A STUDY ON THE SPATIAL-TEMPORAL DYNAMICS OF WORMHOLES IN A BRANEWORLD MODEL

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A STUDY ON THE SPATIAL-TEMPORAL DYNAMICS OF WORMHOLES IN A BRANEWORLD MODEL

ABSTRACT

The dynamic wormhole models that were previously introduced focused on the dynamics of the wormhole itself of either rotating or evolutionary in characteristics and in various frameworks. In this thesis we show the dynamic factor that represents the spatial dynamics in terms of spacetime expansion and contraction in braneworld cosmology framework affects the changes at the throat of the wormhole by either decreasing or increasing the stress energy tensor respectively. This implies an interesting finding concerning the effects of cosmological expansion and contraction of the universe or in general the surrounding space of wormholes that is expanding or contracting from and toward the wormholes respectively. Furthermore, the gravitational lens of a wormhole was also introduced by various researchers. Their treatment was focused basically on the lens signature that describes wormhole geometrical character such as the differences from a black hole or between any various types of wormhole models. The braneworld scenario provides the idea of spacetime with underlying extra-dimensions. The inclusion of extra-dimensional terms in the gravitational lens object spacetime line element will result in some variation in the expression for its gravitational lens deflection angle. In this thesis we investigate such variation by deriving this deflection angle expression. Thus this study not only shows the existence of such variation but also suggests the potential utilization of gravitational lensing to prove the existence of extra dimensions by studying the deflection angle characteristic in accordance with the spacetime expansion rate of the universe.

Keywords: General Relativity, Brane, Gravitational Lens, Wormhole.

PENGAJIAN KEATAS DINAMIK RUANG-MASA LOHONG RUANG DALAM MODEL MEMBRANA ALAM

ABSTRAK

Model-model lohong-ruang dinamik sebelum ini telah diperkenalkan lebih tertumpu kepada kedinamikan lohong-ruang itu sendiri samada perihal lohong-ruang yang berputar atau berevolusi dan di dalam pelbagai kerangka kerja. Dalam tesis ini kami menunjukkan faktor dinamik yang mewakili dinamik ruang dalam terma pengembangan dan pengecutan ruang-masa dalam kerangka kerja kosmologi dunia membrana yang masing-masing memberi kesan kepada perubahan pada rangkungan lohong-ruang samada pengurangan atau penambahan tensor bebanan tenaga. Ini mengimplikasikan suatu penemuan baru yang menarik perihal kesan kosmologi pengembangan dan pengecutan alam atau ruang di sekeliling lohong ruang-lohong ruang yang mengalami pengembangan menjauh dari atau pengecutan kearah lohong ruang tersebut. Selanjutnya, perihal kanta graviti lohong-ruang juga telah di perkenalkan oleh pelbagai penyelidik. Perbincangan tertumpu pada kesan kanta yang memperlihatkan sifat geometri lohong-ruang seperti perbezaan di antara lohong hitam atau di antara pelbagai jenis model-model lohong-ruang. Senario dunia membrana membangkitkan idea ruang-masa yang didasari dimensi-dimensi tambahan. Pemasukan terma dimensi tambahan dalam unsur bentuk ruang-masa objek kanta graviti akan menghasilkan variasi dalam ungkapan sudut lencongan kanta graviti. Maka dalam tesis ini kami telah mengkaji variasi tersebut dengan menerbitkan ungkapan matematik sudut lencongan, maka dalam penyelidikan ini bukan sahaja membuktikan kewujudan variasi ungkapan tersebut malah telah mencadangkan potensi penggunaan kanta graviti dalam pembuktian kewujudan dimensi tambahan dengan mengkaji sifat sudut lencongan selaras dengan kadar pengembangan ruang masa alam.

Kata kunci: Kerelatifan Umum, Membrana, Kanta Graviti, Lohong Ruang.

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$q_{\mu\nu}$	:	Brane metric tensor
χ	:	Brane Newtonian potential
Ψ	:	Brane relativistic potential
P	:	Braneworld wormholes radii
σ	:	Bulk cosmological constant
y, Y	:	Bulk-protruding coordinate, hypersurface coordinate
$E_{\mu\nu}$	:	Bulk traceless entity
$\Gamma^{\alpha}_{\mu\nu}$	:	Christoffel symbol or affine connection
Λ, λ	:	Cosmological constant, vacuum energy
D_{μ}	:	Covariant differentiation
∂_{ν}	:	Differentiation with respect to x^{ν}
$\Theta_{\mu\nu}$	:	Dynamics brane Einstein tensor term
$\xi_{\mu\nu}$	:	Dynamics brane Ricci tensor term
$G_{\mu\nu}$	:	Einstein tensor
$\zeta_{\rho}, \zeta_{p_{rad}}, \zeta_{p_{\perp}}$	:	Energy momentum tensor components dynamic factors
$T^{\mu\nu}, \tau_{\mu\nu}$	:	Energy momentum tensor or stress energy tensor
ς	:	*Exoticity* of exotic matter
$\bar{r}_{*}$	:	Finite radial distance
κ_5^2	:	Five Dimensional bulk energy momentum tensor constant
$\forall$	:	For all
V^{μ}	:	Four vector

u_μ : Four-velocity vector field of flow

G : Gravitational constant

α : Gravitational lens deflection angle

ξ : Gravitational lens impact parameter

G_n : Higher dimensional gravitational constant

s : Interval between any two events

$K_{\mu\nu}$: Intrinsic curvature

K^μ : Killing vector field

$g_{\mu\nu}$: Metric tensor

G^μ_ν : Mixed Einstein tensor

n D : n dimensional

Λ_n : n dimensional brane cosmological constant

${}^{(n)}g_{\mu\nu}$: n dimensional metric tensor

${}^{(n)}R^\alpha_{\mu\beta\nu}$: n dimensional Riemann curvature tensor

n_μ : normal tensor (or vector; tensor of rank 1)

r_0 : Radial distance at the wormhole throat

Φ : Red shift function $\left(\Phi = \Phi(r)\right)$

$R_{\mu\nu}$: Ricci tensor

$R^\alpha_{\mu\beta\nu}$: Riemann curvature tensor

r_s : Schwarzschild radius

b : Shape function $\left(b = b(r)\right)$

$S_{\mu\nu}$: : Stress energy tensor of the *normal matter* in 4 D spacetime

c : The speed of light

$\exists$	:	There exist
M	:	The redshift term of brane metric
N	:	The shape function term of brane
Q	:	The shape function term of bulk
a	:	The surrounding spatial dynamics factor (evolution factor)
$I\left(\phi\right)$	:	Wormhole gravitational lens axial angle function
u	:	Wormhole gravitational lens impact parameter
x^{μ}	:	$x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$ of 3 + 1 (4-D) brane
ADD	:	Arkani-Hamed, Dimopoulus and Dvali
ANEC	:	Averaged Null Energy Condition
ASEC	:	Averaged Strong Energy Condition
AWEC	:	Averaged Weak Energy Condition
DEC	:	Dominant Energy Condition
DGP	:	Dvali-Gabadadze-Porrati
ER	:	Einstein-Rosen
EPR	:	Einstein-Podolsky-Rosen
GR	:	General Relativity
KK	:	Kaluza-Klien
NEC	:	Null Energy Condition
RS	:	Randall-Sundram
SEC	:	Strong Energy Condition
WEC	:	Weak Energy Condition

LIST OF APPENDICES

CHAPTER 1 : INTRODUCTION

1.1 Introduction

Modern physics is based on the two pillars of established theories namely quantum mechanics and theory of general relativity. The former describes the physics at microscopic atomic scale, while the latter describes the macroscopic world of cosmology. These theories are pushing the boundaries of physical sciences by countless research areas that explore the possibilities of physical realities e.g. from the study of the Higgs boson in elementary particles physics to the study of exotic spacetime curvatures characteristic of black hole and wormhole, from the study of what is possibly happening at the core of a black hole or at the throat of a wormhole, or at extremely minutes seconds after the big bang and toward more philosophical like question such as what is the fate of the universe? These feats of research have ultimately motivate physicists to the necessity of exploring more deeper fundamental framework such as combining the two pillars as grand unification of physics. And to date among successful effort of grand unification is the M-theory which consequently triggering research works in its framework that considerably pushing its boundaries.

The solution to the Einstein field equations by Schwarzschild had paved the way to a spherically symmetric solution which inspired the ideas of wormholes. The pioneering works from Einstein-Rosen (Einstein and Rosen, 1935) on relating particle with spacetime, Fuller-Wheeler (Fuller and Wheeler, 1962) on connecting spacetime, Ellis (Ellis, 1973) on particle model in General Relativity as continuation to Einstein-Rosen's works to Morris-Thorne (Morris and Thorne, 1988) were on static spherically symmetric solutions. The generalization of the solution, the works of Teo (Teo, 1998) and Khatsymovsky (Khatsymovsky, 1998) introduced the rotating solution, while Lemos (Lemos et al., 2003) worked on a solution with a cosmological constant. Combining the methodology of the solutions a case study of the generalized solutions for a slowly rotating spherically symmetric wormhole with a cosmological constant was proposed by our previous works (Anuar et al., 2005). However, all these studies were done in the framework of classical general relativity. The emerging of the string and consequently M-theory leads to the introduction of braneworld models in the 1990s. It is based on the idea that the universe is a 3-brane embedded in a five-dimensional bulk. Works by Bronnikov and Kim (Bronnikov and Kim, 2003) were among the pioneers in the attempt to study wormhole in the frameworks of braneworld. More generalized class of braneworld wormholes was introduced recently by Lobo (Lobo, 2007).

Wormhole in general relativity framework violates null energy condition, however braneworld framework naturally support the existence of the exotic structure with topological defect such as wormholes without violating energy condition (Bronnikov et al., 2003). Most wormholes framework do not consider dynamic factor of the surrounding space of either expansion or contraction. Thus a more generalized time dependent dynamic can be embedded in the spacetime metric. Wormholes in dynamic space and its physical characteristic induced by the extra-dimension in braneworld framework may show some differences in its gravitational lens effect in contrast to wormhole cosmology in model based on classical general relativity. Some differences or discrepancies have not been proposed in current gravitational lens calculation. There could be the signature of the existence of large extra-dimension that may have been overlooked while observing gravitational lens.

1.2 Objectives

There are two main objectives of our research. The first objective is to formulate three main expressions in the stress energy tensor that will represent the physical characteristic of the dynamic braneworld which are the energy density, radial pressure and lateral pressure. This will establish the governing terms of the spatial dynamics surrounding the wormhole so that its dynamics effect toward and from the wormhole can be studied. The second objective is more of introducing possible observational method of proving the existence of the spatially dynamics extra-dimension due to its effects on the wormhole. It is the formulation of the extra term expression of deflection angle that is considered as the braneworld wormhole gravitational lens correction due to the bulk term in the braneworld spacetime metric which may provide the signature of extra-dimension illusively embedded in braneworld wormholes gravitational lens curvature. Thus, the result of our work may suggest the application of the derived deflection angle expression as observational method of searching for the existence of higher extra-dimension as predicted in the braneworld cosmological model.

1.3 Motivation

One of the motivations to study wormholes in an M–theory framework, is to search for realistic matter source that will support the ‘exotic’ characteristic of the wormhole spacetime. The theoretical possibilities of the existence of wormhole may generate ‘exoticity’ concerns in cosmology such as the dark energy, exotic matter energy condition violations that imply negative energy density which repulsively expand localized spacetime and even influence the topological characteristic of the universe itself. With M-theory (Schwarz et al., 2007) describing the fundamental framework of physics that can be influenced by extra-dimensional of spacetime, one may explore the concerns of exoticity in order to reduce or eliminate the requirement of exotic matter influence on the physical properties of the wormhole.

In string theory (Zwiebach, 2004), there are basically two types of strings. A string that intra-connecting between two different points on a single brane and a string that inter-connecting between two different points of two different branes (West, 2012). In M-theory, a one-dimensional representation of a string can be expanded to two-dimensional cylindrical like hypersurface representation of a string intra-connecting between two regions on a single brane or inter-connecting between two regions of two different branes (Schwarz et al., 2007). Thus, these representations can be natural state of inter-connecting and intra-connecting two regions of spacetime similar to that of wormhole theories, whereas the latter requires energy condition violation material such as dark energy or cosmological constant in the form of theoretical 'exotic matter' to expand the throat to become a traversable wormhole. So there are several research motivation to explore the possibilities, where in a brane cosmology a traversable wormhole can exist naturally without the requirement of energy condition violation. There are also possible scalar or vector fields that may exist on a brane cosmology that suit the characteristic of exotic matter to 'fuel' the traversable wormhole throat (Barcelo and Visser, 2000). The effects of brane dynamics on toward wormholes and possible observational indication for the existence of extra-dimension due to the extra-dimensional effects on the wormholes in braneworld are the subjects of interest which may contribute to suggestion for experimental evidence of the subject. The subject of wormhole, is currently gaining attention since its relation with the known quantum phenomenon that is the entanglement via ER=EPR (Susskind, 2014). Leonard

Susskind recently has put wormhole topics in the lime light as not just bridge between spaces but also as "bridge" of unification between quantum mechanics and general relativity (Susskind, 2014). The braneworld scenario derived from string and M-theory provides the idea of spacetime with underlying extra-dimensions. It is interesting to study the effect of the underlying extra-dimensions toward the spacetime that by itself has dynamics characteristic where the universe is in dynamics condition as currently in expanding phase or during contraction phase if big crunch theory is taken into consideration. Thus it is compelling to investigate the effect of dynamic factor toward the wormhole characteristics. Gravitational lens is significant as probing method for massive structures of celestial object or in general object that inflicts severe spacetime curvature. Since underlying extra-dimension such as bulk in the braneworld scenario affecting the behavior of spacetime characteristic in the universe thus it is reasonable to study how the extra-dimension affecting the 3 D spatial dimension surrounding wormhole affecting the gravitational lens characteristic due to the wormhole in the braneworld scenario.

1.4 Outline of the thesis

After a brief introduction in Chapter 1, we give a review on the braneworld, wormhole and gravitational lens in Chapter 2. In this chapter, we introduce the extra-dimensional concept based on proposal by Kaluza Klien (KK), Arkani Hamed, Dimopoulos and Dvali (ADD) (Arkani-Hamed et al., 1999) to Randall Sundram (RS) (Randall and Sundram, 1999) braneworld, which will be the basis of our research. We also show elaborately the works of Shiromizu (Shiromizu et al., 2000) Bronnikov (Bronnikov and Kim, 2003) on how the braneworld model naturally provides sustainable material to hold a traversable wormhole intact without the requirement of 'exotic' matter. On wormhole, we discuss the origin of the idea, elaborating the various ideas from non traversable wormhole to traversable wormhole types and show the effect of energy condition violation of wormhole toward wormhole traversability. We also introduce the geometrical concepts of gravitational lens and how the deflection angle is derived from spacetime metric of Schwarzschild objects such as massive star, blackhole to wormhole. Gravitational lens with underlying braneworld model will also be briefly introduced in Chapter 2. We will show how from the deflection angle expression of lens object in braneworld can be separated by general relativity term and the extra terms signifying deflection angle due to brane factor.

The main contributions of this study are presented in Chapter 3 and 4. In Chapter 3 we analyzed on the spacetime metric that is representing brane with one spatial dimensionally higher bulk protruding perpendicularly from 3+1 brane spacetime. We extract the physical characteristic representing the wormholes that exist in such a braneworld model. Confirmation of finiteness and smoothness using Shiromizu (Shiromizu et al., 2000) and Bronnikov (Bronnikov et al., 2003) methods will show that the wormhole is geometrically possible and may exist naturally without requirement of exoticity. Here we highlight an interesting finding regarding the effect of the surrounding dynamics of spacetime toward wormholes. In Chapter 4, we calculate the deflection angle expression representing the wormhole model gravitational lens following the metric framework discussed in Chapter 3. In the expressions, we will include the dynamic factor terms that will represents the dynamics of braneworld spacetime surrounding the wormhole. The theoretical derivation of the deflection angle is another highlight of this study and we propose a new method to validate experimentally the existence of extra-dimension in the universe. Chapter 5 concludes the study with some perspective for future works.

CHAPTER 2 : BACKGROUND AND LITERATURE REVIEW

2.1 Introduction

In the framework of General Relativity, the quest for a realistic mechanism from a source of matter to support the exotic spacetime physical characteristic of a wormhole has met with challenges of energy condition violations . The source of mechanism, even if it is physically possible to exist, is so exotic which then coined as *exotic matter*. To solve the challenges of exotic matter several proposals have been suggested which among them are scalar fields, semi classical gravity, Brans-Dicke theory and some extra dimensional frameworks. Among the many suggestions, one interesting formalism is braneworld cosmology framework which will be the basis of this study. In this cosmological model, the universe is viewed as a 3-brane embedded in a five-dimensional bulk (Lobo, 2007). The idea that contributes toward the physics of wormhole came from the fact that cosmologically inhomogeneities could exist through gravitational instabilities. This implies that cosmological background density fluctuation may allow the conditions for wormholes to exist naturally. It has been shown that the radial dependent equation of state such as the stress energy tensor may self-sustain a wormhole (Morris and Thorne, 1988). The gravitational field concept in braneworld cosmology can be influenced by higher-dimensional space of bulk. This implies that a natural resource of a wormhole geometry can also be influenced by the higher dimensional bulk. These were inspired by the progresses made in string and M theory, which are the foundation of the braneworld ideas.

2.2 Braneworld

The concept of braneworld is that the observable world is analogously a membrane or just a surface to a bulk of multidimensional space with large or infinite extra dimensions (West, 2012). Technically, the membrane or brane is a domain wall in a multidimensional space. The domain wall or brane is where all the standard model fields in which we live in are confined to, which is a 4 D spacetime floating in a higher dimensional bulk. The extra-dimensional could be very large or even infinitely large. This is a large extra-dimensional scenario. This idea is an alternative to the traditional string theory ideas of which the higher dimensional space are compactified on a small scale as suggested by the traditional model of extra-dimensional space by Kaluza and Klien (Wesson, 1999)

There are two main types of braneworld models namely the Arkani Hamed, Dimopoulos and Dvali (ADD) braneworld (Arkani-Hameed et al., 1999) and Randall-Sundram (RS) braneworld (Randall and Sundram, 1999). These models proposed that extra-dimension can be large in contrast to the early notion that extra dimension is a compactified space curled up or wrapped up into a Planck length. The ADD braneworld model is a generalization of the compactified Kaluza-Klien (KK) model where it represents combinations of n flat compact extra-dimensions in which confine all the standard model fields. Only gravity is not constrained by the brane and could propagate through the higher dimensional bulk due to the close string type of graviton that is the quantize form of gravity, whereas all the other fields in standard model are open string type that always attached to the brane. The same constraint characteristics of string are also applied in another braneworld model which was introduced later by Randall and Sundram (Randall and Sundram, 1999). The Randall

Sundram model suggests that the bulk of higher extra-dimension may dynamically influence the brane where it can be gravitationally interact with the bulk. This model interestingly may link the braneworld model with general relativity. Since general relativity theory is originally responsible for the idea of wormhole, thus Randall and Sundram model can be used to expand the theory of wormhole in a framework that is fundamentally based on elegant theory of string and M theory (Schwarz et al., 2007) which is among major concepts that unify physics at the most fundamental level. The first model of RS model (RS1) (Randall and Sundram, 1999) was meant to solve the hierarchy problem. However it was still represented as compactified extra dimension, thus the bulk would just be Planck scale extra-dimension that may only prone to locally influence object in the brane not as what our research intent for a cosmologically type of influence. Then the 2nd Randall and Sundram model (RS2) had captured our curiosity that matches with model that could operate in cosmological aspect.

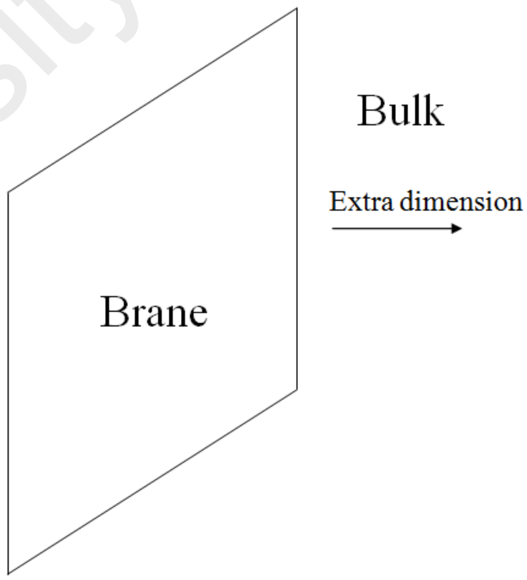

Figure 2.1 : RS2 model of a braneworld.

The RS2 suggest an extra dimension that can be infinite in size thus an alternative to all the previous compactified models of extra-dimension. RS2 model is represented by a single positive tension brane in a non-compact and infinite extra dimension where the metric can be written as a combination of protruding y bulk coordinate and null metric tensor $n_{\mu\nu}$ multiplied with elements of bulk coordinate $|y|/l$ which can be arranged accordingly to represent a specific metric. It can be written as

$$ds^2 = e^{-2|y|/l}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{2.1}$$

For this metric, the Z_2 symmetry is applied for representing the domain wall or the brane. The reflection symmetry is positioned at $y = 0$ (Maarten, 2004)

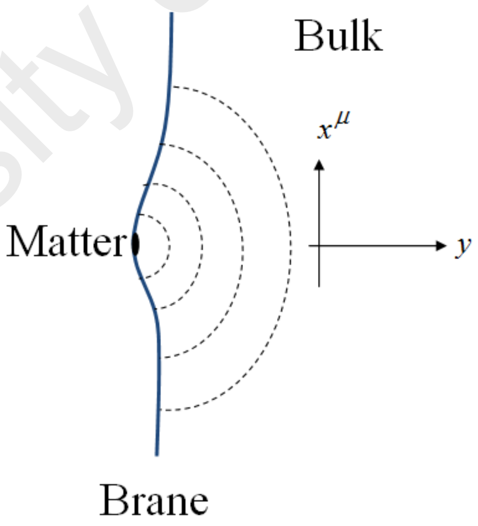

Figure 2.2 : Brane bending due to matter and bulk.

