INTRODUCTION

The probability for a situation may, conceivably, be served by any set of values $x_1, x_2 \ldots$, and corresponding probabilities $p(x_1), p(x_2), \ldots$, where

$$\sum p\,(x_i) = 1.$$

But there are a few particular models which are important because of the large number of instances in which they are useful. The Poisson Distribution is one of such models which is useful in certain situations in which same kind of event occurs repeatedly, but haphazardly, such as is shown in the figure below:—

——x——x——x——x——x——x——x——

(each x represents an occurrence of the event)

Indeed, the Poisson Distribution is one of the few distributions (besides the binomial and normal distributions) of great universality which occurs in a remarkably great variety of problems. It arises frequently in diverse fields. Since demands for service, irrespective of whether on the cashiers or salesmen of a department store, the stock clerk of a factory, the cargo-handling facilities of a port, the maintenance man of a machine shop, as well as the rate at which service is rendered, often lead to random phenomena either approximately or exactly obeying a Poisson Probability Law. Hence the Poisson Distribution is of great utility in fields of commerce, physics, medicine, management science and so forth.

These random events or random phenomena may arise in connection with the occurrence of accidents, errors, breakdowns and other similar calamities. These random events, as seen from above, are quite varied in nature. There are some which do not occur as the outcomes of definite trials of an experiment. Instead, they may occur at random points in time, or at points along a wire or chain, or at points in a plane region or space. Therefore, the Poisson Distribution arises in practice, whenever a process is such that the probability of exactly k events occurring within an interval t is approximately, or exactly given by

$$e^{-\lambda t}\,\frac{(\lambda t^{k})}{k!}$$

where, λt may be interpreted as the mean rate of occurrence (of events) i.e.

λt = average or expected number of occurrences in an interval of length (space) t.

and, setting $t = 1$,

λ = average number of occurrence in a unit interval. Moreover λ is a constant which characterises the particular process concerned.

DERIVATION OF POISSON DISTRIBUTION

Having defined the Poisson Distribution, let us examine the conditions under which one may expect that the number of occurrences of a random event occurring in time or space, obeys a Poisson Probability Law. The Poisson Distribution arises when it may be considered that the following postulates hold:-

Poisson Postulates

1. Events concerning changes in non-overlapping intervals are independent.

2. The probability of a given number of changes in an interval depends on the size, not the location of the interval.

3. For any small positive number h, and any time interval of length h, there exists a positive quantity λ such that:-

   (a) the probability that exactly one event occurs in the interval is approximately equal to λh. This means it is equal to $\lambda h + r_1(h)$ and $r_1(h)/h \to 0$ as $h \to 0$.

   (b) the probability that exactly zero event occurs in the interval is approximately equal to $1-\lambda h$, in the sense that it is equal to $1-\lambda h + r_2(h)$, and $r_2(h)/h \to 0$ as $h \to 0$.

   (c) the probability that two or more events occur in the interval is equal to a quantity $r_3(h)$ such that the quotient $r_3(h)/h \to 0$, as the length h, of the interval $\to 0$.

The positive quantity λ, that is, the λ may be interpreted as the mean rate at which events occur per unit time (space). Therefore, we may refer to λ as the mean rate of occurrence (of events). Putting the basic assumptions of the existence of the parameter λ, in mathematical formulation:-

$$3(a) \quad P_1(h) \doteq \lambda h \quad \text{i.e. } P_1(h) = \lambda h + r_1 h$$

$$(b) \quad P_o(h) \doteq 1-\lambda h \quad \text{i.e. } P_o(h) \doteq 1-\lambda h + r_2(h)$$

3(c) where $n \geq 2$,

$$P_n(h) = r_3(h).$$

The symbol $r_1(h)$ can be read as "some function of a smaller order than h", and is used to denote an unspecified function $E(h)$ having the property that

$$\left(\lim_{h \to 0} \frac{E(h)}{h} = 0\right)$$

This means that $r_1(h)$ represents any quantity which for small h is negligible compared to h. Thus

$$(h^2, e^{-h}, \sin^2 h) \text{ are all } r_1(h).$$

The 1st property of the parameter (I3.a.) states that the probability of exactly I event occurring in a small interval (h) is approximately proportional to the width of the interval (h), when h is small. For the 3rd property stated, (I3. c.) it means that the probability of more than one occurrence of the event in an interval of width, h, is negligible when compared with the probability of a single occurrence of the event, for small (h).

