CHAPTER II

THE POISSON APPROXIMATION

When dealing with Bernoulli trials, one uses the formula $b(k; n, p) = \binom{n}{k}p^k q^{n-k}$; i.e. the probability that n Bernoulli trials with probabilities p for success and q = 1-p for failure, result in k successes and n-k failures ($0 \leq k \leq n$). However when n is relatively large and p is small, with the product

$$\lambda = np$$

is of moderate magnitude, one can gather that the computation of the formula may be laborious. In such cases, it will be more convenient to use an approximation formula to b(k; n, p) which gives good results when p is very small (say, $p < 0.1$) or near zero and n is relatively large (say, $n > 50$).

Let us examine the relation between the Binomial and Poisson distributions by using the bionomial formula through the Poisson postulates (I. 3a, b, c). Suppose that the Poisson postulates hold for changes along a t-axis, considering in particular an interval of size t. Let x be as always the number of changes in an interval of size t. Let λ denote the average number of changes in a unit interval . Then subdivide the interval of width t into n equal parts, each of width t/n. Consider that each of these subintervals as a "trial" and E as the occurrence of a change in a given interval. Actually, there may be more than one change in a subinterval t/n, but by Postulate (I, 3c) this is comparatively unlikely if the subinterval is small.

Hence in each trial, there is either "E" (a change) or "not E" (no change). The probability of "E" is, by Postulate (I.3a), approximately $\lambda t/n$. Therefore, the probability of "not E" is approximately $1 - \lambda t/n$. Using $\lambda t/n$ and $1 - \lambda t/n$ as p and q of the binomialformula respectively, one finds that:

$$P(X = k) = P(k \text{ E's among the n sub-intervals})$$

$$= P(k \text{ "trials" result in } E) \doteq \binom{n}{k}\left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

Rewriting the above approximation with a slight re-arrangement of some of the factors, one obtains:-

$$\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} \frac{(\lambda t)^k}{k!} \left(1 - \frac{\lambda t}{n}\right)^{-k} \left(1 - \frac{\lambda t}{n}\right)^n$$

One would like to know what happens to the approximations as $n \longrightarrow \infty$. When $n \longrightarrow \infty$, then the sub-interval width approaches zero, and the approximations involved get better and better. However as $n \longrightarrow \infty$, the first k factors $\longrightarrow$ I, the next factor is fixed, the next $\longrightarrow$ I, and the last factor becomes

$$\lim_{n \to \infty} \left\{ \left[\left(1 - \frac{\lambda t}{n}\right) \right]^{-n/\lambda t} \right\}^{-\lambda t}$$

Setting $\frac{-\lambda t}{n} = h$, the last factor becomes

$$\left[\lim_{h \to 0} (1 + h)^{1/h} \right]^{-\lambda t} = e^{-\lambda t}$$

as the quantity in the bracket is a definition of e. Therefore, the approximation to the probability that $X = k$ approaches the Poisson Distribution expression

$$e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

Doubtless, this limit-type relationship is an interesting application of the binomial probability formula. However, the practical significance is due to the fact that the approximate equality

$$e^{-\lambda t} \frac{(\lambda t)^k}{k!} \doteq \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

where n is large,

may be read reversely, and the binomial probability may be approximated by the Poisson. If one puts $p = \lambda t/n$ in the above expressions, one obtains:

$$\binom{n}{k} p^k (1 - p)^{n-k} \doteq e^{-np} \frac{(np)^k}{k!}$$

where n is large and

p is small.

When applying the Poisson distribution as a convenient approximation to the binomial distribution, one always states that n must be sufficiently large and p relatively small so that np is of moderate magnitude, in order to get a more accurate or better approximation. One may illustrate the implication and importance of this "thumb-rule".

For simplicity in calculation, one may fix the expected number of successes, $E(B) = np$, at the value 1. Table 2.1[1] below shows 5 different binomial distributions, corresponding to $n = 2$, 5, 10, 20 and 40, with p respectively 1/2, 1/5, 1/10, 1/20, and 1/40, so that in each case $np = 1$.

TABLE 2.1.