The bulk which is one spatial dimension higher than the brane has an influence on brane (universe) cosmological constant Λ. This can be considered as product of high dimensional gravitational constant G_n where n representing the dimension of the bulk. The bulk cosmological constant σ are the main contributor to the cosmological constant Λ, thus $\Lambda = \Lambda(G_n, \sigma)$ as appear in this universe is considered as attachment of brane tension that is fined-tuned against itself. Moreover, an interesting property about the behavior of graviton in 5 D bulk gravity in non compact extra-dimension of the RS2 model. This relates to the normal 4 D brane gravity were also discovered through this model. All these imply the influence of the bulk toward the brane. Therefore, RS2 is enticing for our works due to its interesting extra-dimension model which is geometrically simpler yet elegant concept, thus we had considered the Randall Sundram braneworld second model (RS2) (Randall and Sundram, 1999) as the basis framework for this thesis.

2.3 Braneworld gravity

General relativity is the basis for describing gravity and the physics of wormholes. In the desired limit it shall reproduces the Newton's law of gravity. Therefore by limits approximation the braneworld theory of gravity shall also reproduces signature of general relativity. The limits approximation method in general relativity is known as linearised general relativity. Therefore the same method of limits approximation of linearised braneworld gravity can be shown to reproduce a general relativistic results. The linearised braneworld gravity to the least has shown how the bulk influence the brane to reproduce gravity but however, its only limited to weak field realms. Therefore, not all gravitational phenomena can be described by this method. Gravitational phenomena of the strong field kind such as black hole and perhaps wormhole which is our subject of research require more elegant treatment as provided by Shiromizu, Maeda and Sasaki (Shiromizu, et al., 2000). The treatment not only describes the spacetime curvature due to mass or presence of matter on brane which is in congruence with the classical general relativity, but also shown that condition when matter on brane is absence that is the vacuum condition. This is very interesting concept for the physics of wormhole because it has suggested that the spacetime curvature could exist without the presence of matter whether ordinary or exotic as required by classical general relativity to create spacetime curvature as exotic as wormhole. The bulk by itself can directly influence the brane even without emerging in a form of matter or exotic matter. Thus braneworld wormhole does not require exotic matter for its throat sustainability.

The braneworld gravity model by Shiromizu, Maeda and Sasaki (Shiromizu et al., 2000) shows wider perspective about relationship between the 5 D bulk and the 4 D brane. The approach is about projecting the bulk curvature along the brane. The higher dimensional bulk influencing the brane directly. In more rigorous explanation, our universe as 4 D brane is described as 3+1 brane consist of 3 D spatial and 1 D time. The 3 D spatial brane can be represented by metric $q_{\mu\nu}$ and the 5 D spacetime can be represented by $g_{\mu\nu}$. Thus in general the metric relation can be written as

$$q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \, , \tag{2.2}$$

where n_μ is the vector normal to the 3 D space (the universe). From this metric using the method from Shiromizu, Maeda and Sasaki (Shiromizu et al., 2000) the gravitational equation on 3+1 brane can be derived. This may show an interesting result that can be applied in our wormhole sustainability model.

2.4 Bulk influencing gravity

In this section, we elaborate in detail the derivation method by Shiromizu, Maeda and Sasaki (Shiramizu et al., 2000) to show how the higher dimensional bulk influencing the spacetime curvature in this universe. We start from the Gauss-Codazzi equations that represent 4 dimensional spacetime curvature in term of an intrinsic 5 D geometry and extrinsic curvature (Mannheim, 2005).

The Gauss-Codazzi equations are

$$ {}^{(4)}R^{\alpha}_{\beta\gamma\delta} = {}^{(5)}R^{\mu}_{\nu\rho\sigma} q^{\alpha}_{\mu} q^{\nu}_{\beta} q^{\rho}_{\gamma} q^{\sigma}_{\delta} + K^{\alpha}_{\gamma} K_{\beta\delta} - K^{\alpha}_{\delta} K_{\beta\gamma} \ , \tag{2.3} $$

and

$$ D_{\nu} K^{\nu}_{\mu} - D_{\mu} K = R_{\rho\sigma} n^{\sigma} q^{\rho}_{\mu} \ , \tag{2.4} $$

where D_{μ} is the covariant differentiation with respect to the brane metric $q_{\mu\nu}$ and $K_{\mu\nu}$ is the extrinsic curvature of the brane which is the 3 D spatial space of 4 D spacetime. It is denoted by

$$ K_{\mu\nu} = q^{\alpha}_{\mu} q^{\beta}_{\nu} \nabla_{\alpha} n_{\beta} \ . \tag{2.5} $$

Contracting Equation (2.3) by $\alpha = \gamma$

$$ {}^{(4)}R^{\alpha}_{\beta\alpha\delta} = {}^{(5)}R^{\mu}_{\nu\rho\sigma} q^{\alpha}_{\mu} q^{\nu}_{\beta} q^{\rho}_{\alpha} q^{\sigma}_{\delta} + K^{\alpha}_{\alpha} K_{\beta\delta} - K^{\alpha}_{\delta} K_{\beta\alpha} \ , \tag{2.6} $$

which then becomes

$$ {}^{(4)}R_{\beta\delta} = {}^{(5)}R^{\mu}_{\nu\rho\sigma} q^{\rho}_{\mu} q^{\nu}_{\beta} q^{\sigma}_{\delta} + KK_{\beta\delta} - K^{\alpha}_{\delta} K_{\beta\alpha} \ , \tag{2.7} $$

by Equation (2.2) , we expand the mixed metric tensor of a brane in Equation (2.7) that is $q^{\rho}_{\mu} = g^{\rho\nu} q_{u\nu}$ to be $q^{\rho}_{\mu} = g^{\rho\nu}\left(g_{\mu\nu} - n_u n_\nu\right)$. Thus the result of multiplying convariant and contravariant metric of metric tensor and normal tensors gives the brane metric tensor expression more elaborately as the following

$$ q^{\rho}_{\mu} = g^{\rho}_{\mu} - n_{\mu} n^{\rho} \quad . \tag{2.8} $$

Applying Equation (2.8), it can be shown that Equation (2.7) can be extended as the following

$$ \begin{aligned} {}^{(4)}R_{\beta\delta} &= {}^{(5)}R^{\mu}_{\nu\rho\sigma}\left(g^{\rho}_{\mu} - n_{\mu} n^{\rho}\right) q^{\nu}_{\beta} q^{\sigma}_{\delta} + KK_{\beta\delta} - K^{\alpha}_{\delta} K_{\beta\alpha}, \\ &= {}^{(5)}R^{\mu}_{\nu\rho\sigma} g^{\rho}_{\mu} q^{\nu}_{\beta} q^{\sigma}_{\delta} - R^{\mu}_{\nu\rho\sigma} n_{\mu} q^{\nu}_{\beta} n^{\rho} q^{\sigma}_{\delta} + KK_{\beta\delta} - K^{\alpha}_{\delta} K_{\beta\alpha}, \\ &= {}^{(5)}R_{\nu\sigma} q^{\nu}_{\beta} q^{\sigma}_{\delta} - R^{\mu}_{\nu\rho\sigma} n_{\mu} q^{\nu}_{\beta} n^{\rho} q^{\sigma}_{\delta} + KK_{\beta\delta} - K^{\alpha}_{\delta} K_{\beta\alpha} \quad , \end{aligned} \tag{2.9} $$

Rearrange the indices, Equation (2.9) can be rewritten as

$$ {}^{(4)}R_{\mu\nu} = {}^{(5)}R_{\rho\sigma} q^{\rho}_{\mu} q^{\sigma}_{\nu} - R^{\alpha}_{\beta\gamma\delta} n_{\alpha} q^{\beta}_{\mu} n^{\gamma} q^{\delta}_{\nu} + KK_{\mu\nu} - K^{\alpha}_{\mu} K_{\nu\alpha} \ , \tag{2.10} $$

Then from Equation (2.10) the same pattern can be initiated for the Einstein tensor on 4 D 3 +1 brane where

$$ {}^{(4)}G_{\mu\nu} = {}^{(5)}G_{\rho\sigma} q^{\rho}_{\mu} q^{\sigma}_{\nu} - {}^{(5)}G^{\alpha}_{\beta\gamma\delta} n_{\alpha} q^{\beta}_{\mu} n^{\gamma} q^{\delta}_{\nu} + KK_{\mu\nu} - K^{\alpha}_{\mu} K_{\nu\alpha} \ , \tag{2.11} $$

where the 5 D Einstein tensor components can be written as

$$^{(5)}G_{\rho\sigma} = {}^{(5)}R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma}\, {}^{(5)}R \quad , \tag{2.12}$$

and

$$^{(5)}G^{\alpha}_{\beta\gamma\delta} n_{\alpha} q^{\beta}_{\mu} n^{\gamma} q^{\delta}_{\nu} = {}^{(5)}R^{\alpha}_{\beta\gamma\delta} n_{\alpha} q^{\beta}_{\mu} n^{\gamma} q^{\delta}_{\nu} - \frac{1}{2}\, {}^{(5)}g_{\mu\nu}\, {}^{(5)}R^{\alpha}_{\beta\gamma\delta} n_{\alpha} n^{\beta} n^{\gamma} n^{\delta} \quad , \tag{2.13}$$

so Equation (2.11) can be rewritten as

$$\begin{aligned} {}^{(4)}G_{\mu\nu} = &\left({}^{(5)}R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma}\, {}^{(5)}R \right) q^{\rho}_{\mu} q^{\sigma}_{\nu} - {}^{(5)}R^{\alpha}_{\beta\gamma\delta} n_{\alpha} q^{\beta}_{\mu} n^{\gamma} q^{\delta}_{\nu} + \frac{1}{2}\, {}^{(5)}g_{\mu\nu}\, {}^{(5)}R^{\alpha}_{\beta\gamma\delta} n_{\alpha} n^{\beta} n^{\gamma} n^{\delta} \\ &+ KK_{\mu\nu} - K^{\alpha}_{\mu} K_{\nu\alpha} \quad , \end{aligned} \tag{2.14}$$

since

$$\begin{aligned} \frac{1}{2}\, {}^{(5)}g_{\mu\nu}\, {}^{(5)}R^{\alpha}_{\beta\gamma\delta} n_{\alpha} n^{\beta} n^{\gamma} n^{\delta} &= \frac{1}{2}\left(q_{\mu\nu} + n_{\mu} n_{\nu} \right) R^{\alpha}_{\beta\gamma\delta} n_{\alpha} n^{\beta} n^{\gamma} n^{\delta} \\ &= \frac{1}{2} q_{\mu\nu} R^{\alpha}_{\beta\gamma\delta} n_{\alpha} n^{\beta} n^{\gamma} n^{\delta} + \frac{1}{2} R^{\alpha}_{\beta\gamma\delta} n_{\alpha} n^{\beta} n^{\gamma} n^{\delta} n_{\mu} n_{\nu} \quad . \end{aligned} \tag{2.15}$$

It can be shown from Shiromizu et al., (Shiromizu, et al., 2000)

$$\frac{1}{2}\, {}^{(5)}g_{\mu\nu}\, {}^{(5)}R^{\alpha}_{\beta\gamma\delta} n_{\alpha} n^{\beta} n^{\gamma} n^{\delta} = R_{\rho\sigma} n^{\rho} n^{\sigma} q_{\mu\nu} - \frac{1}{2} q_{\mu\nu} \left(K^{2} - K^{\alpha\beta} K_{\alpha\beta} \right) \quad , \tag{2.16}$$

thus Equation (2.14) becomes

$$\begin{aligned} {}^{(4)}G_{\mu\nu} = &\left({}^{(5)}R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma}\, {}^{(5)}R \right) q^{\rho}_{\mu} q^{\sigma}_{\nu} - \frac{1}{2} q_{\mu\nu} \left(K^{2} - K^{\alpha\beta} K_{\alpha\beta} \right) + R_{\rho\sigma} n^{\rho} n^{\sigma} q_{\mu\nu} \\ &+ KK_{\mu\nu} - K^{\alpha}_{\mu} K_{\nu\alpha} - {}^{(5)}R^{\alpha}_{\beta\gamma\delta} n_{\alpha} q^{\beta}_{\mu} n^{\gamma} q^{\delta}_{\nu} , \end{aligned} \tag{2.17}$$

which shows that the Einstein curvature tensor in 4 D brane is always influenced by the higher dimensional bulk of 5 D. The 5 D Einstein tensor, in relation with the 5 D energy momentum tensor of the bulk $T_{\rho\sigma}$ can be written as

$$^{(5)}R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} {}^{(5)}R = \kappa_5^2 T_{\rho\sigma} \quad . \tag{2.18}$$

so Equation (2.17) becomes

$$^{(4)}G_{\mu\nu} = \kappa_5^2 T_{\rho\sigma} q_\mu^\rho q_\nu^\sigma + R_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} - \frac{1}{2} q_{\mu\nu} \left(K^2 - K^{\alpha\beta} K_{\alpha\beta} \right)$$
$$+ K K_{\mu\nu} - K_\mu^\alpha K_{\nu\alpha} - {}^{(5)}R^\alpha_{\beta\gamma\delta} n_\alpha q_\mu^\beta n^\gamma q_\nu^\delta \quad . \tag{2.19}$$

It can be proven from where we start by defining Ricci tensor in term of stress energy tensor as

$$R_{\rho\sigma} n^\rho n^\sigma = \frac{\kappa_5^2}{3} \left(T_{\rho\sigma} n^\rho n^\sigma - \frac{1}{2} T_\rho^\rho \right),$$

$$R_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} = \frac{\kappa_5^2}{3} \left(T_{\rho\sigma} n^\rho n^\sigma - \frac{1}{2} T_\rho^\rho \right) q_{\mu\nu},$$

$$R_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} = -\frac{\kappa_5^2}{3} T_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} + \frac{2\kappa_5^2}{3} T_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} - \frac{\kappa_5^2}{6} T_\rho^\rho q_{\mu\nu}.$$

Adding up both sides of the equation above with component representing Bulk energy momentum tensor

$$\kappa_5^2 T_{\rho\sigma} q_\mu^\rho q_\nu^\sigma + R_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} = \kappa_5^2 T_{\rho\sigma} q_\mu^\rho q_\nu^\sigma - \frac{\kappa_5^2}{3} T_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} + \frac{2\kappa_5^2}{3} T_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} - \frac{\kappa_5^2}{6} T_\rho^\rho q_{\mu\nu}.$$

Rewritten the mixed brane metric tensor on the right side of the equation above in term of normal tensors to represent the right hand side of the equation as protruding energy momentum tensor from the bulk space that is

$$\kappa_5^2 T_{\rho\sigma} q_\mu^\rho q_\nu^\sigma + R_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} = \kappa_5^2 T_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} - \frac{\kappa_5^2}{3} T_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} + \frac{2\kappa_5^2}{3} T_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} - \frac{\kappa_5^2}{6} T_\rho^\rho q_{\mu\nu}$$

which is

$$\kappa_5^2 T_{\rho\sigma} q_\mu^\rho q_\nu^\sigma + R_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} = \frac{2\kappa_5^2}{3} T_{\rho\sigma} q_\mu^\rho q_\nu^\sigma + \frac{2\kappa_5^2}{3} T_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} - \frac{\kappa_5^2}{6} T_\rho^\rho q_{\mu\nu} \ ,$$

and thus finally becomes

$$\kappa_5^2 T_{\rho\sigma} q_\mu^\rho q_\nu^\sigma + R_{\rho\sigma} n^\rho n^\sigma q_{\mu\nu} = \frac{2\kappa_5^2}{3} \left[T_{\rho\sigma} q_\mu^\rho q_\nu^\sigma + \left(T_{\rho\sigma} n^\rho n^\sigma - \frac{1}{4} T_\rho^\rho \right) q_{\mu\nu} \right] . \qquad (2.20)$$

From Equation (2.19) and Equation (2.20) the 4 dimensional Einstein tensor in the presence of bulk can be written as

$$\begin{aligned} {}^{(4)}G_{\mu\nu} = {} & \frac{2\kappa_5^2}{3} \left[T_{\rho\sigma} q_\mu^\rho q_\nu^\sigma + \left(T_{\rho\sigma} n^\rho n^\sigma - \frac{1}{4} T_\rho^\rho \right) q_{\mu\nu} \right] - \frac{1}{2} q_{\mu\nu} \left(K^2 - K^{\alpha\beta} K_{\alpha\beta} \right) \\ & + K K_{\mu\nu} - K_\mu^\alpha K_{\nu\alpha} - E_{\mu\nu} \quad , \end{aligned} \qquad (2.21)$$

where

$$E_{\mu\nu} = {}^{(5)}R^\alpha_{\beta\gamma\delta} n_\alpha q_\mu^\beta n^\gamma q_\nu^\delta = {}^{(5)}C^\alpha_{\beta\gamma\delta} n_\alpha q_\mu^\beta n^\gamma q_\nu^\delta \quad , \qquad (2.22)$$

which is the projection of the 5 D Weyl tensor onto the brane. This is the tensor term that connects the brane with the bulk which also known as the tidal SET because it will reduce to sets of equations that will not become nullify or close, thus ensuring the presence of bulk effect to the brane even if the brane is at vacuum condition. That is further explained since the $E_{\mu\nu}$ term, Equation (2.22) is traceless which represent the bulk entity at the minimal state in the absence of other terms that representing the bulk (Bronnikov et al. 2003 and Shiromizu et al. 2000). This shows that the spacetime curvature in our 3+1 brane has always been influenced by the bulk even at the minimal level where the Einstein curvature tensor can be related to the bulk term as the following. The Einstein curvature tensor and the Bulk term tensor or defined as the Bulk traceless entity can be written as

$$G_{\mu\nu} = -E_{\mu\nu}. \tag{2.23}$$

From the Codazzi Equation (2.4) and 5 D curvature tensor Equation (2.18)

$$D_\nu K_\mu^\nu - D_\mu K = \kappa_5^2 T_{\rho\sigma} n^\sigma q_\mu^\rho + \frac{1}{2} g_{\rho\sigma} R n^\sigma q_\mu^\rho. \tag{2.24}$$

At the most fundamental state where spacetime is flat that is Ricci tensor is null the Codazzi equation becomes

$$D_\nu K_\mu^\nu - D_\mu K = \kappa_5^2 T_{\rho\sigma} n^\sigma q_\mu^\rho. \tag{2.25}$$

For more detail treatment of braneworld scenario we have to consider a particular symmetry or form of the energy momentum tensor. Let begin with defining hypersurface coordinate that coincide with a brane is when $Y = 0$, thus the general expression describing the coordinate in term of the vector normal to the 3 D space (the universe) can be written as $n_\mu Y^\mu = y$ thus $n_\mu dY^\mu = dy$. This implies that

$$a^\mu = n^\nu \nabla_\nu n^\mu = 0, \tag{2.26}$$

which represents the condition on the coordinate in the extra-dimension direction. Considering that the 5 D metric has the form similar to Equation (2.1) we may write in explicit representation, the metric as components in 4 D spacetime and 1 extra-dimensional higher bulk protruding coordinate y as

$$ds^2 = q_{\mu\nu} dx^\mu dx^\nu + dy^2. \tag{2.27}$$

We may consider 5 D energy momentum tensor in general as

$$T_{\mu\nu} = -\Lambda g_{\mu\nu} + S_{\mu\nu}\delta(y) \quad , \tag{2.28}$$

where Λ is the cosmological constant of the bulk spacetime and

$$S_{\mu\nu} = -\lambda q_{\mu\nu} + \tau_{\mu\nu} \quad , \tag{2.29}$$

consists of vacuum energy λ and energy momentum tensor $\tau_{\mu\nu}$. Since $\tau_{\mu\nu}$ is representing energy momentum tensor on a 3 D brane thus in relation with the vector normal to the brane it can be written as $\tau_{\mu\nu} n^\nu = 0$. $S_{\mu\nu}$ represents stress energy tensor of the normal matter in 4 D spacetime. Normal is considered here, because only gravity is not constrained to the brane whereas the normal matter is constrained or living only on the brane where $Y=0$. Consider junction condition where

$$[X] := \lim_{Y\to+0} X - \lim_{Y\to-0} X = X^{+} - X^{-},$$

$$[q_{\mu\nu}] := \lim_{Y\to+0} q_{\mu\nu} - \lim_{Y\to-0} q_{\mu\nu} = q_{\mu\nu}{}^{+} - q_{\mu\nu}{}^{-},$$

while Y coordinate approaching the brane the vector normal to 3 D space vanishes thus since metric of both sides of the brane are equal fundamentally $g^{+}_{\mu\nu} = g^{-}_{\mu\nu}$ therefore $[q_{\mu\nu}] = 0$. For the extrinsic curvature (Mannheim, 2005)

$$[K_{\mu\nu}] := \lim_{Y\to+0} K_{\mu\nu} - \lim_{Y\to-0} K_{\mu\nu} = K_{\mu\nu}{}^{+} - K_{\mu\nu}{}^{-},$$

$$K_{\mu\nu}{}^{+} - K_{\mu\nu}{}^{-} = -\kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} q_{\mu\nu} S \right),$$

thus $$K_{\mu\nu}{}^{+} = K_{\mu\nu}{}^{-} = -\frac{1}{2}\kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} q_{\mu\nu} S \right) \quad . \qquad (2.30)$$

Substituting Equation (2.30) into Equation (2.21) gives

$$G_{\mu\nu} = \frac{7}{6}\kappa_5^2 T_{\mu\nu} + \frac{K_{\mu\nu}}{2} - E_{\mu\nu}, \qquad (2.31)$$

From Equation (2.28), (2.29) and (2.30), the Equation (2.31) becomes

$$G_{\mu\nu} = \frac{7}{6}\kappa_5^2 \left(\left(\tau_{\mu\nu} - \lambda q_{\mu\nu} \right) \delta(y) - \Lambda g_{\mu\nu} \right) - \frac{\kappa_5^2}{6} \left(\tau_{\mu\nu} - \lambda q_{\mu\nu} \right) - E_{\mu\nu} \quad . \qquad (2.32)$$

