P-player and to the D-neighbors are  $2(1) + 2(1 - r - \alpha)$  and  $1(1 + r - \beta) + 3(0)$  respectively. Their payoffs are identical when the condition  $\beta = 3r + 2\alpha - 3$  is satisfied. Therefore, such a structure is stable when  $\beta = 3r + 2\alpha - 3$  and gives the line- $v$  in Figure 2.3. These two local structures leads to D + P isolated (orange) dots within the AllD phase. The discussion on the phases, the dynamics and the isolated dots in the phase diagram have highlighted the importance of the lattice structure.

## 2.5 Analytical Result and Discussion

### 2.5.1 Pair Approximation Extended to Three Strategies

Pair approximation is a common analytic approach previously applied to the evolutionary snowdrift game on a square and other lattices (Hauert & Doebeli, 2004). Morita extended the pair approximation for two-strategy games to study the effects of degree fluctuations and large clustering in networks (Morita, 2008). Perc formulated the pair approximation

for an evolutionary prisoner's dilemma on spatial networks incorporating a third strategy corresponding to the presence of loners (Perc & Marhl, 2006), the behavior of which is different from the punishers in the present work. The approach invokes an approximation that for a link, say, that connects a C player to a D player, the probability that the C player is also connected to another player with the  $x(x = c, d)$  strategy is given by  $f_{x,c}f_{c,d}/f_c$ , where  $f_{c,d}$  is the density of c-d links in the system and  $f_{x,c}$  carries an analogous meaning, together with a weighing factor for the occurrence of different configurations based on the binomial coefficients. Essentially, the pair approximation handles a three-site correlation (e.g., the three-site connection of  $x - c - d$ ) through two-site correlations given by the link densities. As the detail of the spatial structure is not incorporated, the pair approximation gives results only in qualitative agreement with numerical simulations.

We extended the pair approximation to the case of three strategies. In a square lattice, each node has  $\bar{k} = 4$  nearest neighbors. There are  $3^2$  types of links connecting two sites with three strategies C, D, and P, namely  $c-c$ ,  $c-d$ ,  $c-p$ ,  $d-c$ ,  $d-d$ ,  $d-p$ ,  $p-c$ ,  $p-d$ , and  $p-p$  links. Let  $f_{x,y}$  be the densities of these links in the system, with  $x$  and  $y$  being  $c$ ,  $d$ , and  $p$ . Due to  $f_{x,y} = f_{y,x}$  and normalization condition  $\sum_{x,y} f_{x,y} = 1$  for the nine link densities, we may choose to focus on five link densities,  $f_{c,c}$ ,  $f_{c,d}$ ,  $f_{c,p}$ ,  $f_{d,d}$  and  $f_{d,p}$ , and treat them as the variables for tracing the time evolution of the system. Using these variables, the frequencies of C, D, P can be given by  $f_x = \sum_y f_{x,y}$  where  $x = c, d, p$ .

In the evolutionary snowdrift game, changes take place only when a target player switches the strategy to that of a referencing player. When a target player plays strategy  $x$  and the neighboring referencing player plays strategy  $y$ , the probability of selecting the active  $x$ - $y$  link for consideration of switching is  $f_{x,y}$ . Due to the square lattice structure, there are  $\bar{k} - 1 = 3$  other links that will affect the target player and referencing player's average payoff  $\bar{V}_x$  and  $\bar{V}_y$  respectively, besides the  $x$ - $y$  link. Let there be  $m_x$   $c$ - $x$  links,



$n_x$   $d$ - $x$  links, and  $(3 - m_x - n_x)$   $p$ - $x$  links that are connected to  $x$  besides the  $x$ - $y$  link. Similarly, for the referencing agent with strategy  $y$ , let there be  $m_y$   $c$ - $y$  links,  $n_y$   $d$ - $y$  links, and  $(3 - m_y - n_y)$   $p$ - $y$  links besides the  $x$ - $y$  pair connected to the target player. The switching probability  $\omega_{x,y}$  for the target player to switch his strategy from  $x$  to  $y$  is given by

$$\omega_{x,y}(m_x, n_x, m_y, n_y) = \frac{\bar{V}_y(x, m_y, n_y) - \bar{V}_x(y, m_x, n_x)}{1 + r} \quad (2.3)$$

If  $x$  switches to be  $y$ , there will be corresponding changes in the numbers of the pairs of different kinds by a certain amount. Let  $\Delta n_{x',y'}^{x \rightarrow y}(m_x, n_x)$  be the change in the number of  $x'$ - $y'$  pairs due to the switching event from  $x$  to  $y$  for given  $m_x$  and  $n_x$ . The corresponding change in  $f_{x',y'}$  under such a switching, regardless of the values of  $m_y$  and  $n_y$  surrounding the referencing player, is then given by

$$\Delta f_{x',y'}^{x \rightarrow y}(m_x, n_x) = \frac{\Delta n_{x',y'}^{x \rightarrow y}(m_x, n_x)}{N\bar{k}} f_{x,y} \sum_{m_y, n_y} T_{m_y, n_y}^{(3)} \Omega_y(m_y, n_y) \omega_{x,y}. \quad (2.4)$$

We use  $T_{m,n}^{(3)}$  to represent the coefficient in the trinomial expression, i.e.,

$$T_{m,n}^{(3)} = \binom{3}{m, n, 3 - m - n} = \frac{3!}{m!n!(3 - m - n)!}, \quad (2.5)$$

and it gives the number of configurations amongst  $\bar{k} - 1$  links for given values of  $m$  and  $n$ . The term

$$\Omega_y(m_y, n_y) = \frac{f_{c,y}^{m_y} f_{d,y}^{n_y} f_{p,y}^{3-m_y-n_y}}{f_y^3}, \quad (2.6)$$

gives the probability of occurrence of a configuration in terms of the link densities and frequencies.  $N\bar{k}$  is the total number of pairs in the system.

Taking into account of all the possible values of  $m_x$  and  $n_x$  surrounding the target player due to a particular switching from the  $x$  strategy to the  $y$  strategy, we obtain the

change in  $f_{x',y'}$  as:

$$\Delta f_{x',y'}^{x \rightarrow y} = \sum_{m_x, n_x} T_{m_x, n_x}^{(3)} \Omega_x(m_x, n_x) \Delta f_{x',y'}^{x \rightarrow y}(m_x, n_x). \quad (2.7)$$

Finally, we include the possibility that both  $x$  and  $y$  could be  $c, d$  and  $p$ . Therefore, we arrive at the final expression for  $\Delta f_{x',y'}$  as a result of a switching in strategy as:

$$\begin{aligned} & \Delta f_{x',y'} \\ &= \sum_{x,y} \Delta f_{x',y'}^{x \rightarrow y} \\ &= \frac{1}{N\bar{k}} \sum_{x,y,m_x,n_x} T_{m_x,n_x}^{(3)} \Omega_x(m_x, n_x) \Delta n_{x',y'}^{x \rightarrow y}(m_x, n_x) f_{x,y} \sum_{m_y,n_y} T_{m_y,n_y}^{(3)} \Omega_y(m_y, n_y) \omega_{x,y} \end{aligned} \quad (2.8)$$

where  $x', y' = c, d, p$ . Eq.(2.8) extended the pair approximation to cases with three strategies and  $\bar{k} = 4$ . Based on Eq.(2.8), the equations for  $\Delta f_{c,c}$ ,  $\Delta f_{c,d}$ ,  $\Delta f_{c,p}$ ,  $\Delta f_{d,d}$ , and  $\Delta f_{d,p}$  are explicitly given by the following:

$$\begin{aligned} \Delta f_{c,c}(N\bar{k}) &= \sum_{m_c, n_c} T_{m_c, n_c}^{(3)} (-2m_c) \Omega_c(m_c, n_c) f_{c,d} \sum_{m_d, n_d} T_{m_d, n_d}^{(3)} \Omega_d(m_d, n_d) \omega_{c,d} \\ &+ \sum_{m_d, n_d} T_{m_d, n_d}^{(3)} (2 + 2m_d) \Omega_d(m_d, n_d) f_{d,c} \sum_{m_c, n_c} T_{m_c, n_c}^{(3)} \Omega_c(m_c, n_c) \omega_{d,c} \\ &+ \sum_{m_c, n_c} T_{m_c, n_c}^{(3)} (-2m_c) \Omega_c(m_c, n_c) f_{c,p} \sum_{m_p, n_p} T_{m_p, n_p}^{(3)} \Omega_p(m_p, n_p) \omega_{c,p} \\ &+ \sum_{m_p, n_p} T_{m_p, n_p}^{(3)} (2 + 2m_p) \Omega_p(m_p, n_p) f_{p,c} \sum_{m_c, n_c} T_{m_c, n_c}^{(3)} \Omega_c(m_c, n_c) \omega_{p,c} \end{aligned} \quad (2.9)$$

$$\begin{aligned} \Delta f_{c,d}(N\bar{k}) &= \sum_{m_c, n_c} T_{m_c, n_c}^{(3)} (m_c - n_c - 1) \Omega_c(m_c, n_c) f_{c,d} \sum_{m_d, n_d} T_{m_d, n_d}^{(3)} \Omega_d(m_d, n_d) \omega_{c,d} \\ &+ \sum_{m_c, n_c} T_{m_c, n_c}^{(3)} (-n_c) \Omega_c(m_c, n_c) f_{c,p} \sum_{m_p, n_p} T_{m_p, n_p}^{(3)} \Omega_p(m_p, n_p) \omega_{c,p} \\ &+ \sum_{m_d, n_d} T_{m_d, n_d}^{(3)} (n_d - m_d - 1) \Omega_d(m_d, n_d) f_{d,c} \sum_{m_c, n_c} T_{m_c, n_c}^{(3)} \Omega_c(m_c, n_c) \omega_{d,c} \end{aligned}$$

$$\begin{aligned}
& + \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}(-m_d)\Omega_d(m_d, n_d)f_{d,p} \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}\Omega_p(m_p, n_p)\omega_{d,p} \\
& + \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}n_p\Omega_p(m_p, n_p)f_{p,c} \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}\Omega_c(m_c, n_c)\omega_{p,c} \\
& + \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}m_p\Omega_p(m_p, n_p)f_{p,d} \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}\Omega_d(m_d, n_d)\omega_{p,d} \tag{2.10}
\end{aligned}$$

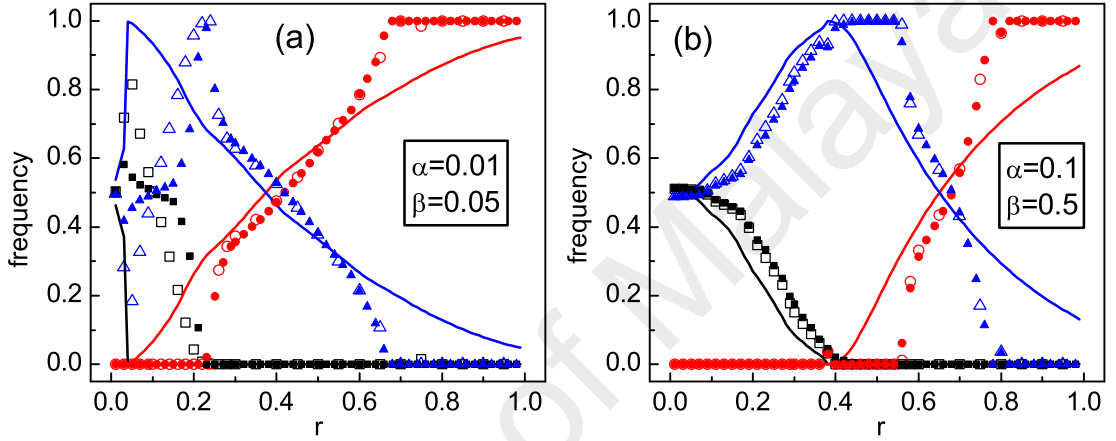
$$\begin{aligned}
\Delta f_{c,p}(N\bar{k}) & = \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}(-3 + m_c + n_c)\Omega_c(m_c, n_c)f_{c,d} \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}\Omega_d(m_d, n_d)\omega_{c,d} \\
& + \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}(-4 + 2m_c + n_c)\Omega_c(m_c, n_c)f_{c,p} \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}\Omega_p(m_p, n_p)\omega_{c,p} \\
& + \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}(3 - m_d - n_d)\Omega_d(m_d, n_d)f_{d,c} \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}\Omega_c(m_c, n_c)\omega_{d,c} \\
& + \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}m_d\Omega_d(m_d, n_d)f_{d,p} \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}\Omega_p(m_p, n_p)\omega_{d,p} \\
& + \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}(2 - 2m_p - n_p)\Omega_p(m_p, n_p)f_{p,c} \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}\Omega_c(m_c, n_c)\omega_{p,c} \\
& + \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}(-m_p)\Omega_p(m_p, n_p)f_{p,d} \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}\Omega_d(m_d, n_d)\omega_{p,d} \tag{2.11}
\end{aligned}$$

$$\begin{aligned}
\Delta f_{d,d}(N\bar{k}) & = \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}(2 + 2n_c)\Omega_c(m_c, n_c)f_{c,d} \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}\Omega_d(m_d, n_d)\omega_{c,d} \\
& + \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}(-2n_d)\Omega_d(m_d, n_d)f_{d,c} \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}\Omega_c(m_c, n_c)\omega_{d,c} \\
& + \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}(-2n_d)\Omega_d(m_d, n_d)f_{d,p} \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}\Omega_p(m_p, n_p)\omega_{d,p} \\
& + \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}(2 + 2n_p)\Omega_p(m_p, n_p)f_{p,d} \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}\Omega_d(m_d, n_d)\omega_{p,d} \tag{2.12}
\end{aligned}$$

$$\begin{aligned}
\Delta f_{d,p}(N\bar{k}) & = \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}(3 - m_c - n_c)\Omega_c(m_c, n_c)f_{c,d} \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}\Omega_d(m_d, n_d)\omega_{c,d} \\
& + \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}n_c\Omega_c(m_c, n_c)f_{c,p} \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}\Omega_p(m_p, n_p)\omega_{c,p} \\
& + \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}(-3 + m_d + n_d)\Omega_d(m_d, n_d)f_{d,c} \sum_{m_c, n_c} T_{m_c, n_c}^{(3)}\Omega_c(m_c, n_c)\omega_{d,c} \\
& + \sum_{m_d, n_d} T_{m_d, n_d}^{(3)}(-4 + m_d + 2n_d)\Omega_d(m_d, n_d)f_{d,p} \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}\Omega_p(m_p, n_p)\omega_{d,p}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}(-n_p) \Omega_p(m_p, n_p) f_{p,c} \sum_{m_c, n_c} T_{m_c, n_c}^{(3)} \Omega_c(m_c, n_c) \omega_{p,c} \\
& + \sum_{m_p, n_p} T_{m_p, n_p}^{(3)}(2 - m_p - 2n_p) \Omega_p(m_p, n_p) f_{p,d} \sum_{m_d, n_d} T_{m_d, n_d}^{(3)} \Omega_d(m_d, n_d) \omega_{p,d}
\end{aligned} \tag{2.13}$$

with  $T_{m,n}^{(3)}$  and  $\Omega$  defined in Eqs.(2.5) and (2.6), respectively. Further extension to lattices with other values of  $\bar{k}$  is straightforward.



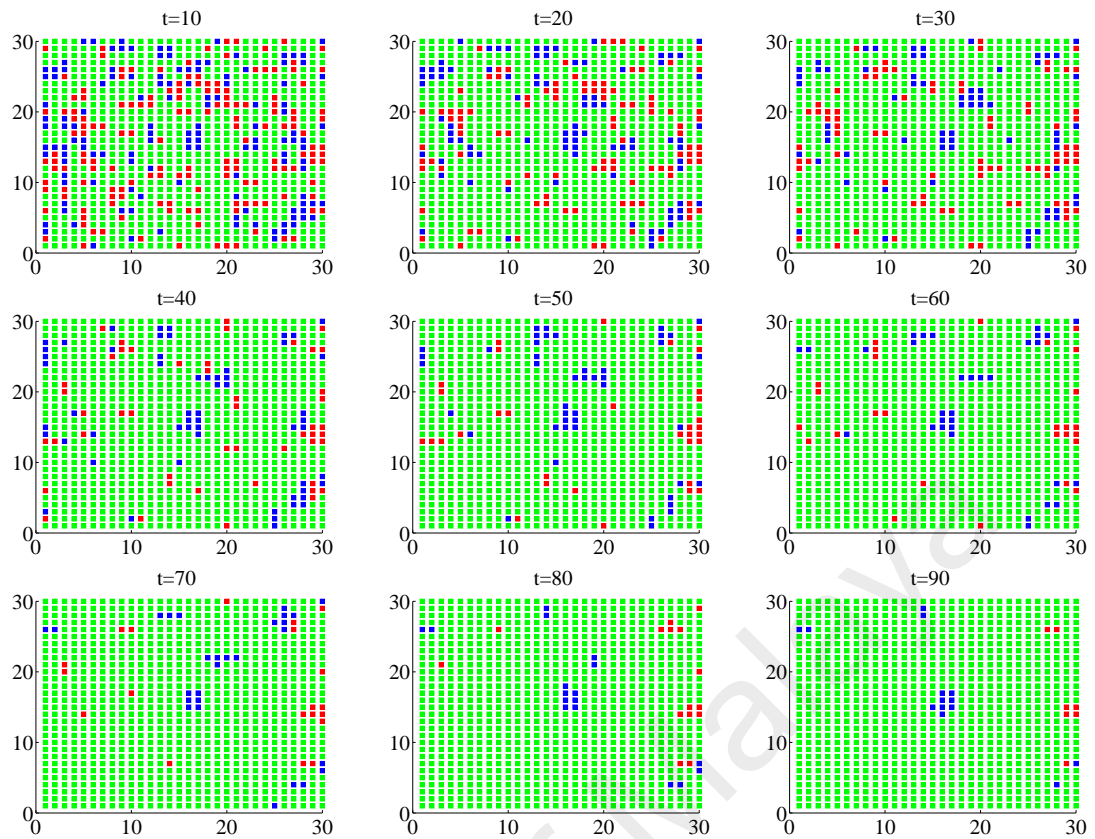
**Figure 2.4:** Comparison of results of the frequencies  $f_c$  (black),  $f_p$  (blue), and  $f_d$  (red) obtained by numerical simulations using the asynchronous updating (open symbols) and synchronous updating (solid symbols) with results of the pair approximation (lines). Parameters are:(a)  $\alpha = 0.01$  and  $\beta = 0.05$  and (b)  $\alpha = 0.1$  and  $\beta = 0.5$ .

Given an initial condition, Eq.(2.8) traces the changes in the link densities as the time evolves. These equations can be used to study the time evolution and the steady state. Eqs.(2.9) - (2.13) can be used to solve for steady states by setting their right-hand sides to zero or by iterating the equations for sufficiently long time. The results in Figure 2.4 are obtained using these equations. We compare the steady state results from Eq.(2.8) with results from numerical simulations in Figure 2.4 to test the validity of the pair approximation. Simulation results for both the asynchronous and synchronous updating schemes are shown. As discussed earlier in association with Figure 2.1, the two updating schemes give results with the same key features. The numerical results are essentially

the same for the case of  $\alpha = 0.1$  and  $\beta = 0.5$  (see Figure 2.4(b)), and some differences are shown in the range of small  $r$  for the case of  $\alpha = 0.01$  and  $\beta = 0.05$  (see Figure 2.4(a)). Results of the pair approximation (three solid lines) capture some of the main features, including the non-monotonic behavior of  $f_p$  and the drop of  $f_c$  to almost zero as a function of  $r$ . However, the agreement is qualitative at best. The pair approximation fails to give the AllD phase at large  $r$  (see Figure 2.4(a) and (b)) and the AllP phase obtained in synchronous updates (see Figure 2.4(b)). In addition,  $f_d$  starts to increase at a lower value of  $r$  than those revealed by the simulation results, and the non-monotonic behavior in  $f_c$  at small  $r$  (see Figures 2.1 and 2.4(a)) is missing.

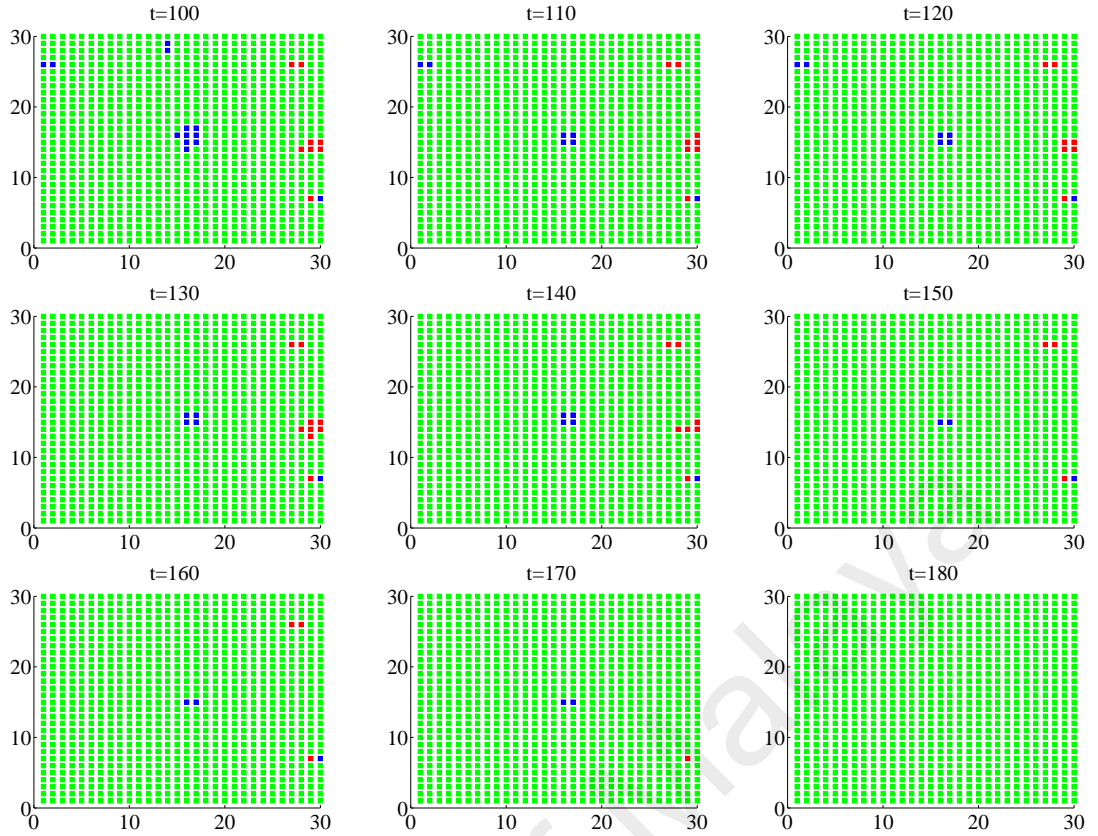
The inadequacy of the pair approximation, both for Snowdrift game in lattices (Hauert & Doebeli, 2004) and in the present case with three strategies, comes mainly from the missing of detailed spatial structures and thus a longer spatial correlation within the approximation. Strategy switching of a target player is related to the payoff of a neighboring referencing player, a value that depends on the neighbor's neighbors of the target player. Therefore, a spatial correlation up to the next-nearest neighbors is essential for understanding a wide class of evolutionary games in spatially structured populations. Such correlation necessarily incorporates the geometry of the underlying lattice. This correlation is not properly treated in the pair approximation, as evidenced by the same set of equations (Eq.(2.8)) for different spatial structures with the same value of  $\bar{k}$  within the pair approximation.

Figures 2.5 and 2.6 show snapshots of cluster of characters for every 10 MCS on the  $30 \times 30$  square lattices with parameters  $r = 0.80$ ,  $\alpha = 0.01$  and  $\beta = 0.05$ . In this model, we assume that at time  $t = 0$ , each configuration, say a  $c$ - $c$  link, has an equal probability of occurring. However, in a structured population, finite or infinite, there is always a probability of clustering of same character types at  $t = 0$ . These small “islands”



**Figure 2.5:** Snapshots of cluster of characters for every 10 MCS on the  $30 \times 30$  square lattices. The parameters are  $r = 0.80$ ,  $\alpha = 0.01$  and  $\beta = 0.05$ . The red squares denote cooperators, green squares denote defectors, and blue squares denote punishers. Continued in Figure 2.6.

will have large impact on the rate of change of characters as time progresses. Consider a cluster of D characters at  $t = 0$  without the presence of P characters amongst its neighbors, the surrounding C characters will be converted by the D characters in the cluster. Thus, the cluster of character D grows larger in size. When its border touches a region with a mixture of C and P characters, the P characters will reduce the payoff of some of the D characters, but due to the large number of D-characters in the neighborhood, and if the punishment is not severe enough, the P characters will not be able to convert all the D-characters into P characters before being converted by the C characters. Once the P characters are converted, the remaining C characters will no longer be protected by the P characters, and eventually all of the C characters will be converted by the D characters as well. In fact, it is highly possible that when the initial small “island” of D characters



**Figure 2.6:** Continued from Figure 2.5. Snapshots of cluster of characters for every 10 MCS on the  $30 \times 30$  square lattices.

expands, its border will be touching a mixture of C and P characters instead of just P or C characters since a region of C and P characters will not change much in its demographics due to the C and P having the same payoff when interacting with each other.

At  $t = 0$ , regions with uniform distributions of C, D and P will see C dominating all the character types, given a high enough punishment and cost to punish. Thus when the “island” of D expands into this region of C characters, the C will be converted. Additionally, since the mixture of C and P will not change much demographically as mentioned, even if the “island” of D expands into a neighborhood with P dominance, the presence of C, with a high enough cost to punish, will result in P being converted by C, which will in turn be dominated by D.

Finally, for an “island” of P surrounded by D, each P in the “island” has a chance of being converted if  $r > (3 - (4 - n_p)\alpha + \beta)/(5 - n_p)$  where  $n_p$  is the number of adjacent P

characters. So, for a large enough  $r$ , “islands” of P will also be eventually converted by the D characters. The above thus explains the importance of clustering of same character types on the final character distribution, a feature which is not present in the pair-approximation. Thus, pair approximation can, at best, capture the numerical results only qualitatively.