BINOMIAL DISTRIBUTION WITH $np = 1$

	$n = 2$ $p = 1/2$	$n = 5$ $p = 1/5$	$n = 10$ $p = 1/10$	$n = 20$ $p = 1/20$	$n = 40$ $p = 1/40$	Poisson Approximation n very large $p = 1/n$
b=0	0.2500	0.3277	0.3487	0.3585	0.3632	0.3679
1	0.5000	0.4096	0.3874	0.3774	0.3725	0.3679
2	0.2500	0.2048	0.1937	0.1887	0.1863	0.1839
3		0.0512	0.0574	0.0596	0.0605	0.0613
4		0.0064	0.0112	0.0133	0.0143	0.0513
5		0.0003	0.0015	0.0022	0.0026	0.0031
6			0.0001	0.0003	0.0004	0.0005
7						0.0001

Comparing the first two rows of the Table, it is obvious that the values of $P(B = 0)$ and $P(B = 1)$ come closer together as n is increased and p decreased. When $n = 2$, $P(B = 1)$ is twice as large as $P(B = 0)$, i.e. $0.5000 = 2 \times 0.2500$. But when $n = 40$, the ratio of $P(B = 1)$ to $P(B = 0)$ is only

$$\frac{0.3725}{0.3632} = 1.026$$

This suggests that when n is very large, $\frac{P(B = 1)}{P(B = 0)}$ might be very close to 1, so that to a good approximation, the same value could be used for both $P(B = 1)$ and $P(B = 0)$. One may easily verify this suggestion. It follows that the formula for binomial probabilities and the fact that $\binom{n}{0} = 1$, and $\binom{n}{1} = n$ that

$$P(B = 0) = q^n, \quad P(B = 1) = npq^{n-1}$$

Since we are dealing with the case $np = 1$, this means

[1] Op. cite. J.L. Hodges, Jr. and E.L. Lehmann, Basic concepts of probability and statistics, San Francisco, London, Amsterdam, Holden-Day Inc. (1964). p. 175.

$$\frac{P(B = 1)}{P(B = 0)} = \frac{npq^{n-1}}{q^n} = \frac{\lambda}{q} \quad \text{..............} \quad (1)$$

If p is close to zero, then $q = 1-p$ is close to 1, therefore $\frac{\lambda}{q}$ is close to 1. This proves the suggestion that the ratio

$\frac{P(B = 1)}{P(B = 0)}$ will be near 1 if p is near zero. Therefore, for large n, and $p = 1/n$ is correspondingly small, we have

$$P(B = 1) \triangleq P(B = 0)$$

Similarly, $\frac{P(B = 2)}{P(B = 1)}$ will be close to 1/2 when n is large and p is small, and $np = 1$. Since $P(B = 2)$ is about half as large as $P(B = 1)$ and $P(B = 1)$ is in turn close to $P(B = 0)$, we have the approximation:-

$$P(B = 2) \triangleq (1/2)P(B = 0)$$

Similarly, one may find that

$P(B = 3) \triangleq 1/3 \cdot 1/2\ P(B=0) = 1/6\ P(B = 0)$
$P(B = 4) \triangleq 1/4 \cdot 1/3 \cdot 1/2\ P(B = 0) = 1/24\ P(B = 0)$ and so forth.
As all possible values of B add up to unity, so

$$\begin{aligned} 1 &= P(B = 0) + P(B = 1) + P(B = 2) + P(B = 3) + \dots \triangleq \\ &\quad P(B = 0) + P(B = 0) + 1/2\ P(B = 0) + 1/6\ P(B = 0) + 1/24P(B = 0) + \dots \\ &= P(B = 0)\ [1 + 1 + 1/2 + 1/6 + 1/24 + \dots] \quad \text{.........} \quad (2) \end{aligned}$$

The factor $[1 + 1 + 1/2 + 1/6 + 1/24 + \dots]$

can be computed numerically to any desired degree of accuracy, as shown in Table 2.2. below:-

TABLE 2.2.

CALCULATION OF $1 + 1 + 1/2 + 1/6 + 1/24 + \dots = e$

1	= 1.0000
1	= 1.0000
1/2	= 0.5000
1/2.1/3 = 1/6	= 0.1667
1/2.1/3.1/4 = 1/24	= 0.0417
1/2 . 1/3. 1/4.1/5 = 1/120	= 0.0083
1/2. 1/3 . 1/4. 1/5.1/6 = 1/720	= 0.0014
1/2.1/3.1/4.1/5.1/6.1/7 = 1/5040	= 0.0002
	2.7183 i.e. e

One may continue to add up as many terms as one wishes, but the remaining terms are so small that they can be termed as negligible.