Returning to the first two Poisson postulates, let us examine the implication of the first postulate, which states that events concerning changes in non-overlapping intervals are independent. This is to say that if an interval of time (space) is divided into n sub-intervals and, for i = 1, 2 ... n, Ai denotes the event that at least one event of the kind under consideration occurs in the ith interval, then for any integer n, A1, A2 ... An are independent events. From what we have discussed, it is obvious that the probability of a change cannot be proportional to the width of the interval for large intervals, since probability is bounded by unity. For small intervals, however, halving the size of the interval halves the probability of finding a change there. Hence the probability of a given number of changes in an interval is subjected to the size of the interval but not to its location.

Under the foregoing assumptions, one may show that the number of occurrences of the event in a period of time (or space) of length (or area or volume) t obeys a Poisson Probability Law with parameter λt, i.e. the probability that exactly k events occur in a time period of length t is

$$(= e^{-\lambda t} \frac{\lambda t^k}{k!})$$

As such, one may describe a sequenceof events which occurs in a time (or space) and satisfying the above assumptions, as the events obeying a Poisson Probability Law at the rate of λ events per unit time (or unit space).

Proof

$$e^{-\lambda t} \frac{\lambda t^k}{k!} \quad \ldots\ldots\ldots\ (1)$$

To prove the above Poisson formula, one must divide the time period of length t into n time periods of length $h = \frac{t}{n}$. Therefore, the probability of k events occurring in the time t is approximately the probability that exactly one event has occurred in exactly k of the n sub-intervals of time of the original time period, t, Under the above assumptions, this is equivalent to the probability of obtaining exactly k successes in n independent repeated Bernoulli trials. It is equal to

$$\binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{h}\right)^{n-k} \quad \ldots\ldots\ldots\ (2)$$

$\binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$ is only an approximation to the probability that k events occur in time t. To get an exact evaluation, one must let the number of sub-intervals $\left(\frac{t}{n}\right)$ increase to infinity, then as $1 \rightarrow 2$, i.e.

$$\binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k} \longrightarrow e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$$\frac{1}{k!} (\lambda t)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k} \frac{(n)k}{n^k} \longrightarrow \frac{1}{k!} (\lambda t)^k e^{-\lambda t}, \text{ as } n \rightarrow \infty,$$

$$= e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

In order that the above Poisson formula gives a legitimate distribution of probability, the sum of the probabilities of all possible values should be unity. This sum of the probabilities is an infinite sum, as shown below:-

$$\frac{\lambda t^0}{0!} e^{-\lambda t} + \frac{\lambda t^1}{1!} e^{-\lambda t} + \frac{\lambda t^2}{2!} e^{-\lambda t} + \frac{\lambda t^3}{3!} e^{-\lambda t} + \ldots + \frac{\lambda t^k}{k!} e^{-\lambda t}$$

$$= e^{-\lambda t} \left(1 + \lambda t + \frac{\lambda t^2}{2!} + \frac{\lambda t^3}{3!} + \ldots + \frac{\lambda t^k}{k!} e^{-\lambda t}\right)$$

But it can be shown that,

$$\left(1 + \lambda t + \frac{\lambda t^2}{2!} + \frac{\lambda t^3}{3!} + \ldots + \frac{\lambda t^k}{k!}\right) = e^{\lambda t}$$

Hence, sum of the probabilities is equal to

$$e^{-\lambda t} \cdot e^{\lambda t} = e^0 = 1$$

$$\text{i.e.} \quad e^{-\lambda t} \sum_{0}^{\infty} \frac{(\lambda t)^k}{k!} = 1$$

Here we have used that for any number x

$$e^x = \sum_{0}^{\infty} \frac{x^k}{k!}$$

Therefore, it will surely apply for $x = \lambda t$.