## 2.6 Conclusion

We have studied the effects of structured population in a Snowdrift game consisting of cooperative punishers. The spatial structure imposes geometrical constraint on how one agent is affected by other agents. Results of extensive numerical simulations, both for the steady state and the dynamics, are presented. Possible phases are identified and discussed, and isolated phases in the  $r - \beta$  space are identified as special local structures of strategies that are stable due to the lattice structure. In a well-mixed population, the altruistic punishers often have lower payoffs compared to the cooperators due to the cost of punishment, and as a result the punishers are suppressed as long as cooperators and defectors are present in the system. In a structured population such as that of a square lattice studied in this chapter, the altruistic punishing strategy can flourish and prevail for appropriate values of the payoff parameters. It implies an enhancement in cooperation due to the presence of the punishing mechanism as the punishing strategy is cooperative in nature. The coexistence of C, D, and P strategies is not found in a well-mixed population. In a square lattice, it is possible for the system to evolve to C + D + P phase at some particular payoff parameters. We extended the commonly used analytic approach of pair approximation from a two-strategy system to a three-strategy system. The results can, at best, capture the main features qualitatively. The theory fails to give the static AllD and AllP phases observed numerically (see Figure 2.4) and in general does not give the frequencies  $f_c$ ,  $f_d$ , and  $f_p$  accurately. Similar discrepancies were also found in applying the pair approximation to structured populations engaging in the Snowdrift game without



punishers. From the formulation of the theory, the discrepancies stem from the improper way of accounting for the spatial correlations up to at least the next-nearest-neighbors as imposed by the lattice structure. In closing, while the focus is on results in a square lattice, the approach used to understand the numerical results is general and thus can be applied to populations of different structures with different values of  $\bar{k}$ . As a final note to close this chapter, it can be concluded that punishment is very effective at promoting cooperation in an evolutionary Snowdrift game with a structured network.

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## CHAPTER 3: LONG-TERM RESEARCH AND DEVELOPMENT INCENTIVES IN A DYNAMIC COURNOT DUOPOLY

### 3.1 Introduction

In Chapter 2, we have studied Snowdrift game incorporating a costly punishing strategy in a population connected through a square lattice. It has been shown that the punishers are very effective in preventing defection if the initial population of the punishers is sufficiently large. In the studies of evolutionary games, bridging the gap between theoretical and empirical research is one of the main challenges for the studies of cooperation. Different problems have been studied in the hope of applying the findings to implement the game theory to a practical scenario, such as the well-known patent war between Apple and Samsung. Therefore, we want to study the role of punishment in a more practical economic model. In the industry, the role of the punisher in a Snowdrift game can be taken up by the patent system as it is possible to protect the profit of a cooperator or innovator via patenting. Thus, we use a more practical Cournot duopoly economic model to investigate the effect of patenting on cooperation and defection. In the Snowdrift game, a person who chooses to invest energy to shovel the snow is termed a cooperator, and in the Cournot duopoly, a person who chooses to invest in research is termed an innovator. Similarly, in the Snowdrift game, a person who chooses not to invest energy to shovel and instead hope that the other person will shovel is called a defector, while in the Cournot duopoly, the person who chooses to use the technology developed by the innovator is called a free-rider. Thus, the concept of cooperation and defection in the evolutionary Snowdrift game considered in Chapter 2 are replaced by innovation and freeriding respectively.

Before there was a patent system, anyone could copy anyone's inventions without fear of patent infringement. The inventor could choose either to share the idea with others, or keep it secret if he or she had an idea with the potential to be widely used and make a lot

of money. But sharing the idea exposes the inventor to the risk of a copycat stealing the inventor's work and profiting. By keeping the idea secret, the inventor could capitalize on it exclusively and obtain maximum benefit from his or her invention .

However, in the aggregate, if every inventor keeps his or her work secret then all inventors and society at large would suffer from a stagnant technology sector. Inventors would waste most of their time duplicating other inventors' work, working in secret to solve problems that another inventor has already (secretly) solved. As a society, it is beneficial for inventors to share their ideas with each other to allow others to build on the current state of the art instead of wasting immense amounts of time by concealing and reduplicating work. Keeping science and technology inventions secret is very undesirable for society, and for each inventor.

The patent system gives inventors who publicly disclose their work special protection to guarantee nobody copies their ideas without permission. As a result, inventors can share their work so society benefits from vibrant technology development, while the inventors' ideas are protected from copycats. Patent system seeks to provide incentives to innovate as well as to disseminate the innovation. Therefore, we also investigate whether patenting is effective in giving enough incentives to firms to innovate, or in the terminology of the Snowdrift game, to cooperate.

In this chapter, we develop a research and development (R&D) Cournot duopoly differential game played by ex-ante asymmetric firms and in which the dynamics of technological diffusion depends on the technology gap between the firms. Technology Diffusion is the process by which an innovation is communicated through certain channels over time to the members of a social system. It is a special type of communication, in that the messages are concerned with new ideas (Rogers, 2003). The Cournot Duopoly (as discussed in Chapter 1, section 1.3.6) is a classic oligopolistic market in which there

are two enterprises producing the same commodity and selling it in the same market. The Cournot duopoly game models a situation in which each firm chooses quantities (outputs) of homogeneous products simultaneously without communication and the market determines the prices at which they are sold. Before making decisions on quantities and prices, each firm has to take into account not only the current strategy of the competitor, but his forthcoming responsive actions as well. In this study, we show that in the long-run equilibrium firms have incentives to innovate as long as the knowledge externalities are bidirectional.

The plan of this chapter is as follows. Section 3.2 is the literature review. In section 3.3, the dynamic Cournot duopoly R&D model is introduced and solved. In Section 3.4, results are presented and discussed. Results are illustrated through the use of numerical simulations. A summary of the results is given in Section 3.5.

## **3.2 Literature Review**

It has been well established that when one firm independently develops a cost reducing innovation, the firm's competitors benefit in the sense that they can use the innovation to reduce their own costs. When such spillover effects are significant, noncooperative firms might be expected to research too little from the standpoint of the industry since each firm tends to ignore the positive externality which its research generates on the cost of its rival firm (see D'Aspremont & Jacquemin, 1988; Henriques, 1990; Simpson & Vonortas, 1994). However, it is also observed that when spillovers are endogenous the firm's disincentive to engage in R&D activity is partially offset. This is because its own R&D can potentially enhance its capacity to absorb its rival's technology (Grunfeld, 2003; Kamien & Zang, 2000; Katsoulacos & Ulph, 1998; Kultti & Takalo, 1998). Moreover, reduced costs of rival firms due to spillovers will lead all firms to compete more intensively in the product market. Empirical findings by Cohen and Levinthal (1989) reinforce the

fact that spillovers have two opposing effects on R&D investment in strategic games: firstly, they increase the firm's incentive to raise its own R&D and, secondly, they create a disincentive for the rival firm to invest in R&D as free riding becomes a better strategy.

D'Aspremont, Jacquemin (1988, 1990) and Kamien et al. (1992) have independently developed game theoretical models to analyze both the cooperative and noncooperative behaviors of firms to engage in R&D activities when spillovers exist. While subsequent research by Henriques (1990), Suzumura (1992), Salant and Shaffer (1998), Simpson and Vonortas (1994), Amir (2000) and many others have extended and generalized their models, very few studies have emphasized on the explicit modeling of spillovers in R&D games. The lack of attention given to the treatment of spillovers can be regarded as a lacuna in this literature as empirical works by Cohen and Levinthal (1989) and Griliches (1992) clearly point out both the complexity and importance of spillovers in R&D models. In fact, Cohen and Levinthal (1989) show that contrary to conventional wisdom, intra-industry spillovers can encourage R&D investment. Moreover, Cameron (2005) observed that as the technology gap between a leader firm and the follower firm narrows, the follower must undertake more formal R&D since its ability to freeride on the leader's R&D reduces. Hence, spillovers are not completely exogenous as assumed in the R&D game literature; they depend on the technology gap between firms. In this study, the aim is to take this relationship between spillovers and technology gap into account.

Katsoulacos and Ulph (1998) were the first to endogenize spillovers in the two stage R&D game. In contrast to previous works which considered the spillover rate as purely exogenous when comparing the cooperative case with the noncooperative regime, they focused on the impact of research joint ventures on innovative performance. The concept of endogenous spillovers was explored further by Kamien and Zang (2000) and generalized by Leahy and Neary (2007) who argued that the firm cannot capture any

spillovers from its rival without engaging in R&D itself. By incorporating absorptive capacity as a strategic variable, they distinguished between two components of spillovers; an exogenous component which represents involuntary spillovers from the firm's R&D activity and an endogenous component that allows the firm to exert control over spillovers. Our notion of spillovers is more general than the one used by these authors as it not only allows for absorptive capacity but also allows the spillover to depend on the technology gap between firms.

Our proposed framework uses the strategic interaction approach of R&D games to develop a dynamic Cournot duopoly model in which firms can invest in process innovations in an environment with imitation via knowledge spillovers. Time is assumed to be continuous and while firms still choose R&D before output as in D'Aspremont and Jacquemin (1988), a differential equation is used to describe how the spillover function (which determines the rate of technology diffusion) evolves over time. We determine whether R&D incentives can be sustained in an environment where technological innovation is almost a public good. We prove the existence of two types of asymmetric equilibria; one in which the leader maintains its technological advantage and one in which the follower catches up with the leader. We find that if the technology diffusion is bidirectional, the equilibrium where both firms invest in R&D at a constant positive rate is stable. Hence, we conclude that the imitation via knowledge spillovers does not deter innovation. While our results are similar to those by Spence (1984) and Bessen and Maskin (2009), our framework differs from theirs as we do not assume that the firms are symmetric (as in Spence, 1984), the technology diffusion rate in our model is not exogenous (as in both Bessen & Maskin, 2009; Spence, 1984) and dynamic strategic interactions with feedback effects are taken into account in our model (unlike Bessen & Maskin, 2009; Spence, 1984).

A precursor paper by Luckraz (2008) considers a similar framework as ours in the context of endogenous growth theory. Our model differs from the model in Luckraz (2008) in the following ways. First, in contrast to Luckraz (2008), here both technology catch-up and leapfrogging are allowed. Secondly, we find some important properties of the steady state equilibrium that Luckraz (2008) was unable to find and finally, unlike Luckraz (2008), we are able to draw more direct conclusions about whether imitation via knowledge spillovers can hinder R&D.

### 3.3 Model

In our model, we assume that the market structure is a duopoly. At each time  $t$ , firm 1 and firm 2 produce an identical product and compete in Cournot fashion in the product market. Our Cournot assumption comes from the fact that we are interested in modelling cost reducing innovations rather than product innovation; hence, we assume product homogeneity just like in d'Aspremont and Jacquemin (1988). The Cournot duopoly game proceeds as follows. In each time  $t$ , the two firms play a two stage Cournot game. Both firms conduct process R&D to reduce their per unit cost of production at the first stage and choose output in the second stage. Each firm's marginal cost of production evolves over time according to an equation of motion and time is assumed to be continuous. While we assume that one firm is the technology leader and the other firm is the laggard, we do not assume as in Luckraz (2008) that the leader is always more productive than the follower. In fact, we impose such a restriction only at  $t = 0$  and hence, technology catch-up and leapfrogging are allowed in this model. This is similar to the situation of the defectors in the Snowdrift game where it is possible for the defectors to have a higher payoff than the cooperators. Moreover, we assume that the technology leader also benefits from some minimal spillovers from the follower but to a lesser extent than the follower benefits from the leader.

More formally we denote time by  $t \in [0, +\infty)$  and assume that for each  $t \in [0, +\infty)$ , firm 1 and firm 2 face a demand function given by  $P_t = A/Q_t$ , where  $A$  is the unit time demand,  $P_t$  is the price function and  $Q_t = q_{1t} + q_{2t}$  is the quantity demanded with  $q_{1t}$  and  $q_{2t}$  are quantities produced by firm 1 and firm 2 respectively at time  $t$ . In order for our demand function and its corresponding welfare function to be well-defined, we need to assume that both price and quantity are bounded. In particular, we assume that there exist  $\bar{P}$  and  $\bar{Q}$  such that  $Q_t \in \left[\frac{A}{\bar{P}}, \bar{Q}\right]$  and  $P_t \in \left[\frac{A}{\bar{Q}}, \bar{P}\right]$ .

The marginal cost of production of firm  $i$  is given by  $c_{it}$  and there are no fixed costs. We assume that firms can invest in R&D to reduce their marginal cost of production. More formally, we assume that for each  $i$ ,  $c_{it}$  is given by

$$c_{it} \equiv \frac{1}{X_{it}} \quad (3.1)$$

where  $X_{it}$  is the productivity level of firm  $i$ . We assume that for each  $t$  and  $i = 1, 2$ ,  $X_{it} \in [1, +\infty)$ . The time derivative of firm  $i$ 's productivity level is given by

$$\dot{X}_{it} = \Lambda_{it}(X_{it}, X_{jt}) R_{it} \quad (3.2)$$

where  $X_{i0}$  is given,  $X_{10} > X_{20} > 1$  and  $R_{it} \in [0, +\infty)$  is the level of R&D conducted by firm  $i$  in time  $t$ . Moreover, firm 1 is the technology leader and firm 2 is the technology follower. Note that  $X_{10} > X_{20}$  implies that the leader is more productive than the follower at  $t = 0$ . We assume that the depreciation rate is zero for simplicity. We also assume that  $\Lambda_{it}(X_{it}, X_{jt}) : [1, +\infty)^2 \rightarrow \mathbb{R}_+$  is given by

$$\Lambda_{it}(X_{it}, X_{jt}) \equiv X_{it}^{1-\sigma_i} X_{jt}^{\sigma_i} \quad (3.3)$$

where  $i, j = 1, 2, i \neq j$  and  $0 < \sigma_1 \leq \sigma_2 < 1/2$  are the technology diffusion parameters.

The technology diffusion parameter,  $\sigma_i$ , plays a crucial part in our model as it reflects the extent to which technological knowledge is a public good in the model. Note that



the technology leakage is involuntary and there is imitation via knowledge spillovers for innovations. On the other hand, each firm needs to undertake some R&D on its own in order to benefit from the technology transfer. While we assume an ex-ante asymmetric setup in which the follower can always free-ride on leader at least as much as the leader can free-ride on the follower, our range of parameter values for  $\sigma_1$  and  $\sigma_2$  also allows us to consider extreme cases like  $\sigma_1 \rightarrow 0$  and  $\sigma_2 \rightarrow 1/2$ , where the laggard firm fully free-rides on the leader or  $\sigma_1 \rightarrow \sigma_2$ . The ex-ante asymmetric assumption allows us to determine whether, with the imitation via knowledge spillovers, the follower will choose a very low level of innovation in equilibrium while freeriding on the leader's R&D.  $\sigma_i$  is assumed to be less than  $1/2$  to reflect the fact that the elasticity of firm  $i$ 's productivity with respect to its own R&D is greater than the elasticity its productivity with respect to its rival's R&D. Thus, the technology diffusion process is imperfect.

Our definition of spillovers is similar to Cohen and Levinthal (1989) together with some extensions. In particular, we define spillovers to include valuable knowledge generated in the research process of the leader and which becomes accessible to the follower if and only if the latter is reverse engineering the innovator's research process. It is important here to note that empirical findings by Cohen and Levinthal (1989) state that spillovers have two opposing effects on R&D investment in strategic games: firstly, they increase the firm's incentive to raise its own R&D and, secondly, they create a disincentive for the rival firm to invest in R&D as freeriding becomes a better strategy.

In practical terms, our assumption that  $\sigma_1 \leq \sigma_2$  will imply that when an industry's market leader is surpassed by the follower, the rate of technological diffusion from the new leader (old follower) to the new follower (old leader) is smaller than that from the new follower to the new leader. The smartphone industry is a good real world example of this assumption. Recently, a Silicon Valley jury ordered Samsung Electronics to pay

Apple \$290 million for copying vital iPhone and iPad features (Canadian Press , November 2013). Despite the fact that Samsung had surpassed Apple to become the new market leader, Apple argued in court that Samsung’s Android-based phones were still copying important iPhone features.

This assumption also has a technical significance as it is important for a full characterization of asymmetry. If on the contrary we had assumed that the order reverses when leapfrogging takes place, the game would look exactly like the game played at the beginning of time (that is, at  $t = 0$ ) since the time horizon is infinite. As the system’s behaviour repeats itself after every round of leapfrogging, this leads to the outcome that the importance of leapfrogging is diminished. On the other hand, with this assumption, the game played by the two firms once leapfrogging takes place is a different game than the one played at the initial time step. This allows us to capture new dynamical behaviour after the first leapfrogging. Hence, with this assumption our model can capture richer dynamics than in a model where the order of the diffusion parameter is reversed. More formally, we have the following:

Let  $\sigma_1(G_t)$  and  $\sigma_2(G_t)$  be the technology diffusion rate of firm 1 and firm 2 respectively in differential game  $\Gamma(\sigma_1(G_t), \sigma_2(G_t))$  with  $G_t$  being the technology gap between the two firms at time  $t$  so that  $\sigma_1(G_t) \leq \sigma_2(G_t)$  if and only if  $G_t > 1$ , where  $\frac{X_{1t}}{X_{2t}} = G_t$ . Assume as before that  $G_0 > 1$ . Suppose that  $\{G_t^*\}_{t=0}^{\infty}$  is the technology gap induced by a closed-loop equilibrium, then either there exists some  $t'$  such that  $G_{t'} < 1$  and  $G_t > 1$  for  $t \in [0, t')$  or such  $t'$  does not exist. If such  $t'$  exists, then the closed-loop equilibrium induced  $\{G_t^*\}_{t=0}^{\infty}$  will be a cyclical equilibrium such that each firm will alternate between innovating and imitating behavior. Thus, the game played over interval  $[0, t')$  repeats itself ad infinitum with firms taking turns to be the leader. Such an equilibrium was studied recently by Luckraz (2013). On the other hand, if such  $t'$  does not exist, then the differen-

tial game will be one where leapfrogging never takes place and will thus be similar to the differential game studied by Luckraz (2008).

We will find it useful to define the following expressions:

$$G_t = \frac{X_{1t}}{X_{2t}}, \quad (3.4)$$

$$\dot{G}_t = \frac{X_{2t}\dot{X}_{1t} - X_{1t}\dot{X}_{2t}}{X_{2t}^2}, \quad (3.5)$$

$$\frac{\dot{X}_{it}}{X_{it}} = \alpha_{it}, \quad (3.6)$$

where  $\alpha_{it}$  is the investment rate of firm  $i$  at time  $t$  and  $\alpha_{it} \in [0, \bar{\alpha}]$  for  $i = 1, 2$ .  $\dot{G}_t$  is the derivative of technology gap between the two firms at time  $t$ . By dividing Eq.(3.5) with Eq.(3.4), we get

$$\frac{\dot{G}_t}{G_t} = \frac{X_{2t}\dot{X}_{1t} - X_{1t}\dot{X}_{2t}}{X_{2t}^2} \left( \frac{X_{2t}}{X_{1t}} \right) = \frac{\dot{X}_{1t}}{X_{1t}} - \frac{\dot{X}_{2t}}{X_{2t}} = \alpha_{1t} - \alpha_{2t}. \quad (3.7)$$

Now using Eqs.(3.2), (3.3) and (3.6), we get

$$R_{1t}(X_{1t}^{1-\sigma_1} X_{2t}^{\sigma_1}) = X_{1t}\alpha_{1t}, \quad (3.8)$$

$$R_{2t}(X_{2t}^{1-\sigma_2} X_{1t}^{\sigma_2}) = X_{2t}\alpha_{2t}. \quad (3.9)$$

By rearranging Eqs.(3.8) and (3.9), we can derive the following useful expressions:

$$R_{1t} = \frac{X_{1t}\alpha_{1t}}{X_{1t}^{1-\sigma_1} X_{2t}^{\sigma_1}} = \alpha_{1t} \frac{X_{1t}^{\sigma_1}}{X_{2t}^{\sigma_1}} = \alpha_{1t} G_t^{\sigma_1}, \quad (3.10)$$

$$R_{2t} = \frac{X_{2t}\alpha_{2t}}{X_{2t}^{1-\sigma_2} X_{1t}^{\sigma_2}} = \alpha_{2t} \frac{X_{2t}^{\sigma_2}}{X_{1t}^{\sigma_2}} = \frac{\alpha_{2t}}{G_t^{\sigma_2}}. \quad (3.11)$$

Finally, we let  $(R_{it})^2 / 2$  be the cost of R&D for each firm where  $i = 1, 2$ . It is important to note that our assumption that  $\alpha_{it}$  is non-negative together with our assumption that the depreciation rate is zero imply that the marginal cost of production,  $c_{it} \equiv \frac{1}{X_{it}}$  will never increase over time. Our justification for this assumption is the following. The marginal

cost,  $c_{it} \equiv \frac{1}{X_{it}}$  reflects the cost per unit given the state-of-the-art technology at time  $t$ . An increasing marginal cost will imply that the state-of-the-art technology gets worse over time. Since it is unlikely that technology worsens over time, our assumption rules out this possibility. The complete dynamic optimization problem is given in the following section.

### 3.3.1 Solving the Model

Firm  $i$  solves an infinite horizon dynamic optimization problem by choosing  $q_{it}$  and  $R_{it}$  to maximize its discounted sum of profits, taking the other firm's strategies ( $q_{jt}$  and  $R_{jt}$ ), the dynamics of the productivity level of each firm and the initial and terminal values of the productivity levels as given. Firm  $i$ 's dynamic optimization problem can be given as

$$V_i \equiv \max_{q_{it}, R_{it}} \int_0^{+\infty} \exp\{-\rho t\} [\pi_i(q_{it}, q_{jt}, X_{it}) - (R_{it})^2 / 2] dt$$

subject to  $\dot{X}_{it} = \Lambda_{it}(X_{it}, X_{jt}) R_{it}$ ,  $X_{i0}$  is given,  $X_{iT} \geq 0$  as  $T \rightarrow +\infty$ ,

for distinct  $i, j = 1, 2$ ,

and where  $\rho \in (0, +\infty)$  is the discount rate and  $\pi_i(q_{it}, q_{jt}, X_{it})$  is the total revenue of firm  $i$  minus its total production cost at time  $t$ . The above optimal control problem is one with two control variables and two state variables. We try to simplify the problem before solving it. First, we observe that  $q_{it}$  does not enter the equation of motion  $\dot{X}_{it} = \Lambda_{it}(X_{it}, X_{jt}) R_{it}$ . Therefore, given our two stage game assumption in each  $t$  and given that the R&D level  $R_{it}$  is chosen before output  $q_{it}$  in each  $t$ , we can find the optimal  $q_{it}$  as a function of  $X_{it}$  and  $X_{jt}$  in each  $t$  to reduce the optimal control problem to one with only one control variable, namely,  $R_{it}$ . Hence, since optimal  $q_{it} \equiv q_i^*(X_{it}, X_{jt})$  does not depend on  $t$  directly, the infinite horizon game with two stages in each  $t$  can be collapsed to an infinite horizon game with only one stage in each  $t$ . We therefore solve for  $q_i^*(X_{it}, X_{jt})$  for each  $t$ . Firm  $i$ 's profit function in any  $t$  is given by

$$\max_{q_{it}} (P_t q_{it} - c_{it} q_{it}) \text{ for } i = 1, 2 \text{ and } j \neq i,$$

where the demand equation,  $P_t = \frac{A}{q_{it} + q_{jt}}$ . The first order condition for firm  $i$  is given by

$$A (q_{it} + q_{jt}) - A q_{it} - c_{it} (q_{it} + q_{jt})^2 = 0.$$

By symmetry, we have

$$A (q_{it} + q_{jt}) - A q_{jt} - c_{jt} (q_{it} + q_{jt})^2 = 0.$$

Solving the above two equations simultaneously, we have

$$q_{it} = \frac{A c_{jt}}{(c_{it} + c_{jt})^2} \text{ for distinct } i, j = 1, 2.$$

After substituting the above in the demand equation to obtain  $P_t$ , we have

$$P_t = \frac{A}{q_{it} + q_{jt}} = \frac{A}{\frac{A c_{jt}}{(c_{it} + c_{jt})^2} + \frac{A c_{it}}{(c_{it} + c_{jt})^2}} = \frac{(c_{it} + c_{jt})^2}{(c_{it} + c_{jt})} = c_{it} + c_{jt}.$$

Hence, the profit function for firm  $i$  at time  $t$  is given by

$$\pi_i = (P_t - c_{it}) q_{it} = [(c_{it} + c_{jt}) - c_{it}] q_{it} = c_{jt} q_{it} = \frac{A c_{jt}^2}{(c_{it} + c_{jt})^2} \text{ for distinct } i, j = 1, 2.$$

Since  $c_{it} \equiv \frac{1}{X_{it}}$ , firm  $i$ 's dynamic optimization problem can be rewritten as

$$V_i \equiv \max_{q_{it}, R_{it}} \int_0^{+\infty} \exp \{-\rho t\} \left[ \frac{A \left( \frac{1}{X_{jt}} \right)^2}{\left( \frac{1}{X_{it}} + \frac{1}{X_{jt}} \right)^2} - (R_{it})^2 / 2 \right] dt$$

subject to  $\dot{X}_{it} = \Lambda_{it} (X_{it}, X_{jt}) R_{it}$ ,  $X_{i0}$  is given,  $X_{iT} \geq 0$  as  $T \rightarrow +\infty$ ,

for distinct  $i, j = 1, 2$ .

The above optimal control problem is one with one control and two state variables.

We next make use of the technology gap to simplify the problem further. Using Eq.(3.4),

we have

$$\frac{A \left( \frac{1}{\bar{X}_{jt}} \right)^2}{\left( \frac{1}{\bar{X}_{it}} + \frac{1}{\bar{X}_{jt}} \right)^2} = \begin{cases} A \left( 1 + \frac{1}{G_t} \right)^{-2} & \text{if } i = 1, j = 2 \\ A (1 + G_t)^{-2} & \text{if } i = 2, j = 1 \end{cases}$$

Finally, Eqs.(3.10) and (3.11) allow us to make a change in variable so that the control variable is now  $\alpha_{it}$ . By the same token, the R&D cost,  $(R_{it})^2 / 2$  can also be written in terms of  $\alpha_{it}$ . Consequently, using Eqs.(3.1), (3.4), (3.6), (3.7), (3.10) and (3.11), we can write down firm 1's and firm 2's objective functions as follows

$$V_1 \equiv \max_{\alpha_{1t} \in [0, \bar{\alpha}]} \int_0^{+\infty} \exp \{-\rho t\} \left[ A \left( 1 + \frac{1}{G_t} \right)^{-2} - \frac{\alpha_{1t}^2 G_t^{2\sigma_1}}{2} \right] dt \quad (3.12)$$

subject to  $\dot{G}_t = (\alpha_{1t} - \alpha_{2t}) G_t$ ,  $G_0 > 1$  is given and  $G_T \geq 0$  as  $T \rightarrow +\infty$

$$V_2 \equiv \max_{\alpha_{2t} \in [0, \bar{\alpha}]} \int_0^{+\infty} \exp \{-\rho t\} \left[ A (1 + G_t)^{-2} - \frac{\alpha_{2t}^2}{2G_t^{2\sigma_2}} \right] dt \quad (3.13)$$

subject to  $\dot{G}_t = (\alpha_{1t} - \alpha_{2t}) G_t$ ,  $G_0 > 1$  is given and  $G_T \geq 0$  as  $T \rightarrow +\infty$ .