Substituting in equation (2), we find that

$$1 \doteq P(B = 0)\ 2.7183$$

$$\therefore\ P(B = 0) \doteq \frac{1}{2.7183} = 0.3679 = e^{-1}$$

That is to say that if n is large so that $p = 1/n$ is small, the probability that there will be no success is approximately 0.3679 or e^{-1}, regardless of the precise values of n and $p = 1/n$. This approximation is already reasonably good when $n = 40$, where the correct value is 0.3632, and the larger n is, the better the approximation will be. Since $P(B = 1) \doteq P(B = 0)$, the same value e^{-1}, i.e. 0.3679 may be used to approximate $P(B = 1)$. Since $P(B = 2) \doteq 1/2P(B = 0)$, the value $1/2(0.3679) = .1839$ may be used to approximate $P(B = 2)$, and so forth. In Table 2.1, the last column shows the approximateon values.

Poisson Approximation - General Case

In the fore-going section only $np = 1$ is considered. But the method is applicable to any fixed value of $E(B) = np$. One may consider the binomial distributions with np fixed at the value

$$np = \lambda$$

The aim is to obtain an approximation to these distributions when p is close to zero, and hence n is very large. As in the case $np = 1$, one has $P(B = 0) = q^n$ $\quad P(B = 1) = npq^{n-1}$

$$\therefore\ \frac{P(B = 1)}{P(B = 0)} = \frac{np}{q}$$

Since q is close to 1, and $np = \lambda$, one has for large n,

$$\frac{P(B = 1)}{P(B = 0)} \doteq \lambda \quad \text{........} \quad (3)$$

$$\text{or}\quad P(B = 1) \doteq \lambda P(B = 0)$$

Similarly, by analogous argument

$$\frac{P(B = 2)}{P(B = 1)} \doteq 1/2\lambda \qquad (4)$$

$$\frac{P(B = 3)}{P(B = 2)} \doteq 1/3\lambda$$

$$\frac{P(B = 4)}{P(B = 5)} \doteq 1/4\lambda$$

$$\frac{P(B = k)}{P(B = k + 1)} \doteq \frac{1}{k}\lambda$$

To illustrate how these relations can be used to obtain approximate probabilities, let λ equal, say, 2. Then it can be found by exactly the same method as for the case $np = \lambda = 1$, that

$$\begin{aligned} P(B = 2) &\doteq 1/2.\ 4P(B = 0) \\ P(B = 3) &\doteq 1/6.\ 8P(B = 0) \\ P(B = 0) &\doteq 1/24.\ 1/16P(B = 0) \text{ and so forth.} \end{aligned} \qquad (5)$$

Adding the probabilities and factoring the common factor $P(B = 0)$, one obtains, in analogy to $1 = P(B = 0)\ [1+1+\frac{1}{2}+1/6+1/24 + \ldots]$

$$1 = P(B = 0)\ [1+2+\tfrac{1}{2}.4 + 1/6.\ 8 + \ldots]$$

$[1+2+\frac{1}{2}.4 + 1/6.\ 8 + \ldots]$ can be calculated to any desired degree of accuracy. Taking only the first twelve terms, one obtains the value 7.3891, hence $P(B = 0) \doteq 0.1353$. The probabilities of the other values can be computed from (3) and (5). By this method, one can obtain the Poisson approximation corresponding to any given value of λ.

Although only the Poisson approximation to the binomial distribution has been discussed, but this approximation is not limited to applications of approximating the binomial distribution, it has a wider range of application. Independent trials with different but small success probabilities may be approximated by the Poisson Distribution. Suppose there are S number of successes on n independent trials with success probabilities $p_1, p_2 \ldots\ p_n$. If these probabilities are small, the Poisson Approximation with

$$\lambda = E(S) = p_1 + p_2 \cdots + p_n$$

will give a high degree of accuracy and work well though the probability of success varies for each trial.