The Equation (2.32) above represents the ordinary Einstein curvature tensor influenced by the higher dimensional bulk of 4+1 brane. Extending Equation (2.32) we have

$$G_{\mu\nu} = \frac{7}{6}\kappa_5^2\tau_{\mu\nu}\delta(y) - \frac{7}{6}\kappa_5^2\lambda q_{\mu\nu}\delta(y) - \frac{7}{6}\kappa_5^2\Lambda g_{\mu\nu} - \frac{\kappa_5^2}{6}\tau_{\mu\nu} + \frac{\kappa_5^2}{6}\lambda q_{\mu\nu} - E_{\mu\nu}. \tag{2.33}$$

Rearrange Equation (2.33)

$$G_{\mu\nu} = -\frac{7}{6}\kappa_5^2\Lambda g_{\mu\nu} - \frac{7}{6}\kappa_5^2\lambda q_{\mu\nu}\delta(y) + \frac{\kappa_5^2}{6}\lambda q_{\mu\nu} + \frac{\kappa_5^2}{6}\left(7\delta(y)-1\right)\tau_{\mu\nu} - E_{\mu\nu},$$

$$G_{\mu\nu} = -\frac{7}{6}\kappa_5^2\Lambda g_{\mu\nu} - \frac{7}{6}\kappa_5^2\lambda g_{\mu\nu}\delta(y) + \frac{7}{6}\kappa_5^2\lambda n_\mu n_\nu\delta(y)$$

$$+\frac{\kappa_5^2}{6}\lambda q_{\mu\nu} + \frac{\kappa_5^2}{6}\left(7\delta(y)-1\right)\tau_{\mu\nu} - E_{\mu\nu} \quad ,$$

$$G_{\mu\nu} = -\frac{7}{6}\kappa_5^2\Lambda g_{\mu\nu} + \frac{7}{6}\kappa_5^2\lambda n_\mu n_\nu\delta(y) - \frac{7}{6}\kappa_5^2\lambda g_{\mu\nu}\delta(y)$$

$$+\frac{\kappa_5^2}{6}\lambda q_{\mu\nu} + \frac{\kappa_5^2}{6}\left(7\delta(y)-1\right)\tau_{\mu\nu} - E_{\mu\nu},$$

$$G_{\mu\nu} = -\frac{7}{6}\kappa_5^2\left(\Lambda g_{\mu\nu} - \lambda n_\mu n_\nu\delta(y)\right) - \frac{7}{6}\kappa_5^2\lambda g_{\mu\nu}\delta(y)$$

$$+\frac{\kappa_5^2}{6}\lambda q_{\mu\nu} + \frac{\kappa_5^2}{6}\left(7\delta(y)-1\right)\tau_{\mu\nu} - E_{\mu\nu}. \tag{2.34}$$

Let $\lambda\delta(y) = \Lambda$ and $\frac{7}{6}\kappa_5^2\Lambda = \Lambda_4$ thus

$$G_{\mu\nu} = -\frac{7}{6}\kappa_5^2\left(\Lambda g_{\mu\nu} - \Lambda n_\mu n_\nu\right) - \frac{7}{6}\kappa_5^2\Lambda g_{\mu\nu}$$

$$+\frac{\kappa_5^2}{6}\lambda q_{\mu\nu} + \frac{\kappa_5^2}{6}\left(7\delta(y)-1\right)\tau_{\mu\nu} - E_{\mu\nu} \quad , \tag{2.35}$$

since from Equation (2.2) then Equation (2.35) can be rewritten as

$$G_{\mu\nu} = -\frac{7}{6}\kappa_5^2\Lambda q_{\mu\nu} - \frac{7}{6}\kappa_5^2\Lambda g_{\mu\nu} + \frac{\kappa_5^2}{6}\lambda q_{\mu\nu} + \frac{\kappa_5^2}{6}\left(7\delta\left(y\right)-1\right)\tau_{\mu\nu} - E_{\mu\nu}, \tag{2.36}$$

then let $\frac{7}{6}\kappa_5^2\Lambda = \Lambda_4$ thus

$$G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} - \Lambda_4 g_{\mu\nu} + \frac{\kappa_5^2}{6}\lambda q_{\mu\nu} + \frac{\kappa_5^2}{6}\left(7\delta\left(y\right)-1\right)\tau_{\mu\nu} - E_{\mu\nu} \quad . \tag{2.37}$$

At the most fundamental level where we consider there exist only pure bulk as the background of the brane that consist nothing else than just membrane that is when $\tau_{\mu\nu} = 0$, $\lambda = 0$ and $\Lambda_4 = 0$ thus it has been shown to reduce to equation (2.23) which implies that the higher dimensional bulk may influence spacetime curvature of the brane without the requirement of matter as well as exotic matter.

2.5 Wormholes

The possibility of the existence of wormholes has long been considered as a topologically non-trivial exact solution to the Einstein field equations. Einstein field equations are derived from general relativity theory that shows the relation between spacetime-curvature and the energy momentum tensor that represent gravitation and matter or energy respectively. The equations are nonlinear and are therefore very complicated. However, Schwarzschild (Dirac, 1996) had considered a special case, which provides an exact solution to the equations; namely, the static spherically symmetric field produced by a spherically symmetric body at rest. Later it has been shown as highlighted by Einstein and Rosen that the Schwarzschild solution has a singularity (Einstein and Rosen, 1935). The idea of singularities was then propagated towards further research on black holes. Not satisfied with the possibility of the existence of singularity, Einstein and Rosen (Einstein and Rosen, 1935), build a geometrical model of a physical elementary “particle” that was everywhere finite and singularity-free to oppose some ideas during the era that material particles might be considered as singularities of the field. Einstein and Rosen solutions provide the mathematical representation of physical space of two identical sheets where a particle is represented by a “bridge” connecting these sheets. This model of elementary particle however was considered a failure but it had generated the idea of Einstein-Rosen bridge and as coined later with the term “wormhole” by Wheeler (Wheeler, 1962) whom ideas concerning “spacetime foam”, were then introduced. Kerr wormhole (Kerr, 1963) can be considered as the firstly introduced rotating wormhole derived from the rotating blackhole spacetime metric. A slowly rotating Kerr wormhole still possesses the event horizon whereas a rapidly rotating Kerr wormhole even though eliminates the event horizon but still possesses the singularity and thus it is non-traversable.

The idea of a traversable wormhole, was firstly introduced by Morris and Thorne (Morris and Thorne, 1988). Unlike the Einstein-Rosen bridge, which was meant to represent an elementary particle, or the Wheeler’s wormhole described as a microscopic charged-carrying wormhole, a traversable wormhole (Karasnikov, 2002) by definition, permits the two way travel of objects. Despite the questionable existence of such wormholes either naturally or constructed by an advanced civilization (Ford and Roman, 2000), their study has generated newly exciting areas of research involving its fundamental properties, the application for faster-than-light travel and the associated problems of causality violation as suggested in Hawking’s paper concerning the chronology protection conjecture (Hawking, 1988 and 1992).

2.6 Wormhole spacetime geometry

The Einstein field equations are complicated due to its non-linearity even for vacuum without cosmological constant. However, the simplest non-trivial solution to the equations is the Schwarzschild geometry, which is a unique spherically symmetric vacuum solution. The solution describes a spherically symmetric body at rest that produce a static spherically symmetric field. The static condition means that, with a static coordinate system, the metric tensor $g_{\mu\nu}$ are independent of the time x^0 or t and also $g_{0\beta}=0$ (where β represents spatial coordinate indices; 1, 2 and 3). The spatial coordinates may be taken to be spherically polar coordinates where $x^1=r$, $x^2=\theta$, $x^3=\varphi$. The most general form for ds^2 compatible with spherical symmetry is (D' Inverno, 1992)

$$ds^2 = Adt^2 - Bdr^2 - Cr^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{2.38}$$

where A, B, and C are functions of r only. Therefore without loss of generality or disturbing the spherical symmetry we can replace A, B and C by any function of r. By virtue of this freedom, the expression for ds^2 may be written as (D' Inverno, 1992)

$$ds^2 = e^{\nu}c^2dt^2 - e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2\,, \tag{2.39}$$

with ν and λ are functions of r only.

From Equation (2.39) we can define the metric tensor

$$g_{\mu\nu} = diag\left[e^{\nu}c^2, -e^{\lambda}, -r^2, -r^2 \sin^2\theta\right] \ , \qquad (2.40)$$

and for the diagonal matrix, the conjugate metric tensor will simply be $g^{\mu\nu} = 1/g_{\mu\nu}$. Thus

$$g^{\mu\nu} = diag\left[e^{-\nu}c^{-2}, -e^{-\lambda}, -r^{-2}, -r^{-2}\sin^{-2}\theta\right] \ . \qquad (2.41)$$

Further calculations are carried out for the affine connection by using (2.41) and we obtain the non-zero affine connections listed below:

$$\begin{aligned}
&\Gamma^0_{00} = \frac{\dot{\nu}}{2} \ ; && \Gamma^0_{01} = \frac{\nu'}{2} \ ; \\
&\Gamma^0_{11} = \frac{e^{\lambda-\nu}}{2c^2}\dot{\lambda} \ ; && \Gamma^1_{00} = \frac{e^{\nu-\lambda}}{2}\nu' c^2 \ ; \\
&\Gamma^1_{01} = \frac{\dot{\lambda}}{2} \ ; && \Gamma^1_{11} = \frac{\lambda'}{2} \ ; \\
&\Gamma^1_{22} = -re^{-\lambda} \ ; && \Gamma^1_{33} = -re^{-\lambda}\sin^2\theta \ ; \\
&\Gamma^2_{12} = \frac{1}{r} \ ; && \Gamma^2_{33} = -\sin\theta\cos\theta \ ; \\
&\Gamma^3_{13} = \frac{1}{r} \ ; && \Gamma^3_{23} = \cot\theta \ . \qquad (2.42)
\end{aligned}$$

Here we denote derivatives with respect to time t and radius r as dot and prime respectively. These expressions are to be substituted in the Ricci tensor expression

$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\gamma_{\mu\nu}\Gamma^\alpha_{\gamma\alpha} - \Gamma^\gamma_{\mu\alpha}\Gamma^\alpha_{\gamma\nu}$ for obtaining the components of Ricci tensor. The results are

$$R_{00} = \frac{e^{\nu-\lambda}c^2}{2}\left(\nu'' + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2} + \frac{2\nu'}{r}\right) - \frac{\ddot{\lambda}}{2} + \frac{\dot{\nu}\dot{\lambda}}{4} - \frac{\dot{\lambda}^2}{4} \ ,$$

$$R_{01} = \frac{\dot{\lambda}}{r} \ ,$$

$$R_{11} = \frac{e^{\lambda-\nu}}{2c^2}\left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2}\right) - \frac{\nu''}{2} + \frac{\lambda'\nu'}{4} - \frac{\nu'^2}{4} + \frac{\lambda'}{r} \ ,$$

$$R_{22} = -e^{-\lambda} + \frac{re^{-\lambda}}{2}\lambda' - \frac{re^{-\lambda}}{2}\nu' + 1 \ ,$$

$$R_{33} = \sin^2\theta R_{22} \ . \qquad (2.43)$$

From these Ricci tensor components we can derive the Ricci scalar by contracting the Ricci tensor $R_{\mu\nu}$ with the conjugate metric tensor $g^{\mu\nu}$; $R = g^{\mu\nu}R_{\mu\nu}$ as to obtain the following equation

$$R = e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2} - \frac{2\lambda'}{r} + \frac{2\nu'}{r} + \frac{2}{r^2}\right) + \frac{e^{-\nu}}{c^2}\left(\frac{\dot{\lambda}\dot{\nu}}{2} - \frac{\dot{\lambda}^2}{2} - \ddot{\lambda}\right) - \frac{2}{r^2} \ . \qquad (2.44)$$

Using $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, we obtain the following Einstein curvature tensor

$$G_{00} = e^{\nu-\lambda}c^2\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{e^{\nu}c^2}{r^2} \ ,$$

$$G_{01} = R_{01} = \frac{\dot{\lambda}}{r} \ ,$$

$$G_{11}=\frac{\nu'}{r}-\frac{e^{\lambda}}{r^2}+\frac{1}{r^2}\ ,$$

$$G_{22}=\frac{e^{-\lambda}}{2}\left(\nu' r-\lambda' r+\nu'' r^2+\frac{\nu'^2 r^2}{2}-\frac{\nu'\lambda' r^2}{2}\right)$$

$$-\frac{e^{-\nu}}{2c^2}\left(\ddot{\lambda} r^2+\frac{\dot{\lambda}^2 r^2}{2}-\frac{\dot{\lambda}\dot{\nu} r^2}{2}\right)$$

$$G_{33}=\sin^2\theta G_{22}\,. \tag{2.45}$$

The mixed Einstein tensor can be derived from $G_\nu^\gamma=g^{\gamma\mu}G_{\mu\nu}$ and the results are

$$G_0^0=e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}\ ,$$

$$G_0^1=-\frac{e^{-\lambda}}{r}\dot{\lambda}\,,$$

$$G_1^1=-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}\ ,$$

$$G_2^2=\frac{e^{-\lambda}}{2}\left(\frac{\lambda'}{r}-\frac{\nu'}{r}-\nu''-\frac{\nu'^2}{2}+\frac{\nu'\lambda'}{2}\right)$$

$$+\frac{e^{-\nu}}{2c^2}\left(\frac{\dot{\lambda}^2}{2}-\frac{\dot{\nu}\dot{\lambda}}{2}+\ddot{\lambda}\right),$$

$$G_3^3=G_2^2\ . \tag{2.46}$$

Any generic static spherically symmetric spacetime can be expressed in general canonical form as (Dirac, 1996)

$$ds^2=-g_{00}(r)dt^2+g_{11}(r)dr^2+r^2d\Omega^2\,.$$

If the spacetime metric contains a horizon it is of convenience for similarity with the Schwarzschild geometry to choose $g_{00}(r)=e^{-2\Phi(r)}\left(1-b(r)/r\right)$ and

$g_{11}(r) = \left(1 - b(r)/r\right)^{-1}$. The horizon can exist in this metric when $b(r) = r$. The presence of horizon in this type of geometry indicates that the geometry may describe a black hole. In the absence of horizon in the geometry it is advantageous to set

$$g_{00}(r) = e^{2\Phi(r)}, \tag{2.47}$$

and thus the spacetime metric will be (Visser, 1995)

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2 . \tag{2.48}$$

Morris and Thorne (Morris and Thorne, 1988) have chosen the form given by (2.48) for a traversable wormhole. The profile of the wormhole spacetime metric is shown in Figure 2.3

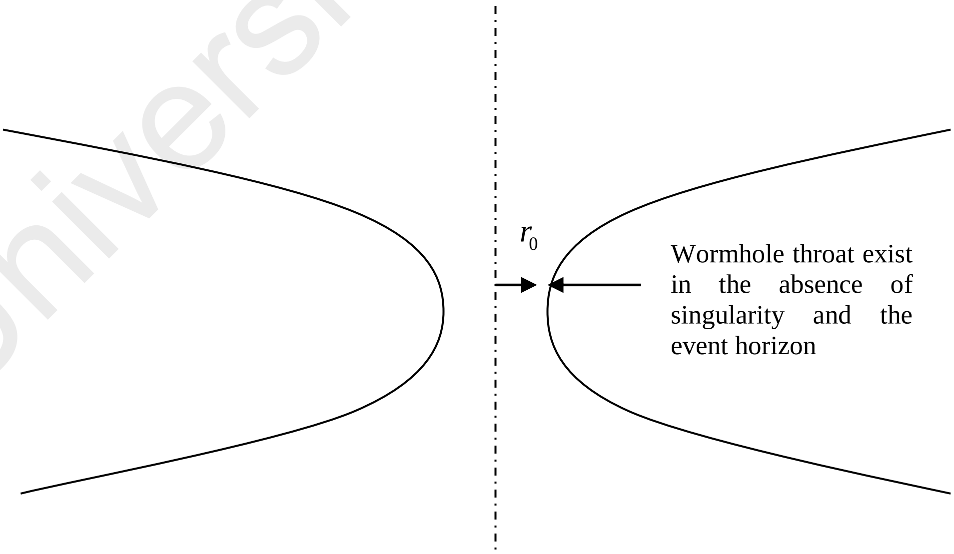

Figure 2.3 : Wormhole spacetime metric profile.

2.7 Traversable wormhole physical characteristics

The Einstein-Rosen bridge and the Wheeler wormhole discussed previously are non traversable. The solution by Einstein and Rosen, even though it has no singularity, however shows the existence of the event horizon. Therefore, the wormhole is non-traversable since once a test particle enters the region of the event horizon there is no way for the particle to escape from that region. A problem for traversability if the event horizon exists is the effect of time dilation similar to the case of a black hole where the observer will see the traveler approaching the event horizon in an infinite amount of time. Wheeler wormhole on the other hand is simply too small. It exists only in a microscopic scale. Kerr wormhole can eliminate the event horizon at high speed rotation but the naked singularity makes it still non-traversable.

Morris and Thorne (Morris and Thorne, 1988) introduced the solution for a traversable wormhole. It is possible, at least in principle, to construct a suitably behaved traversable wormhole spacetime. From the above discussion besides the absence of singularity, the absence of event horizon can be regarded as the minimum requirement for traversability. In their approach, they assumed the existence of a suitable geometry based on the Schwarzschild line element. Riemann tensor (Morgan, 1998 and Srivastava, 1992) associated with this geometry is then calculated. Then, from the Einstein field equations, the stress energy terms which are distributed at the wormhole are deduced as shown below (Karasnikov, 2002):-

The energy density

$$\rho = \frac{c^2}{8\pi G}\frac{b'}{r} , \tag{2.49}$$

the radial pressure

$$p_{rad} = \frac{c^4}{8\pi G}\left(\frac{b}{r^3} - 2\frac{\Phi'}{r}\left(1-\frac{b}{r}\right)\right) \ , \tag{2.50}$$

the lateral pressure

$$p_{\perp} = \frac{c^4}{8\pi G}\left(1-\frac{b}{r}\right)\left(\Phi'' + \Phi'^2 - \frac{b'r-b}{2r^2\left(1-\frac{b}{r}\right)}\Phi' + \frac{b'r-b}{2r^3\left(1-\frac{b}{r}\right)} + \frac{\Phi'}{r}\right) . \tag{2.51}$$

The tangential pressure or lateral pressure $p = p_{\perp}$ is measured in the tangential directions orthogonal to the radial direction. The radial tension is the negative of the radial pressure, that is $\tau = -p_r = -p_{rad}$. The stress energy terms can be depicted in the following wormhole spacetime profile embedding diagram of Figure 2.4.

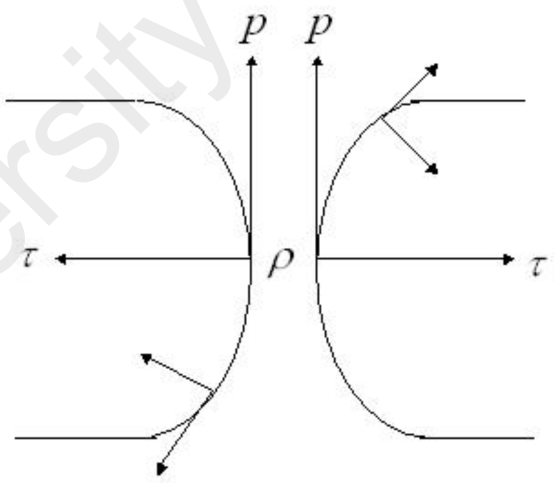

Figure 2.4 : Stress energy terms.

From the stress energy terms, Morris and Thorne found that the energy condition is violated at the throat of the wormhole where the radial tension (opposite of radial pressure) $\tau_0 = -p_0$ is larger than the energy density term ρ_0.

This can be written as

$$\tau_0 - \rho_0 > 0, \tag{2.52}$$

and thus,

$$\rho_0 + p_0 < 0. \tag{2.53}$$

This shows that the energy condition is violated. From the classical perspective, the violation of the energy condition is impossible, however through quantum mechanics, the null energy condition in particular can be violated. Thus the possibility of an existence of a traversable wormhole is always there for us to explore.

In classical general relativity, there are seven types of energy conditions (EC) normally discussed (Visser, 1995). These are: the null (NEC), the weak (WEC), the strong (SEC), the dominant (DEC) energy conditions and the averaged null (ANEC), the averaged weak (AWEC) and the averaged strong (ASEC) energy conditions. From the energy momentum tensor of Hawking-Ellis type (Hawking and Ellis, 1987), the tensor components are given by

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{bmatrix}, \tag{2.54}$$

which consist of the energy density and the three principal pressure i.e. the radial pressure (p_1), radial tension ($-p_1$) and the tangential pressure (p_2 and p_3). The traversable wormhole throat condition violates the WEC, NEC, DEC, and SEC as found by Morris and Thorne. Thus, we define the respective energy conditions as follows.

The null energy condition (NEC) states that for any null vector k^{μ}

$$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0 . \tag{2.55}$$

In term of the principal pressure, it is

$$\rho + p_j \geq 0 \ , \qquad \forall j \ , \quad j = 1, 2, 3. \tag{2.56}$$

The weak energy condition (WEC) states that for any timelike vector

$$T_{\mu\nu}V^{\mu}V^{\nu} \geq 0 . \tag{2.57}$$

Thus, this also implies the null energy condition. In addition, the local energy density as measured by any timelike observer is positive. Therefore, in terms of the principal pressure

$$\rho \geq 0 \qquad \text{and}$$
$$\rho + p_j \geq 0 , \qquad \forall j \ , \quad j = 1, 2, 3. \tag{2.58}$$

The strong energy condition (SEC) states that for any timelike vector

$$\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) V^{\mu}V^{\nu} \geq 0 \ , \tag{2.59}$$

where T is the trace of the stress energy tensor given by $T = T_{\mu\nu} g^{\mu\nu}$. The strong energy condition implies the null energy condition, however, it does not implies in general the weak energy condition. In terms of the principal pressure

$$T = -\rho + \sum_j p_j , \quad \rho + p_j \geq 0 \quad \text{and} \quad \rho + \sum_j p_j \geq 0 ,$$

$$\forall j , \quad j = 1, 2, 3. \tag{2.60}$$

The dominant energy condition (DEC) states that for any timelike vector $T_{\mu\nu} V^{\mu} V^{\nu} \geq 0$ where $T_{\mu\nu} V^{\nu}$ is not spacelike. This shows that the locally measured energy density is always positive and the energy flux is timelike or null. The dominant energy condition implies the weak energy condition and thus the null energy condition.

