GEOMETRY OF WARPED PRODUCT SUBMANIFOLDS OF RIEMANNIAN MANIFOLDS

ABDALQADER MAHMOUD ABDALQADER MUSTAFA

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ABSTRACT

The purpose of this thesis is to present a self-contained study of Riemannian warped product submanifolds. This is accomplished in four major steps; proving existence, deriving basic lemmas, constructing geometric inequalities and applying them to obtai some geometric applications. The whole thesis is divided into nine chapters. The first two chapters are a journey from the origins of this field to the recent results. Here, definitions, basic formulas and open problems are included. It is well known that the existence problem is central in the field of differential geometry, especially in warped product submanifolds. This problem is investigated in the third and the fourth chapters. Moreover, a lot of key results as preparatory lemmas for subsequent chapters can be found in these two chapters. In the second section of chapter five, a benefit has been taken from Nash's embedding theorem to discuss geometrical situations the immersion may possess such as minimality, total geodesic and total umbilical submanifolds. The rest of this work is devoted to establish basic simple relationships between intrinsic and extrinsic invariants. In a hope to provide new solutions to the question asked by Chern (1968), about whether we can find other necessary conditions for an isometric immersion to be minimal or not, Chen (1993, 2002) has considered this problem in his research programs. In this thesis, and following Chen (2002) and Chern (1968), we have hypothesized their open problems in a more general way in the first chapter. As a result, a wider scope of research becomes available. Therefore, new inequalities are constructed by means of new methods, where equality cases are discussed in details.

ABSTRAK

Tujuan karya ini adalah untuk membentangkan kajian serba lengkap mengenai submanifold produk meleding Riemann (Riemannian warped product submanifolds). Ini dicapai dengan tiga langkah utama; membuktikan kewujudan, menghasilkan lema ciri (characteristic lemmas) dan membina ketaksamaan geometri. Seluruh tesis ini dibahagikan kepada tujuh bab. Bab pertama adalah perjalanan dari asal-usul bidang ini sehingga keputusan terkini. Di sini, definisi, formula asas dan masalah terbuka dimasukkan. Adalah diketahui umum bahawa masalah kewujudan adalah penting dalam bidang geometri kebezaan, terutamanya di submanifold produk meleding. Masalah ini disiasat di bab-bab kedua dan ketiga. Selain itu, banyak keputusan penting seperti lema persediaan dihasilkan untuk kegunaan bab-bab seterusnya boleh didapati dalam bab-bab ini. Dalam bahagian pertama bab empat, manfaat yang telah diambil dari teorem penerapan Nash untuk membincangkan keadaan geometri rendaman yang ada padanya seperti minimality, submanifold geodesi seluruh dan umbilik seluruh. Seterusnya, hubungan asas yang mudah antara invarian intrinsik dan ekstrinsik dihasilkan. Ia adalah penting untuk menyatakan bahawa Chen (1993, 2002) adalah pengasas arah penyelidikan, dengan harapan untuk menyediakan penyelesaian tambahan dan perlu untuk soalan yang ditanya oleh Chern (1968) mengenai sama ada kita boleh mencari keadaan yang perlu lain untuk rendaman minimum ataupun tidak. Dalam sekuel ini, dan mengikuti Chen (2002) dan Chern (1968), kami telah membuat hipotesis masalah terbuka mereka dengan cara yang lebih umum dalam bab pertama. Akibatnya, skop yang lebih luas penyelidikan disediakan. Oleh itu, ketidaksamaan baru dibina dengan menggunakan kaedah baru, di mana semua kes kesaksamaan dibincangkan.

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LIST OF SYMBOLS AND ABBREVIATIONS

- $\nabla(f) \equiv$ Gradient of f.
- $\Delta(f) \equiv$ Laplacian of f.
- $||h||^2 \equiv$ Squared norm of the second fundamental form.
- $||\vec{H}||^2 \equiv$ Squared norm of the mean curvature vector.
- $A \equiv$ Shape operator.
- $\nabla_X Z \equiv$ Covariant derivative [*] of Z in the direction of X.
- $\delta \equiv$ Delta invariant or (Chen invariant).
- $Ric \equiv$ Ricci curvature.
- $\xi \equiv$ Characteristic vector field of almost contact manifolds.
- $\theta \equiv$ Slant angle.
- $\tilde{M}^m \equiv m$-dimensional Riemannian ambient manifold.
- $M^n \equiv n$-dimensional Riemannian submanifold.
- $\Gamma(TM^n) \equiv n$-dimensional module of smooth vector fields tangent to M^n.
- $\mathfrak{F}(\tilde{M}^m) \equiv$ Algebra of smooth functions on $\tilde{M}^m$.
- $\zeta \equiv$ Normal vector field belongs to the normal subbundle ν.
- $\nabla^{\perp}_X \zeta \equiv$ Normal connection on the normal tangent bundle.
- $\tilde{K}(X \wedge Y) = \tilde{K}_{XY} \equiv$ Sectional curvature of the plane spanned by the linearly independent vectors X and Y.
- $\tilde{\tau}(T_x M^n)$ and $\tau(T_x M^n) \equiv$ Scalar curvatures of M^n at some $x \in M^n$ with respect to $\tilde{M}^m$ and M^n, respectively.

[*] Throughout this thesis, the vector fields $X,\ Y$ are taken to be tangent to the first factor while the vector fields $Z,\ W$ are considered to be tangent to the second factor of warped product submanifolds, unless otherwise stated.

- $\mathcal{D} \equiv$ Smooth distribution of vector fields.
- $\{e_1, \cdots, e_m\} \equiv$ Local fields of orthonormal frame of $\Gamma(T\tilde{M}^m)$.
- $J \equiv$ Almost Hermitian structure.
- $\phi \equiv (1,\ 1)$-tensor field of almost contact structure.
- $\varphi \equiv$ Isometric immersion.
- $R(X, Y; Z, W) \equiv$ Riemannian curvature tensor.
- $\eta \equiv$ 1-form of almost contact structures.
- $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m \equiv$ Isometric immersion φ from the warped product submanifold $M^n = N_1 \times_f N_2$ into the ambient manifold $\tilde{M}^m$.
- $M^n =_{f_2} N_1 \times_{f_1} N_2 \equiv$ Doubly warped product submanifold.
- $\tilde{g}$ and $g \equiv$ Riemannian metric * on the ambient manifold $\tilde{M}^m$ and the corresponding induced metric on the Riemannian submanifold M^n, respectively.
- PX and $FX \equiv$ Tangential and normal components of JX or ϕX, respectively.
- $\vec{H}_i \equiv$ Partial mean curvature vectors restricted to N_i for $i = 1,\ 2$.
- $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y \equiv$ Tangential and normal components of $(\tilde{\nabla}_X J)Y$ (resp. $(\tilde{\nabla}_X \phi)Y$) in $\tilde{M}^{2m}$ (resp. $\tilde{M}^{2l+1}$), where $X,\ Y \in \Gamma(TM^n)$.
- $\nu \equiv J$-invariant or ϕ-invariant subbundle of the normal bundle $T^{\perp}M^n$.
- $h^r_{ij} \equiv h(e_i, e_j)$-component, which is in the direction of the unit normal vector field e_r.
- $\sum\limits_{1 \leq i \neq j \leq n} h^r_{ij} \equiv$ Summation of h^r_{ij}, where i and j run from 1 to n such that $i \neq j$.

*For simplicity's sake, we use g to refer for the inner products carried by both $\tilde{g}$ and g.

CHAPTER 1: INTRODUCTION

1.1 INTRODUCTION

In this chapter, a brief outlook on this topic is given. At the beginning, the first appearance of warped products in general relativity is mentioned. After that, the concept of warped product manifolds is discussed from a mathematical viewpoint. Since it is the core of this thesis, warped product submanifolds are developed gradually over four milestone theorems; namely, Nash's C^k embedding theorem (Nash, 1956), J. D. Moore's theorem in 1971 (Moore, 1971), S. Nölker's work in 1996 (Nölker, 1996) and B. Y. Chen's paper in 2001, (Chen, 2001). Inspired by the last author, we address the problems of our study and determine its objectives.

1.2 BACKGROUND

The field of warped product manifolds has its origin in the beginning of the last century with the work of Albert Einstein in 1916 when the prominent geometric theory of gravitation was published (Einstein, 1960), which is nowadays known as the general theory of relativity. Soon after the publication of Einstein's theory of general relativity, Karl Schwarzschild found the first exact solution, other than the trivial flat space solution, of the Einstein field equations. It is interesting that the first solution was a warped product spacetime (Letter from K Schwarzschild to A Einstein dated 22 December 1915). More precisely, the Schwarzschild exterior spacetime and the Schwarzschild black hole are defined as the following

Definition 1.2.1. *(O'Neill, 1983). For* $\mathbf{m} > 0$ *let* P_I *and* P_{II} *be the regions* $r > 2\mathbf{m}$ *and* $0 < r < 2\mathbf{m}$ *in the* tr*-half-plane* $\mathbb{R}^1 \times \mathbb{R}^+$*, each furnished with line element* $-\hbar dt^2 + \hbar^{-1}dr^2$*, where* $\hbar(r) = 1 - (2\mathbf{m}/\mathbf{r})$*. If* S^2 *is the unit sphere, then the warped product* $N = P_I \times_r S^2$ *is called Schwarzschild exterior spacetime and* $B = P_{II} \times S^2$ *the Schwarzschild black hole, both of mass* $\mathbf{m}$*.*

Henceforth, the notion of warped products has been playing some important roles in the theory of general relativity as they have been providing the best mathematical models of our universe for now. That is, the warped product scheme was successfully applied in general relativity and semi-Riemannian geometry in order to build basic cosmological models for the universe. For instance, the Robertson-Walker spacetime, the Friedmann cosmological models and the standard static spacetime are given as warped product manifolds. For more cosmological applications, warped product manifolds provide excellent setting to model spacetime near black holes or bodies with large gravitational force. For example, the relativistic model of the Schwarzschild spacetime that describes the outer space around a massive star or a black hole admits a warped product construction (O'Neill, 1983).

In an attempt to construct manifolds of negative curvatures, R. L. Bishop and B. O'Neill introduced the notion of warped product manifolds in 1969 (Bishop & O'Neill, 1996) by homothetically warping the product metric of a product manifold $B \times F$ on the fibers $p \times F$ for each $p \in B$. This generalized product metric appeared in differential geometric studies in a natural way, making the studies of warped product manifolds inevitable with intrinsic geometric point of view.

In 1954, one of the most important contributions in the field of Riemannian submanifolds theory appeared, it is the well-known Nash first embedding theorem, C^1 embedding theorem, published by J. F. Nash (Nash, 1954).

Theorem 1.2.1. *Let (M, g) be a Riemannian manifold and $\varphi : M^m \to \mathbb{R}^n$ a short C^∞-embedding (or immersion) into Euclidean space $\mathbb{R}^n$, where $n \geq m+1$. Then, for arbitrary $\epsilon > 0$ there is an embedding (or immersion) $\varphi_\epsilon : M^m \to \mathbb{R}^n$ which is*

(i) in class C^1;

(ii) isometric: for any two vectors $X,\ Y \in T_xM$ in the tangent space at $x \in M$,

$$g(X, Y) = \langle d\varphi_\epsilon(X), d\varphi_\epsilon(Y)\rangle;$$

(iii) ϵ-close to φ:

$$|\varphi(x) - \varphi_\epsilon(x)| < \epsilon\ \forall\ x \in M.$$

Two years later, a more technical theorem appeared (Nash, 1956). It is the C^k embedding theorem.

Theorem 1.2.2. *Every compact Riemannian n-manifold can be isometrically embedded in any small portion of a Euclidean N-space $\mathbb{E}^N$ with $N = \frac{1}{2}n(3n+11)$. Every non-compact Riemannian n-manifold can be isometrically embedded in any small portion of a Euclidean m-space $\mathbb{E}^m$ with $m = \frac{1}{2}n(n+1)(3n+11)$.*

During this work, when we refer to the Nash's embedding theorem, we mean the C^k embedding theorem of (Nash, 1956).

In view of Nash's theorem, J. D. Moore proved, in 1971, that a Riemannian product immersion does naturally exist whenever it is mixed totally geodesic (Moore, 1971). To see this, suppose that $M_1, \cdots, M_k$ are Riemannian manifolds and that

$$\varphi : M_1 \times \cdots \times M_k \to \mathbb{E}^N$$

is an isometric immersion of the Riemannian product $M_1 \times \cdots \times M_k$ into Euclidean N-space. J. D. Moore proved that if the second fundamental form h of φ has the property that $h(X,Y) = 0$ for X tangent to M_i and Y tangent to M_j, $i \neq j$, then φ is a product immersion; that is, there exist isometric immersions $\varphi_i : M_i \to \mathbb{E}^{m_i},\ 1 \leq i \leq k$, such that

$$\varphi(x_1, \cdots, x_k) = (\varphi_1(x_1), \cdots, \varphi_k(x_k)),$$

when $x_i \in M_i$ for $1 \leq i \leq k$.

The study of differential geometry of warped product submanifolds was intensified after 1996, when S. Nӧlker gave a warped product version of Moore's result. Let $M_0, \cdots, M_k$ be Riemannian manifolds, $M = M_0 \times \cdots \times M_k$ their product, and $\pi_i : M \to M_i$ the canonical projection. If $\rho_1, \cdots, \rho_k : M_0 \to \mathbb{R}_+$ are positive-valued functions, then

$$\langle X, Y \rangle := \langle \pi_{0*}X, \pi_{0*}Y \rangle + \sum_{i=1}^{k} (\rho_i \circ \pi_0)^2 \langle \pi_{i*}X, \pi_{i*}Y \rangle$$

defines a Riemannian metric on M, called a warped product metric. M endowed with this metric is denoted by $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$.

A warped product immersion is defined as follows: Let $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ be a warped product and let $\varphi_i : N_i \to M_i,\ i = 0, \cdots, k$, be isometric immersions, and define $f_i := \rho_i \circ \varphi_0 : N_0 \to \mathbb{R}_+$ for $i = 1, \cdots, k$. Then the map

$$\varphi : N_0 \times_{f_1} N_1 \times \cdots \times_{f_k} N_k \to M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$$

given by $\varphi(x_0, \cdots, x_k) := (\varphi_0(x_0), \varphi_1(x_1), \cdots, \varphi_k(x_k))$ is an isometric immersion, which is called a warped product immersion.

S. Nölker extended Moore's result by showing the natural existence of mixed totally geodesic warped products submanifolds in Riemannian space forms $\tilde{M}^m(c)$ as the following.

Theorem 1.2.3. *Let $\varphi : N_0 \times_{f_1} N_1 \times \cdots \times_{f_k} N_k \to \tilde{M}^m(c)$ be an isometric immersion into a Riemannian manifold of constant curvature c. If h is the second fundamental form of φ and $h(X_i, X_j) = 0$, for all vector fields X_i and X_j, tangent to N_i and N_j respectively, with $i \neq j$, then, locally, φ is a warped product immersion.*

Ever since S. Nölker (Nölker, 1996) gave an explicit description of the warped product representation of Euclidean spaces in 1996, there followed studies of warped product spaces with extrinsic geometric point of views. In 2001, B.Y. Chen introduced the notion of CR-warped product submanifolds in Kaehler manifolds (Chen, 2001). In this paper, he first proved the nonexistence of warped products of the type $N_\perp \times_f N_T$. Reversing the factors, he used a result of S. Hiepko (Hiepko, 1979) to give a characterization theorem of the CR-warped product submanifolds of the type $N_T \times_f N_\perp$ in Kaehler manifolds. Since then, the studies of warped product submanifolds with extrinsic geometric point of view were intensified (Chen, 2013). In this direction of research, the current work aims to continue this sequel of studies.

1.3 MOTIVATIONS AND SCOPE OF THESIS

Three decades after the appearance of the celebrated Nash embedding theorem, its main purpose was materialized by M. Gromov (Gromov, 1985). Here, Riemannian manifolds could always be regarded as Riemannian submanifolds of Euclidean spaces. Inspired by this fact, B. Y. Chen started one of his programs of research in order to study immersibility and non-immersibility of Riemannian warped products in Riemannian manifolds, specially in Riemannian space forms $\tilde{M}^m(c)$ (for example, see (Chen, 2001)-(Chen, 2013)). As a result, he proposed many open problems in this topic (see next section). Recently, a lot of solutions were provided to his problems by many geometers, even though, many gaps still remain. Moreover, many generalizations can be done to save effort and time for potential research. Therefore, this gave us a motivation to fill these gapes and prove such generalizations.

Given a $2m$-dimensional almost complex manifold $\tilde{M}^{2m}$, and a real n-dimensional Riemannian manifold M^n isometrically immersed in $\tilde{M}^{2m}$. It is known that the differential geometry of M^n depends on the behavior of the tangent bundle of M^n relative to the action of the almost complex structure J. Accordingly, we have the following typical classes of submanifolds: CR-submanifolds (Bejancu, 1978), semi-slant submanifolds (Papaghiuc, 1994) and generic submanifolds (Chen, 1981). Analogously, all these kinds of submanifolds have been extended to the setting of almost contact manifolds, (Bejancu, 1986) and (Chen, 2013). Since these classes are the most interesting types of warped product submanifolds, this motivated us to search for their common geometric property. Fortunately, we proved that all of these submanifolds are $\mathcal{D}_i$-minimal warped product submanifolds in both almost Hermitian and almost contact manifolds (see chapter five). Consequently, we are going to impose the $\mathcal{D}_i$-minimality property in the hypothesis of many theorems in this work. Therefore, this will enable us to obtain more general results and will also guide us to establish some new methods for constructing inequalities which were not possible by old techniques (see chapters six, seven and eight).

The concept of minimal submanifolds first appeared in the mid eighteenth century with the work of Euler and Lagrange, even though it has very recently seen major advances that have solved many long standing open conjectures in the field. Thus, it is convenient to study minimal submanifolds. Since our main focus is warped product submanifolds, it becomes more convenient to study the concept of $\mathcal{D}_i$-minimality, because it is the minimality special concept of such submanifolds.

The inevitable motivation was the one that asked by Chen to search for control of extrinsic quantities in relation to intrinsic quantities of Riemannian manifolds. We discussed this problem extensively in the next section. The significance of Riemannian invariants is described by Chen as the following (Chen, 2008):

"Borrowing a term from biology, Riemannian invariants can be considered the DNA of Riemannian manifolds. In particular, curvature invariants are the the most natural and important Riemannian invariants due to their vast applications in other scientific studies.

For instance, the magnitude of a force required to move an object at constant speed, according to Newton's laws, is a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein's general theory of relativity, by the curvatures of the space time. All sorts of shapes, from soap bubbles to red blood cells, seem to be determined by various curvatures (Osserman, 1990). Classically, among extrinsic invariants, the shape operator and the squared mean curvature are the most important ones. Among the main intrinsic invariants, sectional, Ricci and scalar curvatures are the well-known ones." For this, Chen constructed a lot of basic inequalities in terms of intrinsic and extrinsic invariants (see (Chen, 2001)-(Chen, 2013)).

The current thesis aims to continue this sequel of inequalities, specially those which are standing not proved for a long time. Several famous results in differential geometry, such as isoperimetric inequality, Chern-Lashof's inequality, and Gauss-Bonnet's theorem among others, can be regarded as results in this respect. For some recent progress in this direction, see, for instance (Chen, 1999) and references therein.

It is well-known (Chen, 1993) that the following two conditions are necessary for the immersion to be minimal in Euclidean space $\mathbb{E}^m$:

Condition 1: If $\varphi : M^n \to \mathbb{E}^m$ is a minimal immersion from a manifold of positive dimension into a Euclidean m-space, then M^n is non-compact.

Condition 2: If $\varphi : M^n \to \mathbb{E}^m$ is a minimal immersion from a manifold of positive dimension into a Euclidean m-space, then the Ricci tensor of M^n is negative semi-definite.

In (Chen, 1993), and as an answer to the question asked by Chern in page 13 of (Chern, 1968), Chen gave another necessary condition; namely $K(\pi) \geq \frac{1}{2}\tau(T_xM^n)$, where $K(\pi)$ and $\tau(T_xM^n)$ are, respectively, the sectional and the scalar curvatures of M^n, for every plane section $\pi \subset T_xM^n,\ x \in M^n$.

In chapters seven and eight, we give three new necessary conditions for a warped product immersion to be minimal.

1.4 PROBLEMS OF STUDY

This work considers some well-known open problems in differential geometry. Most of these problems arose naturally during the works of B. Y. Chen (Chen, 2002), S. S. Chern (Chern, 1968) and M. Gromov (Gromov, 1985). We also study these problems under a more general setting. In this sequel, problems of this work can be classified into four major categories.

- **Existence and Nonexistence Problems**

Immersibility and non-immersibility of a Riemannian manifold in a Riemannian space form $\tilde{M}^m(c)$ is one of the most fundamental problems in the theory of Riemannian submanifolds. According to Theorem 1.2.2, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimension. Inspired by this theorem, Chen proposed the following problem.

Problem 1.4.1. *(Chen, 2002).* $\forall\, N_1 \times_f N_2 \overset{isometric\ immersion}{\longrightarrow} \mathbb{E}^m$ *or* $\tilde{M}^m(c) \implies ???.$

Taking the ambient manifold to be an almost Hermitian or an almost contact manifold, many nonexistence results, characterization theorems and concrete examples of warped product submanifolds are available, (see, for example (Al-Luhaibi et al., 2009), (Chen, 2001), (Chen, 2013), (Khan et al., 2008), (Khan & Khan, 2009), (Munteanu, 2005), (Mustafa et al., 2013) and (Sahin, 2009)). Motivated by these results, we consider Problem 1.4.1 in a more general way by taking the ambient manifold $\tilde{M}$ to be an arbitrary Riemannian manifold. Thus, for (singly) warped product submanifolds, we have

Problem 1.4.2. $\forall\, N_1 \times_f N_2 \overset{isometric\ immersion}{\longrightarrow} \tilde{M}^m \implies ???$

Or, more generally, for doubly warped product submanifolds, this problem becomes

Problem 1.4.3. $\forall\, {}_{f_2}N_1 \times_{f_1} N_2 \overset{isometric\ immersion}{\longrightarrow} \tilde{M}^m \implies ???$

Some partial solutions to these problems are given in chapters three and four. In Kenmotsu manifolds we constructed several examples ensuring the existence of many warped product submanifolds, another two examples are offered for Kaehler manifolds.

- **Problems of Basic Characteristic Lemmas and Geometric Properties of the Immersions**

It is obvious that the warping function f is one of the most important elements of warped product submanifolds. In fact, it is a particular intrinsic invariant of such submanifolds. On the other hand, the second fundamental form h is the most significant invariant among extrinsic invariants. Therefore, geometers are interested in those relations containing f and h, which can be applied in different tasks (see, for example, references in this survey (Chen, 2013)). However, a lot of gaps are still there. Because of this and of the urgent need of more such results and geometric properties for later work, it is worth to hypothesize the following problem.

Problem 1.4.4. *Given an isometric immersion φ from a warped product submanifold $N_1 \times_f N_2$ into a Riemannian manifold $\tilde{M}^m$. What are the relationships (equations) between the warping function and the second fundamental form of $N_1 \times_f N_2$?*

Now, let $\mathcal{A}$ and $\mathcal{B}$ be any geometric properties which can be imposed on a warped product immersion φ. Since $\mathcal{D}_i$-minimal warped product submanifolds have a central role in this work, we ask the following:

Problem 1.4.5. *Given a $\mathcal{D}_i$-minimal isometric immersion φ from a warped product submanifold $N_1 \times_f N_2$ into a Riemannian manifold $\tilde{M}^m$. Then*

(i) What are the relationships (equations) between the warping function and the second fundamental form of $N_1 \times_f N_2$?

(ii) If φ possesses $\mathcal{A}$, does φ admit $\mathcal{B}$?

Some special solutions for Problem 1.4.4 are thoroughly given in chapters three and four, whereas the second section of chapter five provides some special solutions for Problem 1.4.5, especially Theorem 5.2.1 and Lemma 5.2.6, which are key results in this work.

- **Constructing Inequalities in Terms of Intrinsic and Extrinsic Invariants**

In (Chen, 2002), Chen had determined the goals of this direction of research saying that: "Based on Nash's Theorem, one of my research programs is to search for control of extrinsic quantities in relation to intrinsic quantities of Riemannian manifolds via Nash's Theorem and to search for their applications." Therefore, he asked the following:

Problem 1.4.6. *(Chen, 2002). Let $N_1 \times_f N_2$ be an arbitrary warped product isometrically immersed in $\mathbb{E}^m$ (or in $\tilde{M}^m(c)$) as a Riemannain submanifold. What are the relationships between the warping function f and the extrinsic structure of $N_1 \times_f N_2$?*

All inequalities of chapter five and section two of chapter six are solutions for this problem. Some of these inequalities are given in a general way, where space form cases can be derived easily.

For some technical reasons, very few inequalities like those of section two of chapter six are established. This is because of the 'sensitive' conditions which the Codazzi equation requires in order to carry out the calculations. However, such technical problem is solved in section three of chapter six by using the Gauss equation. There, a new method is established enabling us to construct enough second inequalities of h.

In chapter five, we proved that most of warped product submanifolds of interest belong to the class of $\mathcal{D}_i$-minimal warped product submanifolds. $\mathcal{D}_i$-minimality arose naturally and it is discussed extensively in chapter five. This leads us to propose the following problem which is a special case from the above one.

Problem 1.4.7. *Let $N_1 \times_f N_2$ be a $\mathcal{D}_i$-minimal warped product isometrically immersed in $\mathbb{E}^m$ (or in $\tilde{M}^m(c)$) as a Riemannain submanifold. What are the relationships between the warping function f and the extrinsic structure of $N_1 \times_f N_2$?*

Chapter five and section three of chapter six present a special case solution to this problem as the more general type of such inequalities. Moreover, the technical problems of applying the Codazzi equation are solved by using the Gauss equation. By this, plenty of second inequalities of h are provided for the above two problems.

