CHAPTER III

APPLICATIONS

The Poisson Distribution occurs in a wide range of situations, involving events occurring in time intervals of fixed length, space of fixed volume, areas of fixed size, segments of fixed length, etc. These situations are those where a large number of observations is involved and the probability of an event occurring in any specific observation is very small. Indeed, most of the temporal and spatial distributions follow the Poisson formula.

By temporal distributions, one refers to those distributions dealing with events which may be supposed to occur in equal intervals of time of small but fixed duration. Spatial distributions deal with events which may be supposed to occur in intervals of equal lengths along a straight line, in regions of equal areas and so forth.

Examples of Temporal Distributions Obeying the Poisson Distribution

1. Distribution of numbers of telephone calls received at a particular switch-board per minute for a large number of minutes during a certain hour of the day.

2. Distribution of numbers of death per day due to a specific disease (not in epidemic form) such as heat disease in a certain city for a great number of days.

3. Distribution of numbers of automobiles passing a given point on a highway per minute, for a large number of minute at a given time of the day.

4. Distribution of numbers of α particles emitted by a radioactive substance and received at a certain portion of a plate during intervals, say, of 7.5 seconds, for a great number of these intervals.

Examples of Spatial Distribution following the Poisson Distribution

1. Distribution of numbers of flying-bomb hits on London during the Second World War. (The London area was divided into 576 units of 1/4 square kilometer each, and the number of units which received 0, 1, 2, ... k bomb-hits were recorded and the distribution

was found to follow the Poisson law remarkably well.)

2. Distribution of numbers of typographical errors per page for a great number of pages.

3. Distribution of numbers of bacterial colonies in a given culture on a slide per 0.01 square millimetre for a large numbers of such units.

4. Distributions of the numbers of defective screws per box for a large number of boxes.

Earlier, the parameter λ is referred to as the mean rate of occurrence of events. The following example may show the function as stated:-

Suppose one is observing the times at which automobiles arrive at the toll collector booth on the Ipoh-Kuala Lumpur route. Let us assume that we are informed that the mean rate λ of arrival of automobiles is given by $\lambda = 2$ automobiles per minute. Hence in a time period of length $h = 1$ second $= 1/60$ minute, exactly one car will arrive with approximate probability $\lambda h = 2.1/60 = 1/30$ (according to Poisson postulate 1.3a); whereas exactly zero car will arrive with approximate probability $1-\lambda h = 29/30$ (as stated by Poisson postulate 1.3b).

Similarly, one may compute the probability that a sample of two cubic centimetres of water will contain (i) no bacteria (ii) at least two bacteria, provided it is known that bacteria of a certain kind occur in water at the rate of 1.5 bacteria per cubic centimetre of water.

From the assumptions made earlier, one may conclude that the number of bacteria in a two-cubic-centimetre sample of water obeys a Poisson Probability law with parameter $\lambda t = (2)(1.5) = 3$. Here λ denotes the rate at which bacteria occur in a unit volume and t denotes the volume of the sample of water under consideration. Thus the probability of no bacteria in the sample is e^{-3}, and the probability of two or more bacteria in the sample is $1-e^{-3}$.

The value of the parameter λ, cannot be deduced theoretically. It must be determined empirically as the following example illustrates:

<u>Connections to wrong numbers</u> - Table 3.1 shows statistics of telephone connections to a wrong number.

TABLE 3.1[1]

CONNECTIONS TO WRONG NUMBER

No. of Wrong Connections k	Observed Nos. N_k	Theoretical Values $Np(k; 8.74)$
0-2	1	2.05
3	5	4.76
4	11	10.39
5	14	18.16
6	22	26.45
7	43	33.03
8	31	36.09
9	40	35.04
10	35	30.63
11	20	24.34
12	18	17.72
13	12	11.92
14	7	7.44
15	6	4.33
≥ 16	2	4.65
	267	267.00

where k denotes wrong connections
N_k denotes the number with exactly k wrong connections.

Since

$$T = \sum_{k=0}^{\geq 16} kN_k = (2.1 + 3.5 + 4.11 + \dots + 16.2) = 2334$$

A total of $N = 267$ numbers was observed.

[1]The observations are taken from F. Thorndike, Applications of Poisson's probability summation, The Bell System Technical Journal, Vol. 5(1926), pp. 604-624.