In terms of the principal pressure

$$\rho \geq 0 \quad \text{and} \quad p_j \in [-\rho, +\rho], \qquad \forall j , \quad j = 1, 2, 3. \tag{2.61}$$

Comparing all these principal pressures with Equation (2.53), it shows clearly that the traversable wormhole condition at the throat violates the WEC, NEC, DEC and SEC.

There are examples of physical system theoretically or experimentally that are known to violate the energy condition and thus allows the possibility that the conditions of a traversable wormhole to exist. The case of the Casimir effect is a good experimental example, while the Hawking evaporation is a good theoretical example concerning quantum effects that violates the energy condition. In addition, the observation and theoretical study of the cosmological inflation are the best examples concerning the existence of the negative energy that in principle violates the energy condition.

The definition of the energy condition and the examples of the energy condition violation mentioned above together with the analysis of the traversable wormhole by Morris and Thorne shows that for a wormhole to be a traversable system (Visser, 1995)

$$\exists\ \bar{r}_* \qquad \forall r \in \left(r_0, \bar{r}_*\right), \quad \left(\rho + p\right) < 0 \text{ or } \left(\rho - \tau\right) < 0 \ , \tag{2.62}$$

where $\bar{r}_*$ is at any finite radial distance. Thus, it specifically shows that the energy density at the throat $\rho\left(r_0\right) = \rho_0$ is smaller than the radial tension at the throat where $\tau\left(r_0\right) = \tau_0$, therefore, it can be written as

$$\rho_0 - \tau_0 \leq 0 \ . \tag{2.63}$$

This in particular shows that the null energy condition is violated over the finite range $\left(r_0, \bar{r}_*\right)$ near the traversable wormhole throat which implies the violations of the weak, strong and the dominant energy condition. Therefore for traversability, the throat must be threaded by exotic matter whose definition is the matter that violates the energy conditions. Morris and Thorne (Morris and Thorne, 1988) introduced the dimensionless function “exoticity” (Lemos et al., 2003) which is the ratio of the difference between radial pressure and energy density $\tau - \rho c^2$ that represent the energy condition violation over the modulus the energy density $\left|\rho c^2\right|$ that can be expanded in terms of the wormhole shape function and red shift function as the following

$$\varsigma = \frac{\tau - \rho c^2}{\left|\rho c^2\right|} = \frac{\frac{b}{r} - b' - 2(r-b)\Phi'}{|b'|}. \tag{2.64}$$

In the presence of the cosmological constant Lemos et al. (Lemos et al., 2003) suggested that

$$\varsigma = \frac{\frac{b}{r} - b' - 2(r-b)\Phi'}{\left|b' - \Lambda r^2\right|}. \tag{2.65}$$

This expression shows that when it is positive, the matter presence in the traversable wormhole system is “exotic”.

However subject of "exoticity" can be eliminated by brane cosmology as discussed earlier. Equation (2.23) implies that the spacetime which is the 3+1 brane is a projection of 5 D Weyl tensor. It represents the connection between gravity on the brane and the bulk geometry. Thus the projection of 5 D Weyl tensor $E_{\mu\nu}$ which itself is a traceless tensor may represent the most natural matter to support the wormhole without considering "exoticity" of exotic matter as discussed earlier by Lemos et al. (Lemos et al., 2003). Consider a general spherically symmetric line element with the form of

$$ds^2 = e^{2\gamma(r)} c^2 dt^2 - e^{2\alpha(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \quad . \tag{2.66}$$

The metric tensor of this line element Equation (2.66) provides mixed projected Weyl tensor that directly represent the stress energy tensor in a braneworld (Shiromizu et al., 2000) namely the energy density $\rho = E_t^t$ the radial pressure $p_{rad} = -E_r^r$ and the lateral pressure $p_{\perp} = -E_\theta^\theta = -E_\phi^\phi$ as the following

$$\rho = -\left\{ e^{-2\alpha}\left(\frac{2\alpha'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \right\} \quad , \tag{2.67}$$

$$p_{rad} = -e^{-2\alpha}\left(\frac{2\gamma'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \quad , \tag{2.68}$$

$$p_{\perp} = \frac{e^{-2\alpha}}{2}\left(\frac{\gamma'}{r} - \frac{\alpha'}{r} + \gamma'' + \gamma'^2 - \alpha'\gamma'\right). \tag{2.69}$$

These has shown that the traceless tensor that connects gravity on the brane can be representing the physical characteristic of the spacetime metric directly.

The most recent traversable wormhole in brane framework is the braneworld wormhole model base on DGP (Dvali-Gabadadze-Porrati) (Dvali et al., 2000) braneworld scenario proposed by Ming and Wang (Ming and Wang, 2017). The DGP braneworld traversable wormhole spacetime configurations are supported by the effects of gravity leakage into extra dimensions. Ming and Wang (Ming and Wang, 2017) explored the energy conditions from observational cosmology perspective where they have obtained some general conclusions that are about classes of exotic matter threading the wormhole since the beginning of the evolutionary process of the universe, wormhole spacetime structure restriction by current astrophysical observation and about the evidence from astrophysical observation that support the wormhole structure i.e. the dark energy or the MOG (Modified Gravity) (Clifton et al., 2012) spacetime curvature effects signature. Energy conditions at astrophysical as well as at cosmological scale has been investigated. Energy conditions and the wormhole structure relationship has been derived through observational cosmology perspective where WEC and DEC are satisfied within certain range representing z axis of wormhole. The works of Ming and Wang (Ming and Wang, 2017) has shown the dynamics relationship between the universe of cosmological scale with the energy conditions characteristic that affects wormhole structure at an astrophysical scale supported by observational evidence signifying the relationship. The idea of finding relationship between cosmological scale and astrophysical scale has been introduced also by the works of Ming and Wang (Ming and Wang, 2017) and thus has also inspired the next two chapters of these thesis that deal with the relationship of cosmological dynamics of the universe in brane cosmology (Maarten, 2000) and the stress energy tensor of the wormhole as astrophysical scale supporting by propose observational evidence through gravitational lens characteristic.

2.8 Gravitational lens of wormholes

The bending of light from a distance star way behind the sun proven by astronomers was in fact the first experimental method to verify Einstein's theory of General Relativity (GR) (Dyson et al., 1919). In fact, the deflection of light rays by massive celestial objects have been noticed even by the 17th century astronomers in the era of Newton, and was described by the Newtonian deflection angle. However the Newtonian era deflection angle was not accurate enough and there was not any profound explanation for the reason of the deflection, until Einstein introduced the theory of general relativity that describes precisely the effect that is now known as gravitational lensing. Light follows the shortest path, optimized as a geodesic of spacetime curved or warped through the large energy-momentum tensor representing a massive nearby object such as the sun.

Wormhole, just like any other celestial objects that warp the surrounding spacetime, show the astrophysical signature of gravitational lensing. The expression for the deflection angle due to gravitational lensing, derived from various spacetime line elements, by itself can show some small variation which depends on the character of the lens object spacetime line element, whether it be of a Schwarzchild object with singularity, as with a blackhole, or with flaring out effects, as with a wormhole. The braneworld scenerio derived from string and *M* theory provides the idea of spacetime with underlying extra dimensions. The inclusion of extra-dimensional terms in the lens object spacetime line element will result in some variation in the expression for its gravitational lens deflection angle. Thus in the Chapter 4 we investigate such variation by deriving this deflection angle expression. We will not only show the existence of such variation but also suggest the potential search for the existence of extra dimensions as predicted in the braneworld.

Gravitational lensing occurs in many astrophysical circumstances and is becoming significant in modern cosmology. The effect of the lensing is the distortion of an angle of observation $\hat{\alpha}$ from an angle normally measured without the presence of deflection, θ, that is

$$\hat{\alpha} = \theta - \beta, \tag{2.70}$$

this angle is related to the actual deflection angle α that occurs at the lens in relation with distance components.

It is defined as

$$\hat{\alpha} = \frac{d_{LS}\alpha}{d_{OS}}, \tag{2.71}$$

where d_{LS} and d_{OS} are distances between the lens and the source, and the observer and the source respectively, and α is basic deflection equation given by $\alpha = 4GM(\xi)/c^2\xi$ where ξ is impact parameter in term of distance toward the center of the gravitational lens source. This gives the lens equation as

$$\theta - \beta = \frac{d_{LS}\alpha}{d_{OS}} \quad , \tag{2.72}$$

which came from the ratio that angle of observation over angle of deflection is the same as ratio between distance of the lens from the source over the distance from the observer to the source. This also describes ray-tracing in a perturbed spacetime by showing the relation between the position of the source and position of the image. All these parameters can be depicted in the following diagram.

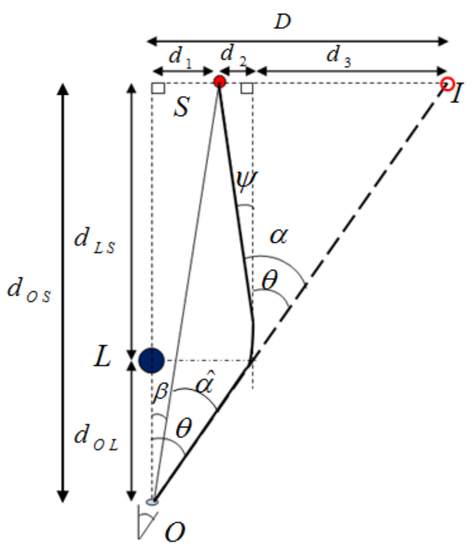

Figure 2.5 : Gravitational lens parameters.

Figure 2.5 shows the setup of the gravitational lensing where light emitted from the source S which could be any far distance celestial object such as a star. The emitted light from the star is deflected by lens L which is in this paper we would consider it as a wormhole in a braneworld framework. Due to the deflection, from the perception of the observer O the star apparently located as an image I at an angle $\hat{\alpha}$ from the original angle β toward the true position of the star, that is the source. We may also derive the trigonometric expression relating the angles of θ, β and $\hat{a}$. From Figure 2.5

$$D = d_1 + d_2 + d_3 \ , \tag{2.73}$$

which trigonometrically shows

$$\frac{d_1}{d_{OS}} = \tan\beta \, , \ \frac{d_2}{d_{LS}} = \tan\psi \, , \frac{d_3}{D} = \frac{d_{LS}}{d_{OS}} \text{ and } \frac{D}{d_{OS}} = \tan\theta \ , \tag{2.74}$$

thus Equation (2.73) can be rewritten as

$$d_{OS}\tan\theta = d_{OS}\tan\beta + d_{LS}\tan\psi + d_{LS}\tan\theta \quad , \tag{2.75}$$

and as $\psi = \alpha - \theta$ so we may write the gravitational lens equation (Virbhadra et al., 2000) as

$$\tan\theta - \tan\beta = \frac{d_{LS}}{d_{OS}}\left[\tan\theta + \tan\left(\alpha - \theta\right)\right] \quad . \tag{2.76}$$

Thus it can be shown trigonometrically the geometric set up of a gravitational lensing. Considering a spherically symmetric spacetime with the generic line element as the following,

$$ds^2 = -A(x)dt^2 + B(x)dx^2 + C(x)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.77}$$

the closest approach of light tracing toward the object that causes the deflection can be defined as

$$x_o = \frac{r_o}{2m} \ . \tag{2.78}$$

where r_0 is the event horizon radius for the case of black hole (Virbhadra and Ellis, 2000) and m is the mass of the object. The deflection angle can be written as function of x_0

$$\alpha(x_o) = I(x_o) - \pi \ . \tag{2.79}$$

With the integral term $I(x_0)$ which actually represents a deflected axial angle close to the object that causes the deflection. To derive the integral equation representing the deflection angle near the lens, we may start by considering the null geodesic character of light emitted from the source traced through the spacetime that is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. So from Equation (2.77), for the null geodesic character of a wormhole shape with respect to time we may write

$$-A(x)dt^2 + C(x)d\theta^2 = 0\,, \tag{2.80}$$

which is represented by time and the wormhole longitudinal or polar angle θ. For null geodesic character of a wormhole spacetime warping with respect to the wormhole radii we may write

$$B(x)dx^2 + C(x)\sin^2\theta d\phi^2 = 0, \tag{2.81}$$

which is represented by the wormhole radii and the wormhole latitudinal or lateral angle (also defined as axial angle ϕ). From Equation (2.80) and Equation (2.81)

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{A(x)}{C(x)} \quad , \tag{2.82}$$

and

$$\left(\frac{d\phi}{dx}\right)^2 = -\frac{B(x)}{C(x)\sin^2\theta}. \tag{2.83}$$

The lens impact parameter is defined as $1/(d\theta/dt) = u$ which is the ratio of metric tensors $u = g_{22}/g_{00}$ so from Equation (2.82) $u = \sqrt{C(x)/A(x)}$. The impact parameter is larger toward the lens when $x \to x_0$ at the region closer toward the source of the lens than at a region a little away from the lens that is $u_o > u_{(region-away)}$ and thus

$$\sqrt{C(x_o)/A(x_o)} > \sqrt{C(x)/A(x)} \quad , \tag{2.84}$$

or we may write $\sqrt{C(x_o)A(x)} > \sqrt{C(x)A(x_o)}$ where the relation between the radius nearer to the lens and radius a little further away from the lens in response to the curvature angle θ as the light from the source S comes nearer toward the lens L can be depicted as

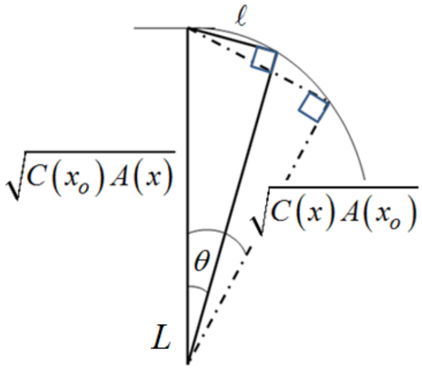

Figure 2.6 : The effect of curvature angle in relation with impact parameter.

thus we may write

$$\ell = \left(C(x_o)A(x) - C(x)A(x_0)\right)^{\frac{1}{2}} \quad , \tag{2.85}$$

and since from Figure 2.6 $\sin\theta = \ell \big/ \sqrt{C(x_o)A(x)}$ thus from Equation (2.85) it can be shown

$$\sin^2\theta = 1 - \frac{C(x)A(x_o)}{C(x_o)A(x)} \tag{2.86}$$

From Equation (2.83) and Equation (2.85), since $I(x_o) = 2\phi$, the integral term can be written with respect to the spacetime line element (2.77) as the following integral expression (Tejiero and Larranaga, 2012)

$$I\left(x_o\right)=2\int_{x_o}^{\infty}\frac{\sqrt{B\left(x\right)}}{\sqrt{C\left(x\right)}\sqrt{\frac{C\left(x\right)A\left(x_o\right)}{C\left(x_o\right)A\left(x\right)}-1}}dx \quad . \tag{2.87}$$

Implying Equation (2.87) on spacetime line elements around gravitational lens objects, the deflection angle terms representing each type of object can be derived. For a Schwarzschild object e.g. a massive planet, a star and a black hole, where its line element is

$$ds^2=-\left(1-2Gm/r\right)dt^2+\left(1-2Gm/r\right)^{-1}dr^2+r^2dr^2+r^2\sin^2\theta d\varphi^2 \tag{2.88}$$

Then by Equations (2.77), (2.79), (2.87) and Equation (2.88) the deflection angle can be shown to be consist of the metric tensor of a line element arranged accordingly as pattern build to represent the deflection angle

$$\alpha=2\int_{r_o}^{\infty}\frac{\left(1-\frac{2Gm}{r}\right)^{-\frac{1}{2}}}{r\sqrt{\frac{r^2}{u^2\left(1-\frac{2Gm}{r}\right)}-1}}dr-\pi \quad , \tag{2.89}$$

where u is the gravitational lens impact parameter that is the ratio of the lensing object's extreme curvature radius r_o (event horizon radius in the case of a black hole) over its redshift function term $\sqrt{1-2Gm/r_o}$, that is $u=r_o\big/\sqrt{\left(1-2Gm/r_o\right)}$. The analysis can be extended to a wormhole spacetime metric given by (Lemos et al., 2003)

$$ds^2 = -e^{2\Phi(r)}c^2dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{2.90}$$

so that the deflection angle for a wormhole can be shown to be

$$\alpha = 2\int_{r_o}^{\infty} \frac{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{r\sqrt{\frac{r^2}{u^2e^{2\Phi}}-1}}dr - \pi \quad , \tag{2.91}$$

where in this case r_o is the radius of the wormhole throat for a wormhole and the redshift function term at the throat is $e^{\Phi(r_o)}$, giving $u = r_o / e^{\Phi(r_o)}$.

2.9 Gravitational lens in braneworld

Gravitational lens can be used as method of determining the characteristic of celestial objects in astrophysical aspect as well as providing signature of the underlying cosmological characteristic behind the astrophysical object such as the universe dynamics of expansion or contraction, modified gravity model, extra-dimensions and braneworld. For example in determining celestial objects characteristic gravitational lens can be used to distinguish between black hole and wormhole (Tsukamoto et al., 2012). In testing the viability of modified or alternative gravity theory as well as brane, gravitational lensing can be used as testing formalism by Keeton (Keeton and Petters, 2005) and Kar (Kar and Pal, 2008) respectively.

Spherically symmetric spacetime metric representing the brane in the weak field limit can be expressed in isotropic coordinates as shown by Kar (Kar and Pal, 2008)

$$ds^2 = -\left(1+\frac{2\chi}{c^2}\right)^2 c^2dt^2 + \left(1-\frac{2\chi-2\Psi}{c^2}\right)d\vec{X}^2, \tag{2.92}$$

where $\chi(r)$ is the Newtonian potential and $\Psi(r)$ is the relativistic potential. $\Psi(r)$ as the correction term from braneworld gravity adds into the equation. From Equation (2.92) using the same calculation method to find the deflection angle, the following expression can be obtained

$$\alpha = \alpha_R + \frac{1}{c^2}\int_o^s \hat{\nabla}_\perp \Psi dl, \tag{2.93}$$

where α_R, is the original value from GR alone. The extra term can be labeled for representing the brane as α_Ψ where

$$\alpha = \alpha_R + \alpha_\Psi \ . \tag{2.94}$$

Since the relation with the integral term I is $\alpha = I - \pi$ then the deflection angle in the braneworld must contain both the GR term and the extra-dimensional term,

$$\alpha = I_R + I_\Psi - \pi \ . \tag{2.95}$$

This shows the braneworld correction, where there exists small deviation of light ray on the brane in comparison to deflection due to general relativity (GR) only. This motivates further elaboration of the extra term which signify the existence of extra dimensions in the braneworld wormhole gravitational lens model.

2.10 Summary

The universe is the braneworld that is the 3+1 brane. The idea of extra-dimensions were originated since the attempt of basic forces unification effort between electromagnetism and gravity as in the Kaluza-Klein (KK) model. Then more comprehensive model which were among effort for unification through string theory between standard model and gravity sparks ideas of explaining gravity hierarchical problems through a compactified extra dimension as of the ADD model. It was the generalization of the compactified KK model. However the most intriguing extra dimension idea that has been the foundation of our studies is the RS braneworld model derived from M theory which is a more comprehensive string theory base idea. What was intriguing, is that the bulk of extra-dimensional space can directly influence the brane represented by the projected Weyl tensor $E_{\mu\nu}$. This has eliminated the requirement of exotic matter to influence the wormhole geometry on brane. In this review we have shown detail derivation of the scheme of how from the concept of braneworld gravity the surrogate of "exoticity" can be shown to produce the same physical characteristic to sustain a traversable wormhole. The brief overview of all the essentials of wormholes from Einstein and Rosen (Einstein and Rosen, 1935) that was the first original wormhole idea to the topic that represent the most challenging aspect of the physics of wormhole that is the violation of the energy condition and the requirement of exotic matter as the solution of the energy conditions violation. We reviewed the discussion about traversable wormhole physical characteristic namely the derivation of the stress energy tensors, their relation with the energy conditions violation and how the brane concept may eliminate the requirement of the exotic matter. The influence of bulk onto the brane may not only manifested by the projection of Weyl tensor upon the stress energy tensors of wormhole but also its relations with the gravitational lens of the warped passage of brane due by the presence of the wormhole. Important concept of gravitational lens were discussed as review and introductory purposes. In the next Chapter 3 and Chapter 4 more elaborate discussions on these aspects will result into interesting findings concerning the effect of surrounding spacetime dynamics and gravitational lens characteristic that signify the existence of extra-dimension.

CHAPTER 3 : EFFECTS OF SPATIAL DYNAMICS ON WORMHOLES

3.1 Introduction

General relativity theory is the driving factor that generates ideas for wormhole physics. In solving for an exact solution in the Einstein field equations in general relativity, Schwarzschild (Dirac, 1996) had shown a solution for a static spherically symmetric object warping the spacetime surrounding it. This solution had paved the ways to the first idea of the black hole where the spherically symmetric object collapse gravitationally to become singularity and creating the region known as event horizon within its Schwarzschild radius. Manipulating the Schwarzschild line element, we can show that the singularity and the event horizon in the solution can be discarded where this methodology had given birth to the idea of traversable wormhole as shown by Morris and Thorne (Morris and Thorne, 1988).

The emergence of the String and consequently M-theory leads to the introduction of braneworld models in the 1990s. It is based on the idea that the universe is a 3+1 brane embedded in a five-dimensional bulk. Works by Bronnikov and Kim (Bronnikov and Kim, 2003) were among the pioneers in the attempt to study wormhole in the frameworks of braneworld. More generalized class of braneworld wormholes was introduced by Lobo (Lobo, 2007).

In the braneworld scenario the standard model fields which constitute the electromagnetism, nuclear strong and weak are confined on the brane but gravity is freely propagating from a brane through the bulk of higher dimension (Rodrigo, 2006). Argument shown by Lobo (Lobo, 2007) has shown that the braneworld gravity provides a natural scenario for the existence of traversable wormhole possibly without the requirement of exotic matter.