Note that since we have already made use of Eq.(3.2) to substitute for  $R_{1t}$  in problem (3.12) and  $R_{2t}$  in problem (3.13), Eq.(3.2) does not enter as a separate constraint in Eq.(3.12) and Eq.(3.13). We assume that the two firms act noncooperatively and that the above can be represented as a continuous time dynamic game in which firm  $i$ 's control variable is given by  $\alpha_{it} \in [0, \bar{\alpha}]$  for  $i = 1, 2$ , the state variable is given by  $G_t \in (0, +\infty)$  and  $G_0 > 1$  is a given initial condition. We could have formulated the problem in such a way that  $R_{it}$  is used as firm  $i$ 's control variable. We choose to use  $\alpha_{it}$  as the control variable for mathematical convenience. Also it is common in the growth literature to use investment as the control variable and stock of knowledge as the state variable (Hayashi, 1982).

The dynamic game is said to have a steady state if and only if  $\lim_{t \rightarrow +\infty} \alpha_{it}$  exists for  $i = 1, 2$ . Denote  $\lim_{t \rightarrow +\infty} \alpha_{it}$  by  $\alpha_i$  for  $i = 1, 2$ . The dynamic game is said to have a steady state equilibrium if and only if there exists some triple  $(G^*, \alpha_1^*, \alpha_2^*)$  that solves the dynamic

system when it is at its steady state.

The  $N$ -tuple  $(\phi_1, \dots, \phi_N)$  of functions  $\phi_i : X \times \{1, \dots, N\} \rightarrow R^{m^i}$  with set of possible strategies,  $X$ , states  $\{1, \dots, N\}$  and set of possible strategies played by the  $m_{th}$  player for  $i$  time steps,  $R^{m^i}$  is called a feedback closed loop Nash equilibrium if, for each  $i \in \{1, 2, \dots, N\}$ , a rule  $u_i(\cdot)$  of the problem above exists for each player and is given by  $u_i(\cdot) = \phi_i(x(i), i)$  with  $x(i)$  is set of control variables in function of  $i$ .

In order to find a state perfect feedback Nash equilibrium, we assume that the functions  $\alpha_1(G_t) : (0, +\infty) \rightarrow [0, \bar{\alpha}]$  and  $\alpha_2(G_t) : (0, +\infty) \rightarrow [0, \bar{\alpha}]$  exist and enter firm 2's and firm 1's current value Hamiltonian equations (see Appendix A) respectively. Indeed, firm 1's and firm 2's current value Hamiltonian equations are given respectively by

$$H_1(G_t, \alpha_{1t}, \lambda_{1t}) = A \left(1 + \frac{1}{G_t}\right)^{-2} - \frac{\alpha_{1t}^2 G_t^{2\sigma_1}}{2} + \lambda_{1t} (\alpha_{1t} - \alpha_2(G_t)) G_t, \quad (3.14)$$

$$H_2(G_t, \alpha_{2t}, \lambda_{2t}) = A (1 + G_t)^{-2} - \frac{\alpha_{2t}^2}{2G_t^{2\sigma_2}} + \lambda_{2t} (\alpha_1(G_t) - \alpha_{2t}) G_t. \quad (3.15)$$

The necessary conditions for the optimal control problem for firm 1 and firm 2 are given by  $\alpha_1(G_t) = \lambda_1(G_t) G_t^{1-2\sigma_1}$  and  $\alpha_2(G_t) = -\lambda_2(G_t) G_t^{1+2\sigma_2}$  respectively where  $\lambda_i(G_t) : (0, +\infty) \rightarrow \mathbb{R}$ . Using the latter to substitute for  $\alpha_1(G_t)$  and  $\alpha_2(G_t)$ , we can derive the following adjoint equations for firm 1 and firm 2:

$$\dot{\lambda}_{1t} = \rho \lambda_{1t} - \left[ \begin{array}{c} \frac{2AG_t}{(1+G_t)^3} - \alpha_{1t}^2 \sigma_1 G_t^{2\sigma_1-1} + \lambda_{1t} \alpha_{1t} + \\ \lambda_{1t} \lambda_2(G_t) (2+2\sigma_2) G_t^{1+2\sigma_2} + \lambda_{1t} \times \frac{d\lambda_2(G_t)}{dG_t} \times G_t^{2+2\sigma_2} \end{array} \right], \quad (3.16)$$

$$\dot{\lambda}_{2t} = \rho \lambda_{2t} - \left[ \begin{array}{c} \frac{-2A}{(1+G_t)^3} + \frac{\sigma_2 \alpha_{2t}^2}{G_t^{2\sigma_2+1}} - \lambda_{2t} \alpha_{2t} + \\ \lambda_1(G_t) \lambda_{2t} (2-2\sigma_1) G_t^{1-2\sigma_1} + \lambda_{2t} \times \frac{d\lambda_1(G_t)}{dG_t} \times G_t^{2-2\sigma_1} \end{array} \right]. \quad (3.17)$$

Note that we do not need to incorporate  $\alpha_{1t}$  as a function of  $G_t$  in firm 1's Hamiltonian

function since its derivative with respect to  $G_t$  will vanish by the envelope theorem in the adjoint equation. Also,  $\alpha_{2t}$  as a function of  $G_t$  does not enter firm 2's Hamiltonian function for the same reason.

Finally, the transversality conditions for the above system are given by

$$\lim_{T \rightarrow +\infty} \exp\{-\rho T\} \lambda_{iT} = 0 \text{ and } \lim_{T \rightarrow +\infty} G_T \geq 0$$

which are equivalent to  $\lim_{T \rightarrow +\infty} \exp\{-\rho T\} \lambda_{iT} G_T = 0$  for  $i = 1, 2$ . The transversality conditions imply that the last term on the right hand side of Eq.(3.16) as well as the last term on the right hand side of Eq.(3.17) vanish as  $t \rightarrow +\infty$ . We will use Eq.(3.16) and Eq.(3.17) to prove the propositions in the following section.

Note that  $Q_t \in \left[\frac{A}{P}, \bar{Q}\right]$  and  $P_t \in \left[\frac{A}{Q}, \bar{P}\right]$  imply that  $A$  is bounded from above by  $\bar{P}\bar{Q}$  and from below by  $\frac{A^2}{PQ}$ . Since the only condition required for the Mangasarian conditions (that the Hamiltonian is jointly concave in the control and the state variables) to be satisfied is that  $A$  is bounded from above and from below,  $\bar{P}\bar{Q}$  can be chosen so that the Mangasarian conditions are satisfied.

### 3.4 Results and Discussion

In this section we present our results. Our first task will be to show that for a subset of the parameter space of the model, the closed-loop system has a stable steady state equilibrium. In this study, by steady state we mean the state in which the state variable grows at a constant rate. A special case of this definition will be a steady state where the state variable's growth rate is zero. A trivial steady state in our model will be one where  $\alpha_{it} = 0$  for  $i = 1, 2$ . If both firms conduct R&D at a positive rate in equilibrium, we would then be able to argue that the incentives to innovate and thereby long-run R&D incentives are not hindered by the imitation via knowledge spillovers. The existence results will be



given in Proposition 3.1 whereas the stability results will be given in Proposition 3.2. In order to test the robustness of this result, we determine the behavior of the firms' investment rates in the neighborhood of the steady state equilibrium in Proposition 3.3. In Proposition 3.4, a characterization of the result with respect to the diffusion rate parameter  $\sigma_i$  will also be given. Such results will help us understand how the optimal R&D behavior of the firms varies with the technology diffusion rate. Finally, we analyze the welfare implications of our results by determining the dynamics of the mark-up ratio in Proposition 3.5.

We begin by stating the following assumption:

**A1.**  $\rho \in (\sigma_i \bar{\alpha}, +\infty)$  for  $i = 1, 2$ , that is, the impatience rate of players is bounded from below by the product of the technology diffusion rate (of both firms) and the maximum feasible investment rate.

The following facts will help us understand the economic rationale behind A1. First, note that A1 implies that if  $\rho > \sigma_i \bar{\alpha}$  for  $i = 1, 2$ , it follows that  $\rho > \sigma_i \alpha_j$  for all  $i, j \in \{1, 2\}$  since  $\alpha_i \leq \bar{\alpha}$  for  $i = 1, 2$ . Second, in economic models the impatience rate is equal to the rate of interest (market rate of return on capital). Third, note that  $\sigma_i \alpha_j$  can be seen as the fraction of firm  $j$ 's investment rate that accrue to firm  $i$ . Thus, the returns on firm  $i$ 's freeriding behavior are given by the growth rate of its own cost reduction due to the R&D of firm  $j$ .

We can now give the economic justification for A1. From the above three observations, one can infer that A1 states that the rate of return on the freeriding behavior of a firm is strictly less than the market rate of return on capital. Although it possible that the rate of return on R&D exceeds the market rate of return ( $\alpha_j > \rho$ ), it can never be the case that  $\sigma_i \alpha_j > \rho$ . Therefore, there are no incentives for firms to engage purely in freeriding behavior in our model. Hence, the relevant subset of the parameter space considered in the model will be the parameter values that satisfy A1 in addition to all conditions given

and justified in the previous section.

We will show in Proposition 3.1 and Proposition 3.2 that if future payoffs are sufficiently discounted, that is, if the firms are sufficiently impatient, there will exist a stable steady state equilibrium in which the follower surpasses the leader in terms of productivity and in which both firms invest in R&D at a constant rate. The sufficiently high discount rate is needed to ensure that the technology leader has enough incentives to conduct R&D in the current state in which it is still maintaining its technological lead and thereby enjoying greater current profits although in some future state it may be surpassed by the follower due to the latter's freeriding behavior which would result in lower future profits for the leader.

First, consider the set  $\Omega = \{\sigma_1, \sigma_2: 0 < \sigma_1 \leq \sigma_2 < 1/2, 3\sigma_1 + 2\sigma_2 > 1 \text{ and } 4\sigma_1^2 + 4\sigma_2^2 + 7\sigma_1\sigma_2 < 1\}$ . Then, we let  $\bar{\alpha}_i$  be the maximum feasible investment rate of firm  $i$ .

#### 3.4.1 Proposition 3.1 (Existence)

(i) If the dynamic game is at its steady state then  $\alpha_1 = \alpha_2$  for a subset of the parameter space that satisfies A1 minus  $\Omega$ . (ii) If  $\sigma_1 + \sigma_2 > 1/2$  ( $< 1/2$ ), then there exists some  $\rho^* \in (\sigma_2 \bar{\alpha}_i, +\infty)$  such that for all  $\rho > \rho^*$  the dynamic game has a steady state equilibrium. Moreover, the equilibrium is asymmetric, that is,  $G_t < 1$  ( $> 1$ ).

**Analytical Proof of Proposition 3.1(i).** Now we present the proof for Proposition 3.1 (i).

Multiplying both sides of Eq.(3.16) and Eq.(3.17) by  $\exp\{-\rho t\}$  and with the transversality condition,  $\lim_{T \rightarrow +\infty} \exp\{-\rho T\} \lambda_{iT} G_T = 0$ , the  $(\lambda_{1t} \times \frac{d\lambda_2(G_t)}{dG_t} \times G_t^{2+2\sigma_2})$  term in Eq.(3.16) and the  $(\lambda_{2t} \times \frac{d\lambda_1(G_t)}{dG_t} \times G_t^{2-2\sigma_1})$  term in Eq.(3.17) vanish as  $t \rightarrow \infty$ . Hence,

$$\dot{\lambda}_{1t} = \rho \lambda_{1t} - \left[ \frac{2AG_t}{(1+G_t)^3} - \alpha_{1t}^2 \sigma_1 G_t^{2\sigma_1-1} + \lambda_{1t} \alpha_{1t} + \lambda_{1t} \lambda_2(G_t) (2+2\sigma_2) G_t^{1+2\sigma_2} \right], \quad (3.18)$$

$$\dot{\lambda}_{2t} = \rho \lambda_{2t} - \left[ \frac{-2A}{(1+G_t)^3} + \frac{\sigma_2 \alpha_{2t}^2}{G_t^{2\sigma_2+1}} - \lambda_{2t} \alpha_{2t} + \lambda_1(G_t) \lambda_{2t} (2-2\sigma_1) G_t^{1-2\sigma_1} \right]. \quad (3.19)$$

We then rewrite the costate variables for firm 1 and firm 2 as

$$\lambda_{1t} = \alpha_{1t} G_t^{-1+2\sigma_1}, \quad (3.20)$$

$$\lambda_{2t} = -\alpha_{2t} G_t^{-1-2\sigma_2}, \quad (3.21)$$

respectively and their derivatives as

$$\dot{\lambda}_{1t} = (2\sigma_1 - 1) \alpha_{1t} G_t^{-2+2\sigma_1} \dot{G}_t + G_t^{-1+2\sigma_1} \dot{\alpha}_{1t}, \quad (3.22)$$

$$\dot{\lambda}_{2t} = (2\sigma_2 + 1) \alpha_{2t} G_t^{-2-2\sigma_2} \dot{G}_t - G_t^{-1-2\sigma_2} \dot{\alpha}_{2t}. \quad (3.23)$$

By substituting Eqs.(3.20), (3.21), (3.22), (3.23), and (3.7) into Eq.(3.18), we have the following

$$\begin{aligned} & (2\sigma_1 - 1) \alpha_{1t} G_t^{-2+2\sigma_1} G_t (\alpha_{1t} - \alpha_{2t}) + G_t^{-1+2\sigma_1} \dot{\alpha}_{1t} \\ &= \rho \alpha_{1t} G_t^{-1+2\sigma_1} - \frac{2AG_t}{(1 + G_t)^3} + \alpha_{1t}^2 \sigma_1 G_t^{2\sigma_1-1} - \alpha_{1t}^2 G_t^{-1+2\sigma_1} \\ & \quad - \left( \alpha_{1t} G_t^{-1+2\sigma_1} \right) \left( -\alpha_{2t} G_t^{-1-2\sigma_2} \right) (2 + 2\sigma_2) G_t^{1+2\sigma_2}, \end{aligned}$$

$$(2\sigma_1 - 1) \alpha_{1t} (\alpha_{1t} - \alpha_{2t}) + \dot{\alpha}_{1t} = \rho \alpha_{1t} - \frac{2AG_t^{2-2\sigma_1}}{(1 + G_t)^3} + \alpha_{1t}^2 \sigma_1 - \alpha_{1t}^2 + \alpha_{1t} \alpha_{2t} (2 + 2\sigma_2),$$

$$\sigma_1 \alpha_{1t}^2 - (1 + 2(\sigma_1 + \sigma_2)) \alpha_{1t} \alpha_{2t} - \rho \alpha_{1t} + \dot{\alpha}_{1t} = -\frac{2AG_t^{2-2\sigma_1}}{(1 + G_t)^3}. \quad (3.24)$$

By substituting Eqs.(3.20), (3.21), (3.22), (3.23), and (3.7) into Eq.(3.19), we have the following

$$\begin{aligned} & (2\sigma_2 + 1) \alpha_{2t} G_t^{-2-2\sigma_2} G_t (\alpha_{1t} - \alpha_{2t}) - G_t^{-1-2\sigma_2} \dot{\alpha}_{2t} \\ &= \rho \left( -\alpha_{2t} G_t^{-1-2\sigma_2} \right) + \frac{2A}{(1 + G_t)^3} - \alpha_{2t}^2 \sigma_2 G_t^{-1-2\sigma_2} - \alpha_{2t}^2 G_t^{-1-2\sigma_2} \\ & \quad - \left( \alpha_{1t} G_t^{-1+2\sigma_1} \right) \left( -\alpha_{2t} G_t^{-1-2\sigma_2} \right) (2 - 2\sigma_1) G_t^{1-2\sigma_1}, \end{aligned}$$

$$(2\sigma_2 + 1) \alpha_{2t} (\alpha_{1t} - \alpha_{2t}) - \dot{\alpha}_{2t} = -\rho \alpha_{2t} + \frac{2AG_t^{1+2\sigma_2}}{(1 + G_t)^3} - \alpha_{2t}^2 \sigma_2 - \alpha_{2t}^2 + \alpha_{1t} \alpha_{2t} (2 - 2\sigma_1),$$

$$-\sigma_2\alpha_{2t}^2 - (1 - 2(\sigma_1 + \sigma_2))\alpha_{1t}\alpha_{2t} + \rho\alpha_{2t} - \dot{\alpha}_{2t} = \frac{2AG_t^{1+2\sigma_2}}{(1 + G_t)^3}. \quad (3.25)$$

Since  $\dot{\alpha}_1 = \dot{\alpha}_2 = 0$  at the steady state, we have

$$\sigma_1\alpha_1^2 - (1 + 2(\sigma_1 + \sigma_2))\alpha_1\alpha_2 - \rho\alpha_1 = -\frac{2AG_t^{2-2\sigma_1}}{(1 + G_t)^3}, \quad (3.26)$$

$$-\sigma_2\alpha_2^2 - (1 - 2(\sigma_1 + \sigma_2))\alpha_1\alpha_2 + \rho\alpha_2 = \frac{2AG_t^{1+2\sigma_2}}{(1 + G_t)^3}. \quad (3.27)$$

We prove Proposition 3.1(i) by contradiction. Suppose  $\alpha_1 \neq \alpha_2$ . Then either (i)  $\alpha_2 > \alpha_1$  or (ii)  $\alpha_2 < \alpha_1$ . Suppose (i) holds. Then from Eq.(3.7), we find that  $\lim_{t \rightarrow +\infty} \frac{\dot{G}_t}{G_t} < 0$ .

If we let  $\lim_{t \rightarrow +\infty} \frac{\dot{G}_t}{G_t} = k < 0$ , then

$$\lim_{t \rightarrow +\infty} \int \frac{dG_t}{G_t} = \lim_{t \rightarrow +\infty} \int k dt,$$

$$\lim_{t \rightarrow +\infty} \ln G_t = \lim_{t \rightarrow +\infty} kt + c, \text{ (where } c \text{ is a constant)}$$

$$\lim_{t \rightarrow +\infty} G_t = \lim_{t \rightarrow +\infty} \exp\{kt\} \cdot \exp\{c\}.$$

From the above, we conclude that  $\lim_{t \rightarrow +\infty} G_t = 0$  at the steady state since  $k < 0$ .

Consequently, Eqs.(3.26) and (3.27) reduce to the following equations at the steady state.

$$\sigma_1\alpha_1^2 - (1 + 2(\sigma_1 + \sigma_2))\alpha_1\alpha_2 = \rho\alpha_1, \quad (3.28)$$

$$-\sigma_2\alpha_2^2 - (1 - 2(\sigma_1 + \sigma_2))\alpha_1\alpha_2 = -\rho\alpha_2. \quad (3.29)$$

Thus (i) leads to Eqs.(3.28) and (3.29). Next suppose that (ii) holds. Then from Eq.(3.7),

we find that  $\lim_{t \rightarrow +\infty} \frac{\dot{G}_t}{G_t} > 0$ . Therefore since  $\lim_{t \rightarrow +\infty} G_t = \lim_{t \rightarrow +\infty} \exp\{kt\} \cdot \exp\{c\}$  still holds and

$k > 0$ , we have  $\lim_{t \rightarrow +\infty} G_t = +\infty$ . Consequently, Eqs.(3.26) and (3.27) reduce to Eqs.(3.28)

and (3.29). Hence, since both (i) and (ii) reduce to Eqs.(3.28) and (3.29), to obtain a

contradiction, we only need to show that given the parameters of the model, no pair

$(\alpha_1^*, \alpha_2^*)$  can be used to solve Eqs.(3.28) and (3.29). From Eqs.(3.28) and (3.29), we have

the following after re-arranging

$$\alpha_1 = \frac{\rho + [1 + 2(\sigma_1 + \sigma_2)]\alpha_2}{\sigma_1}, \quad (3.30)$$

$$\alpha_2 = \frac{\rho - [1 - 2(\sigma_1 + \sigma_2)]\alpha_1}{\sigma_2}. \quad (3.31)$$

Solving Eqs.(3.30) and (3.31) simultaneously, we obtained

$$\alpha_1 = \frac{(2\sigma_1 + 3\sigma_2 + 1)\rho}{\sigma_1\sigma_2 - 4(\sigma_1 + \sigma_2)^2 + 1},$$

$$\alpha_2 = \frac{(3\sigma_1 + 2\sigma_2 - 1)\rho}{\sigma_1\sigma_2 - 4(\sigma_1 + \sigma_2)^2 + 1}.$$

Suppose that  $\sigma_1, \sigma_2$  and  $\rho$  are unrestricted and that  $\alpha_1, \alpha_2$  are not bounded from above, then (3.28) and (3.29) have four possible solutions (three boundary and one interior solution) given by

$$(\alpha_1^*, \alpha_2^*) = \left\{ (0, 0), \left(0, \frac{\rho}{\sigma_2}\right), \left(\frac{\rho}{\sigma_1}, 0\right) \right\} \text{ (boundary solutions),}$$

$$(\alpha_1^*, \alpha_2^*) = \left\{ \left( \frac{(2\sigma_1 + 3\sigma_2 + 1)\rho}{\sigma_1\sigma_2 - 4(\sigma_1 + \sigma_2)^2 + 1}, \frac{(3\sigma_1 + 2\sigma_2 - 1)\rho}{\sigma_1\sigma_2 - 4(\sigma_1 + \sigma_2)^2 + 1} \right) \right\} \text{ (interior solution).}$$

Now since  $\alpha_1 \neq \alpha_2$ ,  $(0, 0)$  cannot be a solution. From A1, we have  $\rho > \sigma_2\bar{\alpha} \geq \sigma_1\bar{\alpha}$  (since  $\sigma_2 \geq \sigma_1$ ). Therefore  $\alpha_{1t} < \frac{\rho}{\sigma_1}$  and  $\alpha_{2t} < \frac{\rho}{\sigma_2}$  for all  $t$ . Hence,  $\left(0, \frac{\rho}{\sigma_2}\right)$  and  $\left(\frac{\rho}{\sigma_1}, 0\right)$  are not feasible solutions. We next consider the interior solution  $(\alpha_1^*, \alpha_2^*) = \left( \frac{(2\sigma_1 + 3\sigma_2 + 1)\rho}{\sigma_1\sigma_2 - 4(\sigma_1 + \sigma_2)^2 + 1}, \frac{(3\sigma_1 + 2\sigma_2 - 1)\rho}{\sigma_1\sigma_2 - 4(\sigma_1 + \sigma_2)^2 + 1} \right)$ . By definition of interior solutions,  $\alpha_1^* > 0$  and  $\alpha_2^* > 0$ . Now since  $\sigma_1, \sigma_2$  and  $\rho$  are all positive, the numerator of  $\alpha_1^*$  must be positive. As a result, the denominator of  $\alpha_1^*$  must be positive too. Consequently, the denominator of  $\alpha_2^*$  will be positive and hence its numerator must be positive as well. Thus, for the interior solution to make sense the following inequalities must hold

$$(3\sigma_1 + 2\sigma_2 - 1)\rho > 0,$$

$$\sigma_1\sigma_2 - 4(\sigma_1 + \sigma_2)^2 + 1 > 0,$$

for all  $\rho \in (\sigma_2\bar{\alpha}, +\infty)$ , and for all  $\sigma_1, \sigma_2 \in \left(0, \frac{1}{2}\right)$  such that  $\sigma_1 \leq \sigma_2$ . However it can be

shown that the above inequalities are satisfied if and only if  $\sigma_1$  and  $\sigma_2$  belong to the following set

$$\Omega = \{\sigma_1, \sigma_2: 0 < \sigma_1 \leq \sigma_2 < 1/2, 3\sigma_1 + 2\sigma_2 > 1 \text{ and } 4\sigma_1^2 + 4\sigma_2^2 + 7\sigma_1\sigma_2 < 1\}.$$

Hence we have a contradiction. Since  $\alpha_1 \neq \alpha_2$  cannot hold, we can say that if the dynamic game is at its steady state then  $\alpha_1 = \alpha_2$  for a subset of the parameter space that satisfies A1 minus  $\Omega$ . The proof is complete. Next, we present the proof for Proposition 3.1 (ii).

**Analytical proof of Proposition 3.1(ii).** From the above Proposition 3.1(i), we know that  $\alpha_1 = \alpha_2$  at the steady state. We can let  $\alpha_1^* = \alpha_2^* = \alpha^*$  and using the fact  $\dot{\alpha}_1 = \dot{\alpha}_2 = 0$  at the steady state, it can be verified that Eq.(3.26) and Eq.(3.27) reduce to the following equations

$$\rho\alpha^* - \frac{2AG_t^{2-2\sigma_1}}{(1+G_t)^3} + (1+\sigma_1+2\sigma_2)(\alpha^*)^2 = 0, \quad (3.32)$$

$$-\rho\alpha^* + \frac{2AG_t^{1+2\sigma_2}}{(1+G_t)^3} + (1-2\sigma_1-\sigma_2)(\alpha^*)^2 = 0. \quad (3.33)$$

First we observe that  $\alpha^* = 0$  is not a possible solution to the above system since both  $A$  and  $G_t$  are positive real numbers. Therefore,  $\alpha^* > 0$ . Solving Eq.(3.32) and Eq.(3.33) simultaneously (summing Eq.(3.32) and Eq.(3.33)) we have

$$\alpha^* = \sqrt{\left(\frac{2AG_t}{(1+G_t)^3}\right) \left(\frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2}\right)}. \quad (3.34)$$

Since from the hypothesis of the proposition we know that  $\sigma_1 + \sigma_2 > 1/2$  (or  $2\sigma_2 > 1 - 2\sigma_1$ ) and  $\alpha^*$  is a real number, the above expression makes sense if and only if  $G_t \leq 1$ . Moreover, if  $G_t = 1$ , Eq.(3.32) and Eq.(3.33) will imply that  $A = 0$  which violates our assumption that  $A \in (0, \infty)$ . Hence, if there exists some  $G_t \in (0, +\infty)$  that

solves Eq.(3.32) and Eq.(3.33), it must belong to the interval  $(0, 1)$ . Replacing Eq.(3.34) in Eq.(3.33) and after simplifying we have

$$\frac{\rho^2 (1 + G_t)^3}{2AG_t} \left( \frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2} \right) = \left( \frac{(1 - 2\sigma_1 - \sigma_2) G_t^{1-2\sigma_1} + G_t^{2\sigma_2} (1 + \sigma_1 + 2\sigma_2)}{2 - \sigma_1 + \sigma_2} \right)^2. \quad (3.35)$$

Eq.(3.35) can be reduced to

$$\frac{(2 - \sigma_1 + \sigma_2) \rho^2}{2A} \left[ \frac{1}{G_t^{2\sigma_1}} - \frac{1}{G_t^{1-2\sigma_2}} \right] = \frac{\left( (1 - 2\sigma_1 - \sigma_2) G_t^{1-2\sigma_1} + G_t^{2\sigma_2} (1 + \sigma_1 + 2\sigma_2) \right)^2}{(1 + G_t)^3}. \quad (3.36)$$

We fix  $G_t = G_L$ , where  $G_L \in (0, 1)$ . Since *RHS* of Eq.(3.36) is not equal to  $+\infty$  for all  $G_t \in (0, 1)$ , we can choose  $\rho^* \in (\sigma_2 \bar{\alpha}_i, +\infty)$  such that the *LHS* of Eq.(3.36)  $>$  *RHS* of Eq.(3.36) when  $G_t = G_L$ . Now consider the *RHS* of Eq.(3.36). Let  $R(G_t) = \frac{\left( (1-2\sigma_1-\sigma_2)G_t^{1-2\sigma_1} + G_t^{2\sigma_2}(1+\sigma_1+2\sigma_2) \right)^2}{(1+G_t)^3}$  be a function from  $(0, +\infty)$  into  $\mathbb{R}$ . It is straightforward to verify that  $R(G_t)$  is a positive real valued function. Note that  $R(G_t)$  is continuous at  $G_t = 1$  and  $R(1) > 0$ . Then, we choose the  $\delta$ -neighborhood of 1 denoted by  $N_\delta^R(1)$  (where  $\delta > 0$ ) such that for all  $G_t \in N_\delta^R(1) \cap (0, 1)$ , we have  $|R(1) - R(G_t)| < \frac{R(1)}{2}$ .