The above two problems are asking about relations including an extrinsic invariant and the warping function as the intrinsic invariant. In the following problems, we concern with intrinsic invariants other than the warping function.

As mentioned before, extrinsic and intrinsic Riemannian invariants have vast applications in other fields of science, (Osserman, 1990). Classically, among extrinsic invariants, the shape operator and the squared mean curvature are the most important ones. Among the main intrinsic invariants, sectional, Ricci and scalar curvatures are the well-known ones (Chen, 2008). This was quite enough for Chen to address the following problem.

Problem 1.4.8. *(Chen, 1999). Establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold.*

From the references of this thesis, we notice that Chen has established many inequalities for Riemannian submanifolds as special case solutions to the above problem. Unfortunately, very few inequalities satisfying the above problem for warped product submanifolds are available. Henceforth, we hypothesize the following problem.

Problem 1.4.9. *Establish simple relationships between the main extrinsic invariants and the main intrinsic invariants (other than the warping function) of a warped product submanifold.*

The final section of chapter eight contains a general inequality for warped product submanifolds which is a special case solution for this problem.

In particular, and restricted to $\mathcal{D}_i$-minimal warped product submanifolds, the above problem is paraphrased as the following

Problem 1.4.10. *Establish simple relationships between the main extrinsic invariants and the main intrinsic invariants (other than the warping function) of a $\mathcal{D}_i$-minimal warped product submanifold.*

Chapter seven is devoted to construct an inequality in terms of Ricci curvature and the mean curvature vector to provide a special solution to this problem. Another special solution is the first inequality of chapter eight.

- **Necessary Conditions for an Isometric Immersion to be Minimal**

S. S. Chern asked in (Chern, 1968) to search for further necessary conditions on the Riemannian metric of a submanifold M^n in order for M^n to admit an isometric minimal immersion into Euclidean space. As mentioned in page six, Chen gave another necessary condition, (Chen, 1993). Later on, he materialized this goal for warped product submanifolds as the following.

Problem 1.4.11. *(Chen, 2002). Given a warped product $N_1 \times_f N_2$, what are the necessary conditions for the warped product to admit a minimal isometric immersion in a Euclidean m-space $\mathbb{E}^m$ (or $\tilde{M}^m(c)$)?*

We give a partial solution to this problem in chapter eight.

Analogously, we ask the following question for $\mathcal{D}_i$-minimal warped product.

Problem 1.4.12. *Given a $\mathcal{D}_i$-minimal warped product $N_1 \times_f N_2$, what are the necessary conditions for the warped product to admit a minimal isometric immersion in an arbitrary Riemannian manifold $\tilde{M}^m$?*

Two new partial solutions for this problem are given in chapters seven and eight.

1.5 OBJECTIVES OF STUDY

In short, our objectives in this thesis are summarized as follows:

- To prove existence or nonexistence of warped product submanifolds into Riemannian ambient manifolds. So, first we determine whether a warped product submanifold exists or not in both almost Hermitian and almost contact manifolds. In case it exists, we characterize its existence in simple characterization theorems, and support this existence by solid examples. This is achieved in Chapter Three and Chapter Four.

- To investigate the basic geometric properties of those warped product submanifolds which do exist in both almost Hermitian and almost contact manifolds, in such a way making it possible to do comparisons between considered structures specially for almost contact manifolds, and to prove essential preparatory lemmas and theorems to fulfill the next goal. Moreover, we aim to discuss the main geometric concepts that a warped product submanifold may possess or inherit, such as totally geodesic, totally umbilical and minimal submanifolds. Some parts of this objective is satisfied in Chapters Three and Four, while the core of this bojective is given in Chapter Five.

- To establish simple relationships between intrinsic and extrinsic invariants for warped product submanifolds. This is accomplished by constructing basic inequalities for such submanifolds involving extrinsic and intrinsic invariants. Moreover, the rich geometry in the equality case is discussed (see Chapters Five to Eight).

- To apply the inequalities in order to derive some geometric applications, specially necessary conditions, for isometric immersions from warped product to Riemannian manifolds, to be minimal. Chapter Seven and Eight concern with this objective.

1.6 LAYOUT OF THESIS

This thesis is organized as the following:

- **Chapter 1**

In this chapter, a brief outlook on this work is given. At the beginning, the appearance of warped products in general relativity is mentioned. After that, the concept of warped product manifolds is discussed from a mathematical point of view. We also address the problems of our study and determine its objectives.

- **Chapter 2**

This chapter is divided into two sections, the first one presents some of the recent significant results in this field. The second section is for definitions and preliminaries. Here, we describe Riemannian manifolds in a way coping with our purposes. So, we focus on the Levi-Civita connection and the curvature tensor. This tensor will be gradually used to define many intrinsic invariant necessary for this work, such as sectional, scalar and Ricci curvatures.

On the other hand, the extrinsic geometry was explored via the second fundamental form of Riemannian submanifolds. For this, a background of submanifold theory is demonstrated, including Gauss formula and equation, Weingarten formula and Codazzi equation.

More significantly, warped products have been discussed form the manifold and the submanifold theories. As a result, it becomes possible to investigate the intrinsic and the extrinsic geometries of such structures.

Finally, we discusses almost Hermitian and almost contact manifolds as particular classes of Riemannian manifolds.

- **Chapters 3 and 4**

These two chapters are devoted to present special solutions for Problems 1.4.1, 1.4.2, 1.4.3 and 1.4.4. Chapter 3 contains two sections, one relates to almost Hermitian manifolds and the other is for almost contact manifolds. In both sections, several existence and nonexistence results are proved. Basic characteristic and geometric results for later work are given in both sections. Chapter 4 can be considered as slant version of Chapter 3. It is divided into three main sections, the first is for semi-slant warped product submanifolds, and the second is for hemi-slant warped product submanifolds. The final section of chapter three presents a special inequality which turns out to be a fundamental existence theorem. Results for both (singly) and doubly warped product submanifolds are presented here, some of them are multi-task results which are helpful in later work. Two simple characterization theorems are proved in these chapters. Moreover, we construct some examples to ensure existence of different types of warped product submanifolds, and to show that our characterization theorem of Chapter 4 is not vacuous, and the integrability condition we imposed is not redundant. In each chapter, we include a table summarizing the main existence and nonexistence results of warped product submanifolds of interest.

- **Chapters 5**

This chapter provides partial solutions for Problems 1.4.5, 1.4.6 and 1.4.7. It has three main sections. The first section shows the existence of a wide class of warped product submanifolds in Riemannian manifolds possessing the $\mathcal{D}_i$-minimality property; namely CR, semi-slant and generic warped product submanifolds, of almost contact and almost Hermitian manifolds of interest. In addition, a nontrivial example of hemi-slant warped product submanifolds is constructed, showing that the $\mathcal{D}_i$-minimality property is wide enough to be considered for further research. In addition, it presents two important results; namely, Theorem 5.2.1 and Lemma 5.2.6, which are used widely to modify the equality case of the first and the second inequalities of h. In the second and the third sections, we consider the first inequality of h. At first, we modify the equality case of this type of inequalities by satisfying the necessity and sufficiency conditions of the equality holding. After that, many inequalities of different structures are established and modified for both almost Hermitian and almost contact manifolds. Moreover, we prove inequalities for semi-slant and hemi-slant warped product submanifolds and in different structures.

- **Chapters 6**

The current chapter provides some special answers for Problems 1.4.6 and 1.4.7. It contains two main sections. In the first one, the second inequality of h is extended for contact

CR-warped product submanifolds of Kenmotsu manifolds. In the second section, we establish a general inequality for $\mathcal{D}_i$-minimal warped product submanifolds of an arbitrary Riemannian manifold, in terms of the second fundamental form and the warping function, and by means of the Gauss equation. This inequality generalizes all inequalities of the first section of this chapter. The table supplied in the third section contains special case inequalities.

- **Chapters 7**

In this chapter, we prove a general inequality involving the Ricci curvature and the mean curvature vector for warped product submanifolds in Riemannian manifolds. We organize this chapter to include four sections. The first two are for the proof of this inequality. The other sections discuss many extensions of the inequality with some applications derived. This chapter provides new solutions for Problems 1.4.8, 1.4.10, 1.4.11 and 1.4.12.

- **Chapters 8**

In this chapter, we construct two inequalities containing the first Chen invariant and the mean curvature vector for warped product submanifolds. In the first inequality, the $\mathcal{D}_i$-minimality is imposed in hypothesis, while the second inequality is for arbitrary Reimannian manifolds. Among others, applications and particular case inequalities are obtained. Inequalities and their applications in this chapter provide solutions for Problems 1.4.8, 1.4.9, 1.4.10, 1.4.11 and 1.4.12.

- **Chapters 9**

This chapter presents some open problems of this field. It is reasonable to say that, most of problems in this chapter are due to the current thesis, which arose naturally during the work. In another line of thought, all problems of this chapter can also be considered as conclusions of this thesis, as well as further research directions. It is expected that they guide our programs of research for many years.

CHAPTER 2: LITERATURE REVIEW

2.1 INTRODUCTION

The current chapter is organized to have two main sections. The first one presents some significant contributions in this field, from which we get motivations to make some progress in this topic. The second section is devoted to discuss preliminaries and concepts necessary for this work, it is divided into four subsections. The first introduces the notion of Riemannian manifolds. In the second, warped products are defined as Riemannian manifolds. The basic equations, definitions and tools of submanifold theory are offered in the third subsection, where warped products are considered as Riemannian submanifolds. In the last subsection, ambient manifolds of interest are listed; namely, almost Hermitian manifolds and almost contact manifolds.

2.2 RECENT SIGNIFICANT RESULTS

As we now know, our inequalities can be classified in two categories. The first one includes inequalities involving the second fundamental form h and the warping function, while the other is for inequalities containing extrinsic and intrinsic invariants other than the warping function.

For the first category, two basic inequalities were established by Chen, then extended for some other settings. The first is for CR-warped product submanifolds of Kaehler manifolds, while the other is for the same warped product but in complex space form. From now on, we will call them the *first inequality of* h, and the *second inequality of* h, respectively, which were given in the following theorems *

Theorem 2.2.1. *(Chen, 2001). Let* $\varphi : M^n = N_T \times_f N_\perp \longrightarrow \tilde{M}^{2m}$ *be an isometric immersion of a* n*-dimensional CR-warped product submanifold into an* $2m$*-dimensional Kaehler manifold* $\tilde{M}^{2m}$*. Then, we have*

(i) $||h||^2 \geq 2n_2||\nabla \ln f||^2$;

*We note that, in (Chen, 2001) Theorem 2.2.1 was published to have four statements, in this work we are interested in the first two statements, generalizing the first and modifying the second.

(ii) *If the equality in (i) holds, then N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2m}$, respectively.*

Combining special case inequalities in (Chen, 2003), we also have

Theorem 2.2.2. *(Chen, 2003). Let $M^n = N_T \times_f N_\perp$ be a CR-warped product submanifold in a complex space form $\tilde{M}^{2m}(c_{Ka})$. Then, we have the following*

$$\frac{1}{2}||h||^2 \geq 2n_1 n_2 \frac{c_{Ka}}{4} + n_2||\nabla \ln f||^2 - n_2\,\Delta(\ln f).$$

In recent research programs, many geometers intend to develop the geometry of warped product submanifolds in case one of its factors is proper slant. In Kaehler manifolds, B. Sahin proved that any semi-slant warped product submanifold is trivial (Sahin, 2006), which obviously means that the purpose of generalizing the CR-warped product has almost failed. Never losing hope, Sahin himself succeeded recently to achieve this goal by proving the existence of mixed totally geodesic hemi-slant warped product submanifolds of the type $N_\theta \times_f N_\perp$ in Kaehler manifolds (Sahin, 2009). Therefore, semi-slant and hemi-slant warped product submanifolds are extensively investigated in this thesis, and for both almost Hermitian and almost contact manifolds.

The following two theorems were firstly proved for Riemannian submanifolds in real space forms by Chen in (Chen, 1993) and (Chen, 1999), they are known nowadays as the Chen first inequality and Ricci-Chen inequality, respectively. Ever since Chen published them, they have been extended for Riemannian submanifolds in various ambient manifolds (Chen, 2013). By contrast, they were not proved for warped product submanifold in any ambient manifold. Thus, the last two chapters of this work are devoted to prove these two theorems in the setting of warped product submanifolds.

The Chen first inequality was first proved in this form

Theorem 2.2.3. *Let M^n be an n-dimensional $(n \geq 2)$ submanifold of a Riemannian manifold $\tilde{M}^m(c)$ of constant sectional curvature c. Then*

$$\inf K \geq \frac{1}{2}\Big\{\tau(T_x M^n) - \frac{n^2(n-2)}{n-1}||\vec{H}||^2 - (n+1)(n-2)c\Big\}, \tag{2.2.1}$$

where K and $\tau(T_x M^n)$ are the sectional curvature and the scalar curvature of M^n, respectively, $x \in M^n$.

Equality holds if and only if, with respect to suitable orthonormal frame fields $e_1, \cdots,$ $e_n, e_{n+1}, \cdots, e_m$, *the shape operators of* M^n *in* $\tilde{M}^m(c)$ *take the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} \mu_1 & 0 & 0 & \cdots & 0 \\ 0 & \mu_2 & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix}, \quad \mu = \mu_1 + \mu_2,$$

$$A_{e_r} = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \cdots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad r = n+2, \cdots, m.$$

The Ricci-Chen inequality was given as follows

Theorem 2.2.4. *Let* $\varphi : M^n \longrightarrow \tilde{M}^m(c)$ *be an isometric immersion of a Riemannian* n*-manifold* M^n *into a Riemannian space form* $\tilde{M}^m(c)$*. Then*

(i) For each unit tangent vector $X \in T_xM^n$*, we have*

$$||\vec{H}||^2(x) \geq \frac{4}{n^2}\{Ric(X) - (n-1)c\},$$

where $||\vec{H}||^2$ *is the squared mean curvature and* $Ric(X)$ *the Ricci curvature of* M^n *at* X*.*

(ii) If $H(x) = 0$*, then a unit tangent vector* X *at* x *satisfies the equality case of* (i) *if and only if* X *lies in the relative null space* $\mathcal{N}_x$ *at* x*.*

(iii) The equality case of (i) *holds identically for all unit tangent vectors at* x *if and only if either* x *is a totally geodesic point or* $n = 2$ *and* x *is a totally umbilical point.*

In fact, theorems of this section are some of the interesting results in this field, many other results will be considered later. On one hand, the above results and other more are either modified or generalized in the setting of warped product submanifolds. On the other hand, new methods, inequalities and natural geometric properties are the most important contributions of this thesis.

2.3 DEFINITIONS AND PRELIMINARIES

Definitions, formulas and basic lemmas are explored briefly in this section. In the first subsection, we introduce the notions of Riemannian manifolds, the Levi-Civita connection and the Riemannian curvature tensor. After that, we discuss the intrinsic geometry of such manifolds. Meaning that, many intrinsic invariants are defined systematically, such as sectional curvature, scalar curvature, Ricci curvature, Riemannian invariants and Chen first invariant. Warped products are defined as Riemannian manifolds in the second subsection. The geometry of Riemannian submanifolds is discussed in the third subsection. After that, warped products are treated as Riemannian submanifolds. The last subsection is devoted for almost Hermitian and almost contact structures.

2.3.1 RIEMANNIAN MANIFOLDS

It is reasonable to embark on this section by the definition of differentiable manifolds (Do Carmo, 1992).

Definition 2.3.1. *A differentiable manifold of dimension m is a Hausdorff paracompact topological space $\tilde{M}^m$ and a family of injective continuous mappings $\mathbf{x}_\alpha : \mathcal{U}_\alpha \subset \mathbb{R}^m \to \tilde{M}^m$ of open sets $\mathcal{U}_\alpha$ of $\mathbb{R}^m$ into $\tilde{M}^m$ such that:*

(i) $\bigcup_\alpha \mathbf{x}_\alpha(\mathcal{U}_\alpha) = \tilde{M}^m$.

(ii) *For any pair $\alpha,\ \beta$, with $\mathbf{x}_\alpha(\mathcal{U}_\alpha) \cap \mathbf{x}_\beta(\mathcal{U}_\beta) = \mathcal{W} \neq \phi$, the sets $\mathbf{x}_\alpha^{-1}(\mathcal{W})$ and $\mathbf{x}_\beta^{-1}(\mathcal{W})$ are open sets in $\mathbb{R}^m$ and the mappings $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ are differentiable.*

(iii) *The family $\{(\mathcal{U}_\alpha, \mathbf{x}_\alpha)\}$ is maximal relative to the conditions (i) and (ii).*

The pair $(\mathcal{U}_\alpha, \mathbf{x}_\alpha)$ (or the mapping $\mathbf{x}_\alpha$) with $x \in \mathbf{x}_\alpha(\mathcal{U}_\alpha)$ is called a *parametrization* (or *system of coordinates*) of $\tilde{M}^m$ at x; $\mathbf{x}_\alpha(\mathcal{U}_\alpha)$ is then called a *coordinate neighborhood* at x. A family $\{(\mathcal{U}_\alpha, \mathbf{x}_\alpha)\}$ satisfying (i) and (ii) is called a *differentiable structure* on $\tilde{M}^m$.

Let $\tilde{M}^m$ be a C^∞ real m-dimensional manifold *. A *linear connection*, $\tilde{\nabla}$, on $\tilde{M}^m$ is a mapping

$$\tilde{\nabla} :\ \Gamma(T\tilde{M}^m)\ \times\ \Gamma(T\tilde{M}^m)\ \to\ \Gamma(T\tilde{M}^m);\ \ (X,\ Y) \to \tilde{\nabla}_X Y,$$

*Throughout this work, we use the symbol ˜ for ambient manifolds, in order to be distinguished from the terminology of submanifolds.

satisfying the following conditions:

$$(i)\ \tilde{\nabla}_{fX+Y}Z = f\tilde{\nabla}_X Z + \tilde{\nabla}_Y Z, \quad (ii)\ \tilde{\nabla}_X(fY + Z) = f\tilde{\nabla}_X Y + \ (Xf)Y + \tilde{\nabla}_X Z,$$

for any $f \in \mathfrak{F}(\tilde{M}^m)$ and $X,\ Y,\ Z \in \Gamma(T\tilde{M}^m)$, where $\mathfrak{F}(\tilde{M}^m)$ and $\Gamma(T\tilde{M}^m)$ denote the algebra of C^∞ functions on $\tilde{M}^m$ and the module of C^∞ sections of the tangent bundle $T\tilde{M}^m$, respectively, (Bejancu, 1978). Indeed, the choice of a linear connection is equivalent to prescribing a way of differentiability on $\tilde{M}^m$. Using $\tilde{\nabla}$, the *covariant derivative* of a $(0,\ 2)$ tensor field G is defined by

$$\tilde{\nabla}_X\big(G(Y,Z)\big) = (\tilde{\nabla}_X G)(Y,Z) + G(\nabla_X Y, Z) + G(Y, \nabla_X Z). \tag{2.3.1}$$

Observe that covariant derivatives for Riemannian tensors of type $(r,\ s)$ could be defined by a similar manner as above, but this is enough for our purpose.

The *torsion tensor* T of a linear connection $\tilde{\nabla}$ is a tensor field T of type $(1,\ 2)$ defined by

$$T(X,\ Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,\ Y],$$

for any $X,\ Y \in \Gamma(T\tilde{M}^m)$, where $[X,\ Y]$ is the Lie bracket of vector fields X and Y defined by

$$[X,\ Y](f) = X(Yf) - Y(Xf),$$

for any $f \in \mathfrak{F}(\tilde{M}^m)$. A *torsion- free connection* is a linear connection with vanishing torsion tensor field.

A tensor field $\tilde{g}$ of type $(0,\ 2)$ is said to be a *Riemannian metric* on $\tilde{M}^m$ if the following conditions are fulfilled:

(i) $\tilde{g}$ is symmetric, i.e., $\tilde{g}(X,\ Y) = \tilde{g}(Y,\ X)$ for any $X,\ Y \in \Gamma(T\tilde{M}^m)$,

(ii) $\tilde{g}$ is positive definite, i.e., $\tilde{g}(X,\ X) \geq 0$ for any $X \in \Gamma(T\tilde{M}^m)$ and $\tilde{g}(X,\ X) = 0$ if and only if $X = 0$.

If the manifold $\tilde{M}^m$ is further endowed with a Riemannian metric $\tilde{g}$, then it is called a *Riemannian manifold*, (Bejancu, 1986) and (O'Neill, 1983).

A tensor field $\tilde{A}$ of type $(r,\ s)$ is said to be *parallel* with respect to the linear connection $\tilde{\nabla}$ if we have

$$\tilde{\nabla}_X \tilde{A} = 0, \quad \forall\, X \in \Gamma(T\tilde{M}^m).$$

A linear connection $\tilde{\nabla}$ on $\tilde{M}^m$ is said to be a *Riemannian connection* if the Riemnnian metric $\tilde{g}$ is parallel with respect to $\tilde{\nabla}$, i.e., if we have

$$X(\tilde{g}(Y,\ Z)) = \tilde{g}(\tilde{\nabla}_X Y,\ Z) + \tilde{g}(Y,\ \tilde{\nabla}_X Z), \tag{2.3.2}$$

for all $X,\ Y,\ Z \in \Gamma(T\tilde{M}^m)$.

On a Riemannian manifold $\tilde{M}^m$ there exists one and only one torsion-free Riemannian connection $\tilde{\nabla}$ called the *Levi-Civita connection*. More formally, this Riemannian connection is characterized concretely by the well-known *Kuzul formula*; namely

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X(\tilde{g}(Y,\ Z)) + Y(\tilde{g}(Z,\ X)) - Z(\tilde{g}(X,\ Y)) \\ &\quad + \tilde{g}([X,\ Y],\ Z) + \tilde{g}([Z,\ X],\ Y) - \tilde{g}([Y,\ Z],\ X), \end{aligned} \tag{2.3.3}$$

for any $X,\ Y,\ Z \in \Gamma(T\tilde{M}^m)$.

The *curvature tensor* $\tilde{R}$ of $\tilde{\nabla}$ is a tensor field of type $(1,\ 3)$ given by

$$\tilde{R}(X,\ Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,\ Y]} Z, \tag{2.3.4}$$

and, the $(0,\ 4)$ tensor field defined by

$$\tilde{R}(X,\ Y,\ Z,\ W) = \tilde{g}(\tilde{R}(X,\ Y)Z,\ W) \tag{2.3.5}$$

is called the *Riemannian curvature tensor*, for any $X,\ Y,\ Z,\ W \in \Gamma(T\tilde{M}^m)$. It is well-known that the Riemannian curvature tensor is a local isometry invariant.

One could use the preceding two equations to show that the Riemannian curvature tensor $\tilde{R}$ admits the following well-known two skew-symmetric properties $\tilde{R}(X, Z) = -\tilde{R}(Z, X)$ and $g(\tilde{R}(X, Z)Y, W) = -g(\tilde{R}(X, Z)W, Y)$. It also satisfies the first Bianchi identity; namely

$$\tilde{R}(X, Z)Y + \tilde{R}(Z, Y)X + \tilde{R}(Y, X)Z = 0.$$

Consequently, these three properties together produce (O'Neill, 1983)

$$\tilde{g}(\tilde{R}(X, Z)Y, W) = \tilde{g}(\tilde{R}(Y, W)X, Z). \tag{2.3.6}$$

We point out that, these properties are useful to derive some relations between the warping function and scalar curvature for warped product submanifolds in Riemannian manifolds, as we will see in Chapter five.

If we choose two linearly independent tangent vectors $X,\ Y \in T_x\tilde{M}^m$, then the *sectional curvature* of the 2-plane π spanned by X and Y is given in terms of the Riemannian curvature tensor $\tilde{R}$ by

$$\tilde{K}(X \wedge Y) = \frac{\tilde{g}(\tilde{R}(X,Y)Y,X)}{\tilde{g}(X,X)\tilde{g}(Y,Y) - (\tilde{g}(X,Y))^2}. \tag{2.3.7}$$

In case that the 2-plane π is spanned by orthogonal unit vectors X and Y from the tangent space $T_x\tilde{M}^m,\ x \in \tilde{M}^m$, the previous definition may be written as

$$\tilde{K}(\pi) = \tilde{K}_{\tilde{M}^m}(X \wedge Y) = \tilde{g}(\tilde{R}(X,\ Y)Y,\ X). \tag{2.3.8}$$

It is worth pointing out that, $\tilde{K}(\pi)$ is independent of the choice of the orthonormal basis $\{X,\ Y\}$ of π, and it determines the Riemannian curvature tensor $\tilde{R}$ completely (O'Neill, 1983). In addition, if $\tilde{K}(\pi)$ is constant for all planes π in $T_x\tilde{M}^m$ and for all points $x \in \tilde{M}^m$, say $\tilde{K}(\pi) = c$, then we call $\tilde{M}^m(c)$ a *real space form*. In fact, space forms are regarded as the simplest important class of Riemannian manifolds. Denoting by $\tilde{M}^m(c)$ a real space form of constant sectional curvature c, the curvature tensor of $\tilde{M}^m(c)$ is expressed as

$$\tilde{R}(X,\ Y)Z = c(\tilde{g}(Y,\ Z)X - \tilde{g}(X,\ Z)Y), \tag{2.3.9}$$

for any $X,\ Y,\ Z \in \Gamma(T\tilde{M}^m(c))$.

Next, consider a local field of orthonormal frames $\{e_1, \cdots, e_m\}$ on $\tilde{M}^m$. A global $(0,\ 2)$ tensor field defined by

$$\tilde{S}(X,\ Y) = \sum_{j=1}^{m}\{\tilde{g}(\tilde{R}(e_j,\ X)Y, e_j)\}, \quad X,\ Y \in T_x\tilde{M}^m \tag{2.3.10}$$

is called the *Ricci tensor field*. If we fix a distinct integer from $\{1, \cdots, m\}$, let us say o, then the *Ricci curvature* of e_o, denoted $\tilde{R}ic(e_o)$, is given by

$$\tilde{R}ic(e_o) = \sum_{\substack{j=1 \\ j\neq o}}^{m} \tilde{K}_{oj}, \tag{2.3.11}$$

where $\tilde{K}_{oj} = \tilde{K}(e_o \wedge e_j)$. From $\tilde{R}ic(e_o) = \tilde{S}(e_o, e_o)$ we observe that the Ricci curvatures determine the Ricci tensor completely.