In this chapter we begin with a form of bulk metric since we consider the bulk is a higher dimensional spacetime that influence the brane (Wong et al., 2011). The extra-dimension term represents coordinate in the bulk that is time independent applied in the metric as protruding position from the brane of 4 D universe inward the 5 D bulk space.

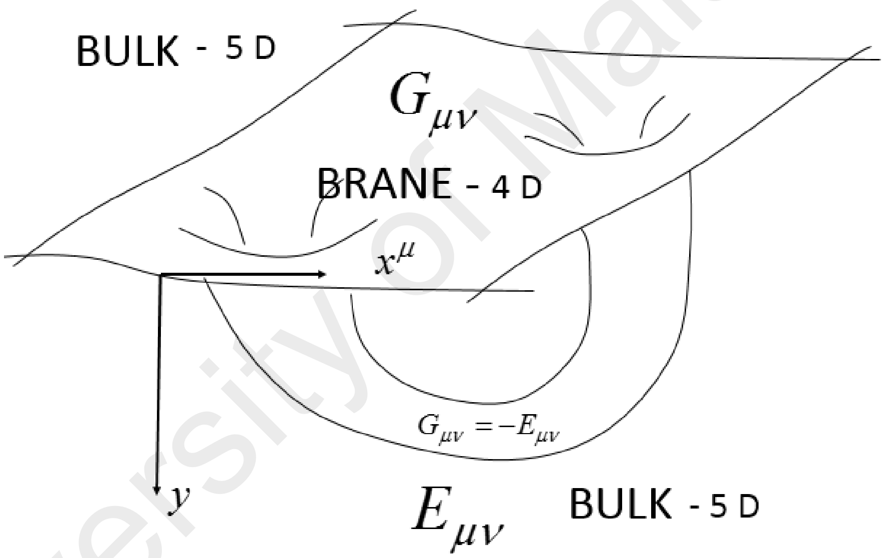

Figure 3.1 : Bulk influencing the brane surrounding the wormhole.

As the x^μ represents 3+1 (4 D) brane terms, the extra-dimensional term y as shown in Figure 3.1 is a wormhole radius-dependent parameter $y = Y(r) \equiv y(r)$ in a (5 D) bulk. By the introduction of a projected Weyl tensor $E_{\mu\nu}$ onto a brane by Bronikov (Bronnikov and Kim, 2003) we shall obtain the stress energy tensor of a wormhole in a braneworld derived from Einstein tensor $G_{\mu\nu}$ which is influenced by the projected Weyl tensor elaborately described in Chapter 2 for Equation (2.23) and as depicted in

Figure 3.1. From the metric we expand the calculation by deriving physical components representing the wormhole model, namely the energy density, radial pressure and lateral pressure. It is a natural criterion to regard the geometry of the wormhole in a brane as finite. Thus the Kretschmann can be extracted from the Riemann tensor so as to ensure the finiteness criterion achieved. From the Ricci scalar we develop a general solution in the form of integral expression that will be very useful in further study of wormhole in a braneworld. Then we consider to include the evolution or dynamic factor of the wormhole and reworks on the geometry from Ricci tensor to Einstein tensor and finally the stress energy tensor which distinguish between the classical (static) and the dynamic terms. This will lead to an interesting properties concerning wormhole characteristic during expansion or contraction of the universe.

3.2 Field equations on the brane

Universe expansion is described as the motion of brane in a higher dimensional bulk (Bowcock et al., 2000 and Mukohyama et al., 2000). Initially, before we get into dynamics characteristic in this chapter, we consider that the bulk is static, thus the bulk term is time independence so that we may compare with the dynamic terms later. The bulk term represents the extra-dimension where in bulk metric the term is embedded in a spacetime metric that behave like ordinary spacetime metric since it has the same signature of red shift and shape function terms. Thus it can be written as follows

$$ds^2 = -M^2 dt^2 + N^2 dr^2 + P^2 d\theta^2 + P^2 \sin^2\theta d\phi^2 + Q^2 \left(Y_{,r}\right)^2 dr^2 , \tag{3.1}$$

where M represents the redshift function of the brane, N and Q represent the shape function term of brane and bulk respectively, P as the braneworld wormhole radii. $y = Y(r)$ represents as the brane position (Wong et al., 2011) in the higher dimensional bulk, thus $dy = Y_{,r}dr$ that is $Y_{,r} = dY/dr$ as Y is a function of r only. Rearrange Equation (3.1) we have metric form that explicitly shows the shape function term

$$ds^2 = -M^2dt^2 + \left(N^2 + Q^2Y_{,r}^2\right)dr^2 + P^2d\theta^2 + P^2\sin^2\theta d\phi^2, \qquad (3.2)$$

with M, N, Q, Y and P are functions of r only since we consider a static bulk. From Equation (3.2) we can define the metric tensor

$$g_{\mu\nu} = diag\left[-M^2, N^2 + Q^2Y_{,r}^2, P^2, P^2\sin^2\theta\right]. \qquad (3.3)$$

and for the diagonal matrix, the conjugate metric tensor will simply be $g^{\mu\nu} = 1/g_{\mu\nu}$.

Further calculations are carried out for the affine connection $\Gamma^{\alpha}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$, and we obtained the non-zero affine connections listed below:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{M_{,r}}{M}, \qquad \Gamma^1_{00} = \frac{MM_{,r}}{N^2 + Q^2Y_{,r}^2},$$

$$\Gamma^1_{11} = \frac{NN_{,r} + QQ_{,r}Y_{,r}^2 + Q^2Y_{,r}Y_{,r,r}}{N^2 + Q^2Y_{,r}^2}, \quad \Gamma^1_{22} = \frac{-PP_{,r}}{N^2 + Q^2Y_{,r}^2},$$

$$\Gamma^1_{33} = \frac{-PP_{,r}\sin^2\theta}{N^2+Q^2Y^2_{,r}} = \Gamma^1_{22}\sin^2\theta \ , \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{P_{,r}}{P} \ ,$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \, , \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{P_{,r}}{P} \, ,$$

$$\Gamma^3_{32} = \cot\theta \quad . \tag{3.4}$$

These expressions are to be substituted in $R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\gamma_{\mu\nu}\Gamma^\alpha_{\gamma\alpha} - \Gamma^\gamma_{\mu\alpha}\Gamma^\alpha_{\gamma\nu}$ for the components of Ricci tensor. The results are

$$R_{00} = \frac{M_{,r,r}}{N^2+Q^2Y^2_{,r}} - \frac{MM_{,r}\left(NN_{,r}+QQ_{,r}Y^2_{,r}+Q^2Y_{,r}Y_{,r,r}\right)}{\left(N^2+Q^2Y^2_{,r}\right)^2} + \frac{2P_{,r}MM_{,r}}{P\left(N^2+Q^2Y^2_{,r}\right)} \, ,$$

$$R_{11} = \frac{NN_{,r}+QQ_{,r}Y^2_{,r}+Q^2Y_{,r}Y_{,r,r}}{N^2+Q^2Y^2_{,r}}\left(\frac{M_{,r}}{M}+\frac{2P_{,r}}{P}\right) - \frac{M_{,r,r}}{M} \, ,$$

$$R_{22} = \frac{PP_{,r}\left(NN_{,r}+QQ_{,r}Y^2_{,r}+Q^2Y_{,r}Y_{,r,r}\right)}{\left(N^2+Q^2Y^2_{,r}\right)^2} - \frac{PP_{,r}M_{,r}}{M\left(N^2+Q^2Y^2_{,r}\right)} - \frac{P_{,r}{}^2}{N^2+Q^2Y^2_{,r}} + 1 \, ,$$

$$R_{33} = R_{22}\sin^2\theta \ . \tag{3.5}$$

From these Ricci tensor components we derive the Ricci scalar by contracting the Ricci tensor $R_{\mu\nu}$, with the conjugate metric tensor $g^{\mu\nu}$ that is

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \quad . \tag{3.6}$$

From Equation (3.5) and metric tensor $g^{\mu\nu}$ of Equation (3.3) we have the Ricci scalar

$$R=-\frac{2M_{,r,r}}{M\left(N^2+Q^2Y_{,r}^2\right)}-\frac{4P_{,r}M_{,r}}{MP\left(N^2+Q^2Y_{,r}^2\right)}+\frac{2}{P^2}\left(1-\frac{P_{,r}^2}{N^2+Q^2Y_{,r}^2}\right)$$

$$+2\frac{\left(NN_{,r}+Y_{,r}Y_{,r,r}Q^2+Y_{,r}^2QQ_{,r}\right)}{\left(N^2+Q^2Y_{,r}^2\right)^2}\left(\frac{M_{,r}}{M}+\frac{2P_{,r}}{P}\right), \tag{3.7}$$

which then resulting the Einstein curvature tensor components by $G_{\mu\nu}=R_{\mu\nu}-g_{\mu\nu}R/2$ where we obtain the non zero expression as the following

$$G_{00}=M^2\left(\frac{P_{,r}}{P}\right)^2\left(1-\frac{P_{,r}^2}{N^2+Q^2Y_{,r}^2}+\frac{2PP_{,r}\left(NN_{,r}+QQ_{,r}Y_{,r}^2+Q^2Y_{,r}Y_{,r,r}\right)}{\left(N^2+Q^2Y_{,r}^2\right)^2}\right),$$

$$G_{11}=\frac{2M_{,r}P_{,r}}{MP}+\frac{1}{P^2}\left(P_{,r}^2-\left(N^2+Q^2Y_{,r}^2\right)\right),$$

$$G_{22}=\frac{1}{M\left(N^2+Q^2Y_{,r}^2\right)}\left(PP_{,r}M_{,r}+M_{,r,r}P^2\right)$$

$$-\frac{\left(NN_{,r}+QQ_{,r}Y_{,r}^2+Q^2Y_{,r}Y_{,r,r}\right)}{\left(N^2+Q^2Y_{,r}^2\right)^2}\left(\frac{M_{,r}P^2}{M}+PP_{,r}\right),$$

$$G_{33}=G_{22}\sin^2\theta\ . \tag{3.8}$$

In braneworld (Randall and Sundram, 1999, Rubakov and Marteens, 2001) framework, a natural resource of wormhole geometry is inspired by the superstring and M theory (Horova and Witten, 1996) where the universe is viewed as domain wall in multidimensional space. Adopted from the Randall-Sundram model, only the gravitational field propagates through the bulk and all other standard model fields remain attached to the brane, which is the universe. When matter on a brane is absence where we consider the condition as vacuum, the field equation is given by relationship of Einstein tensor derived by Shiromizu, Maeda and Sasaki (Shiromizu et al., 2000) and the projected Weyl tensor onto the brane (Bronnikov and Kim, 2003), the equations as shown from Equation (2.11) to (2.22) reduced to (2.23) that is

$$G_{\mu\nu} = -E_{\mu\nu} \quad , \tag{3.9}$$

where $E_{\mu\nu}$ can be a natural matter in a brane world that support a wormhole. Consider a general spherically symmetric wormhole line element of Equation (3.2) and contracting the projected Weyl tensor with metric tensor of Equation (3.2) we will have the mixed projected Weyl tensor that represents the stress energy tensor in a braneworld (Bronnikov and Kim, 2003)

$$E_{\mu}^{\nu} = g^{\nu\beta} E_{\beta\mu} \ . \tag{3.10}$$

The energy density is represented by time component of the mixed tensor,

$$\rho = E_t^t = E_0^0 \quad , \tag{3.11}$$

the radial pressure by the radial component,

$$p_{rad} = -E_r^r = -E_1^1 \quad , \tag{3.12}$$

and the lateral pressure by the angular components

$$p_{\perp} = -E_{\theta}^{\theta} = -E_2^2 = -E_{\phi}^{\phi} = -E_3^3 \quad , \tag{3.13}$$

since $E_{\mu\nu} = -G_{\mu\nu}$ thus $g^{\nu\beta} E_{\beta\mu} = -g^{\nu\beta} G_{\beta\mu}$ implies

$$E_{\mu}^{\nu} = -G_{\mu}^{\nu} \ . \tag{3.14}$$

From the Einstein curvature tensor components of Equation (3.8), thus by Equation (3.14) we derive the physical characteristic expressions. From Equation (3.11) for the energy density

$$\rho = E_0^0 = -G_0^0 . \tag{3.15}$$

By the metric in line element equation of Equation (3.2), the conjugate metric tensor of Equation (3.3), and expanding the mixed Einstein tensor of G_0^0

$G_0^0 = g^{00}G_{00} + g^{01}G_{01} + g^{02}G_{02} + g^{03}G_{03} = g^{00}G_{00}$ where $g^{00} = -\frac{1}{M^2}$ thus by Equation (3.15),

$$\rho = \left(\frac{P_{,r}}{P}\right)^2 \left(\frac{2PP_{,r}\left(NN_{,r} + QQ_{,r}Y_{,r}^2 + Q^2Y_{,r}Y_{,r,r}\right)}{N^2 + Q^2Y_{,r}^2} - \frac{P_{,r}^2}{N^2 + Q^2Y_{,r}^2} + 1\right) . \tag{3.16}$$

From Equation (3.12) for the radial pressure

$$p_{rad} = -E_1^1 = G_1^1 . \tag{3.17}$$

By the conjugate metric tensor of Equation (3.3), and expanding the mixed Einstein of G_1^1, $G_1^1 = g^{10}G_{10} + g^{11}G_{11} + g^{12}G_{12} + g^{13}G_{13} = g^{11}G_{11}$ where $g^{11} = \frac{1}{N^2 + Q^2Y_{,r}^2}$,

thus by Equation (3.17)

$$p_{rad} = \frac{2P_{,r}}{P}\frac{\left(NN_{,r} + QQ_{,r}Y_{,r}^2 + Q^2Y_{,r}Y_{,r,r}\right)}{\left(N^2 + Q^2Y_{,r}^2\right)^2} + \frac{P_{,r}}{P\left(N^2 + Q^2Y_{,r}^2\right)}\left(\frac{2M_{,r}}{M} + \frac{P_{,r}}{P}\right) - \frac{1}{P^2} . \tag{3.18}$$

From Equation (3.13) for the lateral pressure

$$p_{\perp} = -E_2^2 = G_2^2 \ . \tag{3.19}$$

By the conjugate metric tensor of Equation (3.3), and expanding G_2^2,

$$G_2^2 = g^{20}G_{20} + g^{21}G_{21} + g^{22}G_{22} + g^{23}G_{23} = g^{22}G_{22} \text{ where } g^{22} = \frac{1}{P^2} \ ,$$

thus by Equation (3.19),

$$p_{\perp} = \frac{1}{M\left(N^2 + Q^2Y_{,r}^2\right)}\left(\frac{P_{,r}M_{,r}}{P} + M_{,r,r}\right) - \frac{\left(NN_{,r} + QQ_{,r}Y_{,r}^2 + Q^2Y_{,r}Y_{,r,r}\right)}{\left(N^2 + Q^2Y_{,r}^2\right)^2}\left(\frac{M_{,r}}{M} + \frac{P_{,r}}{P}\right) \ , \tag{3.20}$$

where these represents the time independent classical terms of stress energy tensor that are $\rho = \rho_{(classic)}$, $p_{rad} = p_{rad(classic)}$, and $p_{\perp} = p_{\perp(classic)}$, derived from braneworld model of Wong metric tensor (Wong et al., 2011) combined with Bronnikov's analytical method (Bronnikov and Kim 2003) that represent bulk effect toward the brane. We will show later in this chapter that the surrounding spatial dynamics will affect the stress energy tensor of the wormhole.

3.3 Geometry finiteness

To ensure finiteness of wormhole geometry as to avoid singularity to exist we may begin by extracting the Kretschmann scalar from spacetime metric of the wormhole.

The Kretschmann scalar is defined as

$$\mathrm{K} = R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\mu\nu} . \tag{3.21}$$

hence, in term of the metric tensor and Riemann tensor

$$\mathrm{K} = g^{\sigma\gamma} R^{\rho}_{\gamma\mu\nu} g^{\nu\gamma} R^{\mu}_{\gamma\rho\sigma} . \tag{3.22}$$

From the general wormhole spacetime line element Equation (3.2) and by Riemann tensor expression $R^{\sigma}_{\mu\alpha\nu} = \partial_{\alpha}\Gamma^{\sigma}_{\mu\nu} - \partial_{\nu}\Gamma^{\sigma}_{\mu\alpha} + \Gamma^{\gamma}_{\mu\nu}\Gamma^{\sigma}_{\gamma\alpha} - \Gamma^{\gamma}_{\mu\alpha}\Gamma^{\sigma}_{\gamma\nu}$ we have the non-zero terms as the following

$$R^{0}_{101} = -R^{0}_{110} = \frac{N^2 + Q^2 Y^2_{,r}}{P^2_{,r} M^2} R^{1}_{001} \quad , \quad R^{0}_{101} = -\frac{N^2 + Q^2 Y^2_{,r}}{P^2_{,r} M^2} R^{1}_{010} \ ,$$

$$R^{0}_{101} = \frac{1}{M}\left(M_{,r}\left(\frac{NN_{,r} + Y_{,r}Y_{,r,r}Q^2 + Y^2_{,r}QQ_{,r}}{N^2 + Q^2 Y^2_{,r}} \right) - M_{,r,r} \right),$$

$$R^{0}_{202} = -R^{0}_{220} = \left(\frac{P}{P_{,r}} \right)^2 \frac{1}{M^2} R^{2}_{002} = -\left(\frac{P}{P_{,r}} \right)^2 \frac{1}{M^2} R^{2}_{020} \ ,$$

$$R^{0}_{202} = -\frac{M_{,r}PP_{,r}}{M\left(N^2 + Q^2 Y^2_{,r} \right)},$$

$$R^{0}_{303} = -R^{0}_{330} = \left(\frac{P}{P_{,r}} \right)^2 \frac{1}{M^2} R^{3}_{003} \sin^2\theta ,$$

$$R^{0}_{303} = -\left(\frac{P}{P_{,r}} \right)^2 \frac{1}{M^2} R^{3}_{030} \sin^2\theta ,$$

$$R^{0}_{303} = -\frac{M_{,r}PP_{,r}}{M\left(N^2 + Q^2 Y^2_{,r} \right)} \sin^2\theta$$

$$R^{1}_{212} = PP_{,r}\left(\frac{NN_{,r} + Y_{,r}Y_{,r,r}Q^{2} + Y_{,r}^{2}QQ_{,r}}{\left(N^{2} + Q^{2}Y_{,r}^{2}\right)^{2}}\right),$$

$$R^{1}_{313} = PP_{,r}\left(\frac{NN_{,r} + Y_{,r}Y_{,r,r}Q^{2} + Y_{,r}^{2}QQ_{,r}}{\left(N^{2} + Q^{2}Y_{,r}^{2}\right)^{2}}\right)\sin^{2}\theta \quad ,$$

$$R^{2}_{323} = \left(1 - \frac{P_{,r}^{2}}{N^{2} + Q^{2}Y_{,r}^{2}}\right)\sin^{2}\theta \quad . \tag{3.23}$$

Hence, the available Kretschmann scalar Equation (3.22) components that can be derived from these non zero Riemann tensor terms of Equations (3.23) are

$$\mathrm{K}_{1} = \left(g^{11}R^{0}_{101}\right)^{2} = K_{1}^{2} \qquad \text{thus}$$

$$K_{1} = \frac{M_{,r}}{M}\left(\frac{NN_{,r} + Y_{,r}Y_{,r,r}Q^{2} + Y_{,r}^{2}QQ_{,r}}{\left(N^{2} + Q^{2}Y_{,r}^{2}\right)^{2}}\right) - \frac{M_{,r,r}}{M\left(N^{2} + Q^{2}Y_{,r}^{2}\right)} \ ,$$

$$\mathrm{K}_{2} = \left(g^{22}R^{0}_{202}\right)^{2} = \left(g^{33}R^{0}_{303}\right)^{2} = K_{2}^{2} \qquad \text{thus}$$

$$K_{2} = -\frac{P_{,r}M_{,r}}{PM\left(N^{2} + Q^{2}Y_{,r}^{2}\right)} \ ,$$

$$\mathrm{K}_{3} = \left(g^{22}R^{1}_{212}\right)^{2} = \left(g^{33}R^{1}_{313}\right)^{2} = K_{3}^{2} \quad \text{thus}$$

$$K_{3} = \frac{P_{,r}}{P}\left(\frac{NN_{,r} + Y_{,r}Y_{,r,r}Q^{2} + Y_{,r}^{2}QQ_{,r}}{\left(N^{2} + Q^{2}Y_{,r}^{2}\right)^{2}}\right),$$

$$\mathrm{K}_{4} = \left(g^{33}R^{2}_{323}\right)^{2} = K_{4}^{2} \quad \text{thus}$$

$$K_{4} = \frac{1}{P^{2}}\left(1 - \frac{P_{,r}^{2}}{N^{2} + Q^{2}Y_{,r}^{2}}\right) \ . \tag{3.24}$$

Expanding the Kretschmann scalar

$$\mathrm{K} = R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\mu\nu} = R_{0\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{0\nu} + R_{1\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{1\nu} + R_{2\nu}{}^{\rho\nu} R_{\rho\nu}{}^{2\nu} + R_{3\nu}{}^{\rho\nu} R_{\rho\nu}{}^{3\nu}, \qquad (3.25)$$

where after an obviously non-trivial calculation of expanding Equation (3.25), we may have the non-zero result

$$\begin{aligned}
\mathrm{K} = {} & R_{01}{}^{01} R_{01}{}^{01} + R_{01}{}^{10} R_{10}{}^{01} + R_{10}{}^{01} R_{01}{}^{10} + R_{10}{}^{10} R_{10}{}^{10} \\
& + R_{02}{}^{02} R_{02}{}^{02} + R_{02}{}^{20} R_{20}{}^{02} + R_{20}{}^{02} R_{02}{}^{20} + R_{20}{}^{20} R_{20}{}^{20} \\
& + R_{03}{}^{03} R_{03}{}^{03} + R_{03}{}^{30} R_{30}{}^{03} + R_{30}{}^{03} R_{03}{}^{30} + R_{30}{}^{30} R_{30}{}^{30} \\
& + R_{12}{}^{12} R_{12}{}^{12} + R_{12}{}^{21} R_{21}{}^{12} + R_{21}{}^{12} R_{12}{}^{21} + R_{21}{}^{21} R_{21}{}^{21} \\
& + R_{13}{}^{13} R_{13}{}^{13} + R_{13}{}^{31} R_{31}{}^{13} + R_{31}{}^{13} R_{13}{}^{31} + R_{31}{}^{31} R_{31}{}^{31} \\
& + R_{23}{}^{23} R_{23}{}^{23} + R_{23}{}^{32} R_{32}{}^{23} + R_{32}{}^{23} R_{23}{}^{32} + R_{32}{}^{32} R_{32}{}^{32} \ . \qquad (3.26)
\end{aligned}$$

By Riemann tensor identity $R_{abcd} = R_{cdab} = -R_{dcab} = -R_{abdc}$ thus $R_{abcd} = -R_{abdc}$ multiply with metric tensors $g^{\mu c} g^{d\nu}$, we have $g^{\mu c} g^{d\nu} R_{abcd} = -g^{\mu c} g^{d\nu} R_{abdc}$, therefore it is a proof of an identity where

$$R_{ab}{}^{\mu\nu} = -R_{ab}{}^{\nu\mu} \quad . \qquad (3.27)$$

By using this identity and Kretchmann scalar in the form of metric and Riemann tensors, after non-trivial calculation it can be shown that the Kretchmann scalar can be finalized as

$$\mathrm{K} = 4\left(g^{11} R^{0}_{101}\right)^2 + 4\left(g^{22} R^{0}_{202}\right)^2 + 4\left(g^{33} R^{0}_{303}\right)^2 + 4\left(g^{33} R^{1}_{313}\right)^2 + 4\left(g^{22} R^{1}_{212}\right)^2 + 4\left(g^{33} R^{2}_{323}\right)^2 \qquad (3.28)$$

In term of the scalar components by Equation (3.28) it can be shown

$$\mathrm{K} = 4K_1^2 + 8K_2^2 + 8K_3^2 + 4K_4^2 \quad , \tag{3.29}$$

thus $\mathrm{K} < \infty$ as proven by calculating Equation (3.29) using Equation (3.24) which is a necessary and sufficient condition for finiteness of all algebraic curvature invariants.