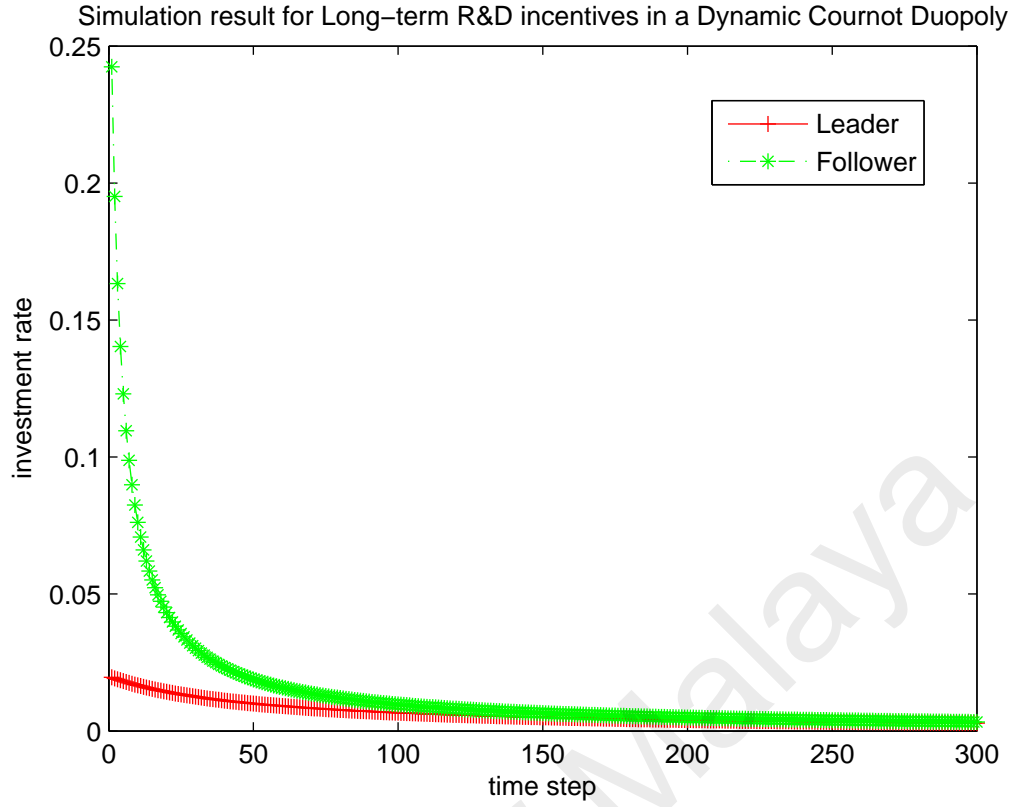
Next, let  $L(G_t) = \frac{(2-\sigma_1+\sigma_2)\rho^2}{2A} \left[ \frac{1}{G_t^{2\sigma_1}} - \frac{1}{G_t^{1-2\sigma_2}} \right]$  be a function from  $(0, +\infty)$  into  $\mathbb{R}$ . Since  $\sigma_1 + \sigma_2 > 1/2$ ,  $L(G_t)$  is positively valued for all  $G_t \in (0, 1)$ . Note that  $L(G_t)$  is continuous at  $G_t = 1$  and  $L(1) = 0$ . Then, we choose the  $\delta'$ -neighborhood of 1 denoted by  $N_{\delta'}^L(1)$  (where  $\delta' > 0$ ) such that for all  $G_t \in N_{\delta'}^L(1) \cap (0, 1)$ , we have  $|L(1) - L(G_t)| < \frac{R(1)}{2}$ . Let  $N^* = N_\delta^R(1) \cap N_{\delta'}^L(1) \cap (0, 1)$ . Pick any  $G_t \in N^*$  and denote it by  $G_U$ . This gives us

$$|L(G_U) - 0| < \frac{R(1)}{2} \text{ and}$$

$$|R(1) - R(G_U)| < \frac{R(1)}{2}.$$

The above two inequalities imply

$$L(G_U) < \frac{R(1)}{2} < R(G_U).$$



**Figure 3.1:** Investment rates of the leader,  $\alpha_{1t}$  and the follower,  $\alpha_{2t}$  versus timesteps.

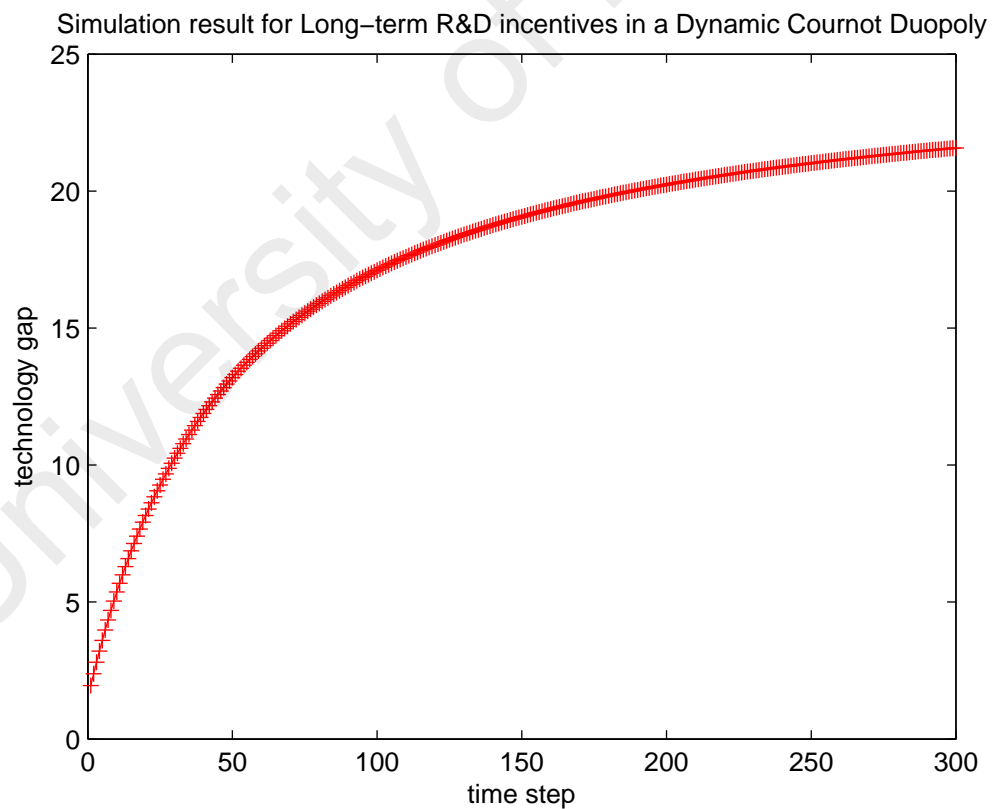
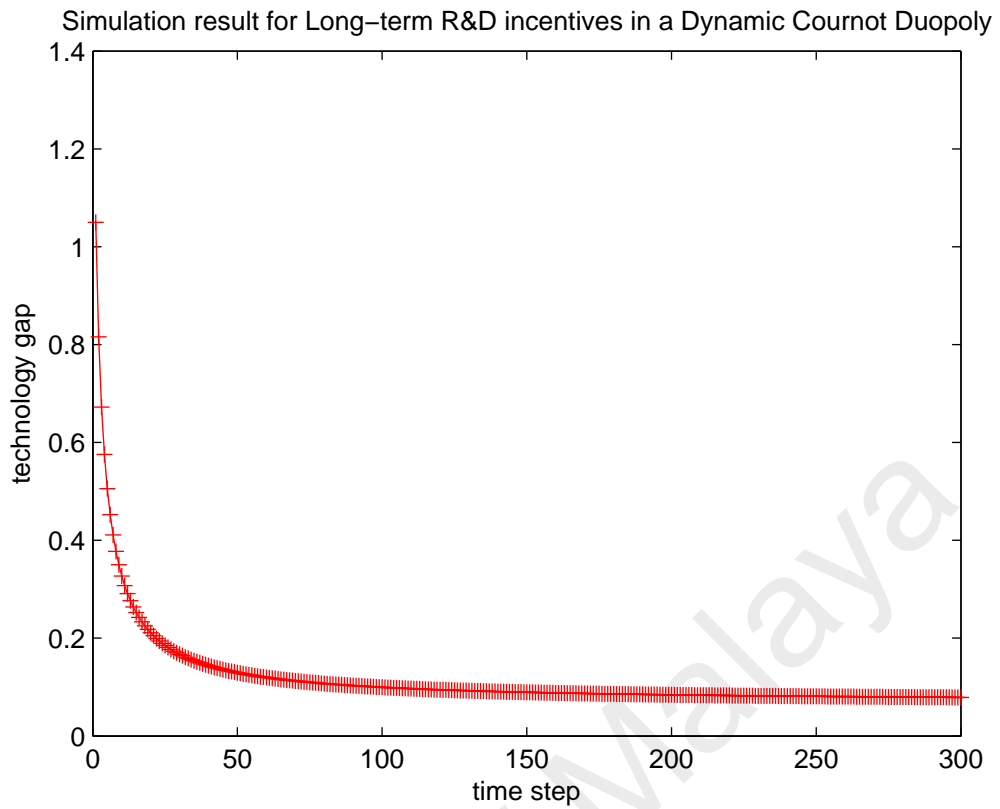
In both cases, at  $G_t = G_U$ ,  $RHS$  of Eq.(3.36)  $>$   $LHS$  of Eq.(3.36). Finally, we define a function  $F : (0, 1) \rightarrow \mathbb{R}$  by

$$F(G_t) = \frac{(2 - \sigma_1 + \sigma_2) \rho^2}{2A} \left[ \frac{1}{G_t^{2\sigma_1}} - \frac{1}{G_t^{1-2\sigma_2}} \right] - \frac{\left( (1 - 2\sigma_1 - \sigma_2) G_t^{1-2\sigma_1} + G_t^{2\sigma_2} (1 + \sigma_1 + 2\sigma_2) \right)^2}{(1 + G_t)^3}. \quad (3.37)$$

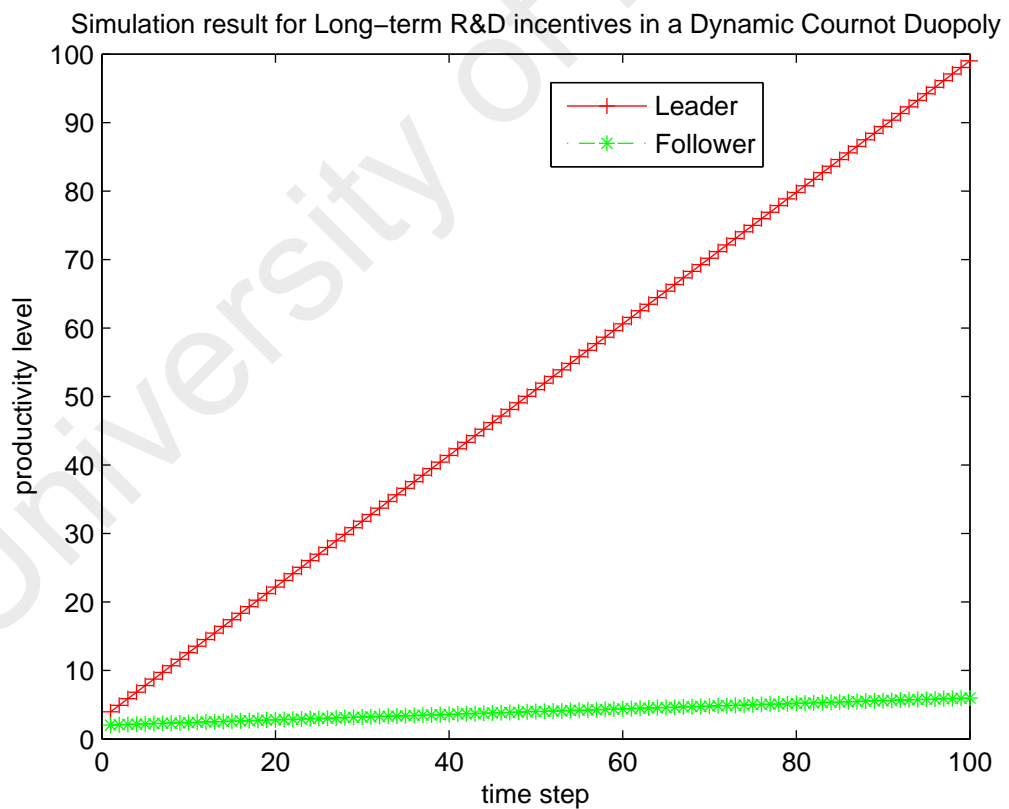
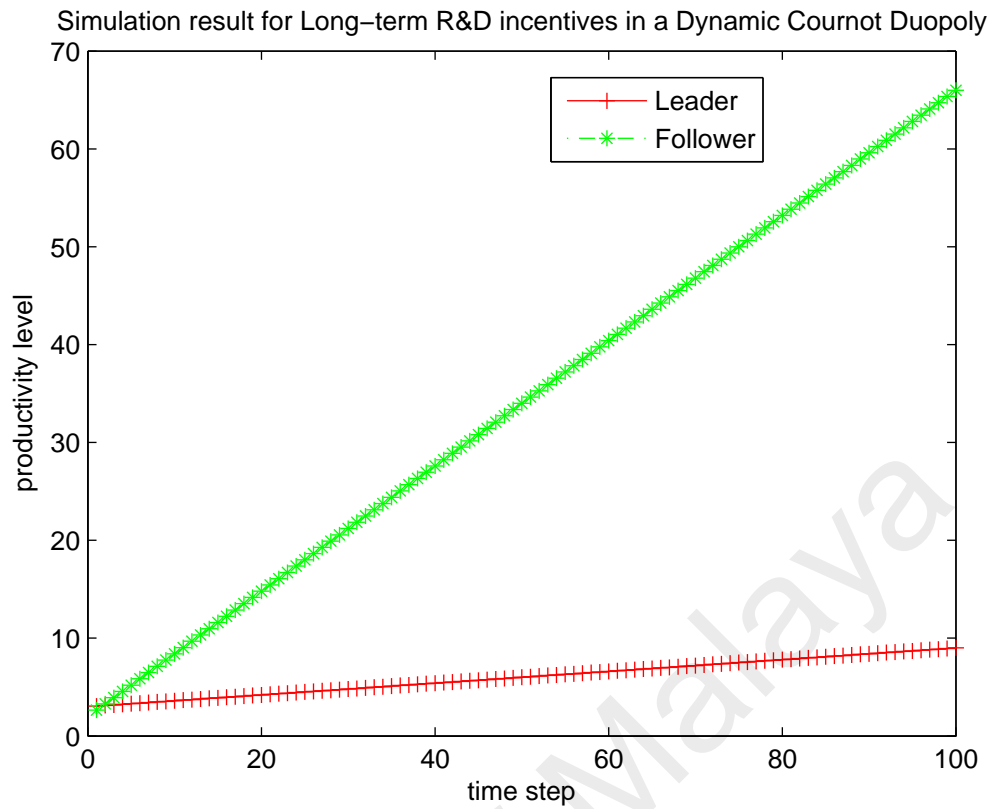
Since  $F(G_t)$  is continuous on  $(0, 1)$  and we know from the previous argument that  $F(G_U) < 0$  and  $F(G_L) > 0$ , so by the intermediate value theorem there exists  $G^*$  such that  $F(G^*) = 0$ . This proves the existence of a pair  $(G^*, \alpha^*)$  that solves the above system at the steady state. The proof for the case where  $\sigma_1 + \sigma_2 < 1/2$  is omitted as it is similar to case where  $\sigma_1 + \sigma_2 > 1/2$ . □

Proposition 3.1(i) states that if a steady state were to exist in this dynamic game, then it needs to have the symmetric investment property, that is, each firm invests in R&D at the





**Figure 3.2:** Technology gap,  $G_t$  versus timesteps for (a)  $\sigma_1 + \sigma_2 > 1/2$  and (b)  $\sigma_1 + \sigma_2 < 1/2$ .



**Figure 3.3:** Productivity levels of the leader,  $X_{1t}$  and the follower,  $X_{2t}$  versus time steps for (a)  $\sigma_1 + \sigma_2 > 1/2$  and (b)  $\sigma_1 + \sigma_2 < 1/2$ .

same rate. Therefore the equilibrium, if it exists, is symmetric in some sense. However, although each firm invests in R&D at the same rate at the steady state, their respective productivity levels are not necessarily equal, that is,  $\alpha_1 = \alpha_2$  does not always imply that  $G_t = 1$  at the steady state. Proposition 3.1(i) shows the establishment of an important long-run relationship between the investment rate of the two firms. A simulation using profit maximizing rules was run with the parameter values  $\sigma_1 = 0.45$ ,  $\sigma_2 = 0.49$ ,  $X_{10} = 3$ ;  $X_{20} = 2$ ;  $\alpha_{10} = 0.02$ ;  $\alpha_{20} = 0.32$  to test the proposition. Figure 3.1 shows that when the simulation is at its steady state,  $\alpha_1$  is equal to  $\alpha_2$ . Hence, the simulation result agrees with Proposition 3.1(i).

Part (ii) of the proposition gives the range of parameter values that guarantee the existence of an equilibrium. Figure 3.2 shows that when the simulation reaches its steady state equilibrium with (a)  $\sigma_1 = 0.45$ ,  $\sigma_2 = 0.49$  ( $\sigma_1 + \sigma_2 > 1/2$ ),  $G_t < 1$  and (b)  $\sigma_1 = 0.20$ ,  $\sigma_2 = 0.25$  ( $\sigma_1 + \sigma_2 < 1/2$ ),  $G_t > 1$ . The figure agrees with Proposition 3.1(ii). Indeed, Proposition 3.1 (ii) shows that two types of steady state equilibria may exist depending on the parameter values. In the first one, the leader maintains its technological lead over the follower and in the second one, the follower catches up with the leader. Thus, although the firms invest at the same rate in equilibrium, their productivity levels differ.

We also observe that if the rate of technology diffusion is high, that is,  $\sigma_1 + \sigma_2 > 1/2$ , then the technology laggard leapfrogs the leader whereas if the technology diffusion rate is low, that is,  $\sigma_1 + \sigma_2 < 1/2$ , the technology leader maintains its technological advantage. Thus, in order for an equilibrium where the leader maintains its technological advantage to exist, the rate of technological diffusion cannot be too high. Figures 3.3(a) and 3.3(b) agree with these observations. This just confirms our intuition.

Now, by part (i) of the proposition, we can conclude that the equilibrium in which both firms invest at a positive rate in equilibrium exists. Hence, neither the leader is discouraged

from innovating despite the lack of appropriability nor does the laggard reduce its own innovation and try to free-ride fully on the leader. The intuition behind this result is that the imitation via knowledge spillovers in our model creates a source of competitive pressure which deters the technology leader from maximizing short-run monopoly and “forces” the leader to innovate further. This type of forcing of a cooperator to cooperate further is interesting given the absence of spillover phenomenon in the Snowdrift game in the previous chapter. In order to complete our analysis, we next give the conditions for the above system to be stable.

By definition, when we say that an equilibrium point is locally stable, we mean that all solutions which begin from an initial condition close to the equilibrium point converge to the equilibrium point as time goes to infinity. An equilibrium point is said to be globally stable if all initial starting conditions lead to it.

### 3.4.2 Proposition 3.2 (Stability)

(i) If  $\sigma_1 + \sigma_2 > 1/2$ , then there exists  $\rho^{**} \in (\rho^*, +\infty)$  such that for all  $\rho > \rho^{**}$ , the dynamic game is locally stable if and only if  $G^* \in \left( \frac{2-2\sigma_1}{1+2\sigma_1}, \frac{1+2\sigma_2}{2-2\sigma_2} \right)$ .

(ii) If  $\sigma_1 + \sigma_2 > 1/2$ ,  $\sigma_1, \sigma_2 > 1/4$  and  $\rho \in (\rho^{**}, +\infty)$ , then there exists  $\sigma_1^* \in (1/4, 1/2)$  such that for all  $\sigma_1 \geq \sigma_1^*$ ,  $G^*$  is a locally stable equilibrium.

**Analytical Proof of Proposition 3.2(i).** First, we recall from Proposition 3.1(ii) that when  $\sigma_1 + \sigma_2 > 1/2$ ,  $G_t \in (0, 1)$  in equilibrium. Since  $\dot{G}_t = 0$  at the steady state from Proposition 3.1(i), the above system is stable if and only if  $\frac{d\dot{G}_t}{dG_t}(G^*) < 0$ . From Eq.(3.7), we find that  $\frac{d\dot{G}_t}{dG_t} = \alpha_{1t} - \alpha_{2t} + G_t \left( \frac{d\alpha_{1t}}{dG_t} - \frac{d\alpha_{2t}}{dG_t} \right)$ . Since from Proposition 3.1 we know that  $\alpha_{1t} - \alpha_{2t} \rightarrow 0$  as  $t \rightarrow +\infty$  at the steady state, it suffices to show that the sign of  $\left( \frac{d\alpha_{1t}}{dG_t} - \frac{d\alpha_{2t}}{dG_t} \right)$  is negative when evaluated at  $G^*$ . Taking the derivative with respect to  $G_t$  on both sides of Eqs.(3.24) and (3.25) and using the fact that  $\frac{d\alpha_{1t}}{dG_t}$  and  $\frac{d\alpha_{2t}}{dG_t}$  tend to zero at the steady state (by definition of steady state), we have the following equations

$$\begin{aligned}
& [2\sigma_1\alpha_{1t} - \rho - \alpha_{2t}(1 + 2(\sigma_1 + \sigma_2))] \frac{d\alpha_{1t}}{dG_t} - \alpha_{1t} [1 + 2(\sigma_1 + \sigma_2)] \frac{d\alpha_{2t}}{dG_t} \\
&= -\frac{d}{dG_t} \left( \frac{2AG_t^{2-2\sigma_1}}{(1+G_t)^3} \right)
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
& [-2\sigma_2\alpha_{2t} + \rho - \alpha_{1t}(1 - 2(\sigma_1 + \sigma_2))] \frac{d\alpha_{2t}}{dG_t} - \alpha_{2t} [1 - 2(\sigma_1 + \sigma_2)] \frac{d\alpha_{1t}}{dG_t} \\
&= \frac{d}{dG_t} \left( \frac{2AG_t^{1+2\sigma_2}}{(1+G_t)^3} \right).
\end{aligned} \tag{3.39}$$

We let  $q_1 \equiv 2\sigma_1\alpha_{1t} - \rho - \alpha_{2t}(1 + 2(\sigma_1 + \sigma_2))$ ,  $q_2 \equiv -\alpha_{1t}(1 + 2(\sigma_1 + \sigma_2))$ ,  $r_1 \equiv -2\sigma_2\alpha_{2t} + \rho - \alpha_{1t}(1 - 2(\sigma_1 + \sigma_2))$ ,  $r_2 \equiv -\alpha_{2t}(1 - 2(\sigma_1 + \sigma_2))$ ,  $m \equiv -\frac{d}{dG_t} \left( \frac{2AG_t^{2-2\sigma_1}}{(1+G_t)^3} \right)$  and  $n \equiv \frac{d}{dG_t} \left( \frac{2AG_t^{1+2\sigma_2}}{(1+G_t)^3} \right)$ . By using these notations, we have

$$q_1 \frac{d\alpha_{1t}}{dG_t} + q_2 \frac{d\alpha_{2t}}{dG_t} = m \tag{3.40}$$

$$r_1 \frac{d\alpha_{2t}}{dG_t} + r_2 \frac{d\alpha_{1t}}{dG_t} = n. \tag{3.41}$$

We solve Eqs. (3.40) and (3.41) simultaneously and obtain the following

$$\frac{d\alpha_{1t}}{dG_t} = \frac{q_2n - r_1m}{r_2q_2 - r_1q_1} \tag{3.42}$$

$$\frac{d\alpha_{2t}}{dG_t} = \frac{q_1n - r_2m}{r_1q_1 - r_2q_2}. \tag{3.43}$$

Recall from the above that  $\frac{dG_t}{dG_t} < 0$  if  $\left( \frac{d\alpha_{1t}}{dG_t} - \frac{d\alpha_{2t}}{dG_t} \right) < 0$ . Thus, we determine the sign of  $\left( \frac{d\alpha_{1t}}{dG_t} - \frac{d\alpha_{2t}}{dG_t} \right)$ . We observe that Eqs. (3.42) and (3.43) imply that

$$\frac{d\alpha_{1t}}{dG_t} - \frac{d\alpha_{2t}}{dG_t} = \frac{(q_1 + q_2)n - (r_1 + r_2)m}{r_2q_2 - r_1q_1}. \tag{3.44}$$