In this context, we shall define another important Riemannian intrinsic invariant called the *scalar curvature* of $\tilde{M}^m$, and denoted by $\tilde{\tau}(T_x\tilde{M}^m)$, which, at some x in $\tilde{M}^m$, is given by

$$\tilde{\tau}(T_x\tilde{M}^m) = \sum_{1\le i<j\le m} \tilde{K}_{ij} = \frac{1}{2}\sum_{i=1}^{m} \tilde{R}ic(e_i), \tag{2.3.12}$$

where $\tilde{K}_{ij} = \tilde{K}(e_i \wedge e_j)$. It is clear that, the first equality in (2.3.12) is congruent to the following equation which will be frequently used in subsequent chapters

$$2\tilde{\tau}(T_x\tilde{M}^m) = \sum_{1\le i\ne j\le m} \tilde{K}_{ij}. \tag{2.3.13}$$

In particular, for a 2-dimensional Riemannian manifold, the scalar curvature is its *Gaussian curvature*. Some times the scalar curvature is defined as $\tilde{t} = \sum_{i=1}^{m} \tilde{S}(e_i, e_i)$; thus $\tilde{t} = 2\tilde{\tau}(T_x\tilde{M}^m)$. However, throughout this work, scalar curvatures are defined as in (2.3.12). Consequently, (2.3.11) together with (2.3.12) yield

$$\tilde{R}ic(e_o) = \tilde{\tau}(T_x\tilde{M}^m) - \sum_{\substack{1\le i<j\le m\\ i,j\ne o}} \tilde{K}_{ij} = \tilde{\tau}(T_x\tilde{M}^m) - \frac{1}{2}\sum_{\substack{1\le i\ne j\le m\\ i,j\ne o}} \tilde{K}_{ij}. \tag{2.3.14}$$

In general, for a k-plane Π_k of $T_x\tilde{M}^m$, let $\{e_1, \cdots, e_k\}$ be an orthonormal basis of Π_k. Then for each fixed $o \in \{1, \cdots, k\}$ the ***k*-Ricci curvature** $\tilde{R}ic_{\Pi_k}(e_o)$ of Π_k at x is defined by (Chen, 1999)

$$\tilde{R}ic_{\Pi_k}(e_o) = \sum_{\substack{j=1\\ j\ne o}}^{k} \tilde{K}_{oj}. \tag{2.3.15}$$

Similarly, the scalar curvature $\tilde{\tau}(\Pi_k)$ of the k-plane Π_k is given by

$$\tilde{\tau}(\Pi_k) = \sum_{1\le i<j\le k} \tilde{K}_{ij}, \tag{2.3.16}$$

where we should note that (see, for example (Bejancu, 1986) and (O'Neill, 1983))

$$2\tilde{\tau}(\Pi_k) = \sum_{1\le j\ne i\le k}^{k} \tilde{K}_{ij} = \sum_{i=1}^{k} \tilde{R}ic_{\Pi_k}(e_i). \tag{2.3.17}$$

Hence, for a fixed integer $o \in \{1, \cdots, k\}$ for $k \le n$, we have (see, for example (Chen, 1999) and (O'Neill, 1983))

$$\tilde{R}ic_{\Pi_k}(e_o) = \tilde{\tau}(\Pi_k) - \sum_{\substack{1\le i<j\le k\\ i,j\ne o}} \tilde{K}_{ij}. \tag{2.3.18}$$

For the subsequent chapters we introduce another two concepts of Riemannian invariants. We first take an integer k such that, $2 \le k \le m$, then the *Riemannian invariant* (Chen, 2008), denoted by Θ, on a Riemannian m-manifold $\tilde{M}^m$ is defined by

$$\Theta_k(x) = \left(\frac{1}{k-1}\right) \inf_{\Pi_k, e_o} \tilde{R}ic_{\Pi_k}(e_o), \qquad x \in \tilde{M}^m, \tag{2.3.19}$$

where Π_k runs over all k-planes in $T_x\tilde{M}^m$ and e_o runs over all unit vectors in Π_k. The second invariant is called the *Chen first invariant*, which is defined as

$$\delta_{\tilde{M}^m}(x) = \tilde{\tau}(T_x\tilde{M}^m) - \inf\{\tilde{K}(\pi) : \pi \subset T_x\tilde{M}^m, x \in \tilde{M}^m, \dim \pi = 2\}. \tag{2.3.20}$$

The δ-invariants are ”very different in nature” from the ”classical” scalar and Ricci curvatures. This is simply due to the fact that both scalar and Ricci curvatures are ”total sum” of sectional curvatures on a Riemannian manifold. In contrast, all of the non-trivial δ-invariants are obtained from the scalar curvature by throwing away a certain amount of sectional curvatures, (Chen, 2008).

Next, we recall two important differential operators of a differentiable function ψ on $\tilde{M}^m$; namely the *gradient* $\tilde{\nabla}\psi$ and the *Laplacian* $\Delta\psi$ of ψ, which are defined, respectively, as follows

$$\tilde{g}(\tilde{\nabla}\psi, X) = X\psi \tag{2.3.21}$$

and

$$\Delta\psi = \sum_{i=1}^{m}((\tilde{\nabla}_{e_i}e_i)\psi - e_ie_i\psi), \tag{2.3.22}$$

for any vector field X tangent to $\tilde{M}^m$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}^m$. As a consequence, we have

$$||\tilde{\nabla}\psi||^2 = \sum_{i=1}^{m}\big(e_i(\psi)\big)^2. \tag{2.3.23}$$

From the integration theory of manifolds, if $\tilde{M}^m$ is orientable compact, then we have

$$\int_{\tilde{M}^m} \Delta f dV = 0, \tag{2.3.24}$$

where dV denotes to the volume element of $\tilde{M}^m$.

An n-dimensional *distribution* on a manifold $\tilde{M}^m$ is a mapping $\mathcal{D}$ defined on $\tilde{M}^m$, which assigns to each point x of $\tilde{M}^m$ an n-dimensional linear subspace $\mathcal{D}_x$ of $T_x\tilde{M}^m$. A vector field X on $\tilde{M}^m$ belongs to $\mathcal{D}$ if we have $X_x \in \mathcal{D}_x$ for each $x \in \tilde{M}^m$. When this happens we write $X \in \Gamma(\mathcal{D})$. The distribution is said to be *differentiable* if for any $x \in \tilde{M}^m$ there exist n differentiable linearly independent vector fields $X_i \in \Gamma(\mathcal{D})$ in a neighborhood of x for $i \in \{i, \cdots, n\}$. From now on, all distributions are supposed to be differentiable of class C^∞. The distribution $\mathcal{D}$ is said to be *involutive* if for all vector fields $X,\ Y \in \Gamma(\mathcal{D})$ we have $[X,\ Y] \in \Gamma(\mathcal{D})$. A submanifold M^n of $\tilde{M}^m$ is said to be an *integral manifold* of $\mathcal{D}$ if for every point $x \in M^n$, $\mathcal{D}_x$ coincides with the tangent space to M^n at x. If there exists no integral manifold of $\mathcal{D}$ which contains M^n, then M^n is called a *maximal integral manifold* or a *leaf* of $\mathcal{D}$. The distribution $\mathcal{D}$ is said to be *integrable* if for every $x \in \tilde{M}^m$ there exists an integral manifold of $\mathcal{D}$ containing x, (Bejancu, 1986).

2.3.2 WARPED PRODUCTS

In an attempt to construct manifolds of negative curvatures, R.L. Bishop and O'Neill (Bishop & O'Neill, 1996) introduced the notion of *warped product manifolds* as follows: Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_{N_1} and g_{N_2}, respectively, and $f > 0$ a C^∞ function on N_1. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \mapsto N_1$ and $\pi_2 : N_1 \times N_2 \mapsto N_2$. Then, the *warped product* $\tilde{M}^m = N_1 \times_f N_2$ is the Riemannian manifold $N_1 \times N_2 = (N_1 \times N_2, \tilde{g})$ equipped with a Riemannian structure such that $\tilde{g} = g_{N_1} + f^2 g_{N_2}$.

To relate the calculus of $N_1 \times N_2$ to that of its factors the crucial notion of *lifting* is introduced as follows. If $f \in \mathfrak{F}(N_1)$, the *lift* of f to $N_1 \times N_2$ is $\tilde{f} = f \circ \pi_1 \in \mathfrak{F}(N_1 \times N_2)$. If $X_p \in T_p(N_1)$ and $q \in N_2$, then the *lift* $X_{(p,q)}$ of X_p to (p, q) is the unique vector in $T_{(p,q)}(N_1)$ such that $d\pi_1(X_{(p,q)}) = X_p$. If $X \in \Gamma(TN_1)$ the *lift* of X to $N_1 \times N_2$ is the vector field X whose value at each (p, q) is the lift of X_p to (p, q). The set of all such *horizontal lifts* X is denoted by $\mathcal{L}(N_1)$. Functions, tangent vectors and vector fields on N_2 are lifted to $N_1 \times N_2$ in the same way using the projection π_2. Note that $\mathcal{L}(N_1)$ and symmetrically the *vertical lifts* $\mathcal{L}(N_2)$ are vector subspaces of $\Gamma\big(T(N_1 \times N_2)\big)$, (O'Neill, 1983).

We recall the following two general results for warped products (O'Neill, 1983).

Proposition 2.3.1. *On $\tilde{M}^m = N_1 \times_f N_2$, if $X,\ Y \in \mathcal{L}(N_1)$ and $Z,\ W \in \mathcal{L}(N_2)$, then*

(i) $\tilde{\nabla}_X Y \in \mathcal{L}(N_1)$ *is the lift of* $\tilde{\nabla}_X Y$ *on* N_1.

(ii) $\tilde{\nabla}_X Z = \tilde{\nabla}_Z X = (Xf/f)Z$.

(iii) $(\tilde{\nabla}_Z W)^\perp = h_{N_2}(Z, W) = -\big(g_{N_2}(Z, W)/f\big)\nabla(f)$.

(iv) $(\tilde{\nabla}_Z W)^T \in \mathcal{L}(N_2)$ *is the lift of* $\nabla^{N_2}_Z W$ *on* N_2,

where g_{N_2}, h_{N_2} and ∇^{N_2} are, respectively, the induced Riemannian metric on N_2, the second fundamental form of N_2 as a submanifold of $\tilde{M}^m$ and the induced Levi-Civita connection on N_2. [*]

It is obvious that, the above proposition leads to the following geometric conclusion.

Corollary 2.3.1. *The leaves $N_1 \times q$ of a warped product are totally geodesic; the fibers $p \times N_2$ are totally umbilical.*

Clearly, the totally geodesy of the leaves follows from (i), while (iii) implies that the fibers are totally umbilical in $\tilde{M}^m$. It is significant to say that, this corollary is one of the key ingredients of this work. Since all our considered submanifolds are warped products. Therefore, this corollary is of fundamental role in proofs and geoemtrical interpretations, especially, in handling the equality cases of our inequalities.

A warped product manifold $\tilde{M}^m = N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant. For a nontrivial warped product $N_1 \times_f N_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibers, respectively. Thus, $\mathcal{D}_1$ is obtained from tangent vectors of N_1 via the horizontal lift and $\mathcal{D}_2$ is obtained by tangent vectors of N_2 via the vertical lift.

In (O'Neill, 1983), warped product manifolds were discussed deeply from an intrinsic geometrical viewpoint. For instance, sectional curvature, scalar curvature, Ricci curvature were extensively investigated. As we discussed in the previous section, all these Riemannian invariants are defined by means of the Riemannian curvature tensor $\tilde{R}$ of $\tilde{M}^m$.

Given $\tilde{M}^m = N_1 \times_f N_2$, let ${}^{N_1}\tilde{R}$ and ${}^{N_2}\tilde{R}$ be the lefts to $\tilde{M}^m$ of the Riemannian curvature tensors of N_1 and N_2. We recall the following central result

Proposition 2.3.2. *(O'Neill, 1983). Let $\tilde{M}^m = N_1 \times_f N_2$ be a warped product with Riemannian curvature tensor $\tilde{R}$. If $X,\ Y,\ U \in \mathcal{L}(N_1)$ and $Z,\ V,\ W \in \mathcal{L}(N_2)$, then*

(1) $\tilde{R}_{XY}U \in \mathcal{L}(N_1)$ is the lift of ${}^{N_1}\tilde{R}_{XY}U$ on N_1.

(2) $\tilde{R}_{VX}Y = (H^f(X,Y)/f)V$, where H^f is the Hessian of f.

(3) $\tilde{R}_{XY}V = \tilde{R}_{VW}X = 0$.

[*] The operators $\perp$, T and $\nabla(f)$ refer to the normal projection, the tangential projection and the gradient of f, respectively.

(4) $\tilde{R}_{XV}W = (\tilde{g}(V,W)/f)\tilde{\nabla}_X(\nabla f)$.

(5) $\tilde{R}_{VW}Z = {}^{N_2}\tilde{R}_{VW}Z - (||\nabla f||^2/f^2)\{\tilde{g}(V,Z)W - \tilde{g}(W,Z)V\}$.

As a generalization of (singly) warped product manifolds, we introduce now *doubly warped product* manifolds (Bonanzinga & Matsumoto, 2004). A *doubly warped product* of Riemannian manifolds (N_1, g_{N_1}) and (N_2, g_{N_2}) with warping functions $f_1 : N_1 \to (0,\infty)$ and $f_2 : N_2 \to (0,\infty)$ is a product manifold $N_1 \times N_2$ endowed with a metric tensor

$$g = f_2^2 g_{N_1} \oplus f_1^2 g_{N_1}.$$

More explicitly, if $X_x,\ Y_x \in T(N_1 \times N_2)$ then

$$\tilde{g}(X_x,\ Y_x) = (f_2 \circ \pi_2)^2 g_{N_1}\Big(d\pi_2(X_x), d\pi_2(Y_x)\Big) + (f_1 \circ \pi_1)^2 g_{N_2}\Big(d\pi_1(X_x), d\pi_1(Y_x)\Big),$$

where $\pi_1 : N_1 \times N_2 \mapsto N_1$ and $\pi_2 : N_1 \times N_2 \mapsto N_2$ are the canonical projections. We denote the *doubly warped product* of Riemannian manifolds (N_1, g_{N_1}) and (N_2, g_{N_2}) by ${}_{f_2}N_1 \times_{f_1} N_2$. If either $f_1 = 1$ or $f_2 = 1$, but not both, then ${}_{f_2}N_1 \times_{f_1} N_2$ becomes (singly) warped product of Riemannian manifolds N_1 and N_2. If $f_1 = f_2 = 1$, then we have a product manifold. If neither f_1 nor f_2 is constant, then we have a proper (nontrivial) doubly warped product manifold. In this case we have

$$\tilde{\nabla}_X Z = X(\ln f_1)Z + Z(\ln f_2)X, \tag{2.3.25}$$

for any $X \in \mathcal{L}(N_1)$ and $Z \in \mathcal{L}(N_2)$.

This was a short introduction about the concept and related results of warped product manifolds. At the end of the next section, warped products will be discussed form the submanifold theory point of view.

Now, let $\{e_1, \cdots, e_{n_1}, e_{n_1+1}, \cdots, e_m\}$ be local fields of orthonormal frame of $\Gamma(T\tilde{M}^m)$ such that n_1, n_2 and m are the dimensions of N_1, N_2 and $\tilde{M}^m$, respectively. Then, for any Riemannian warped product $\tilde{M}^m = N_1 \times_f N_2$, Proposition 2.3.2 (4) implies that the sectional curvature and the warping function are related by (see, for example (Chen, 2002), (Chen, 2008) and (Chen, 2013))

$$\sum_{a=1}^{n_1} \sum_{A=n_1+1}^{m} \tilde{K}(e_a \wedge e_A) = \frac{n_2 \Delta f}{f}. \tag{2.3.26}$$

2.3.3 RIEMANNIAN SUBMANIFOLDS

At first, let us recall the following important two facts regarding Riemannian submanifolds *, (Do Carmo, 1992).

Definition 2.3.2. *Let M^n and $\tilde{M}^m$ be differentiable manifolds. A differentiable mapping $\varphi : M^n \longrightarrow \tilde{M}^m$ is said to be an immersion if $d\varphi_x : T_xM^n \to T_{\varphi(x)}\tilde{M}^m$ is injective for all $x \in M^n$. If, in addition, φ is a homeomorphism onto $\varphi(M^n) \subset \tilde{M}^m$, where $\varphi(M^n)$ has the subspace topology induced from $\tilde{M}^m$, we say that φ is an embedding. If $M^n \subset \tilde{M}^m$ and the inclusion $\boldsymbol{i} : M^n \subset \tilde{M}^m$ is an embedding, we say that M^n is a submanifold of $\tilde{M}^m$.*

It can be seen that if $\varphi : M^n \to \tilde{M}^m$ is an immersion, then $n \leq m$; the difference $m - n$ is called the *codimension* of the immersion φ.

For most local questions of geometry, it is the same to work with either immersions or embeddings. This comes from the following proposition which shows that every immersion is locally (in a certain sense) an embedding.

Proposition 2.3.3. *Let $\varphi : M^n \longrightarrow \tilde{M}^m$, $n \leq m$, be an immersion of the differentiable manifold M^n into the differentiable manifold $\tilde{M}^m$. For every point $x \in M^n$, there exists a neighborhood $\mathfrak{u}$ of x such that the restriction $\varphi|\mathfrak{u} \to \tilde{M}^m$ is an embedding.*

Now, we turn our attention to the differential geometry of the submanifold theory. First, let M^n be n-dimensional Riemannian manifold isometrically immersed in an m-dimensional Riemannian manifold $\tilde{M}^m$. Since we are dealing with a local study, then, by Proposition 2.3.3, we may assume that M^n is embedded in $\tilde{M}^m$. On this infinitesimal scale, Definition 2.3.2 guarantees that M^n is a *Riemannian submanifold* of some nearby points in $\tilde{M}^m$ with induced Riemannian metric g. Then, *Gauss* and *Weingarten formulas* are, respectively, given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.3.27}$$

and

$$\tilde{\nabla}_X \zeta = -A_\zeta X + \nabla^\perp_X \zeta \tag{2.3.28}$$

*From now on, warped products will be considered as Riemannian submanifolds; i.e., $M^n = N_1 \times_f N_2$. The proceeding notation, $\tilde{M}^m = N_1 \times_f N_2$, was for warped product manifolds.

for all $X, Y \in \Gamma(TM^n)$ and $\zeta \in \Gamma(T^{\perp}M^n)$, where $\tilde{\nabla}$ and ∇ denote respectively the Levi-Civita and the *induced* Levi-Civita connections on $\tilde{M}^m$ and M^n, and $\Gamma(TM^n)$ is the module of differentiable sections of the vector bundle TM^n. $\nabla^{\perp}$ is the *normal connection* acting on the normal bundle $T^{\perp}M^n$.

Here, g denotes the *induced Riemannian metric* from $\tilde{g}$ on M^n. For simplicity's sake, the inner products which are carried by g, $\tilde{g}$ or any other induced Riemannian metric are performed via g. However, most of the inner products which will be applied in this thesis are equipped with g, other situations are rarely considered.

Here, it is well-known that the *second fundamental form* h and the *shape operator* A_ζ of M^n are related by

$$g(A_\zeta X, Y) = g(h(X, Y), \zeta) \tag{2.3.29}$$

for all $X, Y \in \Gamma(TM^n)$ and $\zeta \in \Gamma(T^{\perp}M^n)$, (Bejancu, 1986), (O'Neill, 1983). Analytically, we can use (2.3.1) to define the covariant derivative of h, $\tilde{\nabla}h$, with respect to the connection on $TM^n \oplus T^{\perp}M^n$ by

$$(\tilde{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.3.30}$$

Geometrically, M^n is called a *totally geodesic* submanifold in $\tilde{M}^m$ if h vanishes identically. Particularly, the *relative null space*, $\mathcal{N}_x$, of the submanifold M^n in the Riemannian manifold $\tilde{M}^m$ is defined at a point $x \in M^n$ by (Chen, 1999) as

$$\mathcal{N}_x = \{X \in T_x M^n : h(X, Y) = 0 \quad \forall\, Y \in T_x M^n\}. \tag{2.3.31}$$

In a different line of thought, and for any $X \in \Gamma(TM^n)$, $\zeta \in \Gamma(T^{\perp}M^n)$ and a $(1,1)$ tensor field ψ on $\tilde{M}^m$, we write

$$\psi X = PX + FX, \tag{2.3.32}$$

and

$$\psi N = t\zeta + f\zeta, \tag{2.3.33}$$

where PX, $t\zeta$ are the tangential components and FX, $f\zeta$ are the normal components of ψX and $\psi\zeta$, respectively, (Chen, 1990). In the sake of following the common terminology, the tensor field ψ is replaced by ϕ and J in almost contact and almost Hermitian manifolds, respectively. However, the covariant derivatives of the tensor fields ψ, P and F are respectively defined as (Bejancu, 1986)

$$(\tilde{\nabla}_X \psi)Y = \tilde{\nabla}_X \psi Y - \psi \tilde{\nabla}_X Y, \tag{2.3.34}$$

$$(\tilde{\nabla}_X P)Y = \tilde{\nabla}_X PY - P\tilde{\nabla}_X Y \tag{2.3.35}$$

and

$$(\tilde{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\tilde{\nabla}_X Y. \tag{2.3.36}$$

Likewise, we consider a local field of orthonormal frames * $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_m\}$ on $\tilde{M}^m$, such that, restricted to M^n, $\{e_1, \cdots, e_n\}$ are tangent to M^n and $\{e_{n+1}, \cdots, e_m\}$ are normal to M^n. Then, the *mean curvature vector* $\vec{H}(x)$ is introduced as (Bejancu, 1986), (O'Neill, 1983)

$$\vec{H}(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.3.37}$$

On one hand, we say that M^n is a *minimal submanifold* of $\tilde{M}^m$ if $\vec{H} = 0$. On the other hand, one may deduce that M^n is totally umbilical in $\tilde{M}^m$ if and only if $h(X, Y) = g(X, Y)\vec{H}$, for any $X,\ Y \in \Gamma(TM^n)$ (Chen, 2005). It is remarkable to note that the scalar curvature $\tau(x)$ of M^n at x is identical with the scalar curvature of the tangent space T_xM^n of M^n at x; that is, $\tau(x) = \tau(T_xM^n)$ (Chen, 2002).

In general, take an orthonormal basis $\{e_1, \cdots, e_k\}$ for the k-plane Π_k of T_xM^n. Then, the scalar curvature $\tau(\Pi_k)$ of Π_k is given by (Chen, 1999)

$$\tau(\Pi_k) = \sum_{1\le i<j\le k} K(e_i \wedge e_j). \tag{2.3.38}$$

Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_x(\Pi_k)$ of Π_k at x under the exponential map at x. In case Π_2 is a 2-plane, then $\tau(\Pi_2)$ is simply the sectional curvature $K(\Pi_2)$ of Π_2, (Chen, 2002).

In this series, the well-known *equations* of *Gauss* and *Codazzi* are, respectively, given by

$$\begin{aligned} R(X, Y, Z, W) &= \tilde{R}(X, Y, Z, W) \\ &\quad + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned} \tag{2.3.39}$$

*Throughout this work, $M^n = N_1 \times_f N_2$ denotes for the isometrically immersed warped product submanifold in $\tilde{M}^m$. The numbers $m,\ n,\ n_1$, and n_2 are the dimensions of $\tilde{M}^m$, M^n, N_1 and N_2, respectively.

and

$$(\tilde{R}(X,Y)Z)^{\perp} = (\tilde{\nabla}_X h)(Y,Z) - (\tilde{\nabla}_Y h)(X,Z), \tag{2.3.40}$$

for any vectors $X,\ Y,\ Z,\ W \in \Gamma(TM^n)$, where $\tilde{R}$ and R are the curvature tensors of $\tilde{M}^m$ and M^n, respectively, while $(\tilde{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\tilde{R}(X,Y)Z$.

From now on, we refer to the coefficients of the second fundamental form h of M^n with respect to the above local frame by the following notation

$$h^r_{ij} = g(h(e_i,e_j),e_r), \tag{2.3.41}$$

where $i,j \in \{1,...,n\}$, and $r \in \{n+1,...,m\}$. First, by making use of (2.3.41), (2.3.39) and (2.3.8), we get the following

$$K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{r=n+1}^{m} (g(h^r_{ii}\, e_r, h^r_{jj}\, e_r) - g(h^r_{ij}\, e_r, h^r_{ij}\, e_r)). \tag{2.3.42}$$

Equivalently,

$$K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{r=n+1}^{m} (h^r_{ii}h^r_{jj} - (h^r_{ij})^2), \tag{2.3.43}$$

where $\tilde{K}(e_i \wedge e_j)$ denotes the sectional curvature of the 2-plane spanned by e_i and e_j at x in the ambient manifold $\tilde{M}^m$. Secondly, by taking the summation in the above equation over the orthonormal frame of the tangent space of M^n, and due to (2.3.12), we immediately obtain

$$2\tau(T_xM^n) = 2\tilde{\tau}(T_xM^n) + n^2||\vec{H}||^2 - ||h||^2, \tag{2.3.44}$$

where

$$\tilde{\tau}(T_xM^n) = \sum_{1\le i<j\le n} \tilde{K}(e_i \wedge e_j) \tag{2.3.45}$$

denotes the scalar curvature of the n-plane T_xM^n in the ambient manifold $\tilde{M}^m$.