As we consider a spherically symmetric wormhole with spacetime metric of Equation (3.2) situated in a region in the universe where matter on the brane is absence (vacuum), thus the Ricci scalar that represents the 4-dimensional scalar curvature in this region would be zero; $R = 0$ (Dadhich et al., 2002) . As in the Schwarzschild solution (with regards to a general spherically symmetric spacetime) by the Equation (3.2) non zero Ricci tensors of Equation (3.5) we may acquire its Ricci scalar of (3.7) to be expanded, become zero and rewritten as the following

$$\begin{aligned} R = & -\frac{2P_{,r}^2 M_{,r,r}}{M\left(N^2 + Q^2 Y_{,r}^2\right)} - \frac{4P_{,r}^3 M_{,r}}{MP\left(N^2 + Q^2 Y_{,r}^2\right)} \\ & + 2P_{,r}^2 M_{,r} \frac{\left(NN_{,r} + Y_{,r} Y_{,r,r} Q^2 + Y_{,r}^2 QQ_{,r}\right)}{M\left(N^2 + Q^2 Y_{,r}^2\right)^2} \\ & + \frac{2P_{,r}^2}{P^2} - \frac{2P_{,r}^4}{P^2\left(N^2 + Q^2 Y_{,r}^2\right)} \\ & + \frac{4P_{,r}^3\left(NN_{,r} + Y_{,r} Y_{,r,r} Q^2 + Y_{,r}^2 QQ_{,r}\right)}{P\left(N^2 + Q^2 Y_{,r}^2\right)^2} = 0 , \end{aligned} \tag{3.30}$$

where we have taken into consideration that the character of radii P is that

$$P = P(r),\ \ P > 0\,,\ P \approx r\,,\ 1 \to P_{,r}^2\ ,\ \ P_{,r} \to \left(P_{,r}\right)(1) \to \left(P_{,r}\right)P_{,r}^2 \to P_{,r}^3$$

and thus we may also write $P_{,r}^2 \to \left(P_{,r}\right)^2(1) \to \left(P_{,r}\right)^2 P_{,r}^2 \to P_{,r}^4$.

Let generalize Equation (3.2) in the form of

$$ds^2 = -e^{2\gamma(r)}dt^2 + e^{2\alpha(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\ , \tag{3.31}$$

where $e^{\gamma(r)} = M$, $e^{2\alpha(r)} = N^2 + Q^2Y_{,r}^2$ and $r = P/P_{,r}$ Since the non-zero Christoffel of the metric of Equation (3.2) and metric of Equation (3.31) are related as the following

$$\Gamma_{10}^0 = \Gamma_{01}^0 = \frac{M_{,r}}{M} = \gamma'\,, \qquad \Gamma_{00}^1 = \frac{MM_{,r}}{N^2 + Q^2Y_{,r}^2} = e^{2(\gamma-\alpha)}\gamma'\,,$$

$$\Gamma_{11}^1 = \frac{NN_{,r} + Y_{,r}Y_{,r,r}Q^2 + Y_{,r}^2QQ_{,r}}{N^2 + Q^2Y_{,r}^2} = \alpha'\,,$$

$$\Gamma_{22}^1 = -\frac{PP_{,r}}{N^2 + Q^2Y_{,r}^2} = -\frac{r}{e^{2\alpha}}\,, \qquad \Gamma_{33}^1 = \Gamma_{22}^1\sin^2\theta\,,$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{P_{,r}}{P} = \frac{1}{r}\,,$$

$$\Gamma_{33}^2 = -\sin\theta\cos\theta\,, \qquad \Gamma_{32}^3 = \Gamma_{23}^3 = \cot\theta\ . \tag{3.32}$$

Then for integral solution convenience, Equation (3.30) can be rewritten as

$$R = -2e^{-2\alpha}\gamma'' - 2e^{-2\alpha}\gamma'^2 + 2e^{-2\alpha}\gamma'\alpha' + \frac{4}{r}e^{-2\alpha}\alpha'$$

$$-\frac{4}{r}e^{-2\alpha}\gamma' - \frac{2}{r^2}e^{-2\alpha} + \frac{2}{r^2} = 0\ . \tag{3.33}$$

It can be shown that from Christoffel terms above of Equations (3.32), metric tensor of Equation (3.31) relations with Equation (3.7) which has been expanded to equation (3.30) are as the following, where for the 1st term in the equation (3.30) $-\frac{2P_{,r}^2 M_{,r,r}}{M\left(N^2+Q^2Y_{,r}^2\right)} = 2\left(\frac{P_{,r}}{P}\right)\left(\frac{PP_{,r}}{N^2+Q^2Y_{,r}^2}\right)\left(\frac{M_{,r,r}}{M}\right)$ that is in relation with the metric tensor of Equation (3.31) as the term $\frac{M_{,r,r}}{M} = \frac{MM_{,r,r}}{M^2} - \frac{M_{,r}^2}{M^2} + \frac{M_{,r}^2}{M^2} = \gamma'' + \gamma'^2$

and $-\frac{2P_{,r}^2 M_{,r,r}}{M\left(N^2+Q^2Y_{,r}^2\right)} = 2\frac{1}{r}\left(-\frac{r}{e^{2\alpha}}\right)\left(\gamma''+\gamma'^2\right)$

thus, expanding the right side of the equation above we have

$$-\frac{2P_{,r}^2 M_{,r,r}}{M\left(N^2+Q^2Y_{,r}^2\right)} = -2e^{-2\alpha}\gamma'' - 2e^{-2\alpha}\gamma'^2 \quad , \tag{3.34}$$

which is the 1st two terms of Equation (3.33). For the 2nd term in Equation (3.30) $-\frac{4P_{,r}^3 M_{,r}}{MP\left(N^2+Q^2Y_{,r}^2\right)} = 4\left(\frac{M_{,r}}{M}\right)\left(\frac{P_{,r}}{P}\right)^2\left(\frac{-PP_{,r}}{N^2+Q^2Y_{,r}^2}\right)$ where in relation with the metric tensor Equation (3.31) $-\frac{4P_{,r}^3 M_{,r}}{MP\left(N^2+Q^2Y_{,r}^2\right)} = 4\left(\gamma'\right)\left(\frac{1}{r}\right)^2\left(-\frac{r}{e^{2\alpha}}\right)$ thus

$$-\frac{4P_{,r}^3 M_{,r}}{MP\left(N^2+Q^2Y_{,r}^2\right)} = -\frac{4}{r}e^{-2\alpha}\gamma' \quad , \tag{3.35}$$

which is the 5th term of Equation (3.33). For the 3rd term in Equation (3.30)

$$2P_{,r}^2 M_{,r}\frac{\left(NN_{,r}+Y_{,r}Y_{,r,r}Q^2+Y_{,r}^2QQ_{,r}\right)}{M\left(N^2+Q^2Y_{,r}^2\right)^2}$$

$$= 2\left(\frac{P_{,r}}{P}\right)\left(\frac{M_{,r}}{M}\right)\frac{PP_{,r}\left(NN_{,r}+Y_{,r}Y_{,r,r}Q^2+Y_{,r}^2QQ_{,r}\right)}{\left(N^2+Q^2Y_{,r}^2\right)\left(N^2+Q^2Y_{,r}^2\right)} \quad ,$$

where the relation between the metric tensor of Equation (3.31)

$$2P_{,r}^{2}M_{,r}\frac{\left(NN_{,r}+Y_{,r}Y_{,r,r}Q^{2}+Y_{,r}^{2}QQ_{,r}\right)}{M\left(N^{2}+Q^{2}Y_{,r}^{2}\right)^{2}}=2\left(\frac{1}{r}\right)(\gamma')\left(\frac{r}{e^{2\alpha}}\right)\alpha'.$$

Thus

$$2P_{,r}^{2}M_{,r}\frac{\left(NN_{,r}+Y_{,r}Y_{,r,r}Q^{2}+Y_{,r}^{2}QQ_{,r}\right)}{M\left(N^{2}+Q^{2}Y_{,r}^{2}\right)^{2}}=2e^{-2\alpha}\gamma'\alpha' \quad , \tag{3.36}$$

which is the 3rd term of Equation (3.33). Finally for the last three terms of Equation (3.30) ,

$$\frac{2P_{,r}^{2}}{P^{2}}-\frac{2P_{,r}^{4}}{P^{2}\left(N^{2}+Q^{2}Y_{,r}^{2}\right)}+\frac{4P_{,r}^{3}\left(NN_{,r}+Y_{,r}Y_{,r,r}Q^{2}+Y_{,r}^{2}QQ_{,r}\right)}{P\left(N^{2}+Q^{2}Y_{,r}^{2}\right)^{2}}$$

$$=2\left(\frac{P_{,r}}{P}\right)^{2}\left(1-\frac{P_{,r}}{P}\frac{PP_{,r}}{\left(N^{2}+Q^{2}Y_{,r}^{2}\right)}+\frac{2PP_{,r}}{N^{2}+Q^{2}Y_{,r}^{2}}\frac{NN_{,r}+Y_{,r}Y_{,r,r}Q^{2}+Y_{,r}^{2}QQ_{,r}}{N^{2}+Q^{2}Y_{,r}^{2}}\right),$$

where in relation with the metric tensor

$$\frac{2P_{,r}^{2}}{P^{2}}-\frac{2P_{,r}^{4}}{P^{2}\left(N^{2}+Q^{2}Y_{,r}^{2}\right)}+\frac{4P_{,r}^{3}\left(NN_{,r}+Y_{,r}Y_{,r,r}Q^{2}+Y_{,r}^{2}QQ_{,r}\right)}{P\left(N^{2}+Q^{2}Y_{,r}^{2}\right)^{2}}$$

$$=2\left(\frac{1}{r}\right)^{2}\left(1-\frac{1}{r}\left(\frac{r}{e^{2\alpha}}\right)+\frac{2r}{e^{2\alpha}}\alpha'\right),$$

which is the last two terms and the 4th terms of Equation (3.33) that are

$$\frac{2P_{,r}^{2}}{P^{2}}-\frac{2P_{,r}^{4}}{P^{2}\left(N^{2}+Q^{2}Y_{,r}^{2}\right)}+\frac{4P_{,r}^{3}\left(NN_{,r}+Y_{,r}Y_{,r,r}Q^{2}+Y_{,r}^{2}QQ_{,r}\right)}{P\left(N^{2}+Q^{2}Y_{,r}^{2}\right)^{2}}$$

$$=\frac{2}{r^{2}}-\frac{2}{r^{2}}e^{-2\alpha}+\frac{4}{r}e^{-2\alpha}\alpha' \quad . \tag{3.37}$$

Combination of Equation (3.34) to Equation (3.37) have shown that we can transformed Equation (3.30) to (3.33) for conveneince to proceed with further calculation of the integral expression.

To find the integral expression we first require to derive Equation (3.33) as a linear first order equation with respect to $f(r)$ where $f(r) = re^{-2\alpha}$. From Equation (3.33)

$$2e^{-2\alpha}\gamma'' + 2e^{-2\alpha}\gamma'^2 - 2e^{-2\alpha}\gamma'\alpha' - \frac{4}{r}e^{-2\alpha}\alpha' + \frac{4}{r}e^{-2\alpha}\gamma' + \frac{2}{r^2}e^{-2\alpha} = \frac{2}{r^2} \quad , \tag{3.38}$$

hence

$$e^{-2\alpha}r\left(2\gamma''r + 2\gamma'^2 r - 2\gamma'\alpha' r - 4\alpha' + 4\gamma' + \frac{2}{r}\right) = 2 \quad , \tag{3.39}$$

since $f(r) = re^{-2\alpha}$ thus $f' = e^{-2\alpha} - 2e^{-2\alpha}\alpha' r(r)$ and if we multiply with $2 + \gamma' r$ we will have

$$f'(2 + \gamma' r) = f\left(-2\gamma'\alpha' r - 4\alpha' + \gamma' + \frac{2}{r}\right) \quad . \tag{3.40}$$

From Equation (3.39) we may write

$$f\left(2\gamma''r + 2\gamma'^2 r + 3\gamma' - 2\gamma'\alpha' r - 4\alpha' + \gamma' + \frac{2}{r}\right) = 2 , \tag{3.41}$$

then $f\left(2\gamma''r + 2\gamma'^2 r + 3\gamma'\right) + f\left(-2\gamma'\alpha' r - 4\alpha' + \gamma' + \frac{2}{r}\right) = 2$ therefore

$$f\left(2\gamma''r + 2\gamma'^2 r + 3\gamma'\right) + f'(2 + \gamma' r) = 2 \quad , \tag{3.42}$$

which is the Ricci scalar of the wormhole in a braneworld in the form of linear first order differential equation with respect to $f(r)$ thus

$$f'+f\left(\frac{2\gamma''r+2\gamma'^2 r+3\gamma'}{2+\gamma' r}\right)=\frac{2}{2+\gamma' r} \quad . \tag{3.43}$$

Let $a(r)=\dfrac{2\gamma''r+2\gamma'^2 r+3\gamma'}{2+\gamma' r}$ and $b(r)=\dfrac{2}{2+\gamma' r}$ thus (3.43) can be written as $f'+a(r)f=b(r)$ where we multiply by an integrating factor $e^{\int a(r)dr}$ hence $e^{\int a(r)dr}\left(f'+a(r)f\right)=b(r)e^{\int a(r)dr}$ and the left hand side of the equation as a differentiation with respect to r thus $d\left(fe^{\int a(r)dr}\right)\Big/dr=b(r)e^{\int a(r)dr}$. Integrating both sides yields

$$f=e^{-\int a(r)dr}\int b(r)e^{\int a(r)dr}dr \quad . \tag{3.44}$$

For convenience $a(r)$ integral term can be separated from new term $\Gamma(r)$ as

$$\int a(r)dr=\int\frac{2\gamma''r+2\gamma'^2 r}{2+\gamma' r}dr+3\Gamma \quad , \tag{3.45}$$

where $\Gamma(r)=\int\dfrac{\gamma'}{2+\gamma' r}dr$ but the term $\gamma''r+\gamma'^2 r$ can be factorized as

$$r\gamma'\left(\frac{\gamma''}{\gamma'}+\gamma'\right)=r\gamma'\left(\frac{d^2\gamma}{dr^2}\frac{dr}{d\gamma}+\gamma'\right)=r\gamma'\left(\frac{d^2\gamma}{drd\gamma}+\gamma'\right)$$

which implies $r\gamma'\left(\dfrac{d}{dr}\left(\dfrac{d\gamma}{d\gamma}\right)+\gamma'\right)=r\gamma'^2$ and thus $\dfrac{\gamma''r+\gamma'^2 r}{2+\gamma' r}=\dfrac{\gamma'^2 r}{2+\gamma' r}$

it can be shown that the term $\frac{\gamma'^2 r}{2+\gamma' r}dr = \frac{\left(\frac{d\gamma}{dr}\right)^2 r}{2dr + rd\gamma}dr^2 \cong d\gamma$ by considering $2dr + rd\gamma \cong rd\gamma$ where infinitesimal difference of red shift function γ of a wormhole is much bigger than the infinitesimal difference of its corresponding radius, therefore Equation (3.45) becomes

$$\int a(r)dr = 2\int d\gamma + 3\Gamma = 2\gamma + 3\Gamma \ . \tag{3.46}$$

So by Equation (3.44) and Equation (3.46) the integral solution is shown to be

$$f = 2e^{-2\gamma-3\Gamma}\int \frac{e^{2\gamma+3\Gamma}}{2+\gamma' r}dr \ , \tag{3.47}$$

which was similarly concluded but in different manner by Bronikov (Bronnikov and Kim, 2003) as general solution for the condition of $R = 0$

$$f = \frac{2e^{-2\gamma+3\Gamma}}{(2+\gamma' r)^2}\int (2+\gamma' r)e^{2\gamma-3\Gamma}dr \ . \tag{3.48}$$

By proving this general integral solution, it shows the Ricci scalar Equation (3.7) has the smoothness of metric behavior as symmetric wormhole and will provide several other solutions (Bronnikov and Kim, 2003) which concludes that wormholes are not always connected with negative energy density thus implies that the braneworld gravity of the wormhole metric Equation (3.2) provides a natural scenario for the existence of wormhole without the requirement of exotic matter.

3.4 Spatial dynamic effects

Consider a dynamic factor $a(t)$ with spacetime is included in the spacetime metric component of Equation (3.1) representing a dynamic braneworld metric. It can be shown as a separable form of metric tensor terms where we consider uniformity nature of the surrounding space dynamics of expansion or contraction with respect to wormhole physical shape. The spacetime metric components can be written as

$$N(r,t) = a(t)N(r),$$
$$Q(r,t) = a(t)Q(r),$$
$$P(r,t) = a(t)P(r), \tag{3.49}$$

so Equation (3.2) can be rewritten as it has the same signature of red shift and shape function terms

$$ds^2 = -M^2 dt^2 + \left(\left(a(t)N(r)\right)^2 + \left(a(t)Q(r)\right)^2 Y_{,r}^2\right)dr^2$$
$$+\left(a(t)P(r)\right)^2 d\theta^2 + \left(a(t)P(r)\right)^2 \sin^2\theta d\phi^2, \tag{3.50}$$

thus

$$ds^2 = -M^2 dt^2 + a(t)^2\left(N(r)^2 + Q(r)^2 Y_{,r}^2\right)dr^2$$
$$+a(t)^2 P(r)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.51}$$

This spacetime metric represents the spatial dynamics terms of warped space surrounding the wormhole. Equations (3.49) to (3.51) have shown that we have explicitly embedded the time dependence dynamic factor into the shape function and radial terms only. The metric tensor of time term M remain time-independence. Preserving the metric tensor of the time term only to be time-independence indicates that the metric preserves the wormhole geometry itself as time independence whereas letting the shape function and radial terms time dependence indicate that the bulk space influencing the wormhole's surrounding brane may evolve dynamically. Ricci tensor with the dynamics factor can be separated neatly as time-independence (static) or "classic" terms with the dynamic terms $\xi_{\mu\nu}$. They can be represented as the following

$$R_{00} = \frac{1}{a^2} R_{00(classic)} + \xi_{00},$$

$$R_{11} = R_{11(classic)} + \xi_{11},$$

$$R_{22} = R_{22(classic)} + \xi_{22}, \quad \text{and}$$

$$R_{33} = R_{22} \sin^2\theta, \tag{3.52}$$

where

$$\xi_{00} = -\frac{3\ddot{a}}{a},$$

$$\xi_{11} = \frac{\left(N^2 + Q^2 Y_{,r}^2\right)}{M^2}\left(a\ddot{a} + 2\dot{a}^2\right),$$

$$\xi_{22} = \frac{P^2}{M^2}\left(a\ddot{a} + 2\dot{a}^2\right). \tag{3.53}$$

Detail calculations are elaborated further in Appendix A. Einstein tensor with dynamic factor

$$G_{00} = \frac{1}{a^2} G_{00(classic)} + \Theta_{00},$$

$$G_{11} = R_{11(classic)} - \frac{1}{2a^2} g_{11} R_{(classic)} + \Theta_{11},$$

$$G_{22} = R_{22(classic)} - \frac{1}{2a^2} g_{22} R_{(classic)} + \Theta_{22},$$

$$G_{33} = G_{22} \sin^2 \theta, \tag{3.54}$$

where

$$\Theta_{00} = 3\left(\frac{\dot{a}}{a}\right)^2,$$

$$\Theta_{11} = -\frac{N^2 + Q^2 Y_{,r}^2}{M^2}\left(\dot{a}^2 + 2a\ddot{a}\right),$$

$$\Theta_{22} = -\frac{P^2}{M^2}\left(\dot{a}^2 + 2a\ddot{a}\right). \tag{3.55}$$

Detail calculations are elaborated further in Appendix B. Orthonormalized Einstein tensor with dynamic factor by using $G_{\hat{\mu}\hat{\nu}} = \Lambda_{\hat{\mu}}^{\lambda} \Lambda_{\hat{\nu}}^{\gamma} G_{\lambda\gamma}$

where $\Lambda_{\hat{\mu}}^{\lambda} = diag\left[\frac{1}{M}, \frac{1}{a\left(N^2 + Q^2 Y_{,r}^2\right)^{1/2}}, \frac{1}{aP}, \frac{1}{aP\sin\theta}\right]$ which results to

$$G_{\hat{\mu}\hat{\nu}} = \Lambda_{\hat{\mu}}^{\lambda} \Lambda_{\hat{\nu}}^{\gamma} \left\{R_{\lambda\gamma} - \frac{1}{2} g_{\lambda\gamma} R\right\}, \tag{3.56}$$

after operating the Einstein tensor it can be shown to consist of the dynamic terms as the following;

$$G_{\hat{0}\hat{0}} = \frac{1}{a^2} G_{\hat{0}\hat{0}(classic)} + \Theta_{\hat{0}\hat{0}},$$

$$G_{\hat{1}\hat{1}} = \frac{1}{a^2} G_{\hat{1}\hat{1}(classic)} + \Theta_{\hat{1}\hat{1}},$$

$$G_{\hat{2}\hat{2}} = \frac{1}{a^2} G_{\hat{2}2(classic)} + \Theta_{\hat{2}\hat{2}}$$

and $$G_{\hat{3}\hat{3}} = G_{\hat{2}\hat{2}} \quad . \tag{3.57}$$

which generally is $G_{\hat{\mu}\nu'} = \frac{1}{a^2} G_{\hat{\mu}\nu(classic)} + \Theta_{\hat{\mu}\nu'}$. Detail calculations are elaborated further in Appendix C.

Energy momentum tensor represents the following components of stress energy tensor;

$$T_{\hat{0}\hat{0}} = \rho_{(dynamic)},$$

$$T_{\hat{1}\hat{1}} = -p_{rad(dynamic)} \ ,$$

and $$T_{\hat{2}\hat{2}} = T_{\hat{3}\hat{3}} = p_{\perp(dynamic)} \ .$$

These terms defined the components as the energy density $\rho_{(dynamic)}$, radial pressure $p_{rad(dynamic)}$ and lateral pressure $p_{\perp(dynamic)}$ of the wormhole surrounded by dynamic region of space of either expansion or contraction. By Equation (3.57) and Einstein field equation $G_{\hat{\mu}\hat{\nu}} = 8\pi G T_{\hat{\mu}\hat{\nu}}$ the energy momentum tensor components can be represented as

$$T_{\hat{0}\hat{0}} = \frac{1}{a^2}\left(\frac{1}{8\pi G} G_{\hat{0}\hat{0}(classic)}\right) + \frac{1}{8\pi G}\Theta_{\hat{0}\hat{0}} \quad \text{which is}$$

$$\rho_{(dynamic)} = \frac{1}{a^2}\rho_{(classic)} + \zeta_{\rho} \quad , \tag{3.58}$$

that represents energy density with dynamic factor,

$$T_{\hat{1}\hat{1}} = \frac{1}{a^2}\left(\frac{1}{8\pi G} G_{\hat{1}\hat{1}(classic)}\right) + \frac{1}{8\pi G}\Theta_{\hat{1}\hat{1}} \quad \text{which is}$$

$$p_{rad(dynamic)} = \frac{1}{a^2} p_{rad(classic)} + \zeta_{p_{rad}} , \qquad (3.59)$$

that represents radial pressure with dynamic factor,

$$T_{\hat{2}\hat{2}} = T_{\hat{3}\hat{3}} = \frac{1}{a^2}\left(\frac{1}{8\pi G} G_{\hat{2}\hat{2}(classic)}\right) + \frac{1}{8\pi G}\Theta_{\hat{2}\hat{2}} \text{ which is}$$

$$p_{\perp(dynamic)} = \frac{1}{a^2} p_{\perp(classic)} + \zeta_{p_\perp} , \qquad (3.60)$$

that represents lateral pressure with dynamic factor where

$$\zeta_\rho = \frac{1}{8\pi G}\Theta_{\hat{0}\hat{0}} = \frac{1}{8\pi G M^2}\Theta_{00} = \frac{3\dot{a}^2}{a^2 \kappa M^2} ,$$

$$\zeta_{p_{rad}} = \frac{1}{8\pi G}\Theta_{\hat{1}\hat{1}} = \frac{\Theta_{11}}{a^2 8\pi G\left(N^2 + Q^2 Y_{,r}^2\right)} = -\frac{\left(\dot{a}^2 + 2a\ddot{a}\right)}{a^2 \kappa M^2} ,$$

$$\zeta_{p_\perp} = \frac{1}{8\pi G}\Theta_{\hat{2}\hat{2}} = \frac{\Theta_{22}}{a^2 P^2 8\pi G} = -\frac{\left(\dot{a}^2 + 2a\ddot{a}\right)}{a^2 \kappa M^2} , \qquad (3.61)$$

with $\kappa = 8\pi G$ which are residual dynamic factor terms that also contribute to smaller or higher dynamic stress energy tensor components during space expansion or contraction respectively. Detail elaboration of the Einstein tensor dynamics terms relation between $\Theta_{\hat{\mu}\hat{\nu}}$ and $\Theta_{\mu\nu}$ is shown in Appendix C.

3.5 The effects at the wormhole throat

The condition at the throat (Visser and Hochberg, 1997) is where $r = r_0 = b(r_0)$ and $\dot{a} = \ddot{a} = 0$ thus, the stress energy tensor dynamic factor will never become infinite but will be nullified at the throat. At the throat, the dynamic factor does not contributes to the extra residual term of stress energy tensor. There is no change to the dynamic factor with respect to time at the throat, thus the residual terms $\zeta_{\hat{\mu}\hat{\mu}}\big|_{r=r_0} = 0$, thus

$$\zeta_{\hat{0}\hat{0}}\big|_{r=r_0} = \zeta_{\rho}\big|_{r=r_0} = \zeta_{\hat{1}\hat{1}}\big|_{r=r_0} = \zeta_{\tau}\big|_{r=r_0} = \zeta_{\hat{2}\hat{2}}\big|_{r=r_0} = \zeta_{p}\big|_{r=r_0} = 0 \quad . \tag{3.62}$$

From Equations (3.58) to (3.61) the stress energy tensor components at the throat with dynamic factor can be shown reduced to simpler expression in relation to their classical terms as the following

$$\begin{aligned} \rho_{(dynamic)}\big|_{r=r_0} &= \frac{1}{a^2}\,\rho_{(classic)}\Big|_{r=r_0}, \\ p_{rad\,(dynamic)}\big|_{r=r_0} &= \frac{1}{a^2}\,p_{rad(classic)}\Big|_{r=r_0}, \\ p_{\perp(dynamic)}\big|_{r=r_0} &= \frac{1}{a^2}\,p_{\perp(classic)}\Big|_{r=r_0}, \end{aligned} \tag{3.63}$$

where conclusively where $(\rho, p_{rad}, p_{\perp})_{dynamic} < (\rho, p_{rad}, p_{\perp})_{classic}$ when $a(t) > 1$ which is the dynamic-expanding factor when the universe is expanding as in big bang or bounce model (Gielen and Turok., 2016) or more generally the surrounding space of the wormhole is expanding as in regional space expansion model and $(\rho, p_{rad}, p_{\perp})_{dynamic} > (\rho, p_{rad}, p_{\perp})_{classic}$ when $0 < a(t) < 1$ which is the dynamic-contraction factor when the universe is contracting as in big crunch model or the surrounding space of the wormhole is contracting as in regional space contracting model. These can be depicted in the following diagrams

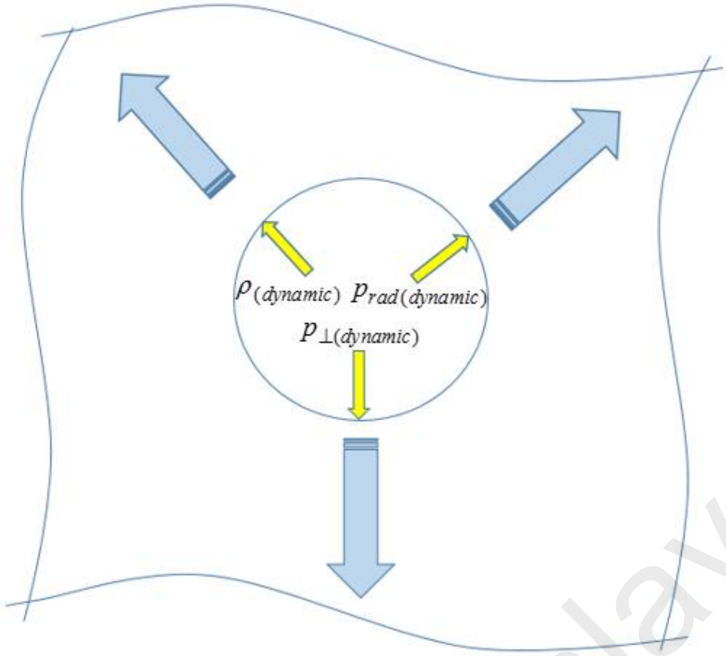

Figure 3.2 : Expanding universe or space.

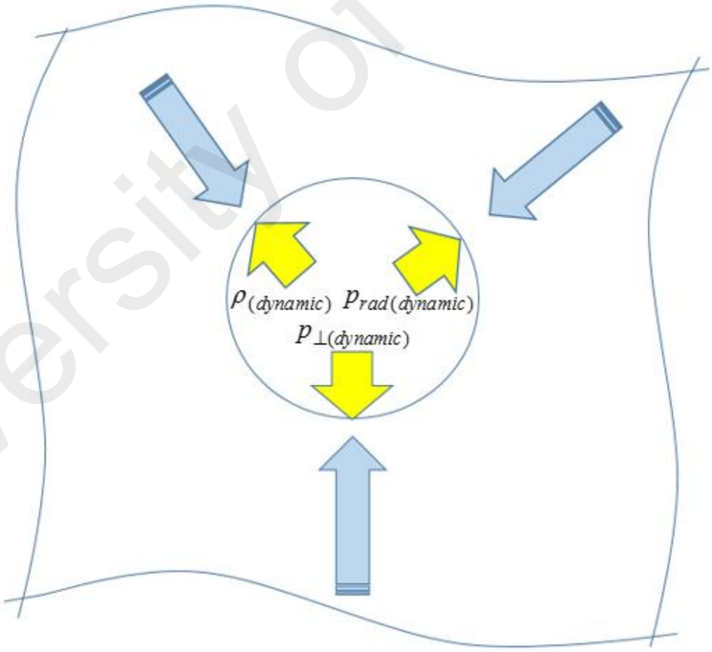

Figure 3.3 : Contracting universe or space.

The stress energy tensor decrease inversely by the dynamic factor as the space of the universe expand as depicted in Figure 3.2 since the dynamic factor may be at $a(t) > 1$, but then it is interesting to consider a contracting space of the universe as depicted in

Figure 3.3 that is collapsing toward big crunch where the dynamics factor may be at $0 < a(t) < 1$. At this condition when universe is contracting the stress energy tensor increase exponentially which significantly affect negatively the wormhole throat sustainability to exist, on the contrary in the condition where the universe is expanding the stress energy tensor decrease thus affect positively the wormhole throat sustainability to exist. These show that a wormhole is easier to exist in an expanding universe rather than in a contracting universe. In general, these are also applicable to similarly affect wormholes that exist in some regions of space that are either expanding (Livio, 2000) or contracting (Gielen and Turok, 2016).

3.6 Conclusion of result

Spacetime metric of a wormhole in a braneworld can also be modeled based on standard general relativity spacetime metric such as that of the Schwarzschild's. In this paper we have shown that this feat can be achieved by equating the projected Weyl tensor with the Einstein tensor where we can obtain the governing equation of a wormhole namely the stress energy tensor in the form of the energy density, radial pressure and lateral pressure which are physical characteristics required for wormhole traversability. As in general relativity the Kretschmann scalar can also be used to treat the finiteness of the spacetime . From the Riemann tensor we derived the general integral equation similar to that of Bronikov which is the solution to the null Ricci scalar expression representing an empty brane scenario, a pure brane condition, that is without any form of dark energy. This equation can be a platform for further characteristic study of wormhole in braneworld. Finally we include the dynamic factor in the wormhole metrics at all parameters related to wormhole radius that concerns with shape function and its spherical geometric terms. In developing the standard calculation it can be shown that the classical terms can be separated nicely with the dynamic terms of the wormhole. By separating the distinguishable terms we have shown that dynamic-expansion factor when $a(t) > 1$ will reduce the amount of stress energy tensors thus a traversable wormhole is easier to exist in an expanding universe (Livio, 2000) which is perhaps the expanding brane. Dynamic-contraction factor when $0 < a(t) < 1$ will increase the amount of stress energy tensors physical characteristic required for wormhole traversability thus a traversable wormhole is difficult to exist in a contracting brane universe. This model of spatial dynamic factor affecting a wormhole can be applied on the cyclic or the bounce universe model (Gielen and Turok, 2016) as there exist some period of expansion (bounce or big bang) and contraction (big crunch) of space in the model. It can also be applied in a cosmological model with regional space expansion and regional space contraction.

CHAPTER : 4 GRAVITATIONAL LENSING SIGNIFYING BRANEWORLD

4.1 Introduction

Gravitational lens is increasingly become importance (Schneider et al. 1992, Narayan and Bartelmann, 1996) as to probe massive structures or celestial objects such as galaxies, galaxy super clusters, neutron stars and super-massive black holes. It also provides precision in the study of illusive subjects such as the distribution of dark matter as well as dark energy in the universe, or can even be used to investigate alternative theories such as modified gravity (Garratini and Lobo, 2013) and to look for signatures of large extra-dimensions (Randall and Sundram, 1999).

The brane cosmology model that formed the basis of our work is the RS II braneworld model because the model provides a realistically flat Minkowski spacetime metric as background, yet with the underlying higher brane interacting large extra-dimensional bulk spacetime. Unlike other brane models, the RS II model in general, allows the bulk geometry to be curved. This allows dynamic interactions between the bulk and the brane making this model very significant for work with high-end concepts in General Relativity such as black holes, and for our case, wormholes.

The interaction of the bulk with the brane may provide interesting signatures of extra dimensions via equations derived for celestial objects in a braneworld framework. Thus it is interesting to investigate mathematically the extra term that emerge when deriving wormhole gravitational lens effect with extra dimensional scenario as in the RS II model. The extra term may provide the signature of extra dimension illusively embedded in the wormhole gravitational lens curvature. In this work, we have started by reviewing the derivation of classical gravitational lens theory which is purely in the framework of GR as well as some previous works in braneworld gravitational lens in Chapter 2. From the general expression we adapt this for black holes in general and for wormholes specifically without the existence of singularity. In this chapter, we expand the derivation by including extra dimensional terms in the wormhole metric in order to obtain the extra terms that would be considered as the braneworld wormhole gravitational lens correction.

4.2 Braneworld gravitational lens extra terms

We start by considering spacetime line element of Equation (3.2) that represents not only spherically symmetric but also the extra dimension term of braneworld. To associate it with a line element characteristic of a wormhole, we consider the red shift function

$$M = e^{\Phi(r)} \ ,$$

shape function, where

$$N^2 + \left(QY_{,r}\right)^2 = \frac{a^2}{1-\frac{b(r)}{r}} \ ,$$

and the radial term

$$P = ar \ . \tag{4.1}$$

Let the shape function $W = \sqrt{N^2 + Q^2 Y_{,r}{}^2}$, hence

$$ds^2 = -M^2 dt^2 + W^2 dr^2 + P^2 d\theta^2 + P^2 \sin^2\theta d\phi^2 \ . \tag{4.2}$$

Let $ds^2 = 0$ as considering light travels from a distance star along light-like path that is the null geodesic so we may write

$$W^2 dr^2 + P^2 \sin^2 \theta d\phi^2 = 0 \ , \tag{4.3}$$

thus it can be shown that

$$\left(\frac{d\phi}{dr}\right)^2 = -\frac{W^2}{P^2 sin^2\theta} . \tag{4.4}$$

For the relation between $d\theta/dt$, we consider the red shift function term $M = e^{\Phi(r)}$ and the expansion factor of the wormhole radius $P = ar$, since we have taken the light-like condition $ds^2 = 0$, thus we may also write

$$-M^2 dt^2 + P^2 d\theta^2 = 0 \ , \tag{4.5}$$

which then yields

$$\left(\frac{d\theta}{dt}\right)^2 \approx \left(\frac{M}{P}\right)^2 \Rightarrow \frac{d\theta}{dt} \approx \frac{M}{P} \quad \text{and} \quad \frac{d\theta}{dt} \approx \frac{M}{P} = \frac{e^{\Phi}}{ar} \ , \tag{4.6}$$

where $ar/e^{\Phi} = u = P_o/M_o$ which is defined as the gravitational lens impact parameter near a wormhole $\xi \to u$. The gravitational lens impact parameter is larger near the wormhole than at region slightly away from the wormhole that is $u > u_{(region-away)}$ where $u_{(region-away)} = P/M$ therefore we may write $P_o/M_o > P/M$ and thus $P_o M > PM_o$

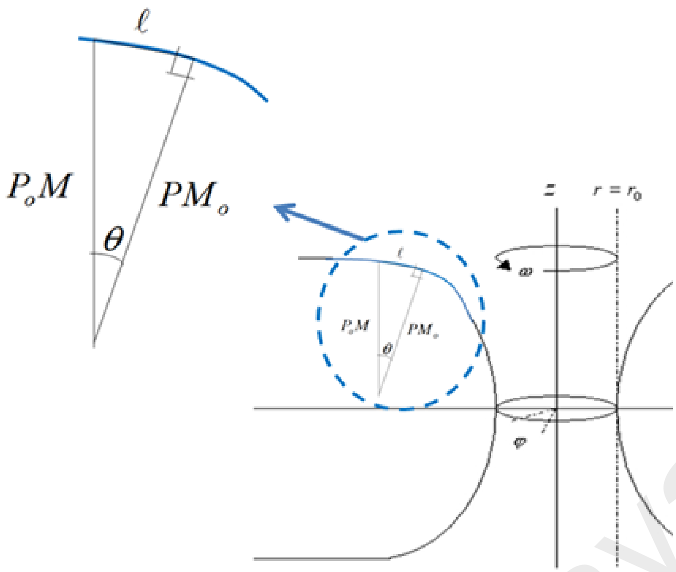

Figure 4.1 : Wormhole throat curvature angle and impact parameter.

From Figure 4.1, the relation between the radius nearer to the wormhole and radius a slightly away from the wormhole in relation with the curvature angle θ (also known as longitudinal or polar angle) can be simplified as $\ell = \left((P_oM)^2 - (PM_o)^2 \right)^{\frac{1}{2}}$. Since $\sin\theta = \ell / P_0 M$ and thus $\sin^2\theta = 1 - P^2/u^2M^2$,therefore from Equation (4.4) the rate of change of the wormhole axial angle ϕ with respect to the wormhole radius is

$$\frac{d\phi}{dr} = \frac{W}{P\left(\frac{P^2}{u^2M^2} - 1\right)^{\frac{1}{2}}} , \tag{4.7}$$

with the shape function term in Equation (4.2), then Equation (4.7) becomes

$$\frac{d\phi}{dr} = \frac{1}{P}\left(\frac{N^2 + Q^2Y_{,r}^{\,2}}{\frac{P^2}{u^2M^2} - 1}\right)^{\frac{1}{2}} , \tag{4.8}$$

thus

$$\phi = \int \frac{1}{P} \left(\frac{N^2 + Q^2 Y_{,r}^{\;2}}{\frac{P^2}{u^2 M^2} - 1} \right)^{\frac{1}{2}} dr \, . \tag{4.9}$$

The wormhole axial angle ϕ can be depicted as in the following Figure 4.2

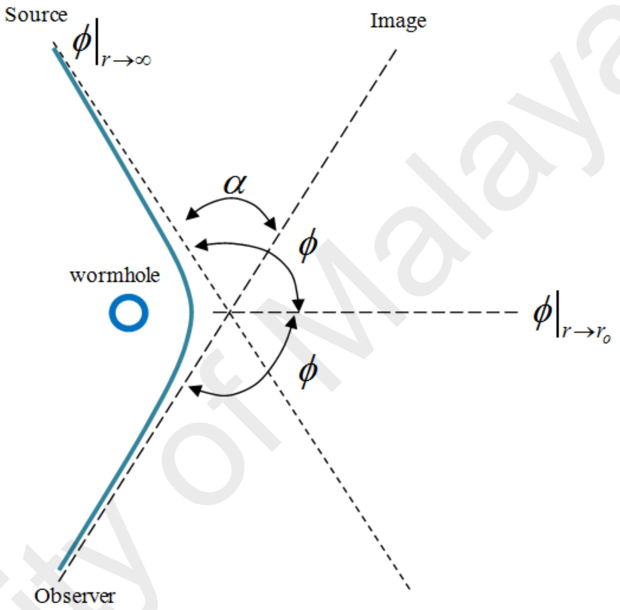

Figure 4.2 : The effect of wormhole axial angle on the deflection angle.