We can choose  $\rho = \hat{\rho}$  to be large enough so that  $q_1 < 0$ ,  $r_1 > 0$ ,  $|r_1q_1| > |r_2q_2|$  and  $r_1 > r_2$ . Let  $\rho^{**} = \max\{\hat{\rho}, \rho^*\}$ . Also note that  $q_2 < 0$  always holds. As a result, the denominator of Eq.(3.44) is positive. Hence,  $m, n > 0$  are sufficient conditions for the sign of  $\left( \frac{d\alpha_{1t}}{dG_t} - \frac{d\alpha_{2t}}{dG_t} \right)$  to be negative. Now,  $m > 0$  if and only if

$$-\frac{d}{dG_t} \left( \frac{2AG_t^{2-2\sigma_1}}{(1+G_t)^3} \right) > 0,$$

$$\frac{-(1+G_t)^3(2-2\sigma_1)(2AG_t^{1-2\sigma_1}) + 3(1+G_t)^2(2AG_t^{2-2\sigma_1})}{(1+G_t)^6} > 0,$$

$$-(1+G_t)(2-2\sigma_1) + 3G_t > 0,$$

$$G_t > \frac{2-2\sigma_1}{1+2\sigma_1}.$$

And that  $n > 0$  if and only if

$$\frac{d}{dG_t} \left( \frac{2AG_t^{1+2\sigma_2}}{(1+G_t)^3} \right) > 0,$$

$$\frac{(1+G_t)^3(1+2\sigma_2)(2AG_t^{2\sigma_2}) - 3(1+G_t)^2(2AG_t^{1+2\sigma_2})}{(1+G_t)^6} > 0,$$

$$(1+G_t)(1+2\sigma_2) - 3G_t > 0,$$

$$2G_t(\sigma_2 - 1) > -(1 - 2\sigma_2),$$

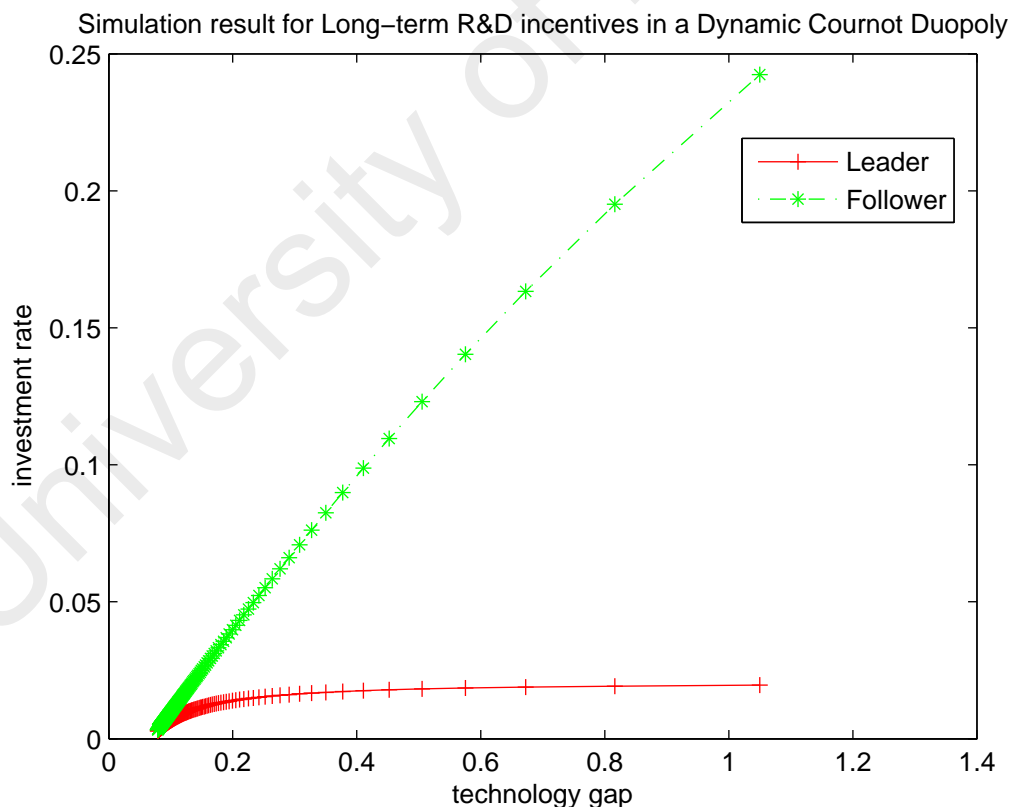
$$G_t < \frac{1+2\sigma_2}{2-2\sigma_2},$$

since  $0 < \sigma_2 < 1/2$  implies that  $\sigma_2 - 1 < 0$ . Moreover,  $\sigma_1 + \sigma_2 > 1/2$  ensures that the interval  $\left( \frac{2-2\sigma_1}{1+2\sigma_1}, \frac{1+2\sigma_2}{2-2\sigma_2} \right)$  exists. Hence, as long as  $\sigma_1 + \sigma_2 > 1/2$ , the system is stable when  $G_t \in \left( \frac{2-2\sigma_1}{1+2\sigma_1}, \frac{1+2\sigma_2}{2-2\sigma_2} \right)$ . Therefore, we can say that if  $\sigma_1 + \sigma_2 > 1/2$ , then there exists  $\rho^{**} \in (\rho^*, +\infty)$  such that for all  $\rho > \rho^{**}$  the dynamic game is locally stable if and only if  $G^* \in \left( \frac{2-2\sigma_1}{1+2\sigma_1}, \frac{1+2\sigma_2}{2-2\sigma_2} \right)$ .

**Analytical Proof of Proposition 3.2(ii).** We recall from Proposition 3.1 (ii) that  $G^* < 1$  if  $\sigma_1 + \sigma_2 > 1/2$ . Hence, we will be looking for a subset of  $(0, 1)$  that contains  $G^*$  for which the system is stable. Now since from Proposition 3.2(i) we know that the system is stable for  $G^* \in \left(\frac{2-2\sigma_1}{1+2\sigma_1}, \frac{1+2\sigma_2}{2-2\sigma_2}\right)$ , we need to show that  $\left(\frac{2-2\sigma_1}{1+2\sigma_1}, \frac{1+2\sigma_2}{2-2\sigma_2}\right) \cap (0, 1)$  is non-empty and that  $G^*$  lies in that intersection. But from the hypothesis of this proposition we know that  $\sigma_1, \sigma_2 > 1/4$ . The latter ensures that  $\frac{2-2\sigma_1}{1+2\sigma_1} < 1$  and that  $\frac{1+2\sigma_2}{2-2\sigma_2} > 1$ . Hence,  $G^*$  is stable if  $G^* \in \left(\frac{2-2\sigma_1}{1+2\sigma_1}, 1\right)$ . Thus, it suffices to show that  $G^*$  belongs to  $\left(\frac{2-2\sigma_1}{1+2\sigma_1}, 1\right)$  for some plausible range of values for the parameter  $\sigma_1$ . Since  $G^* \in (0, 1)$ , we know that there exists an interval  $(a, b)$ , where  $a > 0$  and  $b < 1$ , such that  $G^* \in (a, b)$ . Note that  $\left(\frac{2-2\sigma_1}{1+2\sigma_1}, 1\right)$  decreases to 0 for  $\sigma_1 \in (1/4, 1)$ . Hence, for an arbitrary  $G^*$  and fixing the values of  $a$  and  $b$ , we know that there exists  $\sigma_1 \in (1/4, 1)$  such that  $\left(\frac{2-2\sigma_1}{1+2\sigma_1}, 1\right) \supseteq (a, b)$ . Since the same argument works for any generic  $G^* \in (0, 1)$ , and  $\sigma_1 > 1/4$  from the hypothesis of the proposition,  $\sigma_1^*$  can always be chosen so that  $G^*$  belongs to  $\left(\frac{2-2\sigma_1}{1+2\sigma_1}, 1\right)$ . Therefore, we can say that if  $\sigma_1 + \sigma_2 > 1/2$ ,  $\sigma_1, \sigma_2 > 1/4$  and  $\rho \in (\rho^{**}, +\infty)$ , then there exists  $\sigma_1^* \in (1/4, 1/2)$  such that for all  $\sigma_1 \geq \sigma_1^*$ ,  $G^*$  is a locally stable equilibrium.  $\square$

Proposition 3.2(i) shows that if the steady state technology level  $G^*$  belongs to interval  $\left(\frac{2-2\sigma_1}{1+2\sigma_1}, \frac{1+2\sigma_2}{2-2\sigma_2}\right)$ , then it will be a locally stable equilibrium. Figure 3.4 shows that the simulation result agrees with Proposition 3.2. In Figure 3.4,  $d\alpha_{2t}/dG_t$  is greater than  $d\alpha_{1t}/dG_t$ , which agrees with the proof of Proposition 3.2 above. Proposition 3.2 (ii) shows that the equilibrium found in Proposition 3.1 (ii) indeed belongs to  $\left(\frac{2-2\sigma_1}{1+2\sigma_1}, \frac{1+2\sigma_2}{2-2\sigma_2}\right)$ , and hence is stable. Therefore, the equilibrium in which the technology laggard surpasses the leader is stable as long as the technology diffusion rate is large enough (since  $\sigma_1 + \sigma_2 > 1/2$  implies that  $G_t < 1$  at the steady state from Proposition 3.1(ii)). In fact, since  $\sigma_1^* > 1/4$ , some level of technology diffusion from the laggard to the leader also needs to take place in order for the system to be stable. Intuitively, this prevents the leader from

reducing his investment down to zero when the follower has surpassed the leader. Note that although the follower leapfrogs the leader in equilibrium, we still refer to firm 1 as the leader and firm 2 as the follower due to the ex-ante asymmetric assumption that we imposed on the initial condition  $X_{10} > X_{20} > 1$  and the parameter configuration  $\sigma_1 \leq \sigma_2$ . Proposition 3.2 (ii) also gives the sufficient conditions for a dynamic stable equilibrium in which each firm invests in innovation at a positive rate to exist. The condition we found ( $\sigma_1^* > 1/4$ ) requires that spillovers need to be bidirectional(though not symmetrically bidirectional) and it puts a constraint on how much technology diffusion is admissible in an environment with imitation via knowledge spillovers. Thus, industry level growth driven by R&D and innovations can be sustained with imitation via knowledge spillovers. Figure 3.4 shows that the stability is local and only hold at the steady state.



**Figure 3.4:** Investment rates of the leader,  $\alpha_{1t}$  and the follower,  $\alpha_{2t}$  versus technology gap,  $G_t$  for  $\sigma_1 + \sigma_2 > 1/2$ . These values are at the steady state equilibrium. The stability is local and only holds at the steady state.



At this point it will be useful to comment on the path to and from the stable steady state. Since  $G_0 > 1$ ,  $G_0$  is not stable and because  $G^* < 1$ , at least one leapfrogging must take place on the path to the steady state. This implies that for at least one  $t^\#$  we have  $\alpha_{1t^\#} - \alpha_{2t^\#} < 0$ . However, since we know from Proposition 3.1(i) that  $\alpha_{1t} - \alpha_{2t} = 0$  at the steady state, there must exist some  $\bar{t} > t^\#$  such that either  $\alpha_{1t}$  is rising or  $\alpha_{2t}$  is falling. This also suggests that the dynamics of the control variables may be complicated even on the equilibrium path. While we do not derive the paths of  $\alpha_{1t}$  and  $\alpha_{2t}$  in the transitional dynamics, we give a result that describes the economic intuition driving their behavior. The next proposition shows that the firms' R&D investment rates, that is,  $\alpha_{1t}$  and  $\alpha_{2t}$  are strategic substitutes in the neighborhood of the steady state.

### 3.4.3 Proposition 3.3 (Transitional Dynamics)

*If  $\sigma_1 + \sigma_2 > 1/2$  and  $\rho \in (\rho^{**}, +\infty)$ , then there exists a neighborhood of the steady state equilibrium such that  $\frac{d\alpha_{1t}}{d\alpha_{2t}} < 0$ .*

**Analytical Proof of Proposition 3.3.** First of all, dividing Eq.(3.42) by Eq.(3.43) gives the following expression at the steady state

$$\frac{d\alpha_{1t}}{d\alpha_{2t}} = \frac{r_1m - q_2n}{q_1n - r_2m}. \quad (3.45)$$

Recall that from the proof of Proposition 3.2(i), when  $\rho \in (\rho^{**}, +\infty)$ ,  $r_1 > 0$ ,  $q_2 < 0$  and  $q_1 < 0$ . And  $\sigma_1 + \sigma_2 > 1/2$  implies that  $r_2 > 0$ . It is straightforward to show that  $\frac{d\alpha_{1t}}{d\alpha_{2t}} < 0$  as  $r_1m - q_2n > 0$  and  $q_1n - r_2m < 0$ .  $\square$

Proposition 3.3 shows the effects of both freeriding and competition for greater market share in the neighborhood of the stable equilibrium. Indeed, we observe that if the leader increases its investment rate, the follower responds by lowering its own investment rate as it tends to free-ride more on the leader's research. On the other hand, if the leader reduces

its investment rate, the follower responds by investing more to maintain its market share. Thus, it is never the case that both firms reduce R&D investment in the equilibrium path. This result reinforces the idea that innovations and imitation via knowledge spillovers are compatible.

In order to determine how the equilibrium investment rates change with changes in the technology diffusion parameters, we give the following comparative statics result.

#### 3.4.4 Proposition 3.4 (Comparative Statics)

If  $\sigma_1 + \sigma_2 > 1/2$  and  $\rho \in (\rho^{**}, +\infty)$ , then  $\alpha^*$  (where  $\alpha^* = \alpha_1^* = \alpha_2^*$ ) is increasing in  $\sigma_1$  and decreasing in  $\sigma_2$ .

**Analytical Proof of Proposition 3.4.** From Eq.(3.34), we know that

$$\alpha^* = \sqrt{\left(\frac{2AG_t}{(1+G_t)^3}\right) \left(\frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2}\right)}.$$

Moreover,  $\sigma_1 + \sigma_2 > 1/2$  implies that  $G^* < 1$ . We find the sign of  $\frac{d\alpha^*}{d\sigma_1}$ .

$$\begin{aligned} \operatorname{sgn} \frac{d}{d\sigma_1} \left( \sqrt{\left(\frac{2AG_t}{(1+G_t)^3}\right) \left(\frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2}\right)} \right) &= \operatorname{sgn} \frac{d}{d\sigma_1} \left( \sqrt{\left(\frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2}\right)} \right) \\ &= \operatorname{sgn} \frac{d}{d\sigma_1} \left( \frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2} \right) \\ &\quad (\text{since } \sqrt{(\cdot)} > 0). \end{aligned}$$

$$\frac{d}{d\sigma_1} \left( \frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2} \right) = \frac{(2 - \sigma_1 + \sigma_2)(1 - 2\sigma_1)G_t^{-2\sigma_1} + (G_t^{1-2\sigma_1} - G_t^{2\sigma_2})}{(2 - \sigma_1 + \sigma_2)^2} > 0,$$

since  $\sigma_1 + \sigma_2 > 1/2$  and  $G_t < 1$  implies that  $G_t^{1-2\sigma_1} - G_t^{2\sigma_2} > 0$ .

We next find the sign of  $\frac{d\alpha^*}{d\sigma_2}$ .

$$\operatorname{sgn} \frac{d}{d\sigma_2} \left( \sqrt{\left(\frac{2AG_t}{(1+G_t)^3}\right) \left(\frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2}\right)} \right) = \operatorname{sgn} \frac{d}{d\sigma_2} \left( \frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2} \right).$$

$$\frac{d}{d\sigma_2} \left( \frac{G_t^{1-2\sigma_1} - G_t^{2\sigma_2}}{2 - \sigma_1 + \sigma_2} \right) = \frac{-(2 - \sigma_1 + \sigma_2) 2\sigma_2 G_t^{2\sigma_2-1} - (G_t^{1-2\sigma_1} - G_t^{2\sigma_2})}{(2 - \sigma_1 + \sigma_2)^2} < 0.$$

Therefore, we can say that if  $\sigma_1 + \sigma_2 > 1/2$  and  $\rho \in (\rho^{**}, +\infty)$ , then  $\alpha^*$  (where  $\alpha^* = \alpha_1^* = \alpha_2^*$ ) is increasing in  $\sigma_1$  and decreasing in  $\sigma_2$ . This completes the proof of Proposition 3.4.

□

Proposition 3.4 states that although the equilibrium R&D investment rate increases with the rate of technology diffusion from the follower to the leader, it decreases with the rate of technology diffusion from the leader to the follower. This result gives the same message as in the stability requirement in Proposition 3.2(ii); that in order for a stable equilibrium to exist, some diffusion from the follower to the leader also needs to take place. Since the higher the diffusion from the follower (leader) to the leader (follower), the higher (lower) the value of the equilibrium investment rate, the equilibrium investment rate decreases in the relative diffusion from the leader to the follower. Hence, the bidirectional nature of the spillovers plays an important role in our results. A higher equilibrium investment rate can be achieved only if the extent to which spillovers are bidirectional is greater.

Our final result will look at some welfare implications of the dynamic Cournot game. Our strategy will be to look at behavior of the mark-up ratio (a measure of market inefficiency) and growth rate (see Appendix B) of output.

### 3.4.5 Proposition 3.5 (Welfare)

*Assume that  $\sigma_1 + \sigma_2 > 1/2$  and  $\rho \in (\rho^{**}, +\infty)$ , then the steady state growth rate of  $\frac{P_t - c_{it}}{P_t}$  for  $i = 1, 2$  is zero, whereas the steady state growth rate of  $Q_t$  is equal to  $\alpha^*$ , where  $\alpha^* = \alpha_1^* = \alpha_2^*$ .*

**Analytical Proof of Proposition 3.5.** From Proposition 3.1(i), we know that there exists a steady state characterized by  $\alpha_1 = \alpha_2 = \alpha^*$ . We first compute the growth rate of  $\frac{P_t - c_{it}}{P_t}$ .

For a variable  $y_t$ , denote the growth rate of  $y_t$  by  $g(y_t)$ . Note that  $\frac{P_t - c_{it}}{P_t} = \frac{c_{jt}}{c_{it} + c_{jt}}$  (since  $P_t = c_{it} + c_{jt}$ ). Let  $i = 1$  and  $j = 2$ , and by substitution  $c_{it} = \frac{1}{X_{1t}}$  and  $G_t = \frac{X_{1t}}{X_{2t}}$  from Eqs. (3.1) and (3.4) respectively we have

$$\begin{aligned} \frac{c_{2t}}{c_{1t} + c_{2t}} &= \frac{\frac{1}{X_{2t}}}{\frac{1}{X_{1t}} + \frac{1}{X_{2t}}} \\ &= \frac{X_{1t}}{X_{1t} + X_{2t}} \\ &= \frac{1}{1 + \frac{1}{G_t}}. \end{aligned}$$

Now, the growth rate of  $\frac{c_{2t}}{c_{1t} + c_{2t}}$  is given by the following

$$\begin{aligned} g\left(\frac{c_{2t}}{c_{1t} + c_{2t}}\right) &= g\left(\frac{1}{1 + \frac{1}{G_t}}\right) \\ &= g(1) - g\left(1 + \frac{1}{G_t}\right) \\ &= 0 - g\left(\frac{1}{G_t}\right) \\ &= 0 - (0 - g(G_t)) \\ &= 0 \text{ (since } g(G_t) = 0 \text{ at the steady state).} \end{aligned}$$

We then compute the growth rate of  $Q_t$ . Note that  $Q_t = \frac{A}{c_{it} + c_{jt}}$  (as  $P_t = \frac{A}{Q_t}$ ). Then we have

$$\begin{aligned} g\left(\frac{A}{c_{it} + c_{jt}}\right) &= g(A) - g(c_{it} + c_{jt}) \\ &= 0 - g\left(\frac{X_{jt} + X_{it}}{X_{it}X_{jt}}\right) \\ &= 0 - [\alpha^* - (\alpha^* + \alpha^*)] \\ &= \alpha^*, \end{aligned}$$

where we have made use of the fact that  $g\left(\frac{X_{it} + X_{jt}}{X_{it}X_{jt}}\right) = \alpha^*$ . Since  $X_{it}$  grows at  $\alpha^*$  for each  $i$ , we must have  $\dot{X}_{it} = X_{it}\alpha^*$  (from Eq.(3.6)) and hence,  $\frac{\dot{X}_{it} + \dot{X}_{jt}}{X_{it} + X_{jt}} = \frac{(X_{it} + X_{jt})\alpha^*}{X_{it} + X_{jt}} = \alpha^*$ .  $\square$

Proposition 3.5 shows that while the growth rate of the inefficiency measure tends to zero at the steady state, the growth rate of output is positive and hence sustainable. Thus, although the duopolists conduct R&D perpetually in order to increase their profits, the price mark-up is not increasing at the steady state.

### **3.5 Conclusion**

In this chapter, we proposed a framework to model R&D incentives when both strategic interactions and process innovations with technology diffusion (due to a lack of intellectual property (IP) protection) are considered. While on the one hand growth models that consider strategic interactions tend to downplay the role of R&D spillovers, on the other hand, R&D games studied in the Industrial Organization literature are rarely presented in a continuous time framework and the relation between technology diffusion and the dynamics of the technology gap is not considered. We showed that the presence of bidirectional asymmetric spillovers does not necessarily deter the firms (both leader and follower) from investing at a positive rate. The economic rationale for this observation is that the positive effect of technology diffusion on innovation due to competition (firms fighting for greater market share) is greater than the negative effect due to freeriding. Our results suggest that policy makers and regulators should reconsider the issues relating to imitation and innovation.

As a final note to close this chapter, it can be concluded that innovation or cooperation in an R&D Cournot duopoly can be sustained amongst both innovators and free-riders given a high enough technology diffusion rate, and that the free-riders still need to innovate or cooperate to overtake the innovators or cooperators in terms of productivity.

However, what if the two firms are no longer the only dominant players in the market, and face competition from new players? Will the current features still appear when oligopolism no longer exist? How will the choice to switch between innovation and

freeriding change the dynamics of the game? These questions will be addressed in the next chapter.

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## CHAPTER 4: TECHNOLOGY DIFFUSION, INVESTMENT RATE AND PRODUCTIVITY IN AN EVOLUTIONARY WELL-MIXED N-PERSON RESEARCH AND DEVELOPMENT MODEL

### 4.1 Introduction

In Chapter 3, we have studied the role of punishment in a more practical Cournot duopoly economic model where the role of the punisher in Snowdrift game has been taken up by the patent system. We have developed a research and development (R&D) Cournot duopoly differential game played by ex-ante asymmetric firms with the dynamics of technological diffusion depending on the technology gap between the firms. It was found that if the technology diffusion is bidirectional, both firms invest in R&D at a constant positive rate, and that imitation via knowledge spillovers does not deter innovation. The economic rationale observed is that the positive effect of technology diffusion on innovation (absence of patent system) due to competition (firms fighting for greater market share) is greater than the negative effect due to freeriding. In particular, it showed that patent system in a two player game has not much impact on innovation. In other words, it means that the urge for high productivity prevails over that for freeriding, as in the case of the evolutionary Snowdrift model where players change their strategies based on the performance of their strategies.

Since Cournot models (Touffut, 2007) are commonly used for economic studies, the price-demand relationship is an essential part of the modelling process. Other ingredients that are commonly studied in economics include technology gap, technology diffusion rates, productivity etc. In Chapter 3, the main feature studied in the developed Cournot duopoly model was the investment-technology diffusion-technology gap relationship. The investment rate is the most important feature in the model as it measures both the innovation intensity and productivity of an industry. Although this model was successful in

showing the effect of patenting on innovation, the large number of factors in this model makes it hard to identify precisely the factors that directly affect the investment rate. Therefore, in this chapter, we modified the Cournot duopoly economic model of the research and development industry to allow a better understanding of the underlying factors directly influencing the investment rate of the players and the technological leapfrogging. Technology diffusion rate  $\sigma$  and technology gap  $G$  are the two parameters hypothesized as the underlying factors. Furthermore, the original model only involves two players without adaptive behaviors. By using an evolutionary well-mixed N-player setting, it would be interesting to see how the introduction of the evolutionary feature affects the overall dynamics of the model. Thus, we incorporate the hypothesized two parameters, namely, the technology diffusion rate and technology gap, into an evolutionary well-mixed N-player model, with all of the other parameters from the original model removed.

We also developed a numerical simulation for the model, which can be extended to a lattice model for future studies. A set of differential equations based on the replicator dynamics is used to study the behaviour observed in the simulation result. The replicator dynamics equation was solved by using iterative methods. The technological leapfrogging observed in the original work in Chapter 3 was reproduced in this current evolutionary well-mixed N-player model, thereby verifying the hypothesis that the underlying factors directly influencing the investment rate of the players are the technology diffusion rate and technology gap. In addition, our results show that for  $G \gg 1$  the cutoff value of the technology diffusion rate of the imitator for the leapfrogging to occur is dependant on the investment rate of the population. In particular, the results also show that the investment rate must be sufficiently high for the leapfrogging to occur.

The plan of this chapter is as follows. Section 4.2 is the literature review. Section 4.3 introduces the model in detail. Both analytical and simulation results are presented and



discussed in Section 4.4. Results shown in Chapter 3 are reproduced, together with other interesting results. This is followed by conclusions in Section 4.5.

## 4.2 Literature Review

Nowadays, it has become common for game theory to be presented as the most efficient tool for duopoly market research; although such opinion obviously lacks critical evaluation of game theory's practical application. The game theory approach is usually limited to designing a complicated multi-variable mathematical model, which leads to purely theoretical conclusions based on multiple assumptions, without deepening into possibilities to apply the model in practice. Many authors aim to fill up this niche, suggesting opinion on practical application of game theory in duopoly market research (e.g. Romualdas & Algirdas, 2008).

There are many game theory models commonly applied to analyze duopoly markets (e.g. Romualdas & Algirdas, 2008). The models are employed to estimate market equilibrium, to evaluate gains and losses of each market player and the efficiency of equilibrium at the industry level. The Prisoner's Dilemma model is applied to a hypothetical market entrance game with possible side payments by reformulating the classic model (Skinner & Chamberlin, 2001; Rasmusen, 2006) and using the payoff matrix to show the possible combinations of players' actions and expected payoffs (profits). The problem is solved easily with the iterated dominance technique (Romualdas & Algirdas, 2008). The technique of applying theoretical models to hypothetic market situations accompanied by non-complicated mathematical calculations is used. Therefore, motivated by this work, we simplify the R&D Cournot duopoly model in Chapter 3 to an extent which allows us to study it using an evolutionary well-mixed N-player setting and solve it using non-complicated mathematical methods. The simplified model can also be extended to a lattice model or other complex networks for future studies. Such other networks may

include scale-free, small-world and random networks.

In the developed Cournot duopoly model in Chapter 3, the investment rate is the most important feature in the model as it measures both the innovation intensity and productivity of an industry. It is hard to identify precisely the factors that directly affect the investment rate due to the large number of factors in the model. Given the effects of technology diffusion rate and technological gap on the investment rate in the model, there is a possibility that these two factors will have a direct impact on the investment rates, even in the absence of the price and demand factors. In this way, the possibility of the price, demand and other factors affecting the investment rate is removed. With this removal, the model is simplified to an extent which allows us to create a payoff matrix which focuses on the technology diffusion rates of the innovators and the imitators as well as the technology gap. With the created payoff matrix, we can extend our study by using other network settings like square lattice in Chapter 2 in future works.

In Chapter 3, we have developed and solved an R&D Cournot duopoly model to investigate the effect of patenting on cooperation and defection. The model is very successful in giving insights on whether patenting is effective in giving enough incentives to firms to innovate. In particular, it showed that patenting in a two player game has not much impact on innovation. Hence, in this chapter we study the effect of patenting by using a N-player setting. N-player setting has been widely used in game theory models like the famous Prisoner's Dilemma and Snowdrift game as it represents a real world social interaction. The N-person Snowdrift game resembles the division of labor in group projects in the real world (Chan et al., 2008). It is usually the case that there are free-riders who do not participate much in the work and yet get the credits. Unless no one is willing to take part, the whole group will earn credits from the completion of the project by the work of some active individual(s). It is similar to our R&D Cournot duopoly model where

the imitator try to free-ride on the innovator's R&D.