For a warped product $M^n = N_1 \times_f N_2$, let $\varphi : M^n \to \tilde{M}^m$ be an isometric immersion of $N_1 \times_f N_2$ into an arbitrary Riemannian manifold $\tilde{M}^m$. As usual, let h be the second fundamental form of φ. We call the immersion φ *mixed totally geodesic* if $h(X,Z) = 0$ for any X in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$, (see, for example (Bejancu, 1978) and (Chen, 2002)). In particular, if we denote the restrictions of h to N_1 and N_2 respectively by h_1 and h_2, then for $i = 1$ and 2, we call h_i the *partial second fundamental form* of φ. Automatically, the *partial mean curvature vectors* $\vec{H}_1$ and $\vec{H}_2$ are defined by the following partial traces *

$$\vec{H}_1 = \frac{1}{n_1}\sum_{a=1}^{n_1} h(e_a, e_a), \quad \vec{H}_2 = \frac{1}{n_2}\sum_{A=n_1+1}^{n_1+n_2} h(e_A, e_A) \tag{2.3.46}$$

for some orthonormal frame fields $\{e_1, \cdots, e_{n_1}\}$ and $\{e_{n_1+1}, \cdots, e_{n_1+n_2}\}$ of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

This motivation for the following definition may not be evident at this moment, but it will emerge gradually as we prove its natural existence, then imposing it to have profoundly general results, (see, for example (Bejancu, 1978), (Bejancu, 1986), (Chen, 2002), (Chen, 2005), (Kim et al., 2004) and (Mustafa et al., 2014 & 2015)).

Definition 2.3.3. *An immersion $\varphi : N_1 \times_f N_2 \longrightarrow \tilde{M}^m$ is called $\mathcal{D}_i$-totally geodesic if the partial second fundamental form h_i vanishes identically. If for all $X,\ Y \in \mathcal{D}_i$ we have $h(X,\ Y) = g(X,\ Y)\mathcal{K}$ for some normal vector $\mathcal{K}$, then φ is called $\mathcal{D}_i$-totally umbilical. It is called $\mathcal{D}_i$-minimal if the partial mean curvature vector $\vec{H}_i$ vanishes, for $i = 1$ or 2.*

2.3.4 ALMOST HERMITIAN AND ALMOST CONTACT STRUCTURES

Let $\tilde{M}^{2m}$ be a real C^∞ manifold endowed with an almost complex structure J, i.e. J is a tensor field of type (1,1) such that, at every point $x \in \tilde{M}^{2m}$ we have $J^2 = -I$. Then, the pair $(\tilde{M}^{2m}, J)$ is called an almost complex manifold (see, for example (Bejancu, 1986) and (Moroianu, 2007)). Observe that the differential geometry of $\tilde{M}^{2m}$ depends on the behavior of the tangent bundle of $\tilde{M}^{2m}$ relative to the action of the almost complex structure J. In addition, if the Nijenhuis tensor of J vanishes identically, we then call J a complex structure on $\tilde{M}^{2m}$, and $(\tilde{M}^{2m}, J)$ turns out to be a complex manifold, where the Nijenhuis tensor is defined by

$$[J, J](X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY], \tag{2.3.47}$$

for any $X,\ Y \in \Gamma(T\tilde{M}^{2m})$. In addition, if the almost complex manifold $(\tilde{M}^{2m}, J)$ is furnished with a compatible Riemannian metric $\tilde{g}$, i.e., $\tilde{g}(JX, JY) = \tilde{g}(X, Y)$ for any $X,\ Y \in \Gamma(T\tilde{M}^{2m})$, then $(\tilde{M}^{2m}, J, \tilde{g})$ is called an almost Hermitian manifold.

*Throughout this work, we use the following convention on the range of indices unless otherwise stated, the indices i, j run from 1 to n, the lowercase letters a, b from 1 to n_1, the uppercase letters A, B from n_1 to n and r from n to m.

As we have seen above, the vanishing of the Nijenhuis tensor on almost Hermitian manifolds gives rise to a particular special class of almost Hermitian manifolds called Hermitian manifolds. The Hermitian manifold $(\tilde{M}^{2m}, J, \tilde{g})$ allows one to endow $\tilde{M}^{2m}$ with an alternating 2-form w given by

$$w(X, Y) = \tilde{g}(X, JY)$$

for any $X,\ Y \in \Gamma(T\tilde{M}^{2m})$. This 2-form is called the *associated Kaehler form*. It is always possible to retrieve $\tilde{g}$ from w, more formally

$$\tilde{g}(X, Y) = w(JX, Y).$$

Thus, $\tilde{g}$ now is called a Kaehler metric. In particular, $(\tilde{M}^{2m}, J, \tilde{g})$ becomes a *Kaehler manifold* if w is closed, i.e., $dw = 0$. Equivalently, we say that a Hermitian manifold $(\tilde{M}^{2m}, J, \tilde{g})$ is a Kaehlerian manifold if and only if the complex structure J is parallel with respect to $\tilde{\nabla}$, i.e., whenever the following condition is preserved

$$(\tilde{\nabla}_X J)Y = 0 \tag{2.3.48}$$

for any $X,\ Y \in \Gamma(T\tilde{M}^{2m})$. In particular, if a Kaehler manifold $\tilde{M}^{2m}$ has constant holomorphic sectional curvature c_{Ka} *, then it is called *complex space form*, $\tilde{M}^{2m}(c_{Ka})$. It is well-known that the Riemannian *curvature tensor of a complex space form* $\tilde{M}^{2m}(c_{Ka})$ of constant holomorphic sectional curvature c_{Ka} is given by (Chen, 2003)

$$\begin{aligned}\tilde{R}(X, Y; Z, W) = \frac{c_{Ka}}{4}\{&\tilde{g}(X, W)\tilde{g}(Y, Z) - \tilde{g}(X, Z)\tilde{g}(Y, W) + \tilde{g}(JX, W)\tilde{g}(JY, Z) \\ &-\tilde{g}(JX, Z)\tilde{g}(JY, W) + 2\tilde{g}(X, JY)\tilde{g}(JZ, W)\},\end{aligned} \tag{2.3.49}$$

for any vector fields $X,\ Y,\ Z,\ W \in \Gamma(T\tilde{M}^{2m}(c_{Ka}))$.

In this context, a Hermitian manifold $(\tilde{M}^{2m}, J, \tilde{g})$ with the associated Kaehler 2-form w is called a *locally conformal Kaehler* $(l.c.K)$ manifold if there is a closed 1-form Ω, globally defined on $\tilde{M}^{2m}$, such that $dw = \Omega \wedge w$, (Bonanzinga & Matsumoto, 2004). The closed 1-form Ω is called the *Lee form* of the $l.c.K$ manifold $\tilde{M}^{2m}$. For a $l.c.K$

*In this work, some notation is required to cope with various Riemannian curvature tensors used extensively; the constants $c,\ c_{Ka},\ c_{RK},\ c_S,\ c_{Ke}$ and c_c refer, respectively, to constant sectional curvatures of Riemannian space form, complex space form, generalized complex space form, Sasakian space form, Kenmotsu space form and cosymplectic space form.

manifold $(\tilde{M}^{2m}, J, \tilde{g})$ we define the *Lee vector field* $\lambda = \Omega^\sharp$, where $\sharp$ means rising of indices with respect to $\tilde{g}$; namely $\tilde{g}(X, \lambda) = \Omega(X)$, for all $X \in \Gamma(T\tilde{M}^{2m})$. In the language of covariant derivation, a Hermitian manifold is a $l.c.K$ if and only if it obeys the following tensorial relation

$$(\tilde{\nabla}_X J)Y = [\vartheta(Y)X - \Omega(Y)JX - \tilde{g}(X,Y)A - w(X,Y)\lambda] \tag{2.3.50}$$

for any $X,\ Y \in \Gamma(T\tilde{M}^{2m})$. Here $\vartheta = \Omega \circ J$ and $A = -J\lambda$ are the *anti-Lee form* and the *anti-Lee vector field*, respectively. In terms of the Lee vector field λ, $(2.3.50)$ can be rewritten as

$$(\tilde{\nabla}_X J)Y = [\tilde{g}(\lambda, JV)X - \tilde{g}(\lambda, Y)JX + \tilde{g}(JX, Y)\lambda + \tilde{g}(X, Y)J\lambda]. \tag{2.3.51}$$

In a natural way, it is possible to weaken the condition in $(2.3.48)$ by

$$(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0 \tag{2.3.52}$$

for each $X,\ Y \in \Gamma(T\tilde{M}^{2m})$. Every almost Hermitian manifold satisfying the previous condition is called *nearly Kaehler manifold* (Al-Luhaibi et al., 2009), (Bejancu, 1986).

More generally, there is a class of almost Hermitian manifolds which is finer than nearly Kaehler manifolds, known as RK*-manifolds*, (Al-Luhaibi et al., 2009). More precisely, an RK-manifold $\tilde{M}^{2m}$ is an almost Hermitian manifold for which the curvature tensor $\tilde{R}$ is invariant under J, i.e.,

$$\tilde{R}(JX, JY; JZ, JW) = \tilde{R}(X, Y; Z, W),$$

for any vector fields $X,\ Y,\ Z,\ W \in \Gamma(T\tilde{M}^{2m})$.

Particularly, an almost Hermitian manifold $\tilde{M}^{2m}$ is of *pointwise constant type* if for any $x \in \tilde{M}^{2m}$ and $X \in T_x\tilde{M}^{2m}$

$$\sigma(X, Y) = \sigma(X, Z),$$

where $\sigma(X, Y) = \tilde{R}(X, Y; JX, JY) - \tilde{R}(X, Y; X, Y)$ with Y and Z being tangent vectors at x, orthogonal to X and JX. The manifold $\tilde{M}^{2m}$ is said to be *of constant type* if for any unit vectors $X,\ Y \in \Gamma(T\tilde{M}^{2m})$ with $\tilde{g}(X, Y) = \tilde{g}(JX, Y) = 0,\ \sigma(X, Y)$ is a constant function. It is proven that (see (Al-Luhaibi et al., 2009) and references therein)

Theorem 2.3.1. *Let $\tilde{M}^{2m}$ be an RK-manifold. Then $\tilde{M}^{2m}$ is of pointwise constant type if and only if there exists a function γ on $\tilde{M}^{2m}$ such that*

$$\sigma(X,Y) = \gamma\Big(\tilde{g}(X,X)\tilde{g}(Y,Y) - \big(\tilde{g}(X,Y)\big)^2 - \big(\tilde{g}(X,JY)\big)^2\Big)$$

for any $X,\ Y \in \Gamma(T\tilde{M}^{2m})$. Moreover, $\tilde{M}^{2m}$ is of constant type if and only if the above equality holds for a constant γ. In this case, γ is the constant type of $\tilde{M}^{2m}$.

We end this discussion of almost Hermitian manifolds by the following notion. A *generalized complex space form* is an RK-manifold of constant holomorphic sectional curvature and of constant type, denoted by $\tilde{M}^{2m}(c_{RK},\gamma)$, with curvature tensor $\tilde{R}$ has the following expression

$$\tilde{R}(X,Y;Z,W) = \frac{c_{RK}+3\gamma}{4}\{\tilde{g}(X,W)\tilde{g}(Y,Z) - \tilde{g}(X,Z)\tilde{g}(Y,W)\}$$
$$+\frac{c_{RK}-\gamma}{4}\{\tilde{g}(JX,W)\tilde{g}(JY,Z) - \tilde{g}(JX,Z)\tilde{g}(JY,W) + 2\tilde{g}(X,JY)\tilde{g}(JZ,W)\}, \tag{2.3.53}$$

for any vector fields $X,\ Y,\ Z,\ W \in \Gamma(T\tilde{M}^{2m}(c_{RK}))$.

For an odd dimensional real C^∞ manifold $\tilde{M}^{2l+1}$, let ϕ, ξ, η and $\tilde{g}$ be respectively a $(1,\ 1)$ tensor field, a vector field, a 1-form and a Riemannian metric on $\tilde{M}^{2l+1}$ satisfying

$$\left.\begin{aligned} \phi^2 = -I + \eta\otimes\xi, \quad \phi\xi = 0, \quad \eta\circ\phi = 0, \quad \eta(\xi) = 1,\\ \eta(X) = \tilde{g}(X,\xi), \quad \tilde{g}(\phi X,\phi Y) = \tilde{g}(X,Y) - \eta(X)\eta(Y), \end{aligned}\right\} \tag{2.3.54}$$

for any $X,\ Y \in \Gamma(T\tilde{M}^{2l+1})$. Then we call $(\tilde{M}^{2l+1},\phi,\xi,\eta,\tilde{g})$ an *almost contact metric manifold* and $(\phi,\xi,\eta,\tilde{g})$ an *almost contact metric structure* on $\tilde{M}^{2l+1}$, see (Bejancu, 1986), (Blair, 1971) and (Oubina, 1985).

A fundamental 2-form Φ is defined on $\tilde{M}^{2l+1}$ by $\Phi(X,Y) = \tilde{g}(\phi X,Y)$. An almost contact metric manifold $\tilde{M}^{2l+1}$ is called a contact metric manifold if $\Phi = \frac{1}{2}d\eta$. If the almost contact metric manifold $(\tilde{M}^{2l+1},\phi,\xi,\eta,\tilde{g})$ satisfies $[\phi,\phi] + 2d\eta\otimes\xi = 0$, then $(\tilde{M}^{2l+1},\phi,\xi,\eta,\tilde{g})$ turns out to be a *normal almost contact manifold*, where the Nijenhuis tensor is defined as

$$[\phi,\phi](X,Y) = [\phi X,\phi Y] + \phi^2[X,Y] - \phi[X,\phi Y] - \phi[\phi X,Y] \quad \forall\, X,\ Y \in \Gamma(T\tilde{M}^{2l+1}).$$

For our purpose, we will distinguish four classes of almost contact metric structures; namely, Sasakian, Kenmotsu, cosymplectic and nearly trans-Sasakian structures. At first, an almost contact metric structure is is said to be *Sasakian* whenever it is both contact metric and normal, equivalently (Sasaki, 1960)

$$(\tilde{\nabla}_X \phi)Y = -\tilde{g}(X,Y)\xi + \eta(Y)X. \tag{2.3.55}$$

A 2-plane π in $T_x\tilde{M}^{2l+1}$ of an almost metric manifold $\tilde{M}^{2l+1}$ is called a ϕ-section if $\pi \perp \xi$ and $\phi(\pi) = \pi$. Accordingly, we say that $\tilde{M}^{2l+1}$ is of constant ϕ-sectional curvature if the sectional curvature $\tilde{K}(\pi)$ does not depend on the choice of the ϕ-section π of $T_x\tilde{M}^{2l+1}$ and the choice of a point $x \in \tilde{M}^{2l+1}$. Based on this preparatory concept, a Sasakian manifold $\tilde{M}^{2l+1}$ is said to be a *Sasakian space form* $\tilde{M}^{2l+1}(c_S)$, if the ϕ-sectional curvature is constant c_S along $\tilde{M}^{2l+1}$. Then the associated Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}^{2l+1}(c_S)$ is given by (Bejancu, 1986)

$$\begin{aligned}\tilde{R}(X,Y;Z,W) = \frac{c_S+3}{4}&\Big\{\tilde{g}(X,W)\tilde{g}(Y,Z) - \tilde{g}(X,Z)\tilde{g}(Y,W)\Big\}\\ -\frac{c_S-1}{4}&\Big\{\eta(Z)\Big(\eta(Y)\tilde{g}(X,W) - \eta(X)\tilde{g}(Y,W)\Big)\\ &+\Big(\tilde{g}(Y,Z)\eta(X) - \tilde{g}(X,Z)\eta(Y)\Big)\tilde{g}(\xi,W)\\ -\tilde{g}(\phi X,W)\tilde{g}(\phi Y,Z) &+ \tilde{g}(\phi X,Z)\tilde{g}(\phi Y,W) + 2\tilde{g}(\phi X,Y)\tilde{g}(\phi Z,W)\Big\},\end{aligned} \tag{2.3.56}$$

for any $X,\ Y,\ Z,\ W \in \Gamma(T\tilde{M}^{2l+1}(c_S))$.

An almost contact metric manifold $\tilde{M}^{2l+1}$ is called *Kenmotsu manifold* (Kenmotsu, 1972) if

$$(\tilde{\nabla}_X \phi)Y = \tilde{g}(\phi X,Y)\xi - \eta(Y)\phi X. \tag{2.3.57}$$

By analogy with Sasakian manifolds, a Kenmotsu manifold $\tilde{M}^{2l+1}$ is said to be a *Kenmotsu space form* $\tilde{M}^{2l+1}(c_{Ke})$, if the ϕ-sectional curvature is constant c_{Ke} along $\tilde{M}^{2l+1}$, whose Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}^{2l+1}(c_{Ke})$ is characterized by (Kenmotsu, 1972)

$$\begin{aligned}\tilde{R}(X,Y;Z,W) = \frac{c_{Ke}-3}{4}&\Big\{\tilde{g}(X,W)\tilde{g}(Y,Z) - \tilde{g}(X,Z)\tilde{g}(Y,W)\Big\}\\ -\frac{c_{Ke}+1}{4}&\Big\{\eta(Z)\Big(\eta(Y)\tilde{g}(X,W) - \eta(X)\tilde{g}(Y,W)\Big)\\ &+\Big(\tilde{g}(Y,Z)\eta(X) - \tilde{g}(X,Z)\eta(Y)\Big)\tilde{g}(\xi,W)\end{aligned}$$

$$-\tilde{g}(\phi X,W)\tilde{g}(\phi Y,Z)+\tilde{g}(\phi X,Z)\tilde{g}(\phi Y,W)+2\tilde{g}(\phi X,Y)\tilde{g}(\phi Z,W)\bigg\}, \tag{2.3.58}$$

for any $X,\ Y,\ Z,\ W\in\Gamma(T\tilde{M}^{2l+1}(c_{Ke}))$. We notice that Kenmotsu manifolds are normal but not quasi-Sasakian and, hence, not Sasakian (Blair et al., 1976).

In the case of killing almost contact structure tensors, consider a normal almost contact metric structure $(\phi,\xi,\eta,\tilde{g})$ with both Φ and η are closed. Then, such $(\phi,\xi,\eta,\tilde{g})$ is called *cosymplectic* (Olszak, 1981). Explicitly, cosymplectic manifolds are characterized by normality and the vanishing of Riemannian covariant derivative of ϕ, i.e.,

$$(\tilde{\nabla}_X\phi)Y=0. \tag{2.3.59}$$

A cosymplectic manifold $\tilde{M}^{2l+1}$ is said to be a *cosymplectic space form* $\tilde{M}^{2l+1}(c_c)$, if the ϕ-sectional curvature is constant c_c along $\tilde{M}^{2l+1}$ with Riemannian curvature tensor $\tilde{R}$ expressed by (see, for example (Blair, 1971) and (Blair et al., 1976))

$$\tilde{R}(X,Y;Z,W)=\frac{c_c}{4}\bigg\{\tilde{g}(X,W)\tilde{g}(Y,Z)-\tilde{g}(X,Z)\tilde{g}(Y,W)$$

$$-\eta(Z)\bigg(\eta(Y)\tilde{g}(X,W)-\eta(X)\tilde{g}(Y,W)\bigg)-\bigg(\tilde{g}(Y,Z)\eta(X)-\tilde{g}(X,Z)\eta(Y)\bigg)\tilde{g}(\xi,W)$$

$$+\tilde{g}(\phi X,W)\tilde{g}(\phi Y,Z)-\tilde{g}(\phi X,Z)\tilde{g}(\phi Y,W)-2\tilde{g}(\phi X,Y)\tilde{g}(\phi Z,W)\bigg\}, \tag{2.3.60}$$

for any $X,\ Y,\ Z,\ W\in\Gamma(T\tilde{M}^{2l+1}(c_c))$. Hereafter, we call the almost contact manifold $\tilde{M}^{2l+1}$ a *nearly cosymplectic* manifold if

$$(\tilde{\nabla}_X\phi)Y+(\tilde{\nabla}_Y\phi)X=0. \tag{2.3.61}$$

Based on Gray-Hervella classification of almost Hermitian manifolds (Gray & Hervella, 1980), an almost contact metric structure $(\phi,\xi,\eta,\tilde{g})$ on $\tilde{M}^{2l+1}$ is called a trans-Sasakian structure (Gherghe, 2000) if $(\tilde{M}^{2l+1}\times\mathbb{R},J,\tilde{G})$ belongs to the class W_4 of their classification, where J is the almost complex structure on $\tilde{M}^{2l+1}\times\mathbb{R}$ defined by

$$J(X,ad/dt)=\bigg(\phi X-a\xi,\eta(X)d/dt\bigg)$$

for all vector fields X on $\tilde{M}^{2l+1}$ and smooth functions a on $\tilde{M}^{2l+1}\times\mathbb{R}$, where $\tilde{G}$ is the product metric on $\tilde{M}^{2l+1}\times\mathbb{R}$. This may be expressed by the condition

$$(\tilde{\nabla}_X\phi)Y=\alpha\bigg(\tilde{g}(X,Y)\xi-\eta(Y)X\bigg)+\beta\bigg(\tilde{g}(\phi X,Y)\xi-\eta(Y)\phi X\bigg), \tag{2.3.62}$$

for some smooth functions α and β on $\tilde{M}^{2l+1}$, and we say that the trans-Sasakian structure is of type (α, β). From the above formula it follows that

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta\Big(X - \eta(X)\xi\Big).$$

Up to D. Chinea and C. Gonzalez classification of almost contact structures (Chinea $\&$ Gonzalez, 1990), the class $C_6 \otimes C_5$ coincides with the class of trans-Sasakian structure of type (α, β). Recently, J. C. Marrero proved that a trans-Sasakian manifold of dimension $\geq$5 is either α-Sasakian, β-Kenmotsu or a cosymplectic manifold, (Marrero, 1992).

In (Gherghe, 2000), C. Gherghe introduced nearly trans-Sasakian structure of type (α, β). An almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ on $\tilde{M}^{2l+1}$ is called a *nearly trans-Sasakian* structure (Mustafa et al., 2014 $\&$ 2015) if

$$(\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = \alpha\Big(2\tilde{g}(X,Y)\xi - \eta(Y)X - \eta(X)Y\Big) \\ -\beta\Big(\eta(Y)\phi X + \eta(X)\phi Y\Big). \quad (2.3.63)$$

Evidently, a nearly trans-Sasakian of type (α, β) is nearly-Sasakian, nearly Kenmotsu or nearly cosymplectic according as β = 0, α=1; or α = 0, β=1; or α = β = 0, respectively.

CHAPTER 3: EXISTENCE AND NON-EXISTENCE OF WARPED PRODUCT SUBMANIFOLDS

3.1 INTRODUCTION

This chapter has two significant purposes. The first one is to provide special case solutions for Problems 1.4.2 and 1.4.3, that is to see whether a warped product exists or not in almost Hermitian and almost contact manifolds. In the existence case, we prove some preparatory characteristic results which are necessary for subsequent chapters, and this is the second purpose. Some new examples are given to assert the existence of some important warped product manifolds.

For a submanifold M^n in an almost Hermitian manifold $\tilde{M}^{2m}$ (resp. almost contact manifold $\tilde{M}^{2l+1}$), let $\mathcal{P}_X Y$ denote the tangential component and $\mathcal{Q}_X Y$ the normal one of $(\tilde{\nabla}_X J)Y$ (resp. $(\tilde{\nabla}_X \phi)Y$) in $\tilde{M}^{2m}$ (resp. $\tilde{M}^{2l+1}$), where $X,\ Y \in \Gamma(TM^n)$.

In order to make it a self-contained reference of warped product submanifolds for immersibility and nonimmersibility problems, we hypothesize most of our statements in the current and the next chapters for almost Hermitian and almost contact manifolds, and for warped product submanifolds of type $N_T \times_f N_2$, where N_T and N are holomorphic and Riemannian submanifolds. Meaning that, a lot of particular case results are included in the theorems of the next two chapters.

3.2 WARPED PRODUCT SUBMANIFOLDS OF ALMOST HERMITIAN MANIFOLDS

We begin by considering a warped product submanifold in almost Hermitian manifolds such that one of the factors is holomorphic.

Theorem 3.2.1. *Every warped product submanifold $M^n = N \times_f N_T$ in almost Hermitian manifolds $\tilde{M}^{2m}$ possesses the following*

(i) $g(\mathcal{P}_X Z, W) = 0$;

(ii) $g(\mathcal{P}_Z X, JZ) - g(\mathcal{P}_{JZ} X, Z) = -2(X \ln f)||Z||^2$,

for every vector fields $X \in \Gamma(TN)$ and $Z,\ W \in \Gamma(TN_T)$ such that N and N_T are Riemannian and invariant submanifolds of $\tilde{M}^{2m}$, respectively.

Proof. Taking X and Z as in hypothesis, it is clear that

$$(\tilde{\nabla}_X J)Z = \tilde{\nabla}_X JZ - J\tilde{\nabla}_X Z.$$

Since $Z \in \Gamma(TN_T)$, Proposition 2.3.1 (ii) implies that $\nabla_X JZ = J\nabla_X Z = (X \ln f)JZ$. Thus, making use of $(2.3.27)$, we get

$$(\tilde{\nabla}_X J)Z = h(X, JZ) - Jh(X, Z).$$

Taking the inner product with W, we get (i). For the second part, and by taking advantage of $(2.3.27)$, $(2.3.28)$ and Proposition 2.3.1 (ii), we can write

$$(\tilde{\nabla}_Z J)X + (\tilde{\nabla}_X J)Z = (PX \ln f)Z + h(PX, Z) - A_{FX}Z$$

$$+\nabla^{\perp}_Z FX - (X \ln f)JZ - 2Jh(X, Z) + h(X, JZ).$$

Taking the inner product with JZ in the above equation gives

$$g(\mathcal{P}_Z X + \mathcal{P}_X Z, JZ) = -g(h(Z, JZ), FX) - (X \ln f)||Z||^2.$$

If we substitute JZ for Z in the above equation, then we have

$$-g(\mathcal{P}_{JZ} X + \mathcal{P}_X JZ, Z) = g(h(Z, JZ), FX) - (X \ln f)||Z||^2.$$

By these two equations, we get

$$g(\mathcal{P}_Z X + \mathcal{P}_X Z, JZ) - g(\mathcal{P}_{JZ} X + \mathcal{P}_X JZ, Z) = -2(X \ln f)||Z||^2.$$

Finally, we may apply statement (i) in the above equation to get (ii). $\square$

In particular, if we assume the ambient manifold $\tilde{M}^{2m}$ to be either Kaehler or nearly Kaehler in the theorem above, the nonexistence of proper warped products of the type $N \times_f N_T$ immediately follows. Using (2.3.52) in statement (ii) gives

$$g(\mathcal{P}_X Z, JZ) - g(\mathcal{P}_X JZ, Z) = 2(X \ln f)||Z||^2,$$

if one applies statement (i) on the left hand side of the above equation, he automatically gets $X \ln f = 0$, for every $X \in \Gamma(TN)$. Obviously, this conclusion is true for Kaehler manifolds also. Hence, we can state the following

Corollary 3.2.1. *Warped product submanifolds with holomorphic second factor are Riemannian products, in both Kaehler and nearly Kaehler manifolds.*

It is worth pointing out that, the previous corollary generalizes many nonexistence results in this field, (see, for example (Chen, 2001), (Khan & Khan, 2009) and (Sahin, 2006)).