Thus from the expression $\hat{\lambda} = T/N = 1/N \sum_{k=0}^{\infty} kN_k$

$$\hat{\lambda} = \frac{2334}{267} = 8.7415 = 8.74$$

If it is believed that connections to wrong number obey the Poisson probability law at a mean rate of 8.74 per minute. Then one may conclude that the probability of wrong connections per minute is $e^{-8.74}$. This provide a means of testing the hypothesis that connections to wrong numbers at the rate of 8.74 per minute. Since this is the case, then the probability that in a minute there will be k wrong connections is obtained by:

$$p(k;\ 8.74) = e^{-8.74} \frac{(8.74)^k}{k!},\ \text{where } k = 0, 1, 2 \ldots$$

The expected number in N minutes in which k wrong connections occur, which is equal to $Np(k;\ 8.74)$ may be computed and compared with the observed number of minutes in which k wrong connections have occurred. Hence from Table 3.1 one notes that the theoretical values $Np(k;\ 8.74)$ are rather close to the observed numbers N_k. To be more certain, one may compare the observed and expected numbers by statistical criteria such as the X^2-criterion to judge whether the observations are compatible with the hypothesis that the number of connections to wrong number follows a Poisson probability law at a mean rate of 8.74.

<u>Expected Value (Mean) = Parameter</u>

Let us consider a case in which we may apply the fact that the expected value of the Poisson Distribution is equal to its parameter.

Suppose m is the average number of pulses sticking an object in t seconds. The pulses are independent and random; and the chance of a hit is rare. Since this process obeys the Poisson Distribution, we may conclude that the parameter is equal to its expected value, i.e. m. Hence the probability that k of the pulses hitting the object in t seconds is approximately

$$e^{-m} \frac{m^k}{k!}$$

The following examples show the application of the Poisson Distribution as an approximation to the Binomial Distribution. From the following examples one may observe the ease in computation where the Poisson Distribution is applicable.

<u>Effect of Innoculation</u>: Suppose that the probability that a certain type of innoculation has an adverse effect is 0.005. Suppose one would like to know the probability that 2 out of 1,000 people of

given the innoculation will be adversely affected.

If we let X be number of people out of 1,000 who are so affected, this is binomially distributed. The probability that X = 2 computed by the binomial probability law is:-

$$p(2) = \binom{1,000}{2} (0.005)^2 (1-0.005)^{998}$$
$$= \binom{1,000}{2} (0.005)^2 (0.995)^{998} \quad \ldots\ldots\ldots \quad (a)$$

Rather than compute (a), one may compute instead the approximation given by the Poisson formula, with

$$\lambda t = np = 1,000(0.005) = 5:-$$
$$p(2) \triangleq e^{-5} \frac{5^2}{2!} = \frac{25}{2e^5} \triangleq 0.0842$$

From this the ease of computation by the Poisson formula is apparent. Similarly, one may compute

$$p(\text{no more than 2 are affected}) = p(x = 0, 1, \text{ or } 2)$$
$$= p(0) + p(1) + p(2)$$
$$\triangleq e^{-5} + e^{-5} \frac{5}{1!} + e^{-5} \frac{5^2}{2!}$$
$$\triangleq e^{-5} \left(1 + \frac{5}{1!} + \frac{5^2}{2!}\right) \triangleq 0.125$$

<u>Tossing Coins</u>

The probability of obtaining 5 heads k times when 5 coins are tossed 128 times, by the binomial distribution is:

$$\binom{128}{k} \left(\frac{1}{2^5}\right)^k \left(1 - \frac{1}{2^5}\right)^{128-k}$$

$$= \binom{128}{k} \left(\frac{1}{32}\right)^k \left(\frac{31}{32}\right)^{128-k} \quad \ldots\ldots\ldots\ldots\ldots \quad (a)$$

An approximate value to the above is given by the Poisson Distribution, with $\lambda = np = 128(\frac{1}{32}) = 4$, i.e.

$$e^{-4} \frac{4^k}{k!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (b)$$

When k = 2, the value of (a) = 0.1458
the value of (b) = 0.1465

From this one may conclude that the Poisson Distribution is a good approximation to the binomial distribution. Moreover the Poisson

Distribution enables ease in calculation compared with the binomial distribution.

In recent years, the Poisson probability law has become increasingly important as more and more random phenomena to which the law applies have been studied. In physics, the random emission of electrons from the filament of a vacuum tube, or from a photo-sensitive substance under the influence of light and the spontaneous decomposition of radioactive atomic nuclei lead to phenomena obeying a Poisson probability law. This law occurs frequently in the fields of operations research and management science, since demands for service irrespective of whether on human beings or equipment, and also the rate at which service is rendered, often lead to random phenomena either exactly or approximately obeying a Poisson probability law.