There has been much discussion on the evolutionary metaphor in economics in recent years (Dixon et al., 2002). In biology, successful species or genes tend to become more common because they give rise to more progeny. In the context of social evolution, mechanisms of propagation might also be present: successful firms grow and diversify, their managers circulate, good firms take over bad firms, unsuccessful firms go bust. However, in social evolution there is also the mechanism of imitation: firms tend to imitate the more successful practices of other firms. There is also learning: firms will receive signals from the capital market and elsewhere about how they are performing relative to other firms (this will lead less successful firm types to adapt their behavior). Models with evolutionary updating of the fractions of agents have recently been proposed by many authors in economic and financial models (Hommes and Brock, 1997; Droste et al., 2002; Hommes et al., 2011; Hommes, 2013; Anufriev et al., 2013). In particular, Droste et al. (2002) consider an evolutionary Cournot duopoly with homogeneous goods, linear demand and quadratic production costs. Pairs of firms, each with its own behavioral rule, are randomly matched at every time period to play the game. Hommes et al. (2011) consider a similar evolutionary setup with linear demand and linear production costs but with random matching of  $N$ -firms at a time, which can switch, on the basis of past performances, between costly rational and cheap boundedly rational expectation rules on aggregate output of their rivals. Therefore, motivated by these works, we introduce the evolutionary feature in our simplified Cournot duopoly R&D model. We include evolutionary traits where a player may, with a certain probability, change to a different character class if his payoff is less than a randomly chosen player. The payoff of each strategy plays the central role in determining the frequency of the corresponding population class. The interaction of the  $N$  players will be based on the payoff matrix which focuses on

the technology diffusion rates of the innovators and the imitators as well as the technology gap. The  $N$  players are randomly matched at a time.

In modifying the Cournot duopoly model in Chapter 3 to an evolutionary  $N$ -player setting, one of the goals is to determine the factors causing the leapfrogging. In particular, it is expected that the results in Chapter 3 will be reproduced if the hypothesis that the factors that cause leapfrogging are technology diffusion rate and technology gap is true. This newly developed research and development model also serves to study the effect of the evolutionary feature of an  $N$ -player game on innovation or cooperation amongst the players.

### 4.3 Model

Due to the large number of variables and parameters in the previous model, any investigation of the effect of patenting on an  $N$ -player game becomes difficult. Thus, we propose a simplified model where only the most important features from the said Cournot game (Fehr & Gächter, 2000) are included. In a standard cooperator-defector game, the payoff matrix is given by

$$\begin{array}{c}
 I \quad F \\
 \begin{array}{c}
 I \\
 F
 \end{array}
 \begin{pmatrix}
 R & S \\
 T & P
 \end{pmatrix}.
 \end{array} \tag{4.1}$$

Consider two firms competing in a game. Firm 1 is an innovator ( $I$ ) and invented a product. He then filed for a patent to protect his idea. The profit, taken as the amount of revenue collected from the sale of the product, is denoted by  $b$ . The technology diffusion rate reflects the extent to which the technology knowledge is a public good. The technology leakage is involuntary. If a firm needs a product produced, it needs to either undertake some innovation on its own or to “steal” it from an innovator.  $\sigma_1$  and

$\sigma_2$  are the technology diffusion rates for the innovator ( $I$ ) and imitator ( $F$ ) respectively, and  $0 < \sigma_1 < \sigma_2 < 1/2$ . The technology leakage of the innovator is always greater than that of the imitator. If the innovator shares this project with another innovator, the resultant profit each player receives is given by the total profit minus the cost to carry out research, divided by two. The difference in the technology diffusion rate is equivalent to the cost,  $c$ , to do research, since the amount of technology stolen is equal in monetary value to the research cost. Thus,  $c$  can be replaced by  $\sigma_2 - \sigma_1$ . Since the technological gap  $G$  is beneficial to the innovator in an innovator-imitator interaction, if the innovator is competing with an imitator, the profit the innovator receives is given by the sum of the profit from sales and the technological gap minus the cost for the research. Since the technological gap causes the imitator to lose the benefit, if an imitator is competing with an innovator, the net profit he gets is given by the profit from sales plus the difference in the technology diffusion rates between the two players minus the technological gap. If an imitator meets another imitator, neither of them innovates, so no product is produced and the net profit is zero. Below is the payoff matrix of the new Cournot duopoly game:

$$\begin{array}{cc}
 & \begin{array}{c} I \\ F \end{array} \\
 \begin{array}{c} I \\ F \end{array} & \begin{pmatrix} \frac{b - (\sigma_2 - \sigma_1)}{2} & b - (\sigma_2 - \sigma_1) + G \\ b + (\sigma_2 - \sigma_1) - G & 0 \end{pmatrix}
 \end{array} \quad (4.2)$$

This payoff matrix follows the economics of the Cournot model (Fehr & Gächter, 2000). As discussed in Chapter 3, if  $\sigma_1 + \sigma_2 > 1/2$ , the imitator leapfrogs the innovator, and if  $\sigma_1 + \sigma_2 < 1/2$ , the innovator leads the imitator. Moreover, for both  $\sigma_1 + \sigma_2 > 1/2$  and  $\sigma_1 + \sigma_2 < 1/2$ , there exists a steady state equilibrium which is asymmetric, that is,  $\sigma_1 + \sigma_2 > 1/2$  corresponds to  $G < 1$ , and  $\sigma_1 + \sigma_2 < 1/2$  corresponds to  $G > 1$ . To reproduce the result that the imitator has enough incentive to leapfrog the innovator, we

would need  $\sigma_2 - \sigma_1 > G$  and  $b > 0$ . From the asymmetric property, this requirement can be satisfied. Thus, a suitable configuration for the payoff ordering is

$$b + (\sigma_2 - \sigma_1) - G > b - (\sigma_2 - \sigma_1) + G > \frac{b - (\sigma_2 - \sigma_1)}{2} > 0 \text{ and } T > S > R > P.$$

Different behaviours may emerge from this ordering of payoffs compared to the Prisoner's Dilemma and Snowdrift games, and it would be interesting to investigate whether innovation can emerge from this ordering.

We next make use of the technology gap to simplify the payoff matrix further. In the previous work, we have the steady state investment rate

$$\alpha^* = \sqrt{\frac{2AG(G^{1-2\sigma_1} - G^{2\sigma_2})}{(1+G)^3(2 - \sigma_1 + \sigma_2)}}.$$

Note that at steady state, the investment rates of both players are the same (Smith & Price, 1973). We let  $\alpha^* = \alpha$  and assume that  $\sigma_1 \rightarrow 0$ , where the laggard firm fully free-rides on the leader, therefore giving

$$\sigma_2 = \frac{2AG(G - G^{2\sigma_2})}{\alpha^2(1+G)^3} - 2. \quad (4.3)$$

At steady state,  $\alpha$  and  $A$  are constant, hence we let  $2A/\alpha^2 = K$ , giving

$$\sigma_2 = KG \frac{G - G^{2\sigma_2}}{(1+G)^3} - 2. \quad (4.4)$$

Eq.(4.6) reflects the dependance of the technological diffusion rate between any two players on the technological gap between them.

From Proposition 3.1 (ii), two types of steady-state equilibria may exist depending on the parameter values. In the first one, the leader maintains its technological lead over the

follower and in the second one, the follower catches up with the leader. In addition, if the rate of technology diffusion is high, that is,  $\sigma_1 + \sigma_2 > 1/2$ , then the technology laggard leapfrogs the leader whereas if the technology diffusion rate is low, that is,  $\sigma_1 + \sigma_2 < 1/2$ , the technology leader maintains its technological advantage. Therefore, two cases are considered.

**Case 1:**  $G \gg 1$ . Technology gap between two firms are large. Since  $K$  is a constant, we can conveniently assume unity for its value. We have

$$\sigma_2 = \frac{G - G^{2\sigma_2}}{G^2},$$

$$G^2 \sigma_2 = G - G^{2\sigma_2}.$$

From our study in Chapter 3, for  $G \gg 1$ ,  $\sigma_1 + \sigma_2 < 1/2$ . Hence when the technology diffusion rate is very low,

$$G^2 \sigma_2 - G + 1 = 0. \quad (4.5)$$

By completing the square,

$$G = \frac{1 \pm \sqrt{1 - 4\sigma_2}}{2\sigma_2}, \quad \sigma_2 < \frac{1}{4}. \quad (4.6)$$

To satisfy  $G \gg 1$  for all values of  $\sigma_2 < 1/4$ , Eq.(4.6) has to take the form

$$G = \frac{1 + \sqrt{1 - 4\sigma_2}}{2\sigma_2}, \quad \sigma_2 < \frac{1}{4}. \quad (4.7)$$

This shows that the technology gap between them will be large if the laggard firm's technology diffusion rate is low. Thus, the payoff matrix now becomes

$$\begin{array}{cc} & \begin{array}{c} I \\ F \end{array} \\ \begin{array}{c} I \\ F \end{array} & \begin{pmatrix} \frac{b - \sigma_2}{2} & b - \sigma_2 + \frac{1 + \sqrt{1 - 4\sigma_2}}{2\sigma_2} \\ b + \sigma_2 - \frac{1 + \sqrt{1 - 4\sigma_2}}{2\sigma_2} & 0 \end{pmatrix}, \quad \sigma_2 < \frac{1}{4}. \end{array} \quad (4.8)$$

To simplify the model for more detailed analysis, we introduce a parameter  $r$  and by substituting  $b = 1 + r$  and  $\sigma_2 = 2r$ , giving the following payoff matrix:

$$\begin{array}{c} \\ \\ I \\ F \end{array} \begin{array}{cc} I & F \\ \left( \begin{array}{cc} \frac{1-r}{2} & 1-r + \frac{1+\sqrt{1-8r}}{4r} \\ 1+3r - \frac{1+\sqrt{1-8r}}{4r} & 0 \end{array} \right), & r < \frac{1}{8}. \end{array} \quad (4.9)$$

**Case 2:**  $G \ll 1$ . Technology gap between two firms are small. We have

$$\sigma_2 = KG(G - G^{2\sigma_2}) - 2.$$

From our study in Chapter 3, for  $G \ll 1$ ,  $\sigma_1 + \sigma_2 > 1/2$ . Hence when the technology diffusion rate is very high ( $\sigma_2 \rightarrow 1/2$ ),

$$\begin{aligned} \sigma_2 &= KG(G - 0) - 2, \\ G &= \pm \sqrt{\frac{2 + \sigma_2}{K}}, \quad \sigma_2 > -2. \end{aligned} \quad (4.10)$$

Given that  $\sigma_2 > 0$ , therefore for  $G > 0$ , we have

$$G = \sqrt{\frac{2 + \sigma_2}{K}}, \quad \sigma_2 \in \left(0, \frac{1}{2}\right). \quad (4.11)$$

Thus, the payoff matrix becomes

$$\begin{array}{c} \\ \\ I \\ F \end{array} \begin{array}{cc} I & F \\ \left( \begin{array}{cc} \frac{b - \sigma_2}{2} & b - \sigma_2 + \sqrt{\frac{2 + \sigma_2}{K}} \\ b + \sigma_2 - \sqrt{\frac{2 + \sigma_2}{K}} & 0 \end{array} \right), & \sigma_2 \in \left(0, \frac{1}{2}\right). \end{array} \quad (4.12)$$

Same as in case 1 above, by substituting  $b = 1 + r$  and  $\sigma_2 = 2r$ , we have the following



payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} I & F \end{array} \\ \begin{array}{c} I \\ F \end{array} & \left( \begin{array}{cc} \frac{1-r}{2} & 1-r + \sqrt{\frac{2+2r}{K}} \\ 1+3r - \sqrt{\frac{2+2r}{K}} & 0 \end{array} \right), \quad r > 0. \end{array} \quad (4.13)$$

Note that in Chapter 3, the proof of proposition 3.1(ii) says that:

(i)  $A \in (0, \infty)$  and  $\alpha \in (0, \bar{\alpha})$

(ii) Eq.(4.3) makes sense if and only if  $G > 1$  (for the case  $\sigma_1 + \sigma_2 < 1/2$ ) or  $G < 1$  (for the case  $\sigma_1 + \sigma_2 > 1/2$ ).

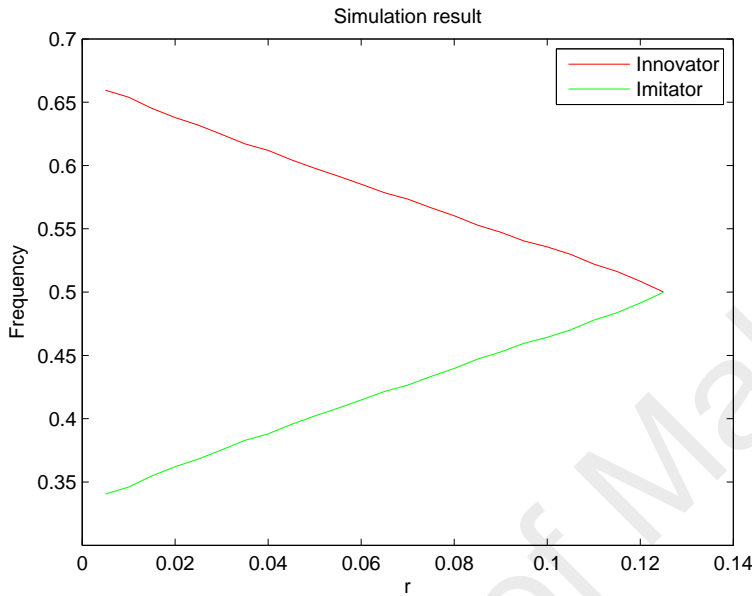
Hence,  $K = 2A/\alpha^2 \in (0, \infty)$ .

## 4.4 Results

### 4.4.1 Simulation Results

To simulate the model, we consider a system of  $N$  ( $N \gg 1$ ) players. At the initial time step, there is a chosen distribution of innovators and imitators. The system evolves like this: at each subsequent time step, a target player  $i$  is randomly chosen to compete with another player randomly chosen from the population. This gives a payoff  $V_i$  to the target player. Then, at the same time step, a referencing player  $j$  is chosen and compete with a randomly selected player in the same way as the target player, and obtain a payoff  $V_j$ . The target player  $i$  then compares his payoff with the referencing player  $j$  for a possible switch in character. If  $V_j > V_i$ , then the target player switches his character to that of his referencing player with probability  $P = \frac{V_j - V_i}{D}$ , where the denominator,  $D$ , is the largest element in the payoff matrix to make sure that the probability  $P$  is less than unity. Else if  $V_j < V_i$ , the target player retains his current character. The time step ends after player  $i$  makes an attempt to update his character, and after all players, on average, have updated their characters; this time interval is referred to as an evolutionary time step. As in previous

studies on stock market dynamics, such as the herding model (Rodgers & Yap, 2002), we can assume that the productivity (or return) of a class of players is directly proportional to the size of the corresponding class. The result for case  $G \gg 1$  is shown in Figure 4.1, and the result for case  $G \ll 1$  is shown in Figure 4.2.



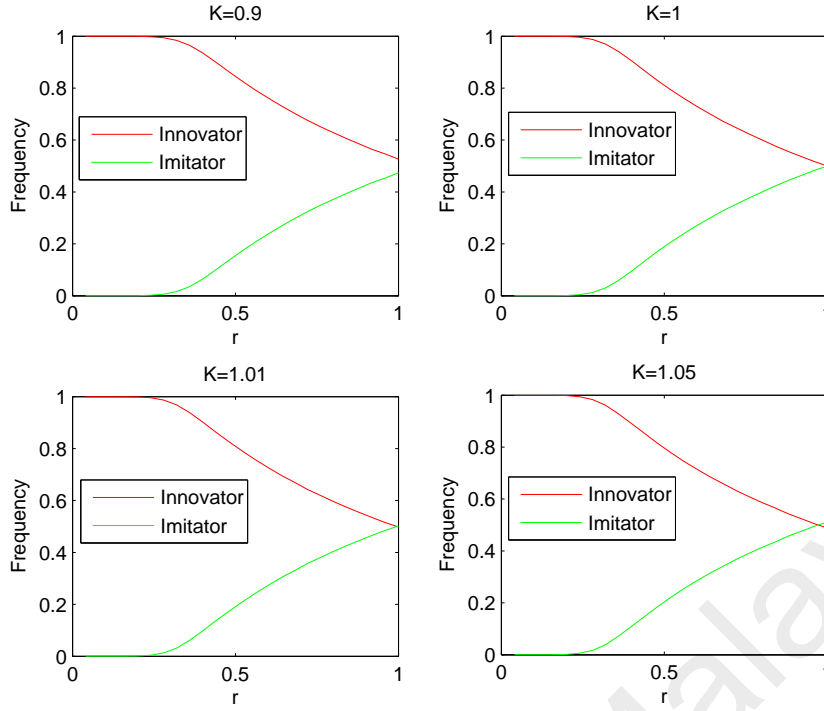
**Figure 4.1:** Simulation result for case  $G \gg 1$ .

#### 4.4.2 Analytical Results

##### 4.4.2.1 Solving of the Replicator Dynamics Equations Using Iterations

In a well-mixed population, a set of differential equations based on the replicator dynamics (Hofbauer & Sigmund, 1998) can be used to study the behaviour observed in Figure 4.1 and Figure 4.2. The basic idea is that if the instantaneous fitness  $F_i(t)$  ( $i = I, F$ ) of a character is higher (lower) than the instantaneous average fitness  $\bar{F}(t)$  in the population, then the frequency of that character will grow (drop). Thus, the time evolution of the frequency of each character,  $f_i$  ( $i = I, F$ ) is governed by the following differential equations:

$$\frac{df_i}{dt} = f_i(t)(F_i(t) - \bar{F}(t)). \quad (4.14)$$



**Figure 4.2:** Simulation result for case  $G \ll 1$ .

Based on the payoff matrices (Eq.(4.9)and Eq.(4.13)), the fitness of each character is given by:

For case 1:  $G \gg 1$ ,

$$F_I(t) = f_I(t) \left( \frac{1-r}{2} \right) + f_F(t) \left( 1-r + \frac{1 + \sqrt{1-8r}}{4r} \right), \quad (4.15)$$

$$F_F(t) = f_I(t) \left( 1+3r - \frac{1 + \sqrt{1-8r}}{4r} \right) + f_F(t)(0). \quad (4.16)$$

For case 2:  $G \ll 1$ ,

$$F_I(t) = f_I(t) \left( \frac{1-r}{2} \right) + f_F(t) \left( 1-r + \sqrt{\frac{2+2r}{K}} \right), \quad (4.17)$$

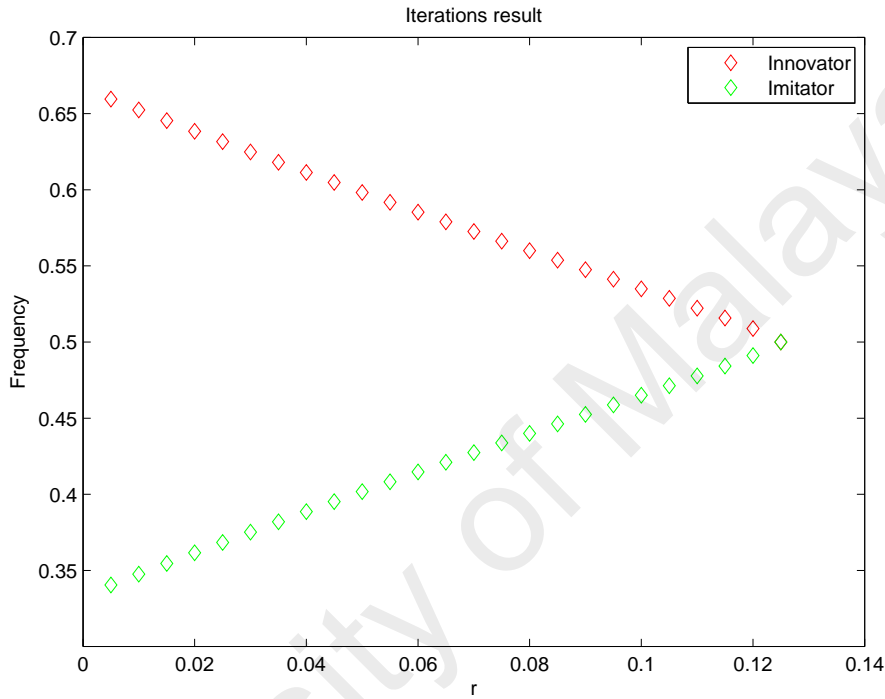
$$F_F(t) = f_I(t) \left( 1+3r - \sqrt{\frac{2+2r}{K}} \right) + f_F(t)(0). \quad (4.18)$$

Since the probability of switching to character  $i$  is  $f_i(t)$ , the instantaneous average fitness

$\bar{F}(t)$  is given by

$$\bar{F}(t) = f_I(t)F_I(t) + f_F(t)F_F(t). \quad (4.19)$$

With a uniform initial distribution of frequencies, as in the figures showing the simulation results, Eqs.(4.15) to (4.18) can be iterated in time to obtain the long time limit of  $f_i$  ( $i = I, F$ ). Figure 4.3 and Figure 4.4 show the steady-state frequencies as obtained by iterating the equations in Eq.(4.14) to convergence for case  $G \gg 1$  and  $G \ll 1$  respectively.



**Figure 4.3:** Iterations result for case  $G \gg 1$ .

#### 4.4.2.2 Solving of the Replicator Dynamics Equations Using Newton's Method

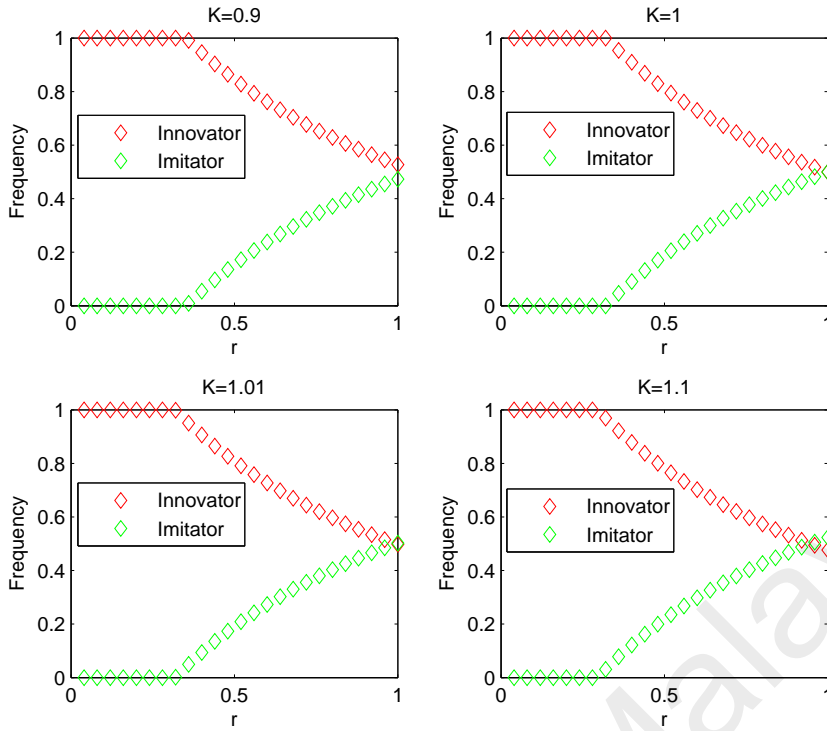
The time evolution of the frequency of each character,  $f_i$  ( $i = I, F$ ), is governed by the differential equations:

$$\frac{df_i}{dt} = f_i(t)(F_i(t) - \bar{F}(t)). \quad (4.20)$$

The fitness of each character  $i$  is given by

$$F_I(t) = f_I(t)P_{II} + f_F(t)P_{IF}, \quad (4.21)$$

$$F_F(t) = f_I(t)P_{FI} + f_F(t)P_{FF}, \quad (4.22)$$



**Figure 4.4:** Iterations result for case  $G \ll 1$ .

where the payoff matrix is

$$\begin{pmatrix} P_{II} & P_{IF} \\ P_{FI} & P_{FF} \end{pmatrix} = \begin{pmatrix} \frac{1-r}{2} & 1-r+G \\ 1+3r-G & 0 \end{pmatrix}. \quad (4.23)$$

The instantaneous average fitness is given by

$$\bar{F}(t) = f_I(t)F_I(t) + f_F(t)F_F(t).$$

For character  $I$ ,

$$\begin{aligned} \frac{df_I}{dt} &= f_I(t)[F_I(t) - \bar{F}(t)] \\ &= f_I(t)[f_I(t)P_{II} + f_F(t)P_{IF} - (f_I(t)F_I(t) + f_F(t)F_F(t))] \\ &= f_I(t)[f_I(t)P_{II} + f_F(t)P_{IF} - f_I(t)(f_I(t)P_{II} + f_F(t)P_{IF}) \\ &\quad - f_F(t)(f_I(t)P_{FI} + f_F(t)P_{FF})] \end{aligned} \quad (4.24)$$

$$\begin{aligned}
&= f_I(t) \left[ f_I(t) \left( \frac{1-r}{2} \right) + f_F(t)(1-r+G) - f_I(t) \left( f_I(t) \left( \frac{1-r}{2} \right) \right. \right. \\
&\quad \left. \left. + f_F(t)(1-r+G) \right) - f_F(t) (f_I(t)(1+3r-G) + f_F(t)(0)) \right] \\
&= f_I(t) \left[ f_I(t) \left( \frac{1-r}{2} \right) + f_F(t)(1-r+G) - f_I^2(t) \left( \frac{1-r}{2} \right) \right. \\
&\quad \left. - f_I(t)f_F(t)(2+2r) \right]. \tag{4.25}
\end{aligned}$$

When  $\frac{df_I}{dt} = 0$ ,

$$\begin{aligned}
&f_I(t) \left[ f_I(t) \left( \frac{1-r}{2} \right) + f_F(t)(1-r+G) - f_I^2(t) \left( \frac{1-r}{2} \right) \right. \\
&\quad \left. - f_I(t)f_F(t)(2+2r) \right] = 0. \tag{4.26}
\end{aligned}$$

Since  $f_I(t) \neq 0$ , we have

$$f_I(t) \left( \frac{1-r}{2} \right) + f_F(t)(1-r+G) - f_I^2(t) \left( \frac{1-r}{2} \right) - f_I(t)f_F(t)(2+2r) = 0. \tag{4.27}$$