By reversing the two factors of the warped product in Theorem 3.2.1, we present the following corresponding theorem for doubly warped product submanifolds.

Theorem 3.2.2. *Let $M^n =_{f_2} N_T \times_{f_1} N$ be a doubly warped product submanifold in an almost Hermitian manifold $\tilde{M}^{2m}$. Then,*

$$g(\mathcal{P}_X Z, JX) - g(\mathcal{P}_{JX} Z, X) = -2(Z \ln f_2)||X||^2,$$

for vector fields $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN)$, where N and N_T are Riemannian and invariant submanifolds of $\tilde{M}^{2m}$, respectively.

Proof. Taking X and Z as in hypothesis. By (2.3.25), (2.3.27) and (2.3.28), it is straightforward to carry out the following calculations

$$(\tilde{\nabla}_X J)Z = (X \ln f_1)PZ + (PZ \ln f_2)X + h(X, PZ) - A_{FZ}X$$

$$+\nabla^{\perp}_X FZ - (X \ln f_1)JZ - (Z \ln f_2)JX - Jh(X, Z).$$

If we take the inner product with JX in the above equation, then

$$g(\mathcal{P}_X Z, JX) = -g(h(X, JX), FZ) - (Z \ln f_2)||X||^2.$$

By replacing JX with X in the above equation we deduce that

$$-g(\mathcal{P}_{JX} Z, X) = g(h(X, JX), FZ) - (Z \ln f_2)||X||^2.$$

Thus, the assertion follows from the above two equations. □

The following corollary can be directly obtained from (2.3.48) and Theorem 3.2.2.

Corollary 3.2.2. *A doubly warped product submanifold with holomorphic first factor is trivial in Kaehler manifolds.*

Combining Corollaries 3.2.1 and 3.2.2 together, one can directly get the next prominent result.

Corollary 3.2.3. *In Kaehler manifolds, there is no proper doubly warped product submanifold such that one of its factors is holomorphic.*

For doubly warped product submanifolds with one of the factors holomorphic, we have already had a negative answer from the preceding corollary. However, the situation is not the same with (singly) warped product submanifolds of holomorphic first factor, and thus we present one of the basic characteristic theorems for subsequent chapters.

Theorem 3.2.3. *Let $M^n = N_T \times_f N$ be a warped product in an almost Hermitian manifold $\tilde{M}^{2m}$. Then, the following hold:*

(i) $g(\mathcal{P}_X Z, Y) = -g(h(X,Y), FZ)$;

(ii) $g(\mathcal{P}_Z X, Z) = (JX \ln f)||Z||^2 + g(h(X,Z), FZ)$;

(iii) $g(\mathcal{P}_Z X, Y) = 0$;

(iv) $$g(\mathcal{P}_Z X, W) + g(\mathcal{P}_W X, Z) = 2(JX \ln f) g(Z,W) + g(h(X,Z), FW) + g(h(X,W), FZ);$$

(v) $g(\mathcal{P}_Z X - \mathcal{P}_X Z, W) - g(\mathcal{P}_W X, Z) = 2(X \ln f) g(Z, PW)$;

(vi) $g(\mathcal{P}_X Z, W) + g(\mathcal{P}_X W, Z) = 0$;

(vii) $g(\mathcal{Q}_X X, J\zeta) + g(\mathcal{Q}_{JX} JX, J\zeta) = -g(h(X,X), \zeta) - g(h(JX,JX), \zeta)$,

for any vector fields $X,\ Y \in \Gamma(TN_T)$, $Z,\ W \in \Gamma(TN)$ and $\zeta \in \Gamma(\nu)$.

Proof. For X and Z as above, we have

$$(\tilde{\nabla}_X J)Z = \tilde{\nabla}_X JZ - J\tilde{\nabla}_X Z. \tag{3.2.1}$$

Equivalently,

$$(\tilde{\nabla}_X J)Z = \tilde{\nabla}_X PZ + \tilde{\nabla}_X FZ - J\tilde{\nabla}_X Z. \tag{3.2.2}$$

Taking the inner product with Y in the above equation gives (i) immediately. Now, by reversing the roles of X and Z in $(3.2.1)$, it follows

$$(\tilde{\nabla}_Z J)X = \tilde{\nabla}_Z JX - J\tilde{\nabla}_Z X. \tag{3.2.3}$$

Taking the inner product with Z in the above equation implies (ii). Subtracting the equation above from $(3.2.2)$, taking into consideration that h is a symmetric form and $\nabla_X Z = \nabla_Z X$, we immediately get

$$(\tilde{\nabla}_X J)Z - (\tilde{\nabla}_Z J)X = \tilde{\nabla}_X PZ + \tilde{\nabla}_X FZ - \tilde{\nabla}_Z JX.$$

Taking the inner product with JY in the above equation yields

$$g(\mathcal{P}_X Z, JY) - g(\mathcal{P}_Z X, JY) = -g(h(X, JY), FZ).$$

Replacing JY by Y in the above equation, gives

$$g(\mathcal{P}_Z X, Y) - g(\mathcal{P}_X Z, Y) = g(h(X, Y), FZ).$$

Applying statement (i) in the above equation proves statement (iii).

Taking the inner product with W in $(3.2.3)$, we will obtain

$$g(\mathcal{P}_Z X, W) = (JX \ln f)g(Z, W) + (X \ln f)g(Z, PW) + g(h(X, Z), FW). \tag{3.2.4}$$

By interchanging the rules of Z and W in the above equation, and due to the fact that $g(Z, PW)$ is skew-symmetric with respect to Z and W, the following holds

$$g(\mathcal{P}_W X, Z) = (JX \ln f)g(Z, W) - (X \ln f)g(Z, PW) + g(h(X, W), FZ). \tag{3.2.5}$$

If we add $(3.2.4)$ and $(3.2.5)$ together, then (iv) follows. While by subtracting $(3.2.5)$ from $(3.2.4)$ we immediately reach

$$g(\mathcal{P}_Z X, W) - g(\mathcal{P}_W X, Z) = 2(X \ln f)g(Z, PW) + g(h(X, Z), FW) - g(h(X, W), FZ). \tag{3.2.6}$$

Moreover, one can take the inner product in $(3.2.2)$ with W to obtain

$$g(\mathcal{P}_X Z, W) = g(h(X, Z), FW) - g(h(X, W), FZ). \tag{3.2.7}$$

Hence, if we subtract (3.2.7) from (3.2.6), we get (v). On the other hand, by using the polarization identity of Z and W in (v), we obtain

$$g(\mathcal{P}_W X - \mathcal{P}_X W, Z) - g(\mathcal{P}_Z X, W) = -2(X \ln f) g(Z, PW).$$

By using statement (v) and the above equation, statement (vi) follows directly.

For (vii), notice that

$$(\tilde{\nabla}_X J)X = \tilde{\nabla}_X JX - J\tilde{\nabla}_X X.$$

First, we take the inner product in the above equation with $J\zeta$ to get

$$g(\mathcal{Q}_X X, J\zeta) = g(h(JX, X), J\zeta) - g(h(X, X), \zeta).$$

After that, we replace JX by X in the above equation to derive

$$g(\mathcal{Q}_{JX} JX, J\zeta) = -g(h(JX, X), J\zeta) - g(h(JX, JX), \zeta).$$

Hence (vii) can be obtained by adding the above two equations. This completes the proof. $\square$

In (Bejancu, 1978), Bejancu initiated the study of the CR-submanifolds of almost Hermitian manifolds by generalizing invariant (holomorphic) and anti-invariant (totally real) submanifolds. He called a submanifold M^n of an almost Hermitian manifold $\tilde{M}^{2m}$ a *CR-submanifold* if there exists on M^n a differentiable holomorphic distribution $\mathcal{D}_T$ whose orthogonal complementary distribution $\mathcal{D}_\perp$ is totally real. In other words, M^n is said to be a CR-submanifold if it is endowed with a pair of orthogonal complementary distributions $(\mathcal{D}_T,\ \mathcal{D}_\perp)$, satisfying the following conditions:

(i) $TM^n = \mathcal{D}_T \oplus \mathcal{D}_\perp$

(ii) $\mathcal{D}_T$ is a holomorphic distribution, i.e., $J\mathcal{D}_T \subseteq TM^n$

(iii) $\mathcal{D}_\perp$ is a totally real distribution, i.e., $J\mathcal{D}_\perp \subseteq T^\perp M^n$.

Denote by ν the maximal J-invariant subbundle of the normal bundle $T^\perp M^n$. Then it is well-known that the normal bundle $T^\perp M^n$ admits the following decomposition

$$T^\perp M^n = F\mathcal{D}_\perp \oplus \nu. \tag{3.2.8}$$

In Kaehler manifolds $\tilde{M}^{2m}$, the warped product $N_T \times_f N_\perp$ is called a CR-*warped product* submanifold, if the submanifolds N_T and $N_\perp$ are integral manifolds of $\mathcal{D}_T$ and $\mathcal{D}_\perp$, respectively. The following prominent nonexistence fact generalizes many nonexistence results in Kaehler manifolds, (see, for example (Khan & Khan, 2009) and (Sahin, 2006)).

Corollary 3.2.4. *In Kaehler manifolds, there is no warped product of type $N_T \times_f N$ other than CR-warped products.*

Proof. We want to show that N is a totally real submanifold when the first factor is holomorphic. Equivalently, it suffices to prove that $PZ = 0$ for every $Z \in \Gamma(TN)$. Evidently, using $(2.3.48)$ in Theorem 3.2.3 (v), we deduce that $X \ln f = 0$ or $g(PZ, W) = 0$, for arbitrary vector fields Z and W tangent to the second factor. This implies either $N_T \times_f N$ is a Riemannian product or $PZ = 0$ for every $Z \in \Gamma(TN)$. Hence if the second factor is not totally real submanifold, then $N_T \times_f N$ is trivial. $\square$

In Kaehler manifolds, a characterization theorem for the CR-warped product submanifold of the type $N_T \times_f N_\perp$ is proved in (Chen, 2001). Here, we construct a concrete example asserting the existence of such warped product submanifold.

Example 3.2.1. *Let $\mathbb{R}^6$ be equipped with the canonical complex structure J, with its Cartesian coordinates $(x_1, \cdots, x_6)$. Then a 3-dimensional submanifold M^3 of $\mathbb{R}^6$ is given by*

$$x_1 = t\cos\theta, \quad x_2 = s\cos\theta, \quad x_5 = t\sin\theta, \quad x_6 = s\sin\theta, \quad x_3 = x_4 = 0.$$

It is clear that M^3 is well-defined with a tangent bundle TM^3 spanned by Z_1, Z_2 and Z_3, such that

$$Z_1 = \cos\theta\,\frac{\partial}{\partial x_1} + \sin\theta\,\frac{\partial}{\partial x_5}, \quad Z_2 = \cos\theta\,\frac{\partial}{\partial x_2} + \sin\theta\,\frac{\partial}{\partial x_6},$$
$$Z_3 = -t\sin\theta\,\frac{\partial}{\partial x_1} - s\sin\theta\,\frac{\partial}{\partial x_2} + t\cos\theta\,\frac{\partial}{\partial x_5} + s\cos\theta\,\frac{\partial}{\partial x_6}.$$

Therefore, $\mathcal{D}_T = span\ \{Z_1,\ Z_2\}$, and $\mathcal{D}_\perp = span\ \{Z_3\}$ are holomorphic and totally real distributions, respectively. Thus, M^3 is a CR-submanifold of $\mathbb{R}^6$. Since it is not difficult to see that $\mathcal{D}_T$ is integrable, then we can denote the integral manifolds of $\mathcal{D}_T$ and $\mathcal{D}_\perp$

respectively by N_T and $N_\perp$. Based on the above tangent bundle, the metric tensor g of M^3 is expressed by

$$g = 2dt^2 + 2ds^2 + (t^2 + s^2)d\theta^2$$

$$= g_{N_T} + (t^2 + s^2)g_{N_\perp}.$$

Obviously, g is a warped metric tensor on M^3. Consequently, M^3 is a CR-warped product submanifold of type $N_T \times_f N_\perp$ in $\mathbb{R}^6$, with warping function $f = \sqrt{t^2 + s^2}$. By means of Gauss formula, we obtain that

$$h(Z_1, Z_1) = h(Z_2, Z_2) = 0.$$

This means that M is a $\mathcal{D}_T$-minimal warped product in $\mathbb{R}^6$.

The following result describes locally a relation of the coefficients of the second fundamental form.

Corollary 3.2.5. *Let $M^n = N_T \times_f N$ be a warped product submanifold in Kaehler or in nearly Kaehler manifolds $\tilde{M}^{2m}$. Then, we have*

$$\sum_{\substack{A,B=1 \\ A \neq B}}^{n_2} g(h(X, e_A), Fe_B) = 0,$$

where $e_1, \cdots, e_{n_2}$ form a local orthonormal frame fields of $\Gamma(TN)$, and X is any vector field tangent to the first factor.

Proof. Using (2.3.48) or (2.3.52) with parts (ii) and (v) of Theorem 3.2.3 gives

$$-2(JX \ln f)g(Z, W) = g(h(X, Z), FW) + g(h(X, W), FZ),$$

for $X \in \Gamma(TN_T)$ and $Z,\ W \in \Gamma(TN)$. Take any two distinct orthogonal unit vectors, say e_v and e_u, from the above frame. Let $Z = e_v$ and $W = e_u$ in the above equation. Then $g(h(X, e_v), Fe_u) = -g(h(X, e_u), Fe_v)$, which gives the result. □

It is reasonable to include the following key result at the end of this section, which plays fascinating roles in subsequent chapters.

Proposition 3.2.1. *Let $M^n = N_T \times_f N$ be isometrically immersed in nearly Kaehler manifolds. Then, the following are fulfilled:*

(i) $g(h(X,Y),FZ)=0;$

(ii) $g(h(X,Z),FZ)=-(JX\ln f)||Z||^2;$

(iii) $g(h(X,X),\zeta)+g(h(JX,JX),\zeta)=0;$

(iv) $g(h(X,Z),FW)=\frac{1}{3}(X\ln f)g(PZ,W)-(JX\ln f)g(Z,W),$

where the vector fields $X,\ Y$ are tangent to the first factor, Z and W are tangent to the second factor and ζ is tangent to the normal subbundle ν.

Proof. In virtue of (2.3.52), the first statement follows directly by using parts (i) and (iii) of Theorem 3.2.3. The second statement is obtained from Theorem3.2.3 (vi), (ii). The third statement is clear from Theorem 3.2.3 (vii) and (2.3.52). For the last statement, we substitute $Z+W$ instead of Z in statement (ii) above, hence we get

$$g(h(X,Z),FW)+g(h(X,W),FZ)=-2(JX\ln f)g(Z,W), \tag{3.2.9}$$

for $X,\ Z$ and W as in the statement above.

Now, making use of $(2.3.27),\ (2.3.28),\ (2.3.29),\ (2.3.52)$ and Proposition 2.3.1 (ii), we carry out the following calculations

$$g(h(X,Z),FW)=g(h(X,Z),JW)=-g(Jh(X,Z),W)=g(J(\nabla_X Z-\tilde{\nabla}_X Z),W)$$

$$=(X\ln f)g(PZ,W)-g(J\tilde{\nabla}_X Z,W)$$

$$=(X\ln f)g(PZ,W)+g((\tilde{\nabla}_X J)Z,W)-g(\tilde{\nabla}_X JZ,W)$$

$$=(X\ln f)g(PZ,W)-g((\tilde{\nabla}_Z J)X,W)-g(\tilde{\nabla}_X PZ,W)-g(\tilde{\nabla}_X FZ,W)$$

$$=(X\ln f)g(PZ,W)+g(J\tilde{\nabla}_Z X)-g(\tilde{\nabla}_Z JX,W)-(X\ln f)g(PZ,W)+g(A_{FZ}X,W)$$

$$=g(J\nabla_Z X,W)+g(Jh(X,Z),W)-(JX\ln f)g(Z,W)+g(h(X,W),FZ)$$

$$=(X\ln f)g(PZ,W)-g(h(X,Z),FW)-(JX\ln f)g(Z,W)+g(h(X,W),FZ).$$

This gives

$$2g(h(X,Z),FW)-g(h(X,W),FZ)=(X\ln f)g(PZ,W)-(JX\ln f)g(Z,W). \tag{3.2.10}$$

Thus, combining (3.2.9) and (3.2.10) together gives statement (iv) directly, which completes the proof. $\square$

In what follows we summarize the immersibility and nonimmersibility cases of Kaehler and nearly Kaehler manifolds according to the preceding results.

Warped Product Submanifold	Kaehler	Nearly Kaehler
$N_\perp \times_f N_T$	X	X
$N_T \times_f N_\perp$	✓	✓
$N_\theta \times_f N_T$	X	X
$N_T \times_f N_\theta$	X	?
$N \times_f N_T$	X	X
$N_T \times_f N$	X	?
$N_\perp \times_f N_\theta$	X	?
$N_\theta \times_f N_\perp$	✓	✓

Table 3.1: Existence and nonexistence of proper warped product submanifolds in Kaehler and nearly Kaehler manifolds.

3.3 WARPED PRODUCT SUBMANIFOLDS OF ALMOST CONTACT MANIFOLDS

It is still an open question whether or not a warped product admits isometric immersions into certain Riemannian manifolds of interest. For instance, many articles have been recently published in almost contact manifolds (see, for example (Khan et al., 2008) and (Munteanu, 2005)). In fact, these papers and a lot others (see references in (Chen, 2013)) provide special case answers for Problems 1.4.2 and 1.4.3. The following theorem generalizes all such nonexistence results as a final answer for doubly warped product submanifolds in almost contact manifolds.

Theorem 3.3.1. *In almost contact manifolds, there does not exist a proper doubly warped product submanifold $M^n =_{f_2} N_1 \times_{f_1} N_2$ such that the characteristic vector field ξ is either tangent to N_1 or N_2.*

Proof. Suppose ξ in $\Gamma(TN_2)$. Then for any $X \in \Gamma(TN_1)$, and by using (2.3.25), we directly calculate

$$2X \ln f_1 = 2X \ln f_1 g(\xi, \xi) = 2g(\tilde{\nabla}_X \xi, \xi) = Xg(\xi, \xi) = X(1) = 0.$$

This means that f_1 is constant. Similarly, it can be shown that f_2 is constant when ξ is tangent to the first factor. Hence, we conclude that a doubly warped product submanifold of almost contact manifolds, in the sense of our hypothesis, is trivial, which completes the proof. □

Considering ξ as in the above hypothesis, this theorem can be simply paraphrased by saying that: doubly warped product submanifolds in almost contact manifolds are but trivial. With this fact, some results concerning inequalities for doubly warped product submanifolds in Kenmotsu manifolds become trivial (see references in (Chen, 2013)).

As a special case of Theorem 3.3.1, we have the following theorem for (singly) warped product submanifolds

Theorem 3.3.2. *There is no warped product submanifolds in almost contact manifolds such that the characteristic vector field ξ is tangent to the second factor.*

The above theorem answers some special cases of Problems 1.4.2 and 1.4.3. On one hand, it generalizes all related nonexistence results of this topic (see, for example (Khan et al., 2008), (Munteanu, 2005), (Mustafa et al., 2013), (Uddin et al., 2014) and (Mustafa et al., 2014)). On the other hand, it guides us to restrict the choice of the factor that ξ should be tangent to in warped product submanifolds of almost contact manifolds.

From now on, the characteristic vector field ξ is supposed to be tangent to the first factor of all warped product submanifolds in almost contact manifolds. Henceforth, we can follow an argument as in the proof of Theorem 3.2.1 to obtain a dual contact version of Theorem 3.2.1.

Theorem 3.3.3. *For each warped product submanifold $N \times_f N_T$ of almost contact manifolds such that ξ is tangent to the first factor, the following are true*

(i) $g(\mathcal{P}_X Z, W) = 0$;

(ii) $g(\mathcal{P}_Z X, JZ) - g(\mathcal{P}_{JZ} X, Z) = -2(X \ln f)||Z||^2$,

for every vector field $X \in \Gamma(TN)$, and $Z,\ W \in \Gamma(TN_T)$.

As a direct application of the preceding theorem, and by using (2.3.63), we state the following remark, which generalizes a lot of nonexistence results in almost contact manifolds (see, for example (Hesegawa & Mihai, 2003) and (Munteanu, 2005)). First, by putting $\beta = 0$ in (2.3.63), we get the structural formula for nearly α-Sasakian manifolds; that is,

$$(\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = \alpha\Big(2\tilde{g}(X,Y)\xi - \eta(Y)X - \eta(X)Y\Big). \tag{3.3.1}$$

Now, we show that the first term on the left hand side of statement (ii) above is zero. From the above equation, we directly get

$$g(\mathcal{P}_Z X, JZ) = -g(\mathcal{P}_X Z, JZ) - \alpha\eta(X)g(Z, JZ).$$

In view of statement (i) of the above theorem, the right hand side of the above equation vanishes identically. Similarly, we can show that $g(\mathcal{P}_{JZ} X, Z) = 0$. Hence, statement (ii) implies that $X \ln f = 0$. This also holds for nearly cosyplectic manifolds, one can prove that using similar analogy like above. Thus, we have the following

Remark 3.3.1. *Warped products of the type $N \times_f N_T$ do not exist in nearly Sasakian and nearly cosymplectic manifolds if ξ is tangent to the first factor, and so for Sasakian and cosymplectic manifolds. However, the situation is different in Kenmotsu manifolds as we will see in the following example and in the next chapter also.*

A submanifold M^n of an almost contact metric manifold $\tilde{M}^{2l+1}$ is said to be a *contact CR-submanifold* if there exist on M^n differentiable distributions $\mathcal{D}_T$ and $\mathcal{D}_\perp$, satisfying the following

(i) $TM^n = \mathcal{D}_T \oplus \mathcal{D}_\perp \oplus \langle\xi\rangle$,

(ii) $\mathcal{D}_T$ is an invariant distribution, i.e., $\phi(\mathcal{D}_T) \subseteq \mathcal{D}_T$,

(iii) $\mathcal{D}_\perp$ is an anti-invariant distribution, i.e., $\phi(\mathcal{D}_\perp) \subseteq T^\perp M^n$.

In Sasakian manifolds, a concrete example of contact CR-warped product submanifolds of the type $N_T \times_f N_\perp$ can be found in (Munteanu, 2005). On the contrary, and in view of Remark 3.3.1, we conclude that warped product submanifolds with second invariant factor are trivial in both Sasakian and cosymplectic manifolds when ξ is tangent to the first factor. In particular, this implies that contact CR-warped product submanifolds of the type $N_\perp \times_f N_T$ reduces to be contact CR-products in Sasakian and cosymplectic manifolds. By contrast, such warped product submanifolds do exist in Kenmotsu manifolds.

To assert the above claim, we provide a counter example that ensures such existence of warped product submanifolds in Kenmotsu manifolds when the second factor is invariant. Besides, we can get an insurance for the existence of contact CR-warped product submanifolds in Kenmotsu manifolds, for both types; $M^n = N_T \times_f N_\perp$ and $M^n = N_\perp \times_f N_T$, when ξ is tangent to the first factor.