Such random phenomena are also found in connection with the occurrence of accidents, errors, breakdowns and other similar calamities. The Poisson distribution is used, for example, in the field of casualty insurance, where the probability that any given house will be destroyed by fire is very small while the insured number of houses is very large. Similarly, the probability that a person will get killed in an automobile is very small while the number of people who ride in cars, is large. The Poisson distribution is also used in problem dealing with the inspection of manufactured products when the probability that any one piece is defective is very small and the lots are very large.

Consequently, the Poisson Distribution may be used to appraise radical divergence from uniformity or from the state of statistical control. Indeed, the Poisson Distribution has played a large part in many industrial sampling problems, particularly in those dealing with large scale mass production. Let us consider the case of defective torch batteries.

Suppose that a manufacturer of torch batteries knows that despite every precaution there is a small unavoidable percentage of defective batteries not up to specification. This has been established at 0.8% or about 8 defective batteries per 1,000. The batteries are packed for retailing in cartons each containing 200. The manufacturer wants to know the chance of a carton containing k defective batteries where k may be 0, 1, 2, ... The manufacturing conditions are such that each carton can be assumed to contain a random sample of production. This is another case where the Poisson Distribution holds. The average number of defectives per carton, λ is given by

$$\lambda = 200 \times 0.008 = 1.6$$

and the chance of a carton containing k defectives is

$$p(k;\lambda) = e^{-\lambda}\frac{\lambda^k}{k!}$$

The results obtained for the different values of k are shown in the Table 3.2 below:-

TABLE 3.2

Number of Defectives in a Carton k		Probability that a Carton Contains k defectives $p(k; \lambda)$
0	. . . .	0.202
1	. . . .	0.323
2	. . . .	0.258
3	. . . .	0.138
4	. . . .	0.055
5	. . . .	0.018
6	. . . .	0.005
7	. . . .	0.001
	Total . .	1.000

In such situations, the manufacturer is usually interested not so much in the individual probabilities of getting different numbers of defectives in a carton; but rather in the chance of getting k or more defectives in a carton. Thus from the above Table, the chance of getting 5 or more defectives is:-

$$0.018 + 0.005 + 0.001 = 0.024 \text{ or}$$

nearly 2½%. This type of information on the product is of great importance. This is because in area of production management, sampling of the manufacture process is often used to maintain the quality of product economically.

Again one could use the Poisson Distribution to gauge the response to advertising:-

Response to Advertisement for Sale of Piano

Suppose a piano is advertised for sale in a newspaper having a circulation of 100,000 readers. Assuming probability that any one reader will respond to the advertisement is

$p = \frac{1}{50,000}$; one may like to know the probability of getting 0,1,2, 3 ... k responses to the advertisement.

Essentially, this is a problem of finding the probabilities of getting "k successes in 100,000 trials" when probability of an individual success is p = 1/50,000. Since p is small, n is large,

$$\lambda = np = 100{,}000 \times 1/50{,}000 = 2.$$

Therefore, on the average, one may expect 2 responses to such an advertisement. The respective probabilities for the different number of responses can be obtained by the Poisson Distribution:

$$p\ (k;\ \ 2) = e^{-2}\ \frac{2^k}{k!}$$

The following Table 3.3 shows the respective probabilities p(k; 2).

TABLE 3.3

No. of Responses k	Probability $p(k;\ \lambda = 2)$
0	0.1353
1	0.2207
2	0.2707
3	0.1804
4	0.0902
5	0.0361
6	0.0120
7	0.0034
8	0.0009
9	0.002

logically, it is possible to obtain the probability of more than 9 responses to the advertisement. But the probability of this is extremely small.

Various biological problems involve the use of the Poisson Distribution, notably those arising when estimating the density of cells or organisms by haemacytometer counts or by the plating method. The same situation is encountered when measuring the density of plants in the wild by the use of quadrat counts. From these limited examples, one may conclude the Poisson Distribution is one of the most useful methods of computing probabilities. It is applicable in diverse fields from operations research, physical experiments such as the computation of probability of radioactive disintegrations of certain substances, to such cases as the frequency of years in which exactly k centenarians will die, or the frequency of loaves with exactly k raisins with λ as the measure of the density of raisins in the dough.