For character  $F$ ,

$$\begin{aligned}
\frac{df_F}{dt} &= f_F(t)[F_F(t) - \bar{F}(t)] \\
&= f_F(t)[f_I(t)P_{FI} + f_F(t)P_{FF} - (f_I(t)F_I(t) + f_F(t)F_F(t))] \\
&= f_F(t)[f_I(t)P_{FI} + f_F(t)P_{FF} - f_I(t)(f_I(t)P_{II} + f_F(t)P_{IF}) \\
&\quad - f_F(t)(f_I(t)P_{FI} + f_F(t)P_{FF})] \\
&= f_F(t) \left[ f_I(t)(1+3r-G) + f_F(t)(0) - f_I(t) \left( f_I(t) \left( \frac{1-r}{2} \right) \right. \right. \\
&\quad \left. \left. + f_F(t)(1-r+G) \right) - f_F(t) (f_I(t)(1+3r-G) + f_F(t)(0)) \right] \\
&= f_F(t) \left[ f_I(t)(1+3r-G) - f_I^2(t) \left( \frac{1-r}{2} \right) - f_I(t)f_F(t)(2+2r) \right]. \tag{4.28}
\end{aligned}$$

When  $\frac{df_F}{dt} = 0$ ,

$$f_F(t) \left[ f_I(t)(1 + 3r - G) - f_I^2(t) \left( \frac{1-r}{2} \right) - f_I(t)f_F(t)(2 + 2r) \right] = 0. \quad (4.29)$$

Since  $f_F(t) \neq 0$ , we have

$$f_I(t)(1 + 3r - G) - f_I^2(t) \left( \frac{1-r}{2} \right) - f_I(t)f_F(t)(2 + 2r) = 0. \quad (4.30)$$

Let  $a = \frac{1-r}{2}$ ,  $b = 1 - r + G$ ,  $c = 2 + 2r$ , and  $d = 1 + 3r - G$ , and substitute into Eqs.(4.27) and (4.30). We obtain

$$-af_I^2(t) + af_I(t) + bf_F(t) - cf_I(t)f_F(t) = 0 \quad (4.31)$$

and

$$-af_I^2(t) + df_I(t) - cf_I(t)f_F(t) = 0. \quad (4.32)$$

We solve Eqs.(4.31) and (4.32) simultaneously by using the Newton's method,

$$\begin{pmatrix} f_{I(n+1)} \\ f_{F(n+1)} \end{pmatrix} = \begin{pmatrix} f_{I(n)} \\ f_{F(n)} \end{pmatrix} - J(f_{I(n)}, f_{F(n)})^{-1} \begin{pmatrix} M(f_{I(n)}(t), f_{F(n)}(t)) \\ N(f_{I(n)}(t), f_{F(n)}(t)) \end{pmatrix}, \quad (4.33)$$

where

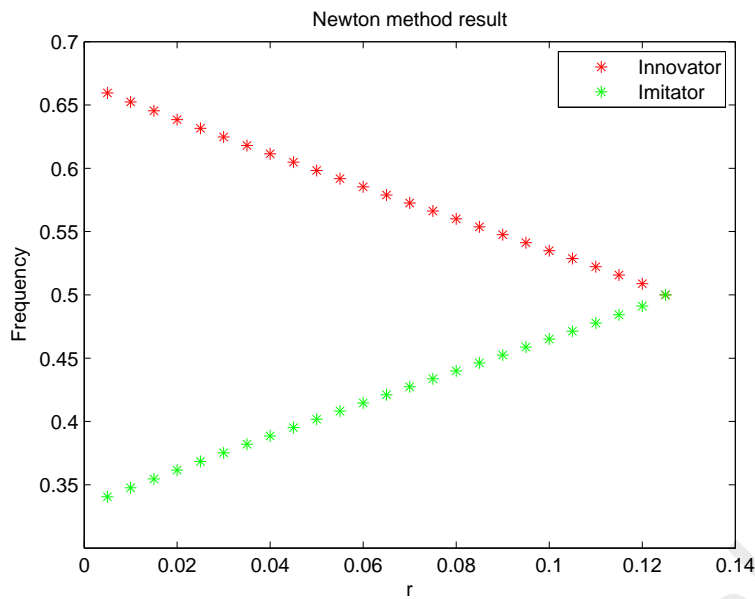
$$\begin{aligned} M(f_{I(n)}(t), f_{F(n)}(t)) &= -af_{I(n)}^2(t) + af_{I(n)}(t) + bf_{F(n)}(t) \\ &\quad - cf_{I(n)}(t)f_{F(n)}(t), \end{aligned} \quad (4.34)$$

$$N(f_{I(n)}(t), f_{F(n)}(t)) = -af_{I(n)}^2(t) + df_{I(n)}(t) - cf_{I(n)}(t)f_{F(n)}(t), \quad (4.35)$$

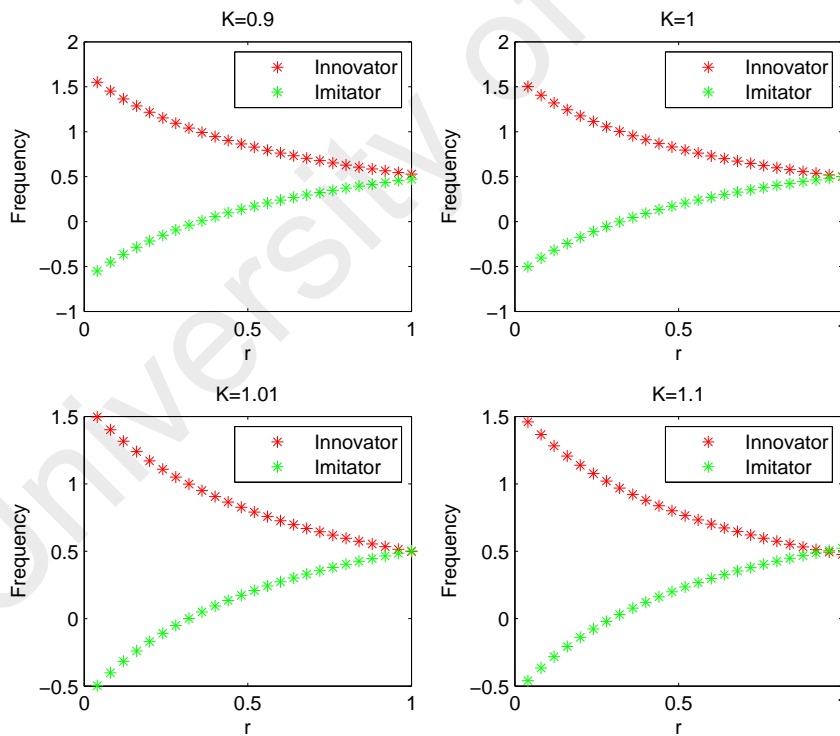
$$J(f_{I(n)}, f_{F(n)}) = \begin{pmatrix} \frac{\partial M}{\partial f_{I(n)}} & \frac{\partial M}{\partial f_{F(n)}} \\ \frac{\partial N}{\partial f_{I(n)}} & \frac{\partial N}{\partial f_{F(n)}} \end{pmatrix}. \quad (4.36)$$

The result for case  $G \gg 1$  is shown in Figure 4.5, and the result for case  $G \ll 1$  is shown

in Figure 4.6.



**Figure 4.5:** Newton's method result for case  $G \gg 1$ .

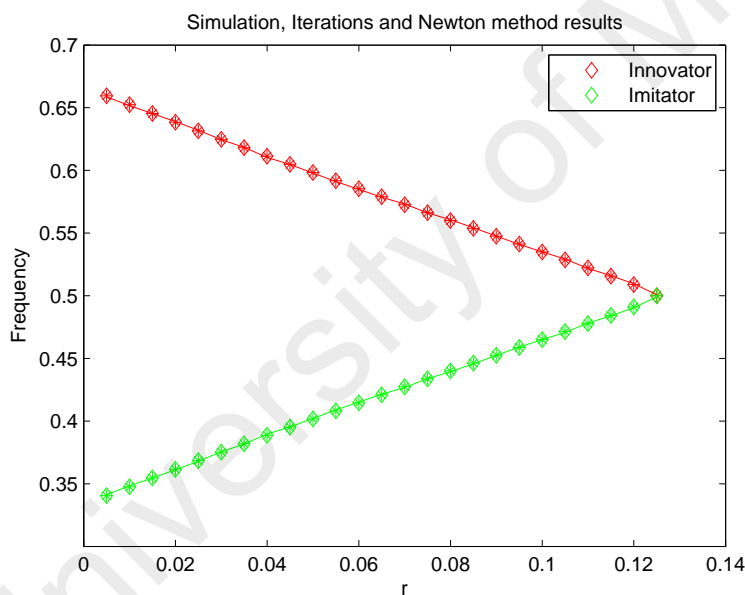


**Figure 4.6:** Newton's method result for case  $G \ll 1$ .

We have compared the simulation results and analytical results in Figures 4.7 and 4.8 for  $G \gg 1$  and  $G \ll 1$  respectively. It can be seen from Figures 4.7 and 4.8 that the

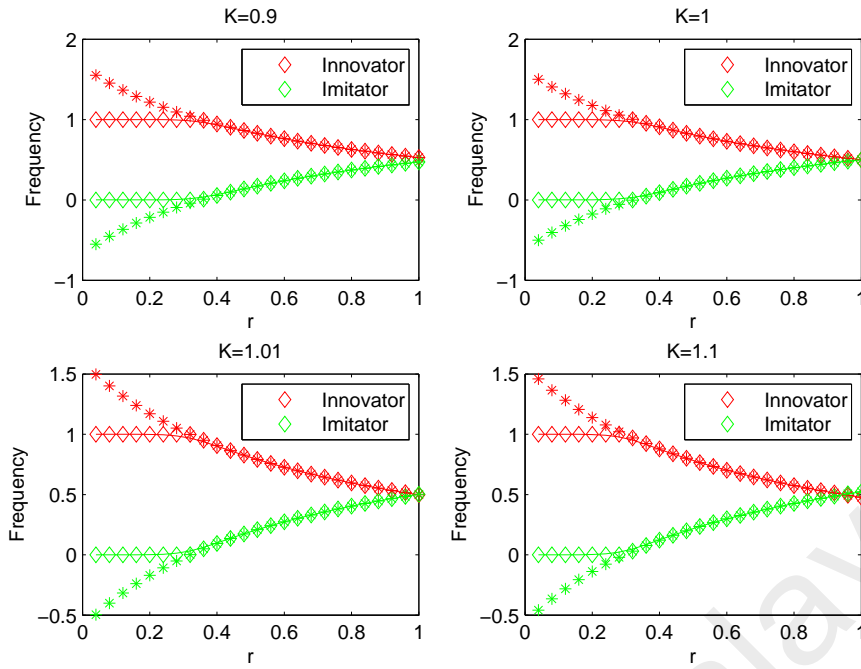


simulation results agree with both the iteration results and the Newton's method results. In the previous study, it was shown that no leapfrogging occurs for  $G \gg 1$ , and leapfrogging only occurs when  $G \ll 1$ . It can be seen from Figure 4.7 that the innovator dominates the imitator for  $G \gg 1$ . This is to say that the imitator can never leapfrog the innovator for this case. It can be seen from Figure 4.8 that the imitator can leapfrog the innovator when the rate of technology diffusion is high and  $G \ll 1$ . Thus, narrowing of the technology gap is required for leapfrogging to occur. We have thus reproduced the results from the previous study in Chapter 3 using an evolutionary N-player well-mixed setting, without the inclusion of price, demand and other factors. This also indicates that the leapfrogging feature is only dependent on the technology diffusion and technology gap.



**Figure 4.7:** Comparison of results obtained by numerical simulation, iteration and Newton's method for case  $G \gg 1$ .

In addition, from Figure 4.8, this work has shown that, for this modified R&D Cournot duopoly model, the cut-off value of the technology diffusion rate of the imitator for the leapfrogging to occur depends on the investment rate of the population. In particular, the cutoff value of  $r$  for the leapfrogging to happen decreases with the increase in the value of  $K$ . The results from Figure 4.8 also show that there is a cutoff in the investment rate



**Figure 4.8:** Comparison of results obtained by numerical simulation, iteration and Newton's method for case  $G \ll 1$ .

of the population,  $K = 2A/\alpha^2 > 1$ , for the leapfrog to occur; the investment rate of the population must be sufficiently high for the imitator to overtake the innovator in terms of research productivity. This also shows that freeriding does not deter the innovation incentives of the firms. Despite freeriding on the innovator, the imitator needs to undertake more innovation to narrow the technology gap before the leapfrogging can be performed by the imitator. To maintain its leadership, further innovation is needed by the innovator. Therefore freeriding contributes to a certain degree of cooperation as both firms take initiatives to improve the technology. In physical terms, this translates to the intuition that too high an investment intensity in the population implies lower technology diffusion. Note that in reality, we have to assume that  $\sigma_1$  is closer to zero than  $\sigma_2$  is closer to 0.5, e.g.  $\sigma_1 = 0.02$  and  $\sigma_2 = 0.499$ . But in this modified version, since  $\sigma_1$  is assumed to be so close to zero that it is assumed to be exactly zero,  $\sigma_2$  must then be greater than 0.5 for  $\sigma_1 + \sigma_2 > 1/2$ . We can look at these two versions of the same system this way: the former is concerned with individual values of  $\sigma_1$  and  $\sigma_2$ , and the latter is with  $\sigma_1 + \sigma_2$ , that is, the

net diffusion (because it is the net diffusion that affects the latter model, individual values of  $\sigma_1$  and  $\sigma_2$  do not affect the results). But the two models still agree on this fact: that if the diffusion rate is high enough, leapfrog will occur, albeit at different cut-offs.

#### 4.4.3 Special Consideration: Consequences of Adopting $r$

In the previous sections, we have shown a presence of leapfrogging for  $\sigma_1 + \sigma_2 > 1/2$  and  $G < 1$ , and an absence of leapfrogging for  $\sigma_1 + \sigma_2 < 1/2$  and  $G > 1$ . In both cases, there exist a constraint for  $r$  since  $\sigma_2 = 2r$ . The parameter  $r$  introduced in this work is used to simplify the model for more detailed analysis. However, there is a need to study the effect of introducing this new parameter on the model as a whole. Thus, it will be useful to investigate any implications of having the constraint  $0 < r < 1/4$ .

By substituting  $\sigma_2 = 2r$  into  $\sigma_2 = \frac{2AG(G-G^2\sigma_2)}{\alpha^2(1+G)^3} - 2$  (from Eq.(4.3)) and letting  $K = 2A/\alpha^2$ , we have

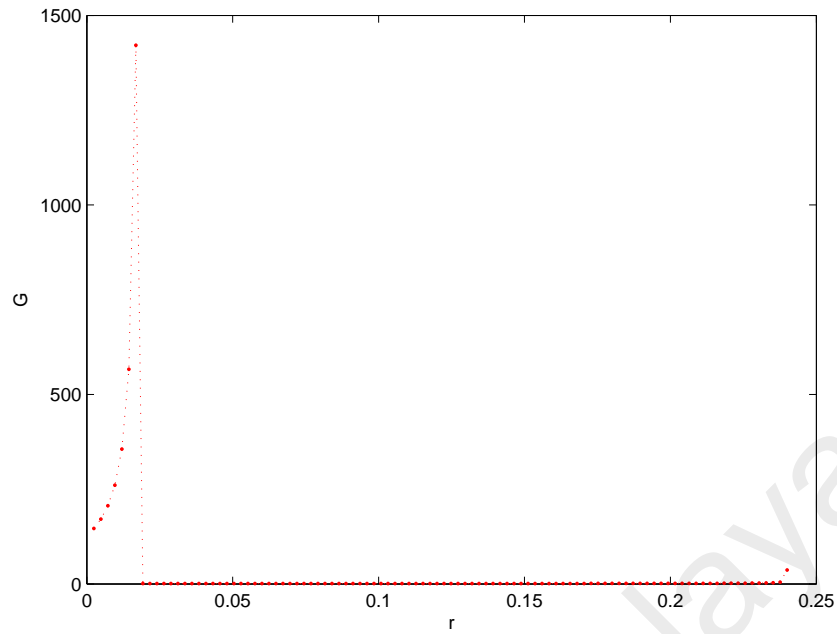
$$(1 + G)^3(2 + 2r) + K(G^{4r+1} - G^2) = 0.$$

Let

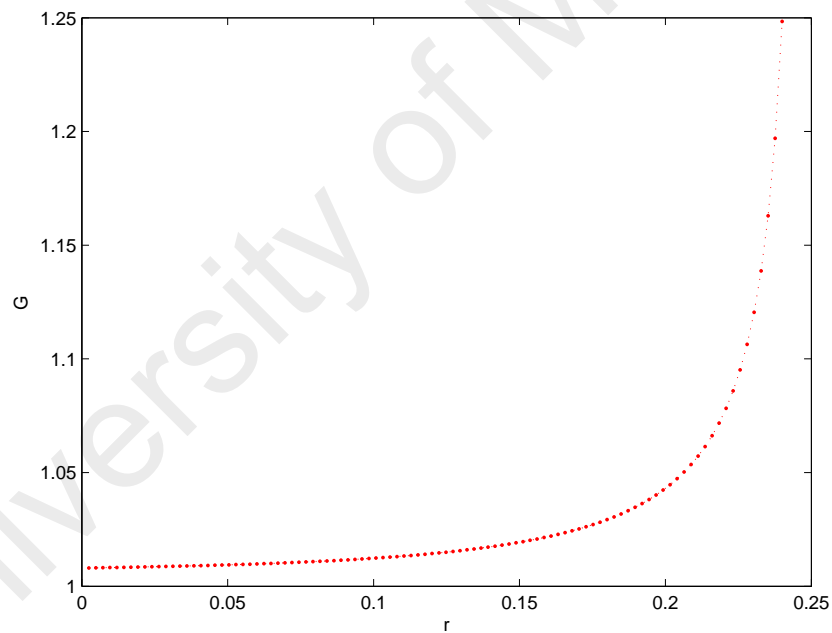
$$f(G, r) = (1 + G)^3(2 + 2r) + K(G^{4r+1} - G^2) = 0. \quad (4.37)$$

Since we are assuming  $\sigma_1 = 0$  (the laggard firm fully free-rides on the leader), we can only consider the case  $\sigma_1 + \sigma_2 < 1/2$  in the previous study because given that  $0 < \sigma_1 < \sigma_2 < 1/2$  and  $\sigma_1 = 0$ , the constraint is that  $\sigma_1 + \sigma_2$  will always be less than  $1/2$ . Also, since  $\sigma_2 = 2r$ , there exists a constraint  $0 < r < 1/4$ .

For  $0 < \sigma_2 < 1/2$  and  $0 < r < 1/4$ ,  $f(G, r)$  can be approximated by a Lagrange interpolating polynomial of degree 3 and then solved to get the values of  $G$  for the respective values of  $r$ .



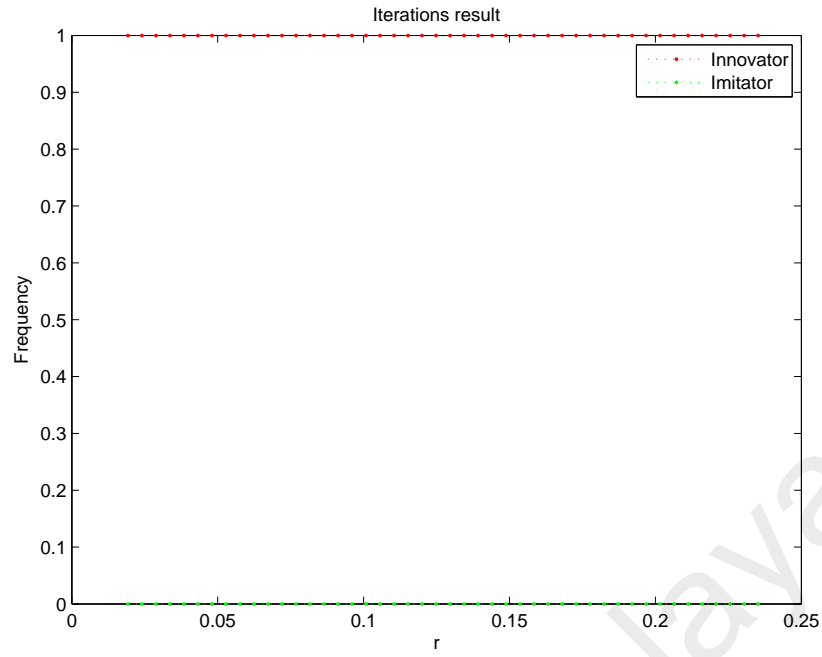
(a)  $K = 265$



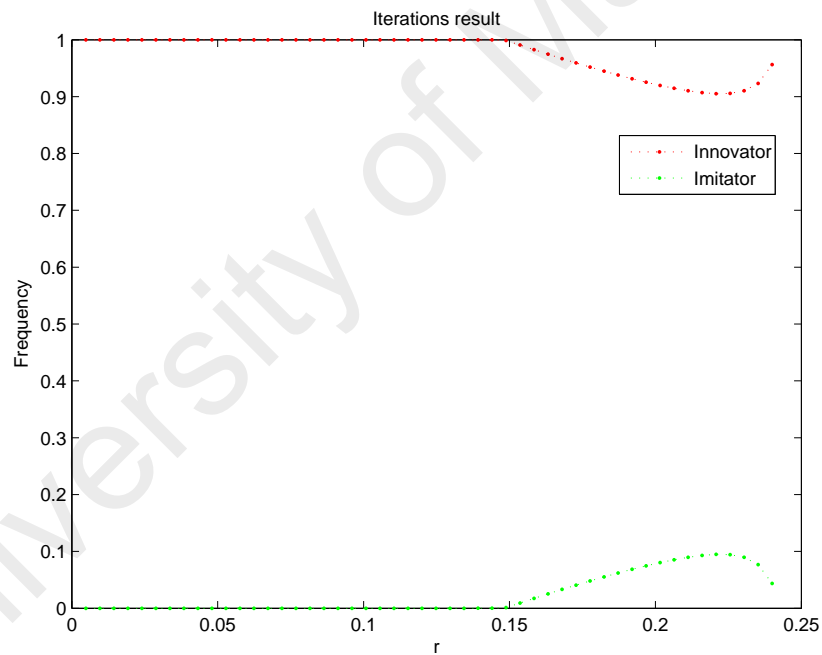
(b)  $K = 2005$

**Figure 4.9:**  $G$  vs  $r$  for (a)  $K = 265$  and (b)  $K = 2005$ .

We found that for  $K \in [265, \infty)$ , Eq.(4.37) will have at least one real root for  $G$  that is greater than unity for  $0 < r < 1/4$ . See Figures 4.9(a) and 4.9(b) for the values of  $G$  for respective values of  $r$ .



(a)  $K = 265$



(b)  $K = 2005$

**Figure 4.10:** Iteration results for (a)  $K = 265$  and (b)  $K = 2005$ .

Additionally, for  $K \in (0, 265)$ , all values of  $G$  are less than unity. Hence, our result for the  $G$  versus  $r$  agrees with the statement in the proof of Proposition 3.1(ii), but with the requirement that  $K \geq 265$ .

Also, for  $K \geq 2005$ , the exponential relationship of  $G$  versus  $r$  is maintained for the entire range of  $0 < r < 1/4$ . Thus, there exists a cut-off for the value of  $K = 2005$  for this exponential paradigm.

Figures 4.10(a) and 4.10(b) show the iteration results. The iterations results are the same for any value of  $K = [265, \infty)$ , where the imitator will never leapfrog the innovator. We have thus reproduced the results from the previous study.

#### 4.5 Conclusion

We have simplified the R&D Cournot duopoly model in Chapter 3 to an extent which allows us to study it using an evolutionary well-mixed N-player setting and solve it using non-complicated iterative and numerical methods, the results of which agree with those in the simulations. The technological leapfrogging observed in the original work in Chapter 3 was reproduced in this current evolutionary well-mixed N-player model, thereby verifying the hypothesis that the underlying factors directly influencing the investment rate of the players and technological leapfrogging are the technology diffusion rate and technology gap. In addition, for this modified R&D Cournot model, we show that for  $G \gg 1$ , the cutoff value of the technology diffusion rate of the imitator for the leapfrogging to occur is dependant on the investment rate of the population, and the investment rate of the population must be sufficiently high for the imitator to overtake the innovator in terms of research productivity. Despite freeriding on the innovator, the imitator needs to undertake more innovation to narrow the technology gap before the leapfrogging can be performed by the imitator. To maintain its leadership, further innovation is needed by the innovator. Therefore freeriding does not deter the innovation incentives of the firms but contributes to a certain degree of cooperation as both firms take initiatives to improve the technology. We simplified our R&D Cournot duopoly model intentionally, with the view that it can be readily generalized to incorporate other interesting and practical features such as real-life

networking effects in the Cournot duopoly game. As a final note to close this chapter, it can be concluded that innovation or cooperation in an adaptive R&D non-oligopolistic economic model can be sustained given high enough technology diffusion and investment rates.

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## CHAPTER 5: CONCLUSION

Why do people cooperate? This question is still not well answered, but (Nowak, 2006) suggested that we cooperate because doing so is usually synergistic. In the long run, cooperation creates more benefits for less cost and makes our lives easier and better. But why don't people always cooperate? We don't do so if we can spare ourselves the effort of working with someone else, but still gain benefits from the cooperator. Since there are good reasons to cooperate and good reasons not to do so, the question "under what conditions will people cooperate?" arises. Despite its seeming simplicity, this question is very complicated to answer, from both a theoretical and an experimental points of view. Indeed, in the words of Ernst Fehr and Simon Gächter (2002), "people frequently cooperate with genetically unrelated strangers, often in large groups, with people they will never meet again, and when reputation gains are small or absent", leaving human cooperation as an "evolutionary puzzle" (Johnson & Bering, 2006).

Game theory, first developed in the 1930s, is a tool for studying cooperation. Traditionally, research in game theory is the study of strategic decision making, which is focused either on whether a rational player should cooperate in a one-off interaction or on looking for "winning solutions" that allow an individual who wants to cooperate to make the best decisions across repeated interactions. In an evolutionary game, we consider players who interact with each other many times, try out different types of strategies over time, and copy the strategies of other players who are more successful. This evolutionary approach to game theory has already led to many useful insights about how to encourage cooperation. It has been known that by punishing defectors appropriately, specific cooperative strategies can do well in an evolutionary setting (for example, see Boyd et al., 2010). Therefore, in this research, we studied the enhancement of cooperation in game theory by using three different models and populations. Specifically, we developed three



models of non-linear systems, and enhanced two types of cooperation, namely public goods cooperation and duopoly cooperation.