Example 3.3.1. *Let $\tilde{M}^9 = \mathbb{R} \times_{e^t} \mathbb{C}^4$ be a Kenmotsu manifold, where $\mathbb{R}$ is the real line, and $\mathbb{C}^4$ is a Kaehler manifold with Kaehlerian structure (G, J). Here, G and J are the restrictions of g and ϕ to $\tilde{M}^9(p)$, respectively, for every $p \in \tilde{M}^9$. Let $(t, x_1, \cdots, x_8)$ be a local coordinates frame of $\tilde{M}^9$ where t and $(x_1, \cdots, x_8)$ denote the local coordinates of $\mathbb{R}$ and $\mathbb{C}^4$, respectively. It is well-known that the Riemannian metric tensor g and the* *vector field ξ are defined on $\tilde{M}^9$ as follows (Kenmotsu, 1972):*

$$g_{(t,x)} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2t}G_(x) \end{pmatrix}, \qquad \xi = \left(\frac{d}{dt}\right).$$

Now, consider the three-dimensional submanifold M^3 of $\mathbb{C}^4$ given by the equations

$$x_1 = e^t v, \quad x_2 = e^t u, \quad x_3 = e^t v, \quad x_4 = e^t u, \quad x_5 = e^t s, \quad x_7 = e^t s, \quad x_6 = x_8 = 0.$$

Observe that the tangent bundle TM^3 is spanned by Z_1, Z_2 and Z_3, where

$$Z_1 = e^t\frac{\partial}{\partial x_1} + e^t\frac{\partial}{\partial x_3}, \quad Z_2 = e^t\frac{\partial}{\partial x_2} + e^t\frac{\partial}{\partial x_4}, \quad Z_3 = e^t\frac{\partial}{\partial x_5} + e^t\frac{\partial}{\partial x_7}.$$

Further, we define the distributions $\mathcal{D}_T$ =span$\{Z_1,\ Z_2\}$, and $\mathcal{D}_\perp$ =span$\{Z_3\}$. It is obvious that $\mathcal{D}_T$ and $\mathcal{D}_\perp$ are holomorphic and totally real distributions on $\mathbb{C}^4$, respectively. Hence, and taking into consideration $\phi(\xi) = 0$, the distributions $\mathcal{D}_\perp \oplus \langle\xi\rangle$ and $\mathcal{D}_T$ are respectively anti-invariant and invariant distributions on $\tilde{M}^9$. Thus, $N^4 = \mathcal{D}_\perp \oplus \langle\xi\rangle \oplus \mathcal{D}_T$ is a contact CR-submanifold in $\tilde{M}^9$. In addition, it is easy to see that both $\mathcal{D}_\perp \oplus \langle\xi\rangle$ and $\mathcal{D}_T$ are integrable. If we denote by $N_\perp$ and N_T the integral manifolds of $\mathcal{D}_\perp \oplus \langle\xi\rangle$ and $\mathcal{D}_T$, respectively, then the metric tensor g of N^4 is

$$g = dt^2 + e^{2t}ds^2 + e^{2t}(dv^2 + du^2) = g_{N_\perp} + e^{2t}g_{N_T}.$$

Therefor, N^4 is a contact CR-warped product submanifold of $\tilde{M}^9$ of the type $N_\perp \times_f N_T$ with warping function $f = e^t$. Moreover, it straight forward to figure out that

$$h(Z_1, Z_1) = h(Z_2, Z_2) = 0.$$

Hence, N^4 is a $\mathcal{D}_2$-minimal warped product submanifold as expected, where $\mathcal{D}_2 = \mathcal{D}_T$..

Likewise, by an analogous procedure to the above we can deduce that $\mathcal{D}_T \oplus \langle\xi\rangle$ is an invariant distribution on $\tilde{M}^9$, and $\mathcal{D}_\perp$ is an anti-invariant. Also, it is not difficult to show integrability of $\mathcal{D}_T \oplus \langle\xi\rangle$. Denoting the integral manifolds of $\mathcal{D}_T \oplus \langle\xi\rangle$ and $\mathcal{D}_\perp$ by N_T and $N_\perp$, respectively, we find that $N^4 = N_T \times_{e^t} N_\perp$ is a non-trivial contact CR-warped product in $\tilde{M}^9$. By calculating the coefficients of h restricted to N_T, we deduce that $N^4 = N_T \times_{e^t} N_\perp$ is a $\mathcal{D}_1$-minimal warped product submanifold as it should be, where $\mathcal{D}_1 = \mathcal{D}_T$.

In this sequel, proper warped product submanifolds of types $N_\theta \times_f N_T$ and $N_T \times_f N_\theta$ do exist in Kenmotsu manifolds, when ξ is tangent to the first factor. Whereas, Remark 3.3.1 informs us that proper warped product submanifolds of type $N_\theta \times_f N_T$ do not exist in both Sasakian and cosymplectic manifolds. Soon we show the nonexistence of $N_T \times_f N_\theta$ in Sasakian and cosymplectic manifolds such that N_θ is proper slant. In Kenmotsu manifolds, examples of warped product submanifolds of both types $N_\theta \times_f N_T$ and $N_T \times_f N_\theta$ will be constructed later, these examples are joined together in Example 4.2.1 which is postponed to the next chapter.

Motivated by Theorem 3.2.3, we prove a dual contact version which will be used in the rest of this work.

Theorem 3.3.4. *Let $M^n = N_T \times_f N$ be a warped product submanifold isometrically immersed in an almost contact manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. Then, we have the following*

(i) $g(\mathcal{P}_X Z, Y) = -g(h(X,Y), FZ)$;

(ii) $g(\mathcal{P}_Z X, Z) = (\phi X \ln f)||Z||^2 + g(h(X,Z), FZ)$;

(iii) $g(\mathcal{P}_Z X, Y) = 0$;

(iv) $$g(\mathcal{P}_Z X, W) + g(\mathcal{P}_W X, Z) = 2(\phi X \ln f) g(Z,W) + g(h(X,Z), FW) + g(h(X,W), FZ);$$

(v) $g(\mathcal{P}_Z X - \mathcal{P}_X Z, W) - g(\mathcal{P}_W X, Z) = 2(X \ln f) g(Z, PW)$;

(vi) $g(\mathcal{P}_X Z, W) + g(\mathcal{P}_X W, Z) = 0$;

(vii) $g(\mathcal{Q}_X X, \phi\zeta) + g(\mathcal{Q}_{\phi X} \phi X, \phi\zeta) = -g(h(X,X),\zeta) - g(h(\phi X, \phi X), \zeta)$,

for arbitrary vector fields $X,\ Y \in \Gamma(TN_T)$, $Z,\ W \in \Gamma(TN)$ and $\zeta \in \Gamma(\nu)$.

Proof. The assertion of statements $(i),\ (ii),\ (iv),\ (v)$ and (vi) can be shown by following similar analogue as that of Theorem 3.2.3. For statement (iii), suppose that X and Z are taken as hypothesis. Then it is obvious that

$$(\tilde{\nabla}_X \phi) Z = \tilde{\nabla}_X PZ + \tilde{\nabla}_X FZ - \phi \tilde{\nabla}_X Z. \tag{3.3.2}$$

Also, for X and Z we have

$$(\tilde{\nabla}_Z \phi)X = \tilde{\nabla}_Z \phi X - \phi \tilde{\nabla}_Z X. \tag{3.3.3}$$

By subtracting (3.3.3) from (3.3.2), we obtain

$$(\tilde{\nabla}_X \phi)Z - (\tilde{\nabla}_Z \phi)X = \tilde{\nabla}_X PZ + \tilde{\nabla}_X FZ - \tilde{\nabla}_Z \phi X.$$

Taking the inner product by ϕY in the above equation, gives

$$g(\mathcal{P}_X Z, \phi Y) - g(\mathcal{P}_Z X, \phi Y) = -g(h(X, \phi Y), FZ).$$

Replacing ϕY by Y yields

$$g(\mathcal{P}_Z X, Y) - \eta(Y) g(\mathcal{P}_Z X, \xi) - g(\mathcal{P}_X Z, Y) + \eta(Y) g(\mathcal{P}_X Z, \xi) =$$

$$g(h(X, Y), FZ) - \eta(Y) g(h(X, \xi), FZ).$$

By using (i) in the above equation we derive

$$g(\mathcal{P}_Z X, Y) = \eta(Y) g(\mathcal{P}_Z X, \xi).$$

Since the right hand side of the above equation vanishes identically, we obtain (iii).

For (vii), if we take $X = \xi$ in the above theorem, then statement (vii) holds directly. Now, for an arbitrary vector field tangential to the first factor and perpendicular to ξ, say X, we have

$$(\tilde{\nabla}_X \phi)X = \tilde{\nabla}_X \phi X - \phi \tilde{\nabla}_X X.$$

First, take the inner product in the above equation with $\phi\zeta$ to get

$$g(\mathcal{Q}_X X, \phi\zeta) = g(h(\phi X, X), \phi\zeta) - g(h(X, X), \zeta).$$

After that, we replace ϕX by X in the above equation to derive

$$g(\mathcal{Q}_{\phi X} \phi X, \phi\zeta) = -g(h(\phi X, X), \phi\zeta) - g(h(\phi X, \phi X), \zeta).$$

Hence (vii) can be obtained by adding the above two equations. □

In virtue of Theorem 3.3.4 (v), we get the following decisive nonexistence result in the setting of almost contact structures, which generalizes several nonexistence results in this field (see references in (Chen, 2013).

Corollary 3.3.1. *In both of Sasakian and cosymplectic manifolds, there is no warped product submanifolds with invariant first factor tangential to ξ, other than contact CR-warped products.*

In particular, this corollary implies the nonexistence of warped product submanifolds of type $N_T \times_f N_\theta$ in Sasakian and cosymplectic manifolds such that N_θ is a proper slant. On the contrary, this is not true for Kenmotsu manifolds as will be shown in Example 4.2.1.

Now, we prepare the following results for later use.

Theorem 3.3.5. *Let $M^n = N_1 \times_f N_2$ be a warped product submanifold isometrically immersed in a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to N_1. Then, the following hold*

(i) $\xi \ln f = \beta$;

(ii) $g(h(\xi, Z), FZ) = -\alpha ||Z||^2$,

for each vector field Z tangent to N_2.

Proof. By (2.3.63), it is straightforward that

$$-\phi\tilde{\nabla}_Z\xi + \tilde{\nabla}_\xi\phi Z - \phi\tilde{\nabla}_\xi Z = -\alpha Z - \beta\phi Z. \tag{3.3.4}$$

For (i), taking the inner product with ϕZ in the above equation, gives

$$-2\,\xi \ln f ||Z||^2 + g(\tilde{\nabla}_\xi\phi Z, \phi Z) = -\beta ||Z||^2,$$

Equivalently,

$$-2\,\xi \ln f ||Z||^2 + \frac{1}{2}\,\xi ||Z||^2 = -\beta ||Z||^2,$$

which implies

$$-2\,\xi \ln f ||Z||^2 + g(\tilde{\nabla}_\xi Z, Z) = -\beta ||Z||^2.$$

Hence, statement (i) follows from the above equation.

Now, we take the inner product with Z in $(3.3.4)$ to derive

$$g(\tilde{\nabla}_\xi\phi Z, Z) + 2\,g(\tilde{\nabla}_\xi Z, \phi Z) = -\alpha ||Z||^2.$$

This can be written as

$$g(\tilde{\nabla}_\xi PZ, Z) + g(\tilde{\nabla}_\xi FZ, Z) + 2\, g(\tilde{\nabla}_\xi Z, PZ) + 2\, g(\tilde{\nabla}_\xi Z, FZ) = -\alpha||Z||^2.$$

Hence, by the Gauss formula and part (ii) of Proposition 2.3.1, we get

$$g(\tilde{\nabla}_\xi FZ, Z) + 2\, g(\tilde{\nabla}_\xi Z, FZ) = -\alpha||Z||^2.$$

Consequently,

$$g(\tilde{\nabla}_\xi Z, FZ) = -\alpha||Z||^2.$$

Statement (ii) follows from the above equation by virtue of Gauss formula. This completes the proof. $\square$

In the spirit of the preceding theorem, It is easy, but important, to distinguish other particular case structures which we contemplate to discuss later. For this, we present the following table which will be useful for constructing first inequalities of h in almost contact manifolds in Chapter Five.

$\tilde{M}^{2l+1}$	$\xi \ln f =$	$g(h(\xi, Z), FZ) =$
Nearly trans-Sasakian	β	$-\alpha\|\|Z\|\|^2$
Nearly α-Sasakian	0	$-\alpha\|\|Z\|\|^2$
Sasakian	0	$-\|\|Z\|\|^2$
Nearly β-Kenmotsu	β	0
Kenmotsu	1	0
Nearly cosymplectic	0	0
Cosymplectic	0	0

Table 3.2: $\xi \ln f$ and $g(h(\xi, Z), FZ)$ for $N_1 \times_f N_2$ in $\tilde{M}^{2l+1}$, such that ξ is tangent to N_1 and Z is tangent to N_2.

Now, assume that the warped product submanifold $N_1 \times_f N_2$ in Theorem 3.3.5 is mixed totally geodesic. Thus, from statement (ii) of the same theorem, we have

$$\alpha||Z||^2 = 0.$$

This implies that, either N_2 is null, or $\alpha = 0$; i.e., $\tilde{M}^{2l+1}$ is not α-Sasakian. Therefore, we get the following significant nonexistence result, which will be useful in inequalities of mixed totally geodesic submanifolds in almost contact manifolds.

Proposition 3.3.1. *There is no mixed totally geodesic warped product submanifold in nearly α-Sasakian manifolds.*

In another line of thought, one can easily verify the following lemma.

Lemma 3.3.1. *Let $N_1 \times_f N_2$ be a warped product submanifold in almost contact manifolds $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. Then, $g((\tilde{\nabla}_\xi \phi)Z, \phi Z) = 0$ for every $Z \in \Gamma(TN_2)$.*

As another important consequence of Theorem 3.3.4, we have the following proposition, which will be extensively used in subsequent chapters.

Proposition 3.3.2. *For any warped product submanifold $M^n = N_T \times_f N$ of nearly trans-Sasakian manifolds with ξ tangent to the first factor, the followings are true*

(1) $g(h(X, Y), FZ) = 0$;

(2) $g(h(X, X), \zeta) + g(h(\phi X, \phi X), \zeta) = 0$;

(3) $g(h(X, Z), FZ) + \alpha\eta(X)||Z||^2 = -(\phi X \ln f)||Z||^2$,

where the vector fields X, Y are tangent to the first factor, Z is tangent to the second and ζ is tangent to the normal subbundle ν.

Proof. Statement (1) follows from (i) and (iii) of Theorem 3.3.4, while (3) is a consequence of (vi) and (ii) of the same theorem. For statement (2) we apply the nearly trans-Sasakian structure for the vector fields X and ξ to obtain the following

$$2\tilde{\nabla}_X \phi X = \alpha\Big(2g(X, X)\xi - 2\eta(X)X\Big) - 2\beta\eta(X)\phi X.$$

Taking the inner product with ζ gives

$$g(h(X, \phi X), \zeta) + g(\phi h(X, X)\zeta) = 0.$$

Replacing X by ϕX gives

$$g(h(-X + \eta(X)\xi, \phi X), \zeta) + g(\phi h(\phi X, \phi X)\zeta) = 0.$$

By these two equations and the fact $h(X, \xi) = 0$, we obtain the result. ☐

By means of Propositions 3.3.1 and 3.3.2, one can easily show that a mixed totally geodesic contact CR-warped product submanifold is indeed trivial in both Sasakian and cosymplectic manifolds. Whereas such submanifolds do exist in Kenmotsu manifolds, this is due to the fact $\xi \ln f = 1$ for all warped product submanifolds of Kenmotsu manifolds when ξ is tangent to the first factor.

In the sequel, we prove necessary and sufficient conditions for a contact CR-submanifold to be locally contact CR-warped product in nearly trans-Sasakian manifolds. For long time, mathematicians had have interest to find an analogue of the classical de Rham theorem to warped products, a result was proved by S. Hiepko (Hiepko, 1979). First, let us recall this result: Let $\mathcal{H}$ be a distribution in the tangent bundle of a Riemannian manifold M^n and let $\mathcal{H}^\perp$ be its orthogonal complementary distribution. Assume that the two distributions are both involutive and the integral manifolds of $\mathcal{H}$ (resp. $\mathcal{H}^\perp$) are extrinsic spheres (resp. totally geodesic). Then, M^n is locally isometric to a warped product $N_1 \times_f N_2$. Moreover, if M^n is simply connected and complete, there exists a global isometry of M^n with a warped product. Using this fundamental method we present the following characterization theorem which has been recently published in (Mustafa et al., 2013).

Theorem 3.3.6. *Every contact CR-submanifold M^n of a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ with an involutive distribution $\mathcal{D}_\perp$ is locally a contact CR-warped product, if and only if the shape operator of M^n satisfies*

$$A_{\phi W}X = -(\phi X\mu)W - \alpha\eta(X)W, \quad X \in \mathcal{D}_T \oplus \langle\xi\rangle, \quad W \in \mathcal{D}_\perp, \tag{3.3.5}$$

for a smooth function μ on M^n, satisfying $V(\mu) = 0$ for each $V \in \mathcal{D}_\perp$.

Proof. First, we will prove that the distribution $\mathcal{D}_T \oplus \langle\xi\rangle$ is integrable and its leaf N_T is totally geodesic. By making use of (2.3.27) and (3.3.5), we obtain

$$g(\phi\tilde{\nabla}_Y X, Z) = -g(\tilde{\nabla}_Y X, \phi Z) = -g(h(X,Y), \phi Z) = 0.$$

Via (2.3.27) we get

$$g(P\nabla_Y X, Z) - g(h(X,Y), \phi Z) = g(\phi\tilde{\nabla}_Y X, Z) = 0.$$

If we apply (3.3.5), we then deduce that

$$g(P\nabla_Y X, Z) = 0, \quad \forall\, X,\ Y \in \mathcal{D}_T \oplus \langle\xi\rangle, \quad \forall\, Z \in \mathcal{D}_\perp.$$

Meaning that; $\mathcal{D}_T \oplus \langle\xi\rangle$ is integrable and its leaf N_T is totally geodesic in M^n.

Further, it is essential to show that $N_\perp$ is totally umbilical; for this we apply (2.3.63) to derive

$$g(\tilde{\nabla}_Z W, \phi X) = -g(\tilde{\nabla}_Z \phi X, W) = -g(\phi\tilde{\nabla}_Z X, W) - g((\tilde{\nabla}_Z\phi)X, W),$$

for each $X \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_\perp)$. By virtue of (2.3.27), the above equation can be written as

$$g(\tilde{\nabla}_Z W, \phi X) = -g(P\nabla_Z X, W) - g(\phi h(X,Z), W) - g((\tilde{\nabla}_Z\phi)X, W).$$

It follows that

$$\begin{aligned} g(\tilde{\nabla}_Z W, \phi X) = -g(P\nabla_Z X, W) - g(\phi h(X,Z), W) - \Big(g((\tilde{\nabla}_Z\phi)X, W) \\ +g((\tilde{\nabla}_X\phi)Z, W)\Big) + g((\tilde{\nabla}_X\phi)Z, W). \end{aligned} \tag{3.3.6}$$

In view of (2.3.63), we conclude that

$$\Big(g((\tilde{\nabla}_Z\phi)X, W) + g((\tilde{\nabla}_X\phi)Z, W)\Big) = -\alpha\eta(X)g(W,Z). \tag{3.3.7}$$

By using (2.3.27), (2.3.28) and (2.3.29), we obtain

$$g((\tilde{\nabla}_X\phi)Z, W) = g(\tilde{\nabla}_X \phi Z, W) - g(\phi\tilde{\nabla}_X Z, W)$$

$$= -g(h(X,W), \phi Z) + g(h(X,Z), \phi W) - g(P\nabla_X Z, W).$$

Since $g(P\nabla_X Z, W) = 0$, so by (3.3.5), it follows that

$$g((\tilde{\nabla}_X\phi)Z, W) = 0. \tag{3.3.8}$$

It is clear that

$$g(P\nabla_Z X, W) = 0. \tag{3.3.9}$$

Thus, by combining relations (3.3.6)-(3.3.9) together, we finally reach that

$$g(\tilde{\nabla}_Z W, \phi X) = g(A_{\phi W} X, Z) + \alpha\eta(X)g(W,Z).$$

Making use of $(3.3.5)$, the above equation is simplified to

$$g(\tilde{\nabla}_Z W, \phi X) = -(\phi X\mu) g(W, Z).$$

Since the distribution $\mathcal{D}_\perp$ is assumed to be integrable, the second fundamental form $h^\perp$ of $N_\perp$ as an immersed submanifold in M^n is explicitly given by the relation

$$g(h^\perp(Z, Y), \phi X) = g(\tilde{\nabla}_Z W, \phi X).$$

By combining the last two equations together, it gives

$$g(h^\perp(Z, Y), \phi X) = -(\phi X\mu) g(W, Z).$$

Meaning that; $N_\perp$ is totally umbilical in M^n.

Notice that the above hypothesis asserts that $V(\mu) = 0$, for each $V \in \mathcal{D}_\perp$, which leads us to conclude that the mean curvature vector field of $\mathcal{D}_\perp$ is nonzero and parallel along $N_\perp$. In addition, it is assumed that the distribution $\mathcal{D}_\perp$ is also integrable, which implies that it is an extrinsic sphere in M^n. Let N_T and $N_\perp$ be integral manifolds of the distributions $\mathcal{D}_T \oplus \langle\xi\rangle$ and $\mathcal{D}_\perp$, respectively. Then, by results obtained in (Hiepko, 1979), M^n is locally a warped product $N_T \times_\mu N_\perp$ with μ as a warping function.

Since the converse is obvious from Theorem 3.3.4 or from Lemma 3.1 (iii) of (Mustafa et al., 2013). This completes the proof. $\square$

Observe that the above theorem generalizes many related recent results, for example contact CR-warped product of cosymplectic, Sasakian and Kenmotsu manifolds can be characterized in a similar way as above (see, for example (Munteanu, 2005)).

CHAPTER 4: WARPED PRODUCTS WITH A SLANT FACTOR

4.1 INTRODUCTION

Two well-known kinds of warped product submanifolds were defined in order to generalize CR-warped products. The first has a proper slant factor and a holomorphic one, while the other contains a proper slant factor and a totally real one. The former is called *semi-slant* warped product submanifold, whereas the latter is the *hemi-slant* warped product submanifold.

This chapter can be thought of as a slant version of the previous chapter, so it inherits some objectives and strategies from the previous one. In this chapter, relative geometric properties for semi-slant and hemi-slant warped product submanifolds are shown, which will be applied to explore different kinds of inequalities in the rest of this work. In addition, a lot of existence and nonexistence results of these warped product submanifolds are proved. Examples of both kinds of semi-slant warped product submanifolds in Kenmotsu manifolds are constructed in Example 4.2.1. More significantly, and for the sake of existence, a special inequality is proved in terms of the gradient of $\ln f$. As a result, two general theorems asserting the existence of any warped product submanifold in both nearly trans-Saskian and Kenmotsu manifolds when ξ is tangent to the first factor are given.

4.2 SEMI-SLANT WARPED PRODUCT SUBMANFOLDS

The study of geometry of slant submanifolds was rapidly developed in the last two decades. The theory of slant immersions in complex geometry was originated by B.Y. Chen in (Chen, 1990) and (Chen, 1990) as a generalization of both holomorphic and totally real submanifolds. The submanifold M^n of an almost Hermitian manifold is called *slant* (Chen, 1990) if for all nonzero vectors X tangent to M^n, the angle between JX and T_xM^n is a constant θ, i.e., it does not depend on the choice of $x \in M^n$ and $X \in T_xM^n$. In fact, this notion has the advantage to explore many significant differential results for any arbitrary angle in $(0,\ \frac{\pi}{2})$, other than 0 and $\frac{\pi}{2}$.

In a Kaehler manifold $\tilde{M}^{2m}$, it is easy to see that M^n is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$P^2 = -\lambda I. \tag{4.2.1}$$

In addition, $\lambda = \cos^2\theta$ (Chen, 1990). The following relations are straightforward consequences of the above equation

$$g(PX, PY) = \cos^2\theta\, g(X, Y) \tag{4.2.2}$$

and

$$g(FX, FY) = \sin^2\theta\, g(X, Y), \tag{4.2.3}$$

for all $X, Y \in \Gamma(TM^n)$.

The study of semi-slant submanifolds was initiated by N. Papaghiuc as a generalization of CR-submanifolds (Papaghiuc, 1994). Given a Kaehlerian manifold $\tilde{M}^{2m}$, according to Papaghiuc, the submanifold M^n is called *semi-slant* if it is endowed with two complementary orthogonal distributions $\mathcal{D}_T$ and $\mathcal{D}_\theta$, where $\mathcal{D}_T$ is holomorphic with respect to J and $\mathcal{D}_\theta$ is slant, i.e., the angle between JX and $(\mathcal{D}_\theta)_x$ is a constant θ for any $X \in (\mathcal{D}_\theta)_x$ and $x \in M^n$.

In this context, the nonexistence of warped product semi-slant submanifolds of the type $M^n = N_\theta \times_f N_T$ was shown in Corollary 3.2.1. Reversing the two factors, we get the following lemma, which can be obtained directly from Proposition 3.2.1 (iv) as a special case lemma. However, we provide another proof of this result.

Lemma 4.2.1. *Let $M^n = N_T \times_f N_\theta$ be a proper semi-slant submanifold of a nearly Kaehler manifold $\tilde{M}^{2m}$. Then, the following hold*

(i) $g(h(X, Z), FW) = \frac{1}{3}(X \ln f) g(PZ, W) - (JX \ln f) g(Z, W)$;

(ii) $g(h(X, PZ), FW) = -\frac{1}{3}\cos^2\theta (X \ln f) g(Z, W) - (JX \ln f) g(PZ, W)$;

(iii) $g(h(X, Z), FPW) = \frac{1}{3}\cos^2\theta (X \ln f) g(Z, W) - (JX \ln f) g(Z, PW)$;

(iv) $g(h(X, PZ), FPW) = -\frac{1}{3}\cos^2\theta (X \ln f) g(Z, PW) - \cos^2\theta (JX \ln f) g(Z, W)$,

for any $Z, W \in \Gamma\big(TN_\theta\big)$ and $X \in \Gamma\big(TN_T\big)$.

Proof. From (2.3.52), we have

$$(JX \ln f)Z + h(JX, Z) - (X \ln f)JZ - Jh(X, Z) + (X \ln f)PZ$$

$$+h(X, PZ) - A_{FZ}X + \nabla^{\perp}_X FZ - (X \ln f)JZ - Jh(X, Z) = 0.$$

Taking the inner product with W in the above equation, gives

$$(JX \ln f)g(Z, W) - (X \ln f)g(PZ, W) + 2g(h(X, Z), FW) - g(h(X, W), FZ) = 0. \tag{4.2.4}$$

Reversing the roles of Z and W in the above equation, we deduce that

$$(JX \ln f)g(Z, W) + (X \ln f)g(PZ, W) + 2g(h(X, W), FZ) - g(h(X, Z), FW) = 0. \tag{4.2.5}$$

If we add (4.2.4) and (4.2.5) together, then we get

$$(JX \ln f)g(Z, W) = -\frac{1}{2}\Big(g\big(h(X, Z), FW\big) + g\big(h(X, W), FZ\big)\Big).$$

Subtracting (4.2.4) from (4.2.5) gives

$$(X \ln f)g(PZ, W) = \frac{3}{2}\Big(g\big(h(X, Z), FW\big) - g\big(h(X, W), FZ\big)\Big).$$

Combining the above two equations proves statement (i). While the rest follow directly by this way. Substituting PZ in statement (i) instead of Z gives statement (ii). Replacing W by PW in (i) proves (iii). Statement (iv) comes from putting PZ and PW in place of Z and W, respectively in statement (i). $\square$

In 1996, Lotta (Lotta, 1996) extended the notion of slant submanifolds to the setting of almost contact metric ambient manifolds. Let M^n be a submanifold of an almost contact metric manifold, suppose the characteristic vector field ξ is tangent to the submanifold M^n. Then the class of *slant submanifolds* in almost contact manifolds is defined as follows:

For each non zero vector X tangent to M^n at x for some point $x \in M^n$ such that X is not proportional to ξ_x, we denote by $0 \leq \angle\theta(X) \leq \pi/2$ the angle between ϕX and T_xM^n, where $\angle\theta(X)$ is called the slant angle. If the slant angle $\angle\theta(X)$ is a constant θ for all $X \in T_xM^n - \langle\xi_x\rangle$ and $x \in M^n$, then M^n is said to be a *slant submanifold*. Obviously, if $\theta = 0$, then M^n is invariant, and if $\theta = \pi/2$, then M^n is an anti-invariant submanifold. A slant submanifold is said to be $proper\ slant$ if it is neither invariant nor anti-invariant submanifold. For these submanifolds, we recall the following characterization (Cabrerizo et al., 1999).

Theorem 4.2.1. *Let M^n be a submanifold of an almost contact metric manifold $\tilde{M}^{2l+1}$ such that $\xi \in \Gamma(TM^n)$. Then M^n is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi).$$

Furthermore, if θ is the slant angle, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequences of the above theorem

$$g(PX, PY) = \cos^2\theta(g(X,Y) - \eta(Y)\eta(X)) \tag{4.2.6}$$

and

$$g(FX, FY) = \sin^2\theta(g(X,Y) - \eta(Y)\eta(X)), \tag{4.2.7}$$

for all $X, Y \in TM^n$.

The notion of *semi-slant* submanifolds was defined in (Cabrerizo et al., 1999) as follows: A submanifold M^n of an almost contact manifold $\tilde{M}^{2l+1}$ is said to be a semi-slant submanifold if there exist two orthogonal distributions $\mathcal{D}_\theta$ and $\mathcal{D}_T$ satisfying

(i) $TM^n = \mathcal{D}_\theta \oplus \mathcal{D}_T \oplus \langle\xi\rangle$;

(ii) $\mathcal{D}_\theta$ is a slant distribution with slant angle $\theta \neq 0$;

(iii) $\mathcal{D}_T$ is an invariant i.e., $\phi\mathcal{D}_T \subseteq TM^n$.

The following theorem will be applied frequently in subsequent chapters, so we state and prove it in a more general way.

Theorem 4.2.2. *Let $M^n = N_T \times_f N$ be a warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ such that N_T and N are, respectively, holomorphic and Riemannian submanifolds, where ξ is tangent to the first factor. Then, we have*

$$g(h(X,Z), FW) = \frac{1}{3}\Big((X\ln f) - \beta\eta(X)\Big)g(PZ,W) - \Big((\phi X\ln f) + \alpha\eta(X)\Big)g(Z,W),$$

for any $Z, W \in \Gamma(TN)$ and $X \in \Gamma(TN_T)$.

Proof. From (2.3.63) we have

$$(\phi X \ln f)Z + h(\phi X, Z) - (X \ln f)\phi Z - \phi h(X, Z) + (X \ln f)PZ + h(X, PZ)$$

$$-A_{FZ}X + \nabla^{\perp}_{X} FZ - (X \ln f)\phi Z - \phi h(X, Z) = -\alpha\eta(X)Z - \beta\eta(X)\phi Z.$$

Taking the inner product with W in the above equation, we obtain

$$(\phi X \ln f)g(Z, W) - (X \ln f)g(PZ, W) + 2g(h(X, Z), FW) - g(h(X, W), FZ)$$

$$= -\alpha\eta(X)g(Z, W) - \beta\eta(X)g(PZ, W).$$

By the polarization of Z and W in the above equation, we deduce that

$$(\phi X \ln f)g(Z, W) + (X \ln f)g(PZ, W) + 2g(h(X, W), FZ) - g(h(X, Z), FW)$$

$$= -\alpha\eta(X)g(Z, W) + \beta\eta(X)g(PZ, W).$$

From the above two equations, we get

$$(\phi X \ln f)g(Z, W) = -\frac{1}{2}\Big(g\big(h(X, Z), FW\big) + g\big(h(X, W), FZ\big) + 2\alpha\eta(X)g(Z, W)\Big)$$

and

$$(X \ln f)g(PZ, W) = \frac{3}{2}\Big(g\big(h(X, Z), FW\big) - g\big(h(X, W), FZ\big)\Big) + \beta\eta(X)g(PZ, W).$$

Thus, the desired result follows form the above two equations. This completes the proof.

□

One particular case of Theorem 4.2.2 is the following lemma.

Lemma 4.2.2. *Let $M^n = N_T \times_f N_\theta$ be a proper semi-slant submanifold of a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. Then, we have*

$$g(h(X, Z), FW) = \frac{1}{3}\Big((X \ln f) - \beta\eta(X)\Big)g(PZ, W) - \Big((\phi X \ln f) + \alpha\eta(X)\Big)g(Z, W),$$

for any $Z, W \in \Gamma\big(TN_\theta\big)$ and $X \in \Gamma\big(TN_T\big)$.

Combining Lemma 4.2.2 with a result in (Mustafa et al., 2014), we reach the following lemma which is a key result for a general inequality in the next chapter.

Lemma 4.2.3. *Let $M^n = N_T \times_f N_\theta$ be a proper semi-slant submanifold of a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to N_T. Then the following hold:*

(i) $\xi lnf = \beta$;

(ii) $g(h(X,Y), FZ) = 0$;

(iii) $g(h(\xi, Z), \phi W) = -\alpha g(Z,W)$;

(iv) $$g(h(X,Z), FW) = \frac{1}{3}\Big((X\ln f) - \beta\eta(X)\Big)g(PZ,W) - \Big((\phi X\ln f) + \alpha\eta(X)\Big)g(Z,W);$$

(v) $$g(h(X,PZ), FW) = -\frac{1}{3}\cos^2\theta\Big((X\ln f) - \beta\eta(X)\Big)g(Z,W) - \Big((\phi X\ln f) + \alpha\eta(X)\Big)g(PZ,W);$$

(vi) $$g(h(X,Z), FPW) = \frac{1}{3}\cos^2\theta\Big((X\ln f) - \beta\eta(X)\Big)g(Z,W) - \Big((\phi X\ln f) + \alpha\eta(X)\Big)g(Z,PW);$$

(vii) $$g(h(X,PZ), FPW) = -\frac{1}{3}\cos^2\theta\Big((X\ln f) - \beta\eta(X)\Big)g(Z,PW) - \cos^2\theta\Big((\phi X\ln f) + \alpha\eta(X)\Big)g(Z,W),$$

for any $Z, W \in \Gamma(TN_\theta)$ *and* $X \in \Gamma(TN_T)$.

Proof. The first three parts can be proved by the same way as in (Mustafa et al., 2014), or the same as in the preceeding results of the previous chapter. Part (iv) is Lemma 4.2.2 itself. Whereas parts (v) and (vi) can be obtained from part (iv) by substituting PZ instead of Z and PW instead of W, respectively. The last part follows also from part (iv) by replacing Z and W by PZ and PW, respectively. □

Remark 3.3.1 and Corollary 3.3.1 lead us to conclude that proper semi-slant warped product submanifolds do not exist in both Sasakian and cosymplectic manifolds. Motivated by the fact that $\xi \ln f = 1$ for all warped product submanifolds of Kenmotsu manifolds, where ξ is tangent to the first factor, we will prove the following characterization theorem.

Theorem 4.2.3. *Let* M^n *be a semi-slant submanifold of a Kenmotsu manifold* $\tilde{M}^{2l+1}$ *such that the slant distribution is integrable. Then,* M^n *is locally a warped product of invariant and slant submanifolds if and only if*

$$A_{FW}X = \{(X\lambda) - \eta(X)\}PW - ((\phi X)\lambda)W, \tag{4.2.8}$$

for any $X \in \Gamma(\mathcal{D}_T \oplus \langle\xi\rangle)$, $W \in \Gamma(\mathcal{D}_\theta)$, and for some function λ on M^n satisfying $Z\lambda = 0$, for every $Z \in \Gamma(\mathcal{D}_\theta)$.

Proof. Since $\mathcal{D}_\theta$ is assumed to be integrable, one can denote by N_θ and h^θ respectively for the integral submanifold of $\mathcal{D}_\theta$, and the second fundamental form of N_θ in M^n. Firstly, suppose that M^n is a semi-slant submanifold satisfying the hypothesis of the theorem. Then for every $Z,\ W \in \Gamma(\mathcal{D}_\theta)$, and $X \in \Gamma(\mathcal{D}_T \oplus \langle\xi\rangle)$, we have

$$\begin{aligned} g(h^\theta(Z,W),\phi X) = g(\nabla_Z W,\phi X) &= -g(\phi\tilde{\nabla}_Z W, X) = g((\tilde{\nabla}_Z\phi)W, X) - g(\tilde{\nabla}_Z \phi W, X) \\ &= g(PZ,W)\eta(X) - g(\tilde{\nabla}_Z PW, X) - g(\tilde{\nabla}_X FW, X) \\ &= g(PZ,W)\eta(X) - g(h^\theta(PW,Z),X) + g(h(X,X),FW). \end{aligned}$$

According to the hypothesis, the above equation reduces to

$$g(h^\theta(Z,W),\phi X) = g(PZ,W)\eta(X) - g(h^\theta(PW,Z),X). \tag{4.2.9}$$

Interchanging X with ϕX, and W with PW in the above equation, it gives

$$-g(h^\theta(Z,PW),X) + \eta(X)\, g(h^\theta(Z,PW),\xi) = \cos^2\theta\, g(h^\theta(W,Z),\phi X). \tag{4.2.10}$$

Now, observe that

$$g(h^\theta(Z,PW),\xi) = g(\tilde{\nabla}_Z PW,\xi) = g(\tilde{\nabla}_Z \phi W,\xi) - g(\tilde{\nabla}_Z FW,\xi). \tag{4.2.11}$$

Because $h(\xi,Z) = 0$ for any submanifold M^n of Kenmotsu manifolds $\tilde{M}^{2l+1}$, where $\xi \in \Gamma(TM^n)$, equation (4.2.11) is congruent to

$$g(h^\theta(Z,PW),\xi) = g(\tilde{\nabla}_Z \phi W,\xi). \tag{4.2.12}$$

Making use of (2.3.57), we deduce that

$$g(h^\theta(Z,PW),\xi) = g(\phi Z,W). \tag{4.2.13}$$

Hence, from (4.2.10) and (4.2.13), we have

$$-g(h^\theta(Z,PW),X) + g(\phi Z,W)\eta(X) = \cos^2\theta\, g(h^\theta(W,Z),\phi X). \tag{4.2.14}$$

Thus, by subtracting (4.2.14) from (4.2.9), we derive

$$\sin^2\theta\, g(h^\theta(W,Z),\phi X) = 0,$$

because $\theta \neq 0$, it is also true that

$$g(h^\theta(W,Z),\phi X) = 0. \tag{4.2.15}$$

On the other hand, we have

$$g(h^\theta(W,Z),\phi X) = g(\tilde{\nabla}_W Z,\phi X) = -g(\phi\tilde{\nabla}_W Z, X) = g((\tilde{\nabla}_W \phi)Z, X) - g(\tilde{\nabla}_W \phi Z, X)$$

$$= g(\phi W, Z)\eta(X) - g(\tilde{\nabla}_W PZ, X) - g(\tilde{\nabla}_W FZ, X)$$

$$= g(PW, Z)\eta(X) - g(h^\theta(W, PZ), X) + g(A_{FZ}W, X). \tag{4.2.16}$$

Hence, from $(4.2.15)$ and $(4.2.16)$, we reach

$$g(h^\theta(W,PZ),X) = g(PW,Z)\eta(X) + g(A_{FZ}W,X).$$

If we replace Z by PZ in the above equation, then it automatically gives

$$-\cos^2\theta\, g(h^\theta(W,Z),X) = \cos^2\theta\, g(W,Z)\eta(X) + g(A_{FPZ}W,X).$$

By means of our hypothesis again, the above equation can be written as

$$-\cos^2\theta\, g(h^\theta(W,Z),X) = 2\cos^2\theta\, g(W,Z)\eta(X) - \cos^2\theta(X\lambda)g(Z,W)$$
$$- (\phi X\lambda)g(PZ,W).$$

If we compare the symmetric and skew-symmetric terms in the above equation, taking into consideration that M^n is a proper semi-slant submanifold, then it produces

$$g(h^\theta(W,Z),X) = (X\lambda)g(Z,W) - 2g(W,Z)\eta(X).$$

The above equation can be written as

$$g(h^\theta(W,Z),X) = g(\nabla\lambda - 2\xi, X)g(Z,W),$$

equivalently,

$$h^\theta(W,Z) = (\nabla\lambda - 2\xi)g(Z,W),$$

for any $Z,\ W \in \Gamma(\mathcal{D}_\theta)$, and $X \in \Gamma(\mathcal{D}_T \oplus \langle\xi\rangle)$.

The above relation shows that the leaves of $\mathcal{D}_\theta$ are totally umbilical in M^n with the mean curvature vector $\nabla\lambda$. Moreover, the condition $Z\lambda = 0$ for any $Z \in \Gamma(\mathcal{D}_\theta)$, implies that the leaves of $\mathcal{D}_\theta$ are extrinsic spheres in M^n; that is, each integral submanifold N_θ of $\mathcal{D}_\theta$ is umbilical and its mean curvature vector field is non zero and parallel along N_θ.

To show the integrability of $\mathcal{D}_T \oplus \langle\xi\rangle$, we first apply (4.2.8) for any $X,\ Y \in \Gamma(\mathcal{D}_T \oplus \langle\xi\rangle)$ and $W \in \Gamma(\mathcal{D}_\theta)$ to obtain that $g(h(X, \phi Y), FW) = g(h(Y, \phi X), FW) = 0$, (since it is trivial from (2.3.54) for $X = \xi$). Consequently, we deduce that $h(X, \phi Y) = h(Y, \phi X)$. Equivalently, it follows from (2.3.36) that $h(X, \phi Y) - h(Y, \phi X) = (\nabla_Y F)X - (\nabla_X F)Y$, which implies that $[X, Y] \in \Gamma(\mathcal{D}_T \oplus \langle\xi\rangle)$ for all $X,\ Y \in \Gamma(\mathcal{D}_T \oplus \langle\xi\rangle)$. Hence $\mathcal{D}_T \oplus \langle\xi\rangle$ is an integrable distribution.

One can easily deduce from $(4.2.8)$ that $g(h(X, Y), FW) = 0$, which means that the $F\mathcal{D}_\theta$-component of $h(X, Y)$ vanishes identically, for the vector fields $X,\ Y$ and W, so by means of $(2.3.57)$ again, we get

$$-F\nabla_X Y = fh(X, Y) - h(X, PY).$$

Since $\nabla_X Y$ is a tangent vector field of M^n, $\phi\nabla_X Y \notin \Gamma(\nu)$, whereas the right hand side of the above equation belongs to ν, hence $F\nabla_X Y = 0$, which implies that $\nabla_X Y \in \Gamma(\mathcal{D}_T \oplus \langle\xi\rangle)$. Thus, each leaf of $\mathcal{D}_T \oplus \langle\xi\rangle$ is totally geodesic.

Hence, by a result of (Hiepko, 1979), M^n is locally a warped product $M^n = N_T \times_\lambda N_\theta$, where N_T and N_θ denote the integral submanifolds of the distributions $\mathcal{D}_T \oplus \langle\xi\rangle$ and $\mathcal{D}_\theta$, respectively, and λ is the warping function.

Conversely, let $M^n = N_T \times_\lambda N_\theta$ be a warped product submanifold in a Kenmotsu manifold $\bar{M}^{2l+1}$. Then in view of (2.3.57), (2.3.27), (2.3.28) and Proposition 2.3.1 (ii), we derive

$$(\phi X \ln\lambda)W + h(\phi X, W) - (X \ln\lambda)\phi W - \phi h(X, W) = -\eta(X)\phi W, \tag{4.2.17}$$

and

$$\begin{aligned}(X \ln\lambda)\phi W + \phi h(X, W) &= -A_{FW}X + \nabla^\perp_X FW \\ &\quad + (X \ln\lambda)PW + h(X, PW),\end{aligned} \tag{4.2.18}$$

for any $X \in \Gamma(\mathcal{D}_T \oplus \langle\xi\rangle)$ and $W \in \Gamma(\mathcal{D}_\theta)$. Then (4.2.8) follows immediately from (4.2.17) and (4.2.18), which ends the proof. $\square$

As mentioned above, in comparison of Sasakian and cosymplectic manifolds on one side and Kenmotsu on the other side, proper semi-slant warped product submanifolds of both types $M^n = N_T \times_f N_\theta$ and $M^n = N_\theta \times_f N_T$ do exist in Kenmotsu manifolds.

For an evidence of this natural existence, we then construct a couple of explicit examples of such warped product submanifolds, giving an evidence that our characterization hypothesis is not vacuous, and the integrability condition imposed on the slant distribution is not redundant.

Example 4.2.1. *Similarly as Example 3.3.1, but considering the four-dimensional submanifolds M^4 of $\mathbb{C}^4$, given by the equations*

$x_1 = e^t\, v, \quad x_2 = e^t\, u, \quad x_3 = e^t\, v, \quad x_4 = e^t\, u, \quad x_5 = \cos\theta_1\, e^t\, s, \quad x_6 = e^t\, r,$ $x_7 = \sin\theta_1\, e^t\, s$ *and* $x_8 = 0$, *where* $\theta_1 \in (0, \frac{\pi}{2})$.

Therefore, the tangent bundle TM^4 is spanned by

$$Z_1 = e^t \frac{\partial}{\partial x_1} + e^t \frac{\partial}{\partial x_3}, \qquad Z_2 = e^t \frac{\partial}{\partial x_2} + e^t \frac{\partial}{\partial x_4},$$
$$Z_3 = \cos\theta_1 e^t \frac{\partial}{\partial x_5} + \sin\theta_1 e^t \frac{\partial}{\partial x_7}, \qquad Z_4 = e^t \frac{\partial}{\partial x_6}.$$

Define the distributions $\mathcal{D}_T$ =span$\{Z_1, Z_2\}$, and $\mathcal{D}_\theta$ =span$\{Z_3, Z_4\}$. Clearly $\mathcal{D}_T$ is a holomorphic distribution on $\mathbb{C}^4$, and $\mathcal{D}_\theta$ is a slant with slant angle $\theta = \theta_1$. Hence, $M^4 = \mathcal{D}_T \oplus \mathcal{D}_\theta$ is a semi-slant submanifold of $\mathbb{C}^4$. As same as the discussion of Example 3.3.1, we can show that $M^5 = N_T \times_f N_\theta$ is a warped product semi-slant submanifold of $\tilde{M}^{2l+1}$ with warping function $f = e^t$, where N_T and N_θ are the integral manifolds of the distributions $\mathcal{D}_T \oplus \langle\xi\rangle$ and $\mathcal{D}_\theta$, respectively. Moreover, using Gauss formula, it is possible to show the $\mathcal{D}_1$-minimality of M^5.

Analogously to Example 3.3.1, if we consider the distributions $\mathcal{D}_\theta \oplus \langle\xi\rangle$ and $\mathcal{D}_T$, then we immediately have another proper 5-dimensional semi-slant warped product submanifold of kind $N_\theta \times_f N_T$, with a slant angle $\theta = \theta_1$ and a warping function $f = e^t$, where N_θ and N_T are the leaves of $\mathcal{D}_\theta \oplus \langle\xi\rangle$ and $\mathcal{D}_T$, respectively.

Notice that the arbitrary slant angle θ guarantees the existence of infinitely number of semi-slant warped product submanifolds. This existence is a new impulse given to semi-slant warped product submanifolds in almost contact manifolds.

4.3 HEMI-SLANT WARPED PRODUCT SUBMANIFOLDS

In this section, we shall study hemi-slant warped product submanifolds in both almost Hermitian and almost contact manifolds.

From Corollaries 3.2.1 and 3.2.4 we concluded there does not exist any warped product submanifold in Kaehler manifolds such that one of the factors is proper slant while the other is totally holomorphic. To come up with a slant generalization, B. Sahin (Sahin, 2009) considered warped product submanifolds such that one of the factors is totally real, while the other is proper. These warped product submanifolds are called hemi-slant warped product submanifolds.

The notion of *hemi-slant submanifolds* of almost Hermitian manifolds is defined as follows (see, for example (Sahin, 2009) and (Bejanco, 1986)): A submanifold M^n of an almost Hermitian manifold $\tilde{M}^{2m}$ is said to be a hemi-slant submanifold if there exist a pair of orthogonal complementary distributions $\mathcal{D}_\theta$ and $\mathcal{D}_\perp$, satisfying

(i) $TM^n = \mathcal{D}_\theta \oplus \mathcal{D}_\perp$;

(ii) $\mathcal{D}_\theta$ is a proper slant distribution with slant angle $\theta \neq \pi/2$;

(iii) $\mathcal{D}_\perp$ is totally real i.e., $J\mathcal{D}_\perp \subseteq T^\perp M^n$.

In view of the above definition, it is clear that every CR-warped product is a particular hemi-slant with $\angle\theta = 0$.

If ν is the maximal invariant subbundle of the normal bundle $T^\perp M^n$ of a hemi-slant submanifold, then the normal bundle $T^\perp M^n$ admits the following decomposition

$$T^\perp M^n = \nu \oplus F\mathcal{D}_\theta \oplus F\mathcal{D}_\perp. \tag{4.3.1}$$

In the sequel, we first discuss the warped product hemi-slant submanifold of type $N_\perp \times_f N_\theta$ in almost Hermitian manifolds $\tilde{M}^{2m}$, with a purpose to prove some characteristic properties.

Theorem 4.3.1. *Let $N_\perp \times_f N_\theta$ be isometrically immersed into an almost Hermitian manifold $\tilde{M}^{2m}$. Then, we have*

$$g(\mathcal{P}_X Z, PZ) + g(\mathcal{P}_Z X, PZ) + g(\mathcal{Q}_X Z, FZ)$$

$$= g(h(X,Z), FPZ) - g(h(Z,PZ), FX) - \cos^2\theta(X\ln f)||Z||^2, \tag{4.3.2}$$

for all vector fields $X \in \Gamma(TN_\perp)$, and $Z \in \Gamma(TN_\theta)$.

Proof. Making use of (2.3.27), (2.3.28), (2.3.32) and (2.3.34), we obtain

$$(\tilde{\nabla}_X J)Z + (\tilde{\nabla}_Z J)X = (X\ln f)PZ + h(X,PZ) - A_{FZ}X + \nabla^\perp_X FZ$$

$$-2(X\ln f)JZ - 2Jh(X,Z) - A_{FX}Z + \nabla^\perp_Z FX.$$

Taking the inner product with JZ in the above equation, gives

$$g(\mathcal{P}_X Z, PZ) + g(\mathcal{P}_Z X, PZ) + g(\mathcal{Q}_X Z, FZ) + g(\mathcal{Q}_Z X, FZ)$$

$$= -g(h(Z,PZ), FX) + \cos^2\theta(X\ln f)||Z||^2 - 2(X\ln f)||Z||^2$$

$$+g(\nabla^\perp_X FZ, FZ) + g(\nabla^\perp_Z FX, FZ). \tag{4.3.3}$$

By similar techniques as above we conclude that

$$g(\nabla^\perp_X FZ, FZ) = g(\tilde{\nabla}_X FZ, FZ) = \frac{1}{2}Xg(FZ,FZ) = \frac{1}{2}\sin^2\theta X g(Z,Z)$$

$$= \sin^2\theta g(\tilde{\nabla}_X Z, Z) = \sin^2\theta(X\ln f)||Z||^2, \tag{4.3.4}$$

and

$$g(\nabla^\perp_Z FX, FZ) = g(\tilde{\nabla}_Z JX, FZ) = g((\tilde{\nabla}_Z J)X, FZ) + g(J\tilde{\nabla}_Z X, FZ)$$

$$= g(\mathcal{Q}_Z X, FZ) + g(J\tilde{\nabla}_Z X, JZ) - g(J\tilde{\nabla}_Z X, PZ)$$

$$= g(\mathcal{Q}_Z X, FZ) + (X\ln f)||Z||^2 - \cos^2\theta(X\ln f)||Z||^2 + g(h(X,Z), FPZ). \tag{4.3.5}$$

Hence, the assertion follows from combining (4.3.3), (4.3.4) and (4.3.5). $\square$

In Kaehler manifolds, we can use similar strategy as above to derive that

$$h(X,PZ) - A_{FZ}X + \nabla^\perp_X FZ - (X\ln f)FZ - Jh(X,Z) = 0.$$

Taking the inner product with FZ in the above equation produces

$$g(h(X,PZ), FZ) = g(h(X,Z), FPZ).$$

Finally, by virtue of (2.3.48), Theorem 4.3.1 and the above equation, we directly reach

Corollary 4.3.1. *Warped product submanifolds of the type $N_\perp \times_f N_\theta$ do not exist in Kaehler manifolds.*

With notation as above, one can verify that warped product submanifolds of the type $N_\perp \times_f N_\theta$ in nearly Kaehler manifolds admit

$$g(\mathcal{Q}_X Z, FZ) = g(h(X, PZ), FZ) - g(h(X, Z), FPZ).$$

Fitting $(2.3.52)$ and the above equation together in Theorem 4.3.1, we have

Corollary 4.3.2. *There is no mixed totally geodesic warped product submanifold of the type $N_\perp \times_f N_\theta$ in nearly Kaehler manifolds.*

From Corollary 4.3.1, we directly deduce that there is no proper warped product submanifold of the type $M^n = N_\perp \times_f N_\theta$ in Kaehler manifolds. On the contrary, warped products of the type $N_\theta \times_f N_\perp$ do exist in Kaehler manifolds. A concrete example of the latter type is constructed in the next chapter, Example 5.2.1. For more other examples, we refer to (Sahin, 2009).

Those examples show the natural existence of warped product submanifolds of type $N_\theta \times_f N_\perp$ in Kaehler manifolds. Therefore, it is interesting to investigate them in a more general setting. Hence, we first state the following

Lemma 4.3.1. *Let $N_\theta \times_f N_\perp$ be isometrically immersed in a nearly Kaehler manifold $\tilde{M}^{2m}$. Then, the following hold*

(i) $g(h(Z, W), FX) = (PX \ln f) g(Z, W) + g(h(X, Z), FW);$

(ii) $g(h(X, Y), FZ) = 2g(h(X, Z), FY) - g(h(Y, Z), FX),$

for arbitrary vector fields $X,\ Y \in \Gamma(TN_\theta)$, and $Z,\ W \in \Gamma(TN_\perp)$.

Proof. By means of $(2.3.52)$, it is possible to obtain the following

$$(PX \ln f)Z + h(PX, Z) - A_{FX}Z + \nabla^\perp_Z FX - 2(X \ln f)FZ - 2Jh(X, Z)$$
$$-A_{FZ}X + \nabla^\perp_X FZ = 0. \tag{4.3.6}$$

Taking the inner product with W, we have

$$(PX \ln f) g(Z, W) - g(h(Z, W), FX) + 2g(h(X, Z), FW) - g(h(X, W), FZ) = 0.$$

Changing the roles of Z and W in the above equation gives

$$(PX \ln f)g(Z, W) - g(h(Z, W), FX) + 2g(h(X, W), FZ) - g(h(X, Z), FW) = 0.$$

Combining the above two equations together, we deduce that

$$g(h(X, Z), FW) = g(h(X, W), FZ).$$

Hence, (i) follows from the preceding two equations. Statement (ii) follows by taking the inner product with Y in $(4.3.6)$, which finishes the proof. □

Analogously, we are now going to discuss warped product hemi-slant submanifolds of the type $N_\perp \times_f N_\theta$ in almost contact manifolds $\tilde{M}^{2l+1}$. Firstly, we have

Theorem 4.3.2. *Let $N_\perp \times_f N_\theta$ be isometrically immersed into an almost contact manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. Then, we have*

$$g(\mathcal{P}_X Z, PZ) + g(\mathcal{P}_Z X, PZ) + g(\mathcal{Q}_X Z, FZ)$$

$$= g(h(X, Z), FPZ) - g(h(Z, PZ), FX) - \cos^2\theta (X \ln f)||Z||^2, \tag{4.3.7}$$

for all vector fields $X \in \Gamma(TN_\perp)$, and $Z \in \Gamma(TN_\theta)$.

Proof. Making use of (2.3.27), (2.3.28), (2.3.32) and (2.3.34), we derive

$$(\tilde{\nabla}_X \phi)Z + (\tilde{\nabla}_Z \phi)X = (X \ln f)PZ + h(X, PZ) - A_{FZ}X + \nabla^\perp_X FZ$$

$$-2(X \ln f)\phi Z - 2\phi h(X, Z) - A_{FX}Z + \nabla^\perp_Z FX.$$

Taking the inner product with ϕZ in the above equation, gives

$$g(\mathcal{P}_X Z, PZ) + g(\mathcal{P}_Z X, PZ) + g(\mathcal{Q}_X Z, FZ) + g(\mathcal{Q}_Z X, FZ)$$

$$= -g(h(Z, PZ), FX) + \cos^2\theta (X \ln f)||Z||^2 - 2(X \ln f)||Z||^2$$

$$+g(\nabla^\perp_X FZ, FZ) + g(\nabla^\perp_Z FX, FZ). \tag{4.3.8}$$

By similar techniques as above, it is easy to verify that

$$g(\nabla^\perp_X FZ, FZ) = g(\tilde{\nabla}_X FZ, FZ) = \frac{1}{2}Xg(FZ, FZ) = \frac{1}{2}\sin^2\theta Xg(Z, Z)$$

$$= \sin^2\theta g(\tilde{\nabla}_X Z, Z) = \sin^2\theta (X \ln f)||Z||^2, \tag{4.3.9}$$

and

$$g(\nabla^{\perp}_{Z}FX, FZ) = g(\tilde{\nabla}_{Z}\phi X, FZ) = g((\tilde{\nabla}_{Z}\phi)X, FZ) + g(\phi\tilde{\nabla}_{Z}X, FZ)$$

$$= g(\mathcal{Q}_{Z}X, FZ) + g(\phi\tilde{\nabla}_{Z}X, \phi Z) - g(\phi\tilde{\nabla}_{Z}X, PZ)$$

$$= g(\mathcal{Q}_{Z}X, FZ) + (X\ln f)||Z||^{2} - \cos^{2}\theta(X\ln f)||Z||^{2} + g(h(X,Z), FPZ). \quad (4.3.10)$$

As a result, combining (4.3.8), (4.3.9) and (4.3.10) together proves the theorem. $\square$

Following similar strategy as in the above proof, it is easy to show that the following equation

$$h(X, PZ) - A_{FZ}X + \nabla^{\perp}_{X}FZ - (X\ln f)FZ - Jh(X,Z) = 0$$

is satisfied for both cosymplectic and Sasakian manifolds, whenever ξ is tangent to the first factor.

Taking the inner product with FZ in the above equation, we get

$$g(h(X, PZ), FZ) = g(h(X,Z), FPZ).$$

Consequently, making use of (2.3.55), (2.3.59), Theorem 4.3.2 and the above equation, we get

Corollary 4.3.3. *Warped product submanifolds of the type $N_{\perp} \times_{f} N_{\theta}$ do not exist in Sasakian and cosymplectic manifolds.*

From Proposition 3.3.1, one can conclude that there is no mixed totally geodesic warped product submanifold of the type $N_{\perp} \times_{f} N_{\theta}$ in nearly Sasakian manifolds. By the similar analogy as in Corollary 4.3.2, we can easily prove that such manifolds are trivial in nearly cosymplectic manifolds. Therefore, we state the following

Corollary 4.3.4. *There is no mixed totally geodesic warped product submanifold of the type $N_{\perp} \times_{f} N_{\theta}$ in nearly Sasakian and nearly cosymplectic manifolds.*

Likewise, for warped product hemi-slant submanifolds of type $N_{\theta} \times_{f} N_{\perp}$ in almost contact manifolds, we state the following.

Lemma 4.3.2. *Let $N_{\theta} \times_{f} N_{\perp}$ be isometrically immersed in a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. Then, we have*

(i) $g(h(Z,W),FX) = (PX \ln f)g(Z,W) + g(h(X,Z),FW)$
$+ \alpha\eta(X)g(Z,W);$

(ii) $g(h(X,Y),FZ) = 2g(h(X,Z),FY) - g(h(Y,Z),FX),$

where $X,\ Y \in \Gamma(TN_\theta)$*, and* $Z,\ W \in \Gamma(TN_\perp)$.

Proof. By means of (2.3.52), we get

$$(PX \ln f)Z + h(PX,Z) - A_{FX}Z + \nabla^\perp_Z FX - 2(X \ln f)FZ - 2Jh(X,Z)$$
$$-A_{FZ}X + \nabla^\perp_X FZ = -\alpha\eta(X)Z - \beta\eta(X)\phi Z. \tag{4.3.11}$$

Taking the inner product with W, gives

$$(PX \ln f)g(Z,W) - g(h(Z,W),FX) + 2g(h(X,Z),FW) - g(h(X,W),FZ) =$$
$$-\alpha\eta(X)g(Z,W).$$

Changing the roles of Z and W in the above equation gives

$$(PX \ln f)g(Z,W) - g(h(Z,W),FX) + 2g(h(X,W),FZ) - g(h(X,Z),FW) =$$
$$-\alpha\eta(X)g(Z,W).$$

In view of the above two equations, we deduce that

$$g(h(X,Z),FW) = g(h(X,W),FZ).$$

Hence, (i) follows from the preceding two equations, whereas statement (ii) follows by taking the inner product with Y in (4.3.11). $\square$

In (Uddin et al., 2012), we studied hemi-slant warped product submanifolds in nearly cosymplectic manifolds under the name of pseudo-slant. In the same paper, the following results were obtained.

Theorem 4.3.3. *Let* $M^n = N_\perp \times_f N_\theta$ *be a warped product submanifold of a nearly cosymplectic manifold* $\tilde{M}^{2l+1}$ *such that* ξ *is tangent to the first factor. Then* M^n *is a Riemannian product of* $N_\perp$ *and* N_θ *if and only if* $\mathcal{P}_X PX \in \Gamma(TN_\theta)$*, for any* $X \in \Gamma(TN_\theta)$*, where* N_θ *and* $N_\perp$ *are proper slant and anti-invariant submanifolds of* $\tilde{M}^{2l+1}$*, respectively.*

Theorem 4.3.4. *Let $M^n = N_\theta \times_f N_\perp$ be a warped product submanifold of a nearly cosymplectic manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. Then M^n is a Riemannian product of N_θ and $N_\perp$ if and only if*

$$g(h(X,Z),FZ) = g(h(Z,Z),FX),$$

for any $X \in \Gamma(TN_\theta)$ and $Z \in \Gamma(TN_\perp)$, where N_θ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\tilde{M}^{2l+1}$, respectively.

4.4 SPECIAL INEQUALITY FOR THE EXISTENCE OF WARPED PRODUCT SUBMANIFOLDS IN ALMOST CONTACT MANIFOLDS

The following inequality is for existence purposes. Even though it differs from all other inequalities of this thesis, it is one of the most important results. This is because it proves that every warped product submanifold in nearly trans-Saskian manifold is indeed a proper one when ξ is tangent to the first factor.

Theorem 4.4.1. *Let $M^n = N_1 \times_f N_2$ be a warped product submanifold of a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. Then, we have*

(i) $||\nabla \ln f||^2 \geq \beta^2$.

(ii) The equality of (i) holds identically if and only if $X \ln f = 0$, for all $X \in \Gamma(TN_1)$ such that X is orthogonal to ξ.

(iii) In particular, if N_1 is an invariant submanifold, then the equality of (i) holds identically if and only if $h(X,Z) \in \nu$ for all X and Z tangent to the first and the second factors, respectively, where X is orthogonal to ξ.

Proof. For some differentiable function, ψ, on $\tilde{M}^{2l+1}$, we first recall $(2.3.23)$ form chapter two, that is;

$$||\tilde{\nabla}\psi||^2 = \sum_{i=1}^{m}\big(e_i(\psi)\big)^2. \tag{4.4.1}$$

Since f and $\ln f$ act on N_1, the above relation implies

$$||\nabla \ln f||^2 = \sum_{a=1}^{n_1}\big(e_a(\ln f)\big)^2, \tag{4.4.2}$$

where $\{e_1 = \xi, \cdots, e_{n_1}\}$ is a local orthonormal frame of $\Gamma(TN_1)$.

In view of this adapted frame, we can expand $(4.4.2)$ as follows

$$||\nabla \ln f||^2 = (\xi(\ln f))^2 + \sum_{a=2}^{n_1} \big(e_a(\ln f)\big)^2. \tag{4.4.3}$$

It is well-known that $\xi \ln f = \beta$ for warped product submanifolds as in the hypothesis, (see Theorem 3.3.5 (i)). Using this fact to evaluate the first term on the right hand side of $(4.4.3)$, it automatically yields

$$||\nabla \ln f||^2 = \beta^2 + \sum_{a=2}^{n_1} \big(e_a(\ln f)\big)^2. \tag{4.4.4}$$

Consequently, the inequality of statement (i) follows immediately from the above relation.

For statement (ii), the inequality of (i) holds identically if and only if

$$\sum_{a=2}^{n_1} \big(e_a(\ln f)\big)^2 = 0. \tag{4.4.5}$$

It is obvious that the above equation provides a necessary and sufficient condition for the equality sign of (i). This proves statement (ii).

Now, for the last statement let N_1 to be an invariant submanifold in $\tilde{M}^{2l+1}$. Then, from Theorem 4.2.2, we have

$$g(h(X,Z), FW) = \frac{1}{3}\Big((X \ln f) - \beta\eta(X)\Big)g(PZ, W) - \Big((\phi X \ln f) + \alpha\eta(X)\Big)g(Z, W),$$

for any $Z, W \in \Gamma\big(TN_2\big)$ and $X \in \Gamma\big(TN_1\big)$.

In view of $(4.4.5)$, if the equality holds then the above equation becomes

$$g(h(X,Z), FW) = 0.$$

This means that $h(X,Z) \in \nu$ for all X and Z tangent to the first and the second factors, respectively, where X is orthogonal to ξ.

Conversely, if $h(X,Z) \in \nu$ for all X and Z as above, then we deduce that

$$\frac{1}{3}(X \ln f)g(PZ, W) - (\phi X \ln f)g(Z, W) = 0,$$

for any $Z, W \in \Gamma\big(TN_2\big)$ and $X \in \Gamma\big(TN_1\big)$, X orthogonal to ξ.

Comparing the symmetric or the skew-symmetric terms of the above equation gives

$$X \ln f = 0,$$

for all $X \in \Gamma\big(TN_1\big)$, X orthogonal to ξ. Hence, the equality holds and completes the proof. $\square$

Even though we get a nice geometric description for a necessary and sufficient condition for the equality case of (i) holding, the inequality of (i) itself is enough to get the following prominent existence theorems in nearly trans-Sasakian and nearly β-Kenmotsu manifolds.

Theorem 4.4.2. *There exist proper warped product submanifolds, $M^n = N_1 \times_f N_2$, in nearly trans-Sasakian manifolds such that $\xi \in \Gamma(TN_1)$.*

In particular,

Theorem 4.4.3. *There exist proper warped product submanifolds, $M^n = N_1 \times_f N_2$, in nearly β-Kenmotsu manifolds such that $\xi \in \Gamma(TN_1)$.*

Clearly, the above two theorems are direct consequences of the above inequality, which is valid for general warped product submanifolds in nearly trans-Sasakian and then in nearly β-Kenmotsu manifolds.

Taking an advantage of Theorem 3.3.2 of the previous chapter and the above two theorems, we reach the following

Theorem 4.4.4. *In nearly trans-Sasakian manifolds, there exist proper warped product submanifolds if ξ is tangent to the first factor, whereas they are trivial if ξ is tangent to the second.*

Particularly,

Theorem 4.4.5. *In nearly β-Kenmotsu manifolds, there exist proper warped product submanifolds if ξ is tangent to the first factor, whereas they are trivial if ξ is tangent to the second.*

Based on the fact that addressed on Theorem 3.3.5 (i), we conclude that all warped product submanifolds in nearly β-Kenmotsu and nearly trans-Sasakian manifolds do exist whenever ξ is tangent to the first factor, which implies their existence in nearly Kenmotsu and Kenmotsu manifolds also. This comes from $\xi \ln f = \beta$ in the nearly trans-Sasakian and nearly β-Kenmotsu manifolds. These observations have been materialized in this section via Theorems 4.4.1, 4.4.2, 4.4.3, 4.4.4 and 4.4.5. Thus, the following table actually contains those cases of existence and nonexistence problems of some almost contact manifolds of interest.

The abbreviations of manifolds are: Sas.≡ Sasakian, Ken.≡ Kenmotsu, Cos.≡ cosymplectic, n.Sas.≡ nearly Sasakian, n.Ken≡ nearly Kenmotsu, n.cos.≡ nearly cosymplectic and n.t.S.≡ nearly trans-Sasakian.

Type	Sas.	Ken.	Cos.	n.Sas.	n.Ken.	n.cos.	n.t.S.
$N_\perp \times_f N_T$	X	✓	X	X	✓	X	✓
$N_T \times_f N_\perp$	✓	✓	✓	✓	✓	✓	✓
$N_\theta \times_f N_T$	X	✓	X	X	✓	X	✓
$N_T \times_f N_\theta$	X	✓	X	?	✓	?	✓
$N \times_f N_T$	X	✓	X	X	✓	X	✓
$N_T \times_f N$	X	✓	X	?	✓	?	✓
$N_\perp \times_f N_\theta$	X	✓	X	?	✓	?	✓
$N_\theta \times_f N_\perp$	?	✓	?	?	✓	?	✓

Table 4.1: Existence and nonexistence of warped product submanifolds in almost contact manifolds with ξ tangent to the first factor.

CHAPTER 5: $\mathcal{D}_i$-MINIMALITY AND THE FIRST INEQUALITY OF h

5.1 INTRODUCTION

Given a warped product submanifold $N_1 \times_f N_2$. We recall some relative terminology from Chapter Two, as usual $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the distributions given by the vectors tangent to leaves and fibers, respectively. This means that $\mathcal{D}_1$ and $\mathcal{D}_2$ are respectively obtained from tangent vectors of N_1 and N_2 via horizontal and vertical lifts, respectively. In the second section of the current chapter, it is proven that $\mathcal{D}_i$-minimality is possessed by a wide class of warped product submanifolds, some of these warped product submanifolds were shown to have this geometric property in (Mustafa et al., 2014) and (Mustafa et al., 2015). After that, some results concerning $\mathcal{D}_i$-minimal warped product submanifolds are obtained, to be diversely applied in subsequent chapters; namely Theorem 5.2.1 and Lemma 5.2.6.

In Chapter Two, we called the inequality proved in (Chen, 2001) the first inequality of h. In the third section of this chapter, we first use the results of the second section to modify all inequalities of the first kind in both almost Hermitian and almost contact manifolds. At the same time, we give new generalizations of many inequalities for CR, semi-slant and hemi-slant warped product submanifolds.

It is worth pointing out that, the first inequality of h was first initiated by Chen for CR-warped product submanifolds in Kaehler manifolds (Chen, 2001). After many extensions of such inequality in the setting of contact CR-warped product submanifolds of almost contact manifolds (see, for example (Arsalan et al., 2005) and (Munteanu, 2005)), we gave an inequality which is still the main general inequality of this kind till date (Mustafa et al., 2013).

In the setting of semi-slant warped product submanifolds, we also initiated the study of such type of inequalities for nearly cosymplectic manifolds (Uddin et al., 2014), then we gave a more general one in (Mustafa et al.,2014). In hemi-slant warped product submanifolds, a slightly different inequality was proved by Sahin in Kaehler manifolds (Sahin, 2009). We contemplate to modify and generalize these kinds of inequalities in this chapter.

5.2 $\mathcal{D}_i$-MINIMALITY OF WARPED PRODUCT SUBMANIFOLDS

In the sense of Definition 2.3.3, we are going to show the natural existence of $\mathcal{D}_i$-minimal warped product submanifolds in both almost Hermitian and almost contact manifolds. In these manifolds, $\mathcal{D}_i$-minimality is proved for both $i = 1$ and $i = 2$.

Firstly, in view of Proposition 3.2.1, we can directly derive

Corollary 5.2.1. *Let $M^n = N_T \times_f N_\perp$ be a CR-warped product submanifold into an arbitrary Kaehler manifold $\tilde{M}^{2m}$. Then, the following hold*

(i) $g(h(X, X), FZ) = 0$*;*

(ii) $g(h(X, X), \zeta) = -g(h(JX, JX), \zeta)$*,*

for all vector fields $X,\ Z$ and ζ tangent to $N_T,\ N_\perp$ and ν, respectively.

Secondly, we provide the next key result which will be referred to frequently during this section.

Lemma 5.2.1. *Every CR-warped product submanifold of the type $M^n = N_T \times_f N_\perp$ is a $\mathcal{D}_1$-minimal warped product submanifold in Kaehler manifolds, where $\mathcal{D}_1 = \mathcal{D}_T$.*

Proof. Consider the following local field of orthonormal frames of the Kaehler manifold $\tilde{M}^{2m}$: $\{e_1, \cdots, e_s, e_{s+1} = Je_1, \cdots, e_{n_1} = e_{2s} = Je_s, e_{n_1+1} = e_1^\star, \cdots, e_{n_1+n_2} = e_n = e_q^\star, e_{n+1} = Je_1^\star, \cdots, e_{n+n_2} = Je_q^\star, e_{n+n_2+1} = \bar{e}_1, \cdots, e_{2m} = \bar{e}_{2l=\gamma}\}$ such that $\{e_1, \cdots, e_s, e_{s+1} = Je_1, \cdots, e_{n_1} = e_{2s} = Je_s\}$, $\{e_{n_1+1} = e_1^\star, \cdots, e_{n_1+n_2} = e_n = e_q^\star\}$, $\{e_{n+1} = Je_1^\star, \cdots, e_{n+n_2} = Je_q^\star\}$ and $\{e_{n+n_2+1} = \bar{e}_1, \cdots, e_{2m} = \bar{e}_{2l=\gamma}\}$ are the local fields of orthonormal frames of $\Gamma(TN_T)$, $\Gamma(TN_\perp)$, $\Gamma(JTN_\perp)$ and $\Gamma(\nu)$, respectively.

Using the terminology in (2.3.41), it is straightforward to have

$$\sum_{r=n+1}^{2m} \sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1}^{2m} \left(h_{11}^r + \cdots + h_{n_1 n_1}^r\right).$$

In view of (3.2.8), the right hand side summation can be decomposed as

$$\sum_{r=n+1}^{2m} \sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1}^{2m-\gamma} \left(h_{11}^r + \cdots + h_{n_1 n_1}^r\right) + \sum_{r=n+1+q}^{2m} \left(h_{11}^r + \cdots + h_{n_1 n_1}^r\right).$$

Taking into account part (i) of Corollary 5.2.1, the first summation on the right hand side of the above equation vanishes, whereas we expand the other summation in view of the above orthonormal frames to get

$$\sum_{r=n+1}^{2m}\sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1+q}^{2m}\big(h_{11}^r + \cdots + h_{ss}^r + h_{s+1s+1}^r + \cdots + h_{2s2s}^r\big).$$

Equivalently,

$$\sum_{r=n+1}^{2m}\sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1+q}^{2m}\Big(g(h(e_1,e_1),e_r) + \cdots + g(h(e_s,e_s),e_r) \\ + g(h(Je_1,Je_1),e_r) + \cdots + g(h(Je_s,Je_s),e_r)\Big).$$

Now, if we apply part (ii) of Corollary 5.2.1 on the above equation, then it automatically gives

$$\sum_{r=n+1}^{2m}\sum_{a=1}^{n_1} h_{aa}^r = \sum_{r=n+1+q}^{2m}\Big(g(h(e_1,e_1),e_r) + \cdots + g(h(e_s,e_s),e_r) \\ - g(h(e_1,e_1),e_r) - \cdots - g(h(e_s,e_s),e_r)\Big) \\ = 0.$$

Clearly, this proves the vanishing of the coefficients h_{aa}^r under summation, for $a \in \{1,\cdots,n_1\}$ and $r \in \{n+1,\cdots,2m\}$. Therefore, the partial mean curvature vector $\vec{H}$ defined in (2.3.46) does vanish. Hence, in the sense of Definition 2.3.3, we get the assertion. $\square$

Remark 5.2.1. *Putting $\mathcal{D}_1 = \mathcal{D}_T$, then by following the above scheme typically one can show that warped product submanifolds of types $M^n = N_T \times_f N_\perp$, $M^n = N_T \times_f N_\theta$ and $M^n = N_T \times_f N$ are $\mathcal{D}_1$-minimal in nearly Kaehelr and in $l.c.K.$ manifolds when the Lee vector field λ is tangent to M^n.*

Now, we will show the existence of $\mathcal{D}_2$-minimal warped product submanifolds in almost Hermitian manifolds. For this, we consider locally conformal Kaehler, $l.c.K.$, manifolds to be the ambient manifolds $\tilde{M}^{2m}$. Firstly, we have:

Lemma 5.2.2. *Let $M^n = N \times_f N_T$ be a warped product submanifold in a $l.c.K.$ manifold $\tilde{M}^{2m}$ such that the Lee vector field λ is tangent to M^n, where N_T and N are holomorphic and Riemannian submanifolds of $\tilde{M}^{2m}$. Then we have* *(i)* $g(h(X,X),FZ)=0$;

(ii) $g(h(X,X),\zeta)+g(h(JX,JX),\zeta)=0$,

where X, Z and ζ are vector fields on N_T, N and ν, respectively.

Proof. In view of $(2.3.51)$, we directly obtain

$$(PZ\ln f)X+h(PZ,X)-A_{FZ}X+\nabla^{\perp}_{X}FZ$$

$$-(Z\ln f)JX-Jh(X,Z)=g(\lambda,PZ)X-g(\lambda,Z)JX.$$

Taking the inner product with X and JX respectively on the above equation, yields

$$(PZ\ln f)||X||^2-g(h(X,X),FZ)=g(\lambda,PZ)||X||^2$$

and

$$Z\ln f=g(\lambda,Z).$$

From the above two equations we get (i). For the other part, observe that $(2.3.51)$ gives

$$(\tilde{\nabla}_X J)X=g(\lambda,JX)X-g(\lambda,X)JX+J\lambda||X||^2.$$

Taking the inner product with $J\zeta$ produces

$$g(h(JX,X),J\zeta)-g(h(X,X),\zeta)=0.$$

Interchanging the roles of X and JX in the above equation, we obtain

$$-g(h(JX,X),J\zeta)-g(h(JX,JX),\zeta)=0.$$

Combining the above two equations, (ii) immediately follows. □

As a result, if we follow an analogous argument as in the proof of Lemma 5.2.1, then we can show that $M^n=N\times_f N_T$ is a $\mathcal{D}_2$-minimal warped product submanifold in $l.c.K.$ manifolds, where $\mathcal{D}_2=\mathcal{D}_T$. Moreover, similar discussion like above can show the existence of other kinds of $\mathcal{D}_2$-minimal warped product submanifolds in $