From Figure 4.2 it is obvious that the deflection angle α can be written as a function of the wormhole axial angle

$$\alpha = 2\phi - \pi \, . \tag{4.10}$$

By Equation (4.9) and Equation (4.10) we may write the expression for the deflection angle as

$$\alpha = 2\int_{r_o}^{\infty} \frac{1}{P}\left(\frac{N^2 + Q^2 Y_{,r}^{\,2}}{\frac{P^2}{u^2 M^2} - 1} \right)^{\frac{1}{2}} dr - \pi \,. \tag{4.11}$$

We may rewrite Equation (4.11) as deflection angle in terms of the wormhole axial angle function $I(\phi)$ as

$$\alpha = I(\phi) - \pi \,, \tag{4.12}$$

where the axial angle function is

$$I(\phi) = 2\int_{r_o}^{\infty} \frac{1}{P}\left(\frac{N^2 + Q^2 Y_{,r}^{\,2}}{\frac{P^2}{u^2 M^2} - 1} \right)^{\frac{1}{2}} dr \,. \tag{4.13}$$

We factorize the $\left(N^2 + Q^2 Y_{,r}^{\,2}\right)^{\frac{1}{2}}$ term as $N\left(1 + \frac{Q^2}{N^2} Y_{,r}^{\,2}\right)^{\frac{1}{2}}$ and expand $\left(1 + \frac{Q^2}{N^2} Y_{,r}^{\,2}\right)^{\frac{1}{2}}$ by binomial series

$$\left(1 + \left(\frac{QY_{,r}}{N}\right)^2\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\left(\frac{QY_{,r}}{N}\right)^2 - \frac{1}{8}\left(\frac{QY_{,r}}{N}\right)^4 + \frac{1}{16}\left(\frac{QY_{,r}}{N}\right)^6 + \ldots.. \tag{4.14}$$

Thus by Equation (4.14), Equation (4.11) can be rewritten as

$$\alpha = 2\int_{r_o}^{\infty} \frac{1}{P\sqrt{\frac{P^2}{u^2 M^2} - 1}} \left(N + \frac{1}{2}\frac{\left(QY_{,r}\right)^2}{N} - \frac{1}{8}\frac{\left(QY_{,r}\right)^4}{N^3} + \frac{1}{16}\frac{\left(QY_{,r}\right)^6}{N^5} + \ldots.. \right) dr - \pi \,, \tag{4.15}$$

which can be rewritten to show the respective classical (without brane) and brany (the effect of brane) contributions.

$$\alpha = 2\int_{r_o}^{\infty} \frac{N}{P\sqrt{\frac{P^2}{u^2M^2}-1}}\,dr$$

$$+2\int_{r_o}^{\infty} \frac{1}{P\sqrt{\frac{P^2}{u^2M^2}-1}}\left(\frac{1}{2}\frac{\left(QY_{,r}\right)^2}{N}-\frac{1}{8}\frac{\left(QY_{,r}\right)^4}{N^3}+\frac{1}{16}\frac{\left(QY_{,r}\right)^6}{N^5}+.....\right)dr-\pi\cdot \tag{4.16}$$

This shows that the wormhole axial function has two terms, which are the "classical" and the "brany", respectively

$$I\left(\phi\right)=I\left(\phi\right)_{class}+I\left(\phi\right)_{brany}\ , \tag{4.17}$$

where $I\left(\phi\right)_{class} = 2\displaystyle\int_{r_o}^{\infty} \frac{N}{P\sqrt{\frac{P^2}{u^2M^2}-1}}\,dr$, and thus

$$I\left(\phi\right)_{brany} = 2\int_{r_o}^{\infty} \frac{1}{P\sqrt{\frac{P^2}{u^2M^2}-1}}\left(\frac{1}{2}\frac{\left(QY_{,r}\right)^2}{N}-\frac{1}{8}\frac{\left(QY_{,r}\right)^4}{N^3}+\frac{1}{16}\frac{\left(QY_{,r}\right)^6}{N^5}+.....\right)dr\ . \tag{4.18}$$

From the shape function and radii terms in the line element Equation (4.2) and radii Equation (4.1) we may let

$$N^2=\frac{1}{1-\frac{b\left(r\right)}{r}}\ . \tag{4.19}$$

as purely representing classical general relativistic wormhole and

$$\left(QY_{,r}\right)^2 = \frac{a^2-1}{1-\frac{b(r)}{r}} \ , \tag{4.20}$$

as purely representing the extra-dimensional term of the braneworld wormhole. Hence a complete deflection angle integral expression due to a wormhole in a braneworld is

$$\alpha = 2\int_{r_o}^{\infty} \frac{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{ar\sqrt{\frac{a^2r^2}{u^2e^{2\Phi(r)}}-1}}dr + 2\int_{r_o}^{\infty} \frac{1}{ar\sqrt{\frac{a^2r^2}{u^2e^{2\Phi(r)}}-1}} \left(\frac{1}{2}\frac{\left(a^2-1\right)}{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}\left(1-\frac{b(r)}{r}\right)} \right.$$

$$-\frac{1}{8}\frac{\left(a^2-1\right)^2}{\left(1-\frac{b(r)}{r}\right)^{-\frac{3}{2}}\left(1-\frac{b(r)}{r}\right)^2} + \frac{1}{16}\frac{\left(a^2-1\right)^3}{\left(1-\frac{b(r)}{r}\right)^{-\frac{5}{2}}\left(1-\frac{b(r)}{r}\right)^3}$$

$$\left. -\frac{5}{128}\frac{\left(a^2-1\right)^4}{\left(1-\frac{b(r)}{r}\right)^{-\frac{7}{2}}\left(1-\frac{b(r)}{r}\right)^4} + ... \right) dr - \pi \ , \tag{4.21}$$

which can be rewritten as follows

$$\alpha = 2\int_{r_o}^{\infty} \frac{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{ar\sqrt{\frac{a^2r^2}{u^2e^{2\Phi(r)}}-1}} \left(1 + \frac{a^2-1}{2} - \frac{\left(a^2-1\right)^2}{8} + \frac{\left(a^2-1\right)^3}{16}\right.$$

$$\left. -\frac{5\left(a^2-1\right)^4}{128} + \frac{7\left(a^2-1\right)^5}{256} - ....\right) dr - \pi \ . \tag{4.22}$$

Let $\beta = a^2 - 1$ and

$$\lambda = \frac{\beta}{2} - \frac{\beta^2}{8} + \frac{\beta^3}{16} - \frac{5\beta^4}{128} + \frac{7\beta^5}{256} - .... \pm O(\beta). \tag{4.23}$$

Hence Equation (4.22) can be rewritten as the deflection angle of the braneworld wormhole

$$\alpha_{brane} = 2(1+\lambda) \int_{r_o}^{\infty} \frac{\left(1 - \frac{b(r)}{r}\right)^{-\frac{1}{2}}}{ar\sqrt{\frac{a^2 r^2}{u^2 e^{2\Phi(r)}} - 1}} dr - \pi \ . \tag{4.24}$$

If evolution or spatial dynamic factor $a \to 1$ therefore $\lambda \to 0$ meaning that there is no evolution at all, then the expression will resort back to the deflection expression for a purely general relativity background without the need for the existence of extra-dimensional braneworld terms, as the classical deflection angle expression of Equation (2.91). Hence, this also shows that the evolutionary (dynamics) model by itself suggests the existence of the extra-dimensional braneworld. The dynamic factor or expansion rate $a = a(t)$ can be very slightly higher than 1 at the wormhole vicinity. This can be shown by considering the simplest relation with Hubble constant H that is

$$H = \frac{\dot{a}(t)}{a(t)}, \tag{4.25}$$

Thus, in term of the spacetime expansion rate (dynamic factor) we may write

$$a(t) = e^{Ht} . \tag{4.26}$$

So, if there exist Hubble constant, as in the dynamics of the universe, there exist an expansion factor which at the very least is slightly higher than 1. Now as $\beta = e^{2Ht} - 1$ then we may write

$$\lambda = \frac{e^{2Ht}-1}{2} - \frac{\left(e^{2Ht}-1\right)^2}{8} + \frac{\left(e^{2Ht}-1\right)^3}{16} + ...O\left(Ht\right) \ . \tag{4.27}$$

which shows that the existence of Hubble constant indicates a dynamic universe and thus by itself, signify a braneworld cosmology.

4.3 Numerical estimation

We may expand Equation (4.24) accordingly under the boundary conditions of the wormhole, for the interior region where the domain of the integral is (r_0, r_s) and the exterior region of the wormhole where the domain of the integral is (r_s, ∞) as

$$\alpha_{brane} = 2\left(1+\lambda\right)\left(\int_{r_o}^{r_s} \frac{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{ar\sqrt{\frac{a^2r^2}{u^2e^{2\Phi(r)}}-1}}dr + \int_{r_s}^{\infty} \frac{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{ar\sqrt{\frac{a^2r^2}{u^2e^{2\Phi(r)}}-1}}dr\right) - \pi \ . \tag{4.28}$$

This consist of the "classic" general relativity (GR) term and the brane extra term

$$\alpha_{brane} = \alpha_{GR} + \alpha_{braneExtra} , \tag{4.29}$$

where the GR term is

$$\alpha_{GR} = 2\left(\int_{r_o}^{r_s} \frac{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{ar\sqrt{\frac{a^2r^2}{u^2e^{2\Phi(r)}}-1}}dr + \int_{r_s}^{\infty} \frac{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{ar\sqrt{\frac{a^2r^2}{u^2e^{2\Phi(r)}}-1}}dr \right) - \pi \quad . \tag{4.30}$$

and the brane extra term is

$$\alpha_{braneExtra} = 2\lambda\left(\int_{r_o}^{r_s} \frac{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{ar\sqrt{\frac{a^2r^2}{u^2e^{2\Phi(r)}}-1}}dr + \int_{r_s}^{\infty} \frac{\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}}{ar\sqrt{\frac{a^2r^2}{u^2e^{2\Phi(r)}}-1}}dr \right) \quad . \tag{4.31}$$

The wormhole interior region is represented by the radial domain $r_o < r < r_s$ that is the region between the wormhole throat and the wormhole Schwarzschild radius. In this region the shape function can be written as

$$b(r) = B\left(1-e^{-r/(r_s-r)}\right) \;, \tag{4.32}$$

where

$$B = \frac{r_o}{1-e^{-r_o/(r_s-r_0)}} \,, \tag{4.33}$$

and the red shift function can be written as

$$\Phi(r) = \frac{1}{2}\ln\left(1-\frac{b(r)}{r}+\frac{\varepsilon}{r^2}\right) \qquad , \tag{4.34}$$

thus from Equations (4.32), (4.33) and (4.34)

$$\Phi(r) = \frac{1}{2}\ln\left(1 - \frac{r_o}{r}\left(\frac{1-e^{-r/(r_s - r)}}{1-e^{-r_o/(r_s - r_0)}}\right) + \frac{\varepsilon}{r^2}\right) \quad . \tag{4.35}$$

The wormhole exterior region is represented by the radial domain $r_s < r < \infty$ that is the region just outside the wormhole Schwarzschild radius to the open and flat spacetime. In this region, the shape function is a constant and represented as

$$b(r) = b(r_s) = B = \frac{r_o}{1-e^{-r_o/(r_s - r_0)}} \ , \tag{4.36}$$

and the red shift function is represented by

$$\Phi(r) = \frac{1}{2}\ln\left(1 - \frac{B}{r}\right) \quad , \tag{4.37}$$

thus from Equation (4.36) and Equation (4.37)

$$\Phi(r) = \frac{1}{2}\ln\left(1 - \frac{r_o}{r\left(1-e^{-r_o/(r_s - r_0)}\right)}\right) . \tag{4.38}$$

The wormhole regions and its corresponding parameters $B = b(r_s)$, $r_0 = b(r_0)$ that are the shape function value at Schwarzschild radius and throat radius respectively are depicted in the following Figure 4.3. The figure shows the region between the wormhole throat radius and the Schwarzschild radius is the interior region while the exterior region of the wormhole is region beyond its Schwarzschild radius toward the surrounding space.

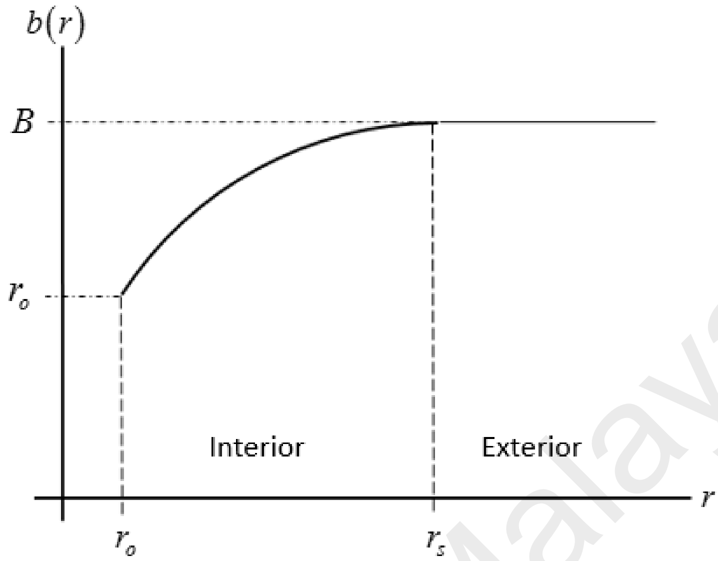

Figure 4.3 : Wormhole regions corresponding parameters.

By assuming numerical value of the wormhole throat, its Schwarzschild radius (m) and red shift function error term as the following; $r_o = 5250$, $r_s = 10000$, and $\varepsilon = 1$, where the ratio between those radiuses must always in accordance to wormhole geometric characteristic which is not to represent a severely curved spacetime as black hole or too lightly curved as ordinary Schwarzschild celestial object. The impact parameter is defined as $u = r_o / e^{\Phi(r_o)}$ thus by Equation (4.34)

$$u = \frac{r_o}{\sqrt{1 - \frac{b(r_o)}{r_o} + \frac{\varepsilon}{r_o^2}}} \quad , \tag{4.39}$$

since $b(r_0) = r_0$ thus

$$u = \frac{r_o^2}{\sqrt{\varepsilon}} \ . \tag{4.40}$$

Using Equation (4.28), Equation (4.30) and Equation (4.31) we may tabled up the relationship between the brane deflection angle with its classical general relativity terms and its extra brane terms as the following

a	α_{GR}	$\alpha_{braneExtra}$	α_{brane}
1.01	0.592	0.074	0.667
1.02	0.519	0.146	0.666
1.03	0.449	0.215	0.664
1.04	0.380	0.281	0.661
1.05	0.313	0.345	0.659
1.06	0.248	0.407	0.655
1.07	0.185	0.465	0.651
1.08	0.124	0.522	0.646
1.09	0.064	0.577	0.641
1.1	0.006	0.629	0.636

Table 4.1 : Deflection angles (radian) with respect to space expansion rate.

Using this numerical values from Table 4.1 the relation of deflection angle α, with respect to expansion rate a as shown in Figure 4.4

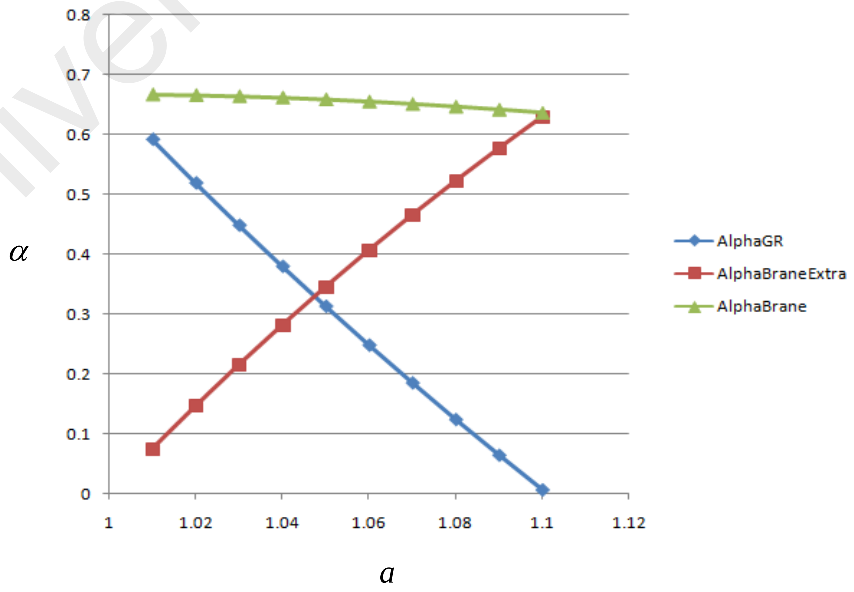

Figure 4.4 : Deflection angles correspond to the increases of spatial expansion rate.

From Table 4.1 at the low spacetime expansion rate, the result shows that there exist only small variation between the classical GR base deflection angle and the brane extra term deflection angle. However as we explore various spacetime expansion rates, the influence of brane extra-dimension is increasingly significant as the expansion rate is increases as depicted in Figure 4.4. Table 4.1 also interestingly shows that as the rate of expansion increases, if the deflection angle calculation is totally based on classical general relativity (GR), the deflection angle will be reduced significantly. This implies as if gravitational lensing is gradually becomes a minimal occurrence in a purely classical general relativity base model in a region or period of accelerating expanding spacetime, whereas if the deflection angle calculation is based on our brane deflection angle formulation of Equation (4.28) the gravitational lensing is still remain a significant occurrence to be observed even at an increasing rate of expansion.

4.4 Conclusion of result

The deflection angle of a gravitational lens showed the character of the lens object since it is derived from its spacetime line element expression. Some slight variation of the angle expression, which depends on its mass and its spacetime curvature, is presented. If the mass is assumed to be fixed, the surrounding spacetime characteristic of any two different lens object will show the difference e.g. in the case between a black hole with singularity and event horizon, and a traversable wormhole without singularity and event horizon, will show the difference. As the spacetime characteristic plays an important role on deflection angle hence it is discernible that the extra-dimensional braneworld will have influence on the spacetime curvature and thus on the gravitational lensing, with the deflection angle showing differences by some extra terms. Our derivation has shown that the existence of the extra-dimension braneworld influencing the extra term in deflection angle expression is closely related to and in fact dependent on the spacetime evolution factor or the spatial dynamic expansion rate. Moreover if we consider different expansion rates, where as the rate increases it is shown that the main contribution of gravitational lensing is due to the brane terms. This is plausible, as we can relate the evolution factor with the accelerating expansion of the universe fueled by dark energy while the existence of dark energy (Amendola and Tsujikawa, 2010) is very much influenced by the existence of large extra-dimension as in the RS II model braneworld.

CHAPTER 5 : SUMMARY AND CONCLUSION

5.1 Summary

In the framework of braneworld cosmology we have introduced a case study of a spherically symmetric wormhole in a surrounding spacetime represented by 3+1 brane "floating" in one additional extra higher dimension perpendicularly projected from the brane's "plane" described firstly by Wong (Wong et al., 2011). It is indeed originated from the pioneering works of Bronnikov, RS II model until the recent works of Wong et al. We have derived the field equation of the wormhole and used Shiromizu et al. postulation that directly equate the projection of Weyl tensor with the wormhole physical characteristic itself namely the energy density, radial pressure and the lateral pressure. The geometry finiteness and smoothness were also tested using Kretchmann scalar and Bronnikov's integral solution for using these concept upon Wong type wormhole on 3+1 brane floating in 4 D spatial bulk which has proven that the Shiromizu and Bronnikov postulates for our model still work, where the finiteness and smoothness were preserved thus indeed it shows to be naturally imparted with the extra-dimensional bulk effect to maintain wormhole traversable sustainability without the requirement of spacetime "exoticity". The spatial dynamics terms of space surrounding the wormhole were considered by just explicitly embedding time dependence dynamic factor into the shape function terms and the radial term only, while preserving the metric tensor of time term to indicate the metric preserving the wormhole geometry as time independence but yet letting the shape function and radial terms to represent bulk space influencing the wormhole's surrounding brane that evolve dynamically which is therefore as far as the wormhole geometry is concerned the wormhole itself is time independent.

It is interesting to consider the observable evidence of the existence of extra-dimension manifested by the curvature-intense celestial object as our works on wormholes. Light path in the vacuum of space is purely manifesting the "shape" of spacetime. The existence of extra-dimension may influence the shape. Gravitational lens has been the first prediction of the Einstein theory of General Relativity which was by itself the first theory to predict the existence of the first extra-dimensional concept known to physics historically with its introduction to the idea of warped spacetime describing gravity. Thus the idea that gravitational lens may predict or signify the existence of extra-dimension is expected.

5.2 Conclusions

As the result of deriving the physical characteristic from the spacetime metric of spherically symmetric wormhole in a surrounding spacetime represented by 3+1 brane, it has been shown to be distinguishable neatly between the classical terms of general gelativity and brane terms with its surrounding brane-bulk influence dynamics. This has resulted into the interesting findings where there exist correlation between the surrounding wormhole spatial dynamics and the wormhole physical characteristics required for wormhole traversability. The dynamic expansion factor $a(t) > 1$ will reduce the amount of stress energy tensor namely the energy density, radial pressure and the lateral pressure implying that the formation of traversable wormholes is easier in an expanding spatial region.

In cosmological aspect, during the period where universe is expanding e.g. during the inflationary period or as per currently, which seems like the period where the universe is acceleratingly expanding, the formation of traversable wormholes are easier to occur. The dynamic contraction factor when $0 < a(t) < 1$ on the other hand will increase the amount of stress energy tensors physical characteristics thus implying that the formation of traversable wormhole is harder in a shrinking spatial region or in cosmological aspect, during the period of contracting universe e.g. the big crunch. This model of spatial dynamic factor affecting wormholes can be applied on the cyclic or the bounce universe model as there exist some period of expansion (bounce or big bang) and contraction (big crunch) of space in the model. It can also be applied in a cosmological model with regional space expansion and regional space contraction.

As for the idea that gravitational lens may signify the existence of extra-dimension, the main observable parameter that can be used to detect the fringes of the gravitational lens resulted by the presence of bulk underlying the brane is the deflection angle. The deflection angle of gravitational lens shows the character of the lens object since it is derived from its spacetime metric expression which is the underlying warp passage of light travelling in vacuum space. As the spacetime metric physical characteristic of the red shift, the shape function and radial terms contribute to the deflection angle expression, hence the extra-dimension term that is specifically embedded in the shape function term will definitely affects the spacetime curvature which therefore may influence the gravitational lensing. As the result to the extra-dimensional term influence, the deflection angle shows the slight differences by some extra terms.

Our derivation has shown not only that the existence of the spatial extra-dimension of braneworld influencing the extra term in deflection angle expression but also the effects of spatial dynamic factor. The result implies interesting findings where it can be shown that the effects of different expansion rates correlate with the influence of the extra-dimension on the gravitational lens deflection angle in which as the expansion rate increases, the result suggest that the main contribution of gravitational lensing is more due to the brane terms. This is very much related to the idea that accelerating expansion of the universe is fueled by dark energy while the existence of dark energy can also be manifested by the existence of large extra-dimension as in the RS II braneworld model.

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