In reality, some individuals are willing to pay a cost so as to punish others who do not behave well (defectors). Such a punishing strategy, which can be regarded as a form of direct or indirect reciprocity, is an important mechanism for promoting cooperation. In Chapter 2, we studied the role of punishment and its effects on promoting cooperation in a structured population. Using a recent model of Snowdrift game that incorporates a costly punishing strategy in a well-mixed population (Xu et al., 2011), we developed a Snowdrift game that incorporates a costly punishing strategy in a population connected through a square lattice. In a well-mixed population, the altruistic punishers often have lower payoffs compared to the cooperators due to the cost of punishment, and as a result, the punishers are suppressed as long as cooperators and defectors are present in the system. In contrast to a well-mixed population, for appropriate values of the payoff parameters in a population connected through a square lattice, the altruistic punishing strategy can flourish and prevail. This implies that there is an enhancement in cooperation due to the presence of the punishing mechanism as the punishing strategy is cooperative in nature. The collapse of cooperation occurs only when the ratio of costs to benefits becomes too high. In addition, the coexistence of cooperation (C), defection (D), and punishment (P) strategies (C+D+P phase) is not found in a well-mixed population, but in a square lattice, it is possible for the system to evolve to C+D+P phase for some payoff parameters. The evolutionary cooperation showed in our model is a public goods cooperation for the sake of societal benefit. We used pair approximation as our analytic approach and we extended it from a two-strategy system to a three-strategy system. We showed that the pair approximation can, at best, capture the numerical results only qualitatively due to the improper way of including spatial correlation imposed by the lattice structure.

In the studies of evolutionary games, bridging the gap between theoretical and empirical research is one of the main challenges for the study of cooperation. Different problems have been studied in the hope of applying the findings to a practical scenario. Therefore, in Chapter 3, we studied the role of punishment in a more practical Cournot duopoly economic model. In the industry, the role of the punisher in the Snowdrift game can be taken up by a patent system that grants an innovator monopoly rights over the use of an innovation for a given period of time, thereby making it possible to protect the profits of a cooperator or innovator via patenting. We developed a research and development (R&D) Cournot duopoly differential game played by ex-ante asymmetric firms, in which the dynamics of technological diffusion depend on the technology gap between the firms. We studied the effect of patenting on cooperation and defection in the sustainment of long-term R&D incentives, and whether R&D incentives can be sustained in an environment where technological innovation is almost a public good. We proved the existence of two types of asymmetric equilibria: one in which the technology leader maintains its technological advantage, and the other in which the technology follower catches up with the leader. We found that if the technology diffusion is bidirectional, the equilibrium where both firms invest in R&D at a constant positive rate is stable. In the long-run equilibrium, firms have incentives to innovate as long as the knowledge externalities are bidirectional. Hence, we concluded that imitation via knowledge spillovers does not deter innovation. We proposed a framework to model R&D incentives when both strategic interactions and process innovations with technology diffusion (due to a lack of intellectual property protection) are considered. In contrast to the evolutionary cooperation showed in Chapter 2, which is for the sake of societal benefit, the duopoly cooperation showed in this model is for the sake of economic benefit.

In Chapter 4, we simplified the R&D Cournot duopoly differential game model in Chapter 3 to an extent that allows us to study it using an evolutionary well-mixed N-player setting, and to identify precisely the factors that directly affect the firms' investment rate and technological leapfrogging. The technological leapfrogging observed in Chapter 3 was reproduced, thereby verifying the hypothesis that the underlying factors directly influencing the investment rate of the firms are the technology diffusion rate and technology gap. In addition, for this modified R&D Cournot model, we showed that for technology gap  $G \gg 1$ , the cutoff value of the technology diffusion rate of the imitator for the leapfrogging to occur is dependant on the investment rate of the population, that is, the investment rate of the population must be sufficiently high for the imitator to overtake the innovator in terms of research productivity. Despite freeriding (no patent system) on the innovator, the imitator needs to undertake more innovation to narrow the technology gap before the leapfrogging can be performed by the imitator. Further innovation is needed by the innovator in order to maintain its leadership. Therefore, freeriding does not deter the innovation incentives of the firms, and contributes to cooperation, to a certain degree, as both firms take initiatives to improve their technologies. We simplified our R&D Cournot duopoly model intentionally, with the view that it can be readily generalized to incorporate other interesting and practical features such as real-life networking effects in the Cournot duopoly game. We have also developed a numerical simulation for the model, which can be extended to a lattice model for future studies. We used a set of differential equations based on replicator dynamics to study the behaviour observed in the simulation results, and solved these equations using iterative methods.

In summary, we have shown that firstly, punishment is very effective in promoting cooperation in an N-player evolutionary game with a structured network. Secondly, we have also shown that cooperation in a two-player oligopolistic differential game can

prevail given a high enough technology diffusion rate. Thirdly, we have also shown that cooperation in an N-player non-oligopolistic evolutionary game with a well-mixed network can be sustained given high enough technology diffusion and investment rates. Given that the aim of this thesis is to investigate the dynamics of cooperation, it would be useful to find the common factor(s) promoting cooperation in the various games studied in this work. It can be seen that the appropriate infrastructure for interactions must be present for the factor promoting cooperation for the particular game to work efficiently. In the case of the Snowdrift game, it is the structured network; while in the cases of the R&D games, it is the ease with which technology is diffused. Although achieved through different means, the aim of all players in all three games is the same, which is to obtain higher benefits. Specifically, the players in the N-player evolutionary Snowdrift and non-oligopolistic R&D games strive for a higher payoff, while the firms in the Cournot duopoly R&D game fight for a greater market share.

In the real world, many examples have shown that cooperation can contribute to mutual benefit. One example is the Android mobile operating system. Android is the most widely used mobile operating system in the world, and many mobile phone manufacturers, for example Samsung, Asus, and Huawei, use Android in their products. Although Google is the maker of Android, Android itself is an open source platform, and anyone may use it or customize it for free (see <https://source.android.com>). This makes Android an economical choice for the creation of a cost-competitive smartphone. Moreover, Android is designed to empower developers to produce innovative applications that cater to different sectors of our daily life. Android not only enhances innovation in smartphones, but it also improves the quality of life of its users (Farkade & Kaware, 2015).

We have found that it is impossible to guarantee cooperation amongst members of a group in the long run. We can expect a lot of cooperation, on average, given the right

amount of payoffs. The challenge, then, in developing a real world model that can ensure cooperation at equilibrium, is determining the optimal payoffs and how to capture the complexity of human interactions. Cooperation is an important trait in the society. After all, quoting from a Polish proverb, “TWO HEADS ARE BETTER THAN ONE”.

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## REFERENCES

- Amir, R. (2000). Modelling imperfectly appropriable R&D via spillovers. *International Journal of Industrial Organization*, 18(7), 1013-1032.
- Antal, T., Redner, S., & Sood, V. (2006). Evolutionary dynamics on degree-heterogeneous graphs. *Physical Review Letters*, 96, 188104.
- Anufriev, M., Kopanyi, D., & Tuinstra, J. (2013). Learning cycles in Bertrand competition with differentiated commodities and competing learning rules. *Journal of Economic Dynamics and Control*, 37(12), 2562-2581.
- Aumann, R. J., Maschler, M., & Stearns, R. (1995). *Repeated games with incomplete information*. Massachusetts, US: MIT Press.
- Axelrod, R. (1984). *The evolution of cooperation*. New York, NY: Basic Books.
- Bajari, P., Hong, H., & Ryan, S. P. (2010). Identification and estimation of a discrete game of complete information. *Econometrica*, 78(5), 1529-1568.
- Beck, J. (2008). *Combinatorial games: Tic-tac-toe theory*. Cambridge, England: Cambridge University Press.
- Bellhouse, D. (2007). The problem of Waldegrave. *Electronic Journal for History of Probability and Statistics*, 3(2).
- Bertrand, J. (1883). Review of Cournot's 'rechercher sur la theorie mathematique de la richesse'. *Journal des Savants*, 67, 499-508.
- Bessen, J. E., & Maskin, E. (2009). Sequential innovation, patents and imitation. *RAND Journal of Economics*, 40, 611-635.
- Binmore, K. (2007). *Playing for real: A text on game theory*. Oxford, England: Oxford University Press.
- Borel, E. (1921). La theorie des jeu et les equations integrales a noyau symetrique gauche. *Comptes Rendus de l'Academie des Science*, 173, 1304-1308.

- Boyd, R., Gintis, H., & Bowles, S. (2010). Coordinated punishment of defectors sustains cooperation and can proliferate when rare. *Science*, 328, 617-620.
- Cameron, G. (2005). The sun also rises: Productivity convergence between Japan and the USA. *Journal of Economic Growth*, 10, 387-408.
- Carlsson, H., & van Damme, E. (1993). Equilibrium selection in Stag-Hunt games. In K. Binmore, A. Kirman, & P. Tani (Eds.), *Frontiers of game theory* (pp. 237-254). Massachusetts, US: MIT Press.
- Chan, C. H., Yin, H. P., Hui, P. M., & Zheng, D. F. (2008). Evolution of cooperation in well-mixed N-person Snowdrift games. *Physica A*, 387(12), 2919–2925.
- Chan, N. W. H., Xu, C., Tey, S. K., Yap, Y. J., & Hui, P. M. (2013). Evolutionary Snowdrift game incorporating costly punishment in structured populations. *Physica A: Statistical Mechanics and its Applications*, 392(1), 168–176.
- Clutton-Brock, T., & Parker, G. A. (1995). Punishment in animal societies. *Nature*, 373, 209-216.
- Cohen, W., & Levinthal, D. (1989). Innovation and learning: The two faces of R&D. *The Economic Journal*, 99, 569-596.
- Cournot, A. A. (1838). *Researches into the mathematical principles of the theory of wealth*. New York, NY: Macmillan.
- Curiel, I. (1997). *Cooperative game theory and applications: Cooperative games arising from combinatorial optimization problems* (Vol. Theory and Decision Library C (Book 16)). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- D'Aspremont, C., & Jacquemin, A. (1988). Cooperative and noncooperative R&D in duopoly with spillovers. *American Economic Review*, 78, 1133-1137.
- D'Aspremont, C., & Jacquemin, A. (1990). Erratum. *American Economic Review*, 80, 641-642.
- Dawes, R. M., McTavish, J., & Shaklee, H. (1977). Behavior, communication, and assumptions about other people's behavior in a commons dilemma situation. *Journal of Personality and Social Psychology*, 35(1), 1-11.

- Dixon, H. D., Wallis, S., & Moss, S. (2002). Axelrod meets Cournot: Oligopoly and the evolutionary metaphor. *Computational Economics*, 20(3), 139-156.
- Doebeli, M., & Hauert, C. (2005). Models of cooperation based on the Prisoner's Dilemma and the Snowdrift game. *Ecology Letters*, 8, 748-766.
- Doebeli, M., Hauert, C., & Killingback, T. (2004). The evolutionary origin of cooperators and defectors. *Science*, 306, 859-862.
- Dreber, A., Rand, D. G., Fudenberg, D., & Nowak, M. A. (2008). Winners don't punish. *Nature*, 452, 348-351.
- Drechsel, J. (2010). *Cooperative lot sizing games in supply chains*. Berlin, Germany: Springer-Verlag.
- Droste, E., Hommes, C., & Tuinstra, J. (2002). Endogenous fluctuations under evolutionary pressure in Cournot competition. *Games and Economic Behavior*, 40(2), 232-269.
- Farkade, A. M., & Kaware, S. R. (2015). The Android -a widely growing mobile operating system with its mobile based applications. *International Journal of Computer Science and Mobile Applications*, 3(1), 39-45.
- Fehr, E., & Gächter, S. (2000). Cooperation and punishment in public goods experiments. *American Economic Review*, 90, 980-994.
- Fehr, E., & Gächter, S. (2002). Altruistic punishment in humans. *Nature*, 415, 137-140.
- Geckil, I. K., & Anderson, P. L. (2009). *Applied game theory and strategic behavior*. Boca Raton, FL: CRC Press.
- Gibbs, S. (2015, July 21). Facebook, Google, Dell, HP, eBay back Samsung in patent war with Apple. *theguardian*. Retrieved from <https://www.theguardian.com/technology/2015/jul/21/facebook-google-dell-hp-ebay-samsung-apple>
- Griliches, Z. (1992). The search for R&D spillovers. *Scandinavian Journal of Economics*, 94, 29-47.



- Grunfeld, L. (2003). Meet me halfway but don't rush: Absorptive capacity and strategic investment revisited. *International Journal of Industrial Organization*, 21(8), 1091-1109.
- Hardin, R. (1971). Collective action as an agreeable  $n$ -Prisoner's Dilemma. *Systems Research and Behavioral Science*, 16(5), 472-481.
- Harsanyi, J. (1967). Games with incomplete information played by "Bayesian" players, Part i-iii. Part i. The basic model. *Management Science*, 14, 159-182.
- Harsanyi, J. (1968a). Games with incomplete information played by "Bayesian" players. Part ii. Bayesian equilibrium points. *Management Science*, 14, 320-334.
- Harsanyi, J. (1968b). Games with incomplete information played by "Bayesian" players. Part iii. The basic probability distribution of the game. *Management Science*, 14, 486-502.
- Hauert, C., & Doebeli, M. (2004). Spatial structure often inhibits the evolution of cooperation in the Snowdrift game. *Nature*, 428, 643-646.
- Hauert, C., & Szabo, G. (2005). Game theory and physics. *American Journal of Physics*, 73(5), 405-414.
- Haurie, A., & Krawczyk, J. (2000). An introduction to dynamic games. *Lecture Notes*.
- Hayashi, F. (1982). Tobin's marginal  $q$  and average  $q$ : A neoclassical interpretation. *Econometrica*, 50, 213-224.
- Henriques, I. (1990). Cooperative and noncooperative R&D in duopoly with spillovers: Comment. *American Economic Review*, 80(3), 638-640.
- Hofbauer, J., & Sigmund, K. (1998). *Evolutionary games and population dynamics*. Cambridge, England: Cambridge University Press.
- Hommes, C. (2013). *Behavioral rationality and heterogeneous expectations in complex economic systems*. Cambridge, England: Cambridge University Press.
- Hommes, C., Ochea, M. I., & Tuinstra, J. (2011). On the stability of the Cournot

equilibrium: An evolutionary approach. *CeNDEF working papers, No. 11-10*.

Hommes, C. H., & Brock, W. A. (1997). A rational route to randomness. *Econometrica*, 65(5), 1059-1095.

Isaacs, R. (1999). *Differential games: A mathematical theory with applications to warfare and pursuit, control and optimization*. New York, NY: Dover Publications.

Johnson, D., & Bering, J. (2006). Hand of god, mind of man: Punishment and cognition in the evolution of cooperation. *Evolutionary Psychology*, 4, 219-233.

Joon, A. Y. (2013, Nov 21). Samsung ordered to pay Apple \$290m for copycat features. *CBCNews*. Retrieved from <http://www.cbc.ca/news/business/samsung-ordered-to-pay-apple-290m-for-copycatfeatures-1.2435645>.

Kagel, J. H., & Roth, A. E. (1995). *The handbook of experimental economics*. Princeton, NJ: Princeton University Press.

Kamien, M., Muller, E., & Zang, I. (1992). Research joint ventures and R&D cartels. *American Economic Review*, 82(5), 1293-1306.

Kamien, M. I., & Zang, I. (2000). Meet me halfway: Research joint ventures and absorptive capacity. *International Journal of Industrial Organization*, 18(7), 995-1012.

Katsoulacos, Y., & Ulph, D. (1998). Endogenous spillovers and the performance of research joint ventures. *Journal of Industrial Economics*, 46(3), 333-357.

Kirk, D. E. (2012). *Optimal control theory: An introduction*. New York, NY: Courier Corporation.

Kulttia, K., & Takalob, T. (1998). R&D spillovers and information exchange. *Economics Letters*, 61(1), 121-123.

Leahy, D., & Neary, J. P. (2007). Absorptive capacity, R&D spillovers, and public policy. *International Journal of Industrial Organization*, 25(5), 1089-1108.

Ling, Y., Liu, J., Zhu, P., & Wei, G. (2014). Rigorous punishment promotes cooperation in Prisoner's Dilemma game. In G. D. Caro & G. Theraulaz (Eds.), *Bio-inspired mod-*

*els of network, information, and computing systems. BIONETICS 2012. Lecture notes of the institute for computer sciences, social informatics and telecommunications engineering (Vol. 134). New York, NY: Springer.*

Luce, R. D., & Raiffa, H. (1957). *Games and decisions: Introduction and critical survey*. New York, NY: Courier Corporation.

Luckraz, S. (2007). Dynamic noncooperative R&D in duopoly with spillovers and technology gap. *Game Theory and Applications*, 11, 139-161.

Luckraz, S. (2008). Process spillovers and growth. *Journal of Optimization Theory and Applications*, 139, 315-335.

Luckraz, S. (2011). R&D games in a Cournot duopoly with isoelastic demand function: A comment. *Economic Modelling*, 28, 2873-2876.

Luckraz, S. (2013). On innovation cycles in a finite discrete R&D game. *Economic Modelling*, 30, 510-513.

Mazalov, V. (2014). *Mathematical game theory and applications*. New York, NY: Wiley.

Morita, S. (2008). Extended pair approximation of evolutionary game on complex networks. *Progress of Theoretical Physics*, 119(1), 29-38.

Nash, J. (1950). Equilibrium points in N-person games. *Proceedings of the National Academy of Sciences of the United States of America*, 36(1), 48-49.

Nash, J. (1951). Non-cooperative games. *Annals of Mathematics*, 54, 286-295.

Novshek, W. (1985). On the existence of Cournot equilibrium. *Review of Economic Studies*, 52, 85-98.

Nowak, M. A. (2006). Five rules for the evolution of cooperation. *Science*, 314, 1560-1563.

Nowak, M. A. (2008). Generosity: A winner's advice. *Nature*, 456, 579.

- Nowak, M. A., & Highfield, R. (2011). *Supercooperators: Altruism, evolution, and why we need each other to succeed*. New York, NY: Simon and Schuster.
- Ohtsuki, H., Hauert, C., Lieberman, E., & Nowak, M. A. (2006). A simple rule for the evolution of cooperation on graphs and social networks. *Nature*, *441*, 502-505.
- Ohtsuki, H., Iwasa, Y., & Nowak, M. A. (2009). Indirect reciprocity provides only a narrow margin of efficiency for costly punishment. *Nature*, *457*, 79-82.
- Ohtsuki, H., Nowak, M. A., & Pacheco, J. (2007). Breaking the symmetry between interaction and replacement in evolutionary dynamics on graphs. *Physical Review Letters*, *98*, 108106.
- Ostrom, E. J., Walker, J., & Gardner, R. (1992). Covenants with and without a sword: Self-governance is possible. *American Political Science Review*, *86*, 404-417.
- Ozdoglar, A. (2015). Strategic form games and Nash equilibrium. In J. Baillieul & T. Samad (Eds.), *Encyclopedia of systems and control* (pp. 1364-1371). London, England: Springer-Verlag.
- Pacheco, J. M., A, A. T., & Nowak, M. A. (2006). Coevolution of strategy and structure in complex networks with dynamical linking. *Physical Review Letters*, *97*, 258103.
- Perc, M., & Marhl, M. (2006). Evolutionary and dynamical coherence resonances in the pair approximated Prisoner's Dilemma game. *New Journal of Physics*, *8*, 142.
- Poundstone, W. (1992). *Prisoner's Dilemma*. New York, NY: Doubleday.
- Prisner, E. (2014). Theory 2: Simultaneous games. In *Game theory through examples* (pp. 4-18). US: Mathematical Association of America.
- Rand, D. G., Dreber, A., Ellingsen, T., Fudenberg, D., & Nowak, M. A. (2009). Positive interactions promote public cooperation. *Science*, *325*, 1272-1275.
- Rapaport, A., & Chammah, A. M. (1965). *Prisoner's Dilemma: A study in conflict and cooperation*. Michigan, MI: University of Michigan Press.
- Rapaport, A., & Chammah, A. M. (1966). The game of Chicken. *American Behavioral*

*Scientist*, 10(3), 10-28.

- Rasmusen, E. (2006). *Games and information: An introduction to game theory (4th ed.)*. Oxford, England: Blackwell Publishers.
- Rodgers, G. J., & Yap, Y. J. (2002). Growth and addition in a herding model. *The European Physical Journal B - Condensed Matter and Complex Systems*, 28(1), 129-132.
- Rogers, E. M. (2003). *Diffusion of innovations (5th ed.)*. New York, NY: Free Press.
- Romualdas, G., & Algirdas, K. (2008). Application of game theory for duopoly market analysis. *Journal of Business Economics and Management*, 9(3), 207-217.
- Roughgarden, T. (2010). Algorithmic game theory. *Communications of the ACM*, 53(7), 78-86.
- Rousseau, J. J. (1992). *Discourse on the origin of inequality* (Cress, D. A., Trans.). Indianapolis, Ind: Hackett Publishing Company.
- Salant, S. W., & Shaffer, G. (1998). Optimal asymmetric strategies in research joint ventures. *International Journal of Industrial Organization*, 16(2), 195-208.
- Samuelson, L., & Zhang, J. (1992). Evolutionary stability in asymmetric games. *Journal of economic theory*, 57(2), 363-391.
- Santos, F. C., & Pacheco, J. M. (2005). Scale-free networks provide a unifying framework for the emergence of cooperation. *Physical Review Letters*, 95(9), 098104.
- Schelling, T. C. (1960). *The strategy of conflict*. Harvard, England: Harvard University Press.
- Selten, R. (1965). Spieltheoretische behandlung eines oligopolmodells mit nachfrage-tragheit: Teil I: Bestimmung des dynamischen preisgleichgewichts. *Zeitschrift für die gesamte Staatrwissenschaft*, 2, 301-324.
- Shapley, L. S. (1964). Some topics in two-person games. In M. Dresher, L. S. Shapley, & A. W. Tucker (Eds.), *Advances in game theory* (vol. 52, pp. 1-28). Princeton, NJ:

Princeton University Press.

- Shubik, M. (1955). A comparison of treatments of a duopoly problem, part ii. *Econometrica*, 23(4), 417-431.
- Sigmund, K., & Nowak, M. A. (1999). Evolutionary game theory. *Current Biology*, 9(14), 503-505.
- Simpson, R. D., & Vonortas, N. S. (1994). Cournot equilibrium with imperfectly appropriable R&D. *The Journal of Industrial Economics*, 42(1), 79-92.
- Skinner, A., & Chamberlin, E. H. (2001). Oligopoly and oligopolistic interdependence: The issue of space. In S. G. Medema, J. E. Biddle, & J. B. Davis (Eds.), (pp. 105-124). Abingdon, England: Routledge.
- Skyrms, B. (2004). *The Stag Hunt and the evolution of social structure*. Cambridge, England: Cambridge University Press.
- Smith, J. M. (1982). *Evolution and the theory of games*. Cambridge, England: Cambridge University Press.
- Smith, J. M., & Price, G. R. (1973). The logic of animal conflict. *Nature*, 246, 15-18.
- Smith, J. M., & Szathmari, E. (1995). *The major transitions in evolution*. Oxford, England: Freeman.
- Spence, M. (1984). Cost reduction, competition, and industry performance. *Econometrica*, 52(1), 101-122.
- Straffin, P. D. (1993). *Game theory and strategy*. US: The Mathematical Association of America.
- Sugden, R. (1986). *The economics of rights, cooperation and welfare*. Oxford, England: Blackwell Publishers.
- Suzumura, K. (1992). Cooperative and noncooperative R&D in oligopoly with spillovers. *American Economic Review*, 82, 1307-1320.

- Szolnoki, A., & Perc, M. (2013). Effectiveness of conditional punishment for the evolution of public cooperation. *Journal of Theoretical Biology*, 325, 34-41.
- Tanimoto, J. (2015). *Fundamentals of evolutionary game theory and its applications*. Japan: Springer.
- Taylor, P., & Jonker, L. (1978). Evolutionary stable strategies and game dynamics. *Mathematical Biosciences*, 40, 145-156.
- Terhune, K. (1968). Motives, situation, and interpersonal conflict within Prisoner's Dilemma. *Journal of Personality and Social Psychology*, 8(3, Part 2), 1-24.
- Touffut, J. P. (2007). *Augustin Cournot: Modelling economics*. Cheltenham, England: Edward Elgar Publishing.
- Varian, H. R. (2010). *Intermediate microeconomics: A modern approach (8th ed.)*. New York, NY: W W Norton & Co.
- von Neumann, J. (1928). Zur theorie der gesellschaftsspiele. *Mathematische Annalen*, 100, 295-320.
- von Neumann, J., & Morgenstern, O. (1944). *Theory of games and economic behavior*. Princeton, NJ: Princeton University Press.
- von Stackelberg, H. (1934). *Marktform und gleichgewicht*. New York, NY: Springer-Verlag Wien.
- Wang, W. X., Ren, J., Chen, G., & Wang, B. H. (2006). Memory-based Snowdrift game on networks. *Physical Review E*, 74(5), 056113.
- Webb, J. N. (2007). Games with continuous strategy sets. In *Game theory: Decisions, interaction and evolution* (pp. 107-117). London, England: Springer.
- Weibull, J. W. (1997). *Evolutionary game theory, reprint edition*. Massachusetts, US: MIT Press.
- Wu, Z. X., Xu, X. J., Chen, Y., & Wang, Y. H. (2005). Spatial Prisoner's Dilemma game with volunteering in Newman-Watts small-world networks. *Physical Review E*,

71, 037103.

- Xu, C., Hui, P. M., & Zheng, D. F. (2007). Networking effects on evolutionary Snowdrift game in networks with fixed degrees. *Physica A: Statistical Mechanics and its Applications*, 2(15), 773-780.
- Xu, C., Ji, M., Yap, Y. J., Zhang, D., & Hui, P. M. (2011). Costly punishment and cooperation in the evolutionary Snowdrift game. *Physica A: Statistical Mechanics and its Applications*, 390(9), 1607-1614.
- Yamagishi, T. (1986). The provision of a sanctioning system as a public good. *Journal of Personality and Social Psychology*, 51(1), 110-116.
- Yap, Y. J., Luckraz, S., & Tey, S. K. (2014). Long-term research and development incentives in a dynamic Cournot duopoly. *Economic Modelling*, 39, 8-18.
- Zheng, D. F., Yin, H. P., Chan, C. H., & Hui, P. M. (2007). Cooperative behavior in a model of evolutionary Snowdrift games with N-person interactions. *Europhysics Letters*, 80(1), 18002.
- Zhong, L. X., Zheng, D. F., Zheng, B., Xu, C., & Hui, P. M. (2006). Networking effects on cooperation in evolutionary Snowdrift game. *Europhysics Letters*, 76(4), 724-730.



## LIST OF PUBLICATIONS

1. Chan, N. W. H., Xu, C., Tey, S. K., Yap, Y. J., & Hui, P. M. (2013). Evolutionary snowdrift game incorporating costly punishment in structured populations. *Physica A: Statistical Mechanics and its Applications*, 392(1), 168–176.
2. Yap, Y. J., Luckraz, S., & Tey, S. K. (2014). Long-term research and development incentives in a dynamic cournot duopoly. *Economic Modelling*, 39, 8-18.

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