The Poisson Distribution function is obtained by adding up the proper number of Poisson probabilities:-

$$F(x) = P(X \leq x) = \sum_{k \leq x} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$$p(k, \lambda t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

is defined as the probability of finding exactly k occurrences in an fixed interval of length t. Thus the probability of no occurrences in an interval of length t is

$$p(0, \lambda t) = e^{-\lambda t};$$

and the probability of one or more occurrences is

$$1 - e^{-\lambda t} \quad \ldots\ldots\ldots\ldots (3)$$

The parameter λ is a physical constant which determines the density of occurrences on the t-axis. Although one is able to state assumptions under which a random phenomenon will obey a Poisson probability law with some parameter λ, the value of this physical constant λ cannot be deduced theoretically. It must be determined empirically, and this determination is a statistical problem. If λ is large, the chance of finding no occurrence is small as (3) shows. Suppose that a physical experiment is repeated N number of times, when N is very large. Each time one counts the number of events in an interval of fixed length t. For each integer k = 0, 1, 2, 3, ... let N_k be the number of times that exactly k events are observed.

Then $N_0 + N_1 + N_2 + \ldots N_k + \ldots = N$

The total number of events observed in the N experiments (N intervals of length t) is

$$0.N_0 + 1.N_1 + 2.N_2 + \ldots + kN_k + \ldots = T \quad \ldots\ldots\ldots (4)$$

Hence T/N is the average, i.e. it represents the observed average number of events occurring per time interval of length t. Since N is large, one can expect that

$$N_k \doteq N_p(k;\ \lambda t) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (5)$$

Substituting from equation (5) into (4), one finds

$$T \doteq N \left\{p(1;\ \lambda t) + 2p(2;\ \lambda t) + 3p(3;\ \lambda t) + \ldots + kp(k;\ \lambda t)\right\}$$

$$= Ne^{-\lambda t}\ \lambda t \left\{1 + \frac{\lambda t}{1} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^k}{k!} + \ldots\right\}$$

$$= Ne^{-\lambda t} \cdot e^{\lambda t} = N\lambda t$$

Therefore, $\lambda t \doteq T/N$

As an estimate $\hat{\lambda}$ of the value of the parameter λ, one takes

$$\hat{\lambda} = T/N = 1/N \sum_{k=0}^{\infty} kN_k$$

Hence if one believes that the random phenomenon under observation obeys a Poisson probability law with parameter $\hat{\lambda}$, then one may compute the probability $p(k;\ \hat{\lambda})$ that in a time interval of length t, exactly k successes will occur. This relation, (λt), gives one a means of estimating λ from observations and of comparing theory with experiments.

Expected Value and Variance of the Poisson Distribution

$$E(X) = \sum_{0}^{\infty} f(x)px$$

$$= e^{-\lambda t} \sum_{1}^{\infty} \frac{(\lambda t)^k}{(k-1)!}$$

$$= e^{-\lambda t} \cdot e^{\lambda t} \cdot \lambda t. = \lambda t$$

Therefore, λt = the mean or expected number of occurrences (or changes) in an interval of length t.

Setting $t = 1$,

λ = the mean or expected number of occurrences in a unit interval.

With this interpretation of (λ) and λt, one observes that specifying the expected number of occurrences in an interval of any size completely determines the distribution of probability among the various possible number of occurrences in an interval of any other size. Thus, if the size of the basic interval under consideration is doubled, the expected number of occurrences is also doubled.

The variance of a Poisson Distribution may be computed as follows:-

Expectation of $(X^2 - X)$:-

$$E(X^2 - X) = \sum_{0}^{\infty} (k^2 - k)\, f(k)$$

$$= e^{-\lambda t} (\lambda t)^2 \sum_{0}^{\infty} \frac{(\lambda t)^{k-2}}{(k-2)!}$$

$$= (\lambda t)^2$$

Hence the variance is

$$\sigma^2 = E(X^2) - \mu^2$$

$$= E(X^2 - X) + E(X) - [E(X)]^2$$

$$= (\lambda t)^2 + t - (\lambda t)^2$$

$$= \lambda t.$$

Thus one observes that the Poisson Distribution has the distinct characteristic of having its expected value equal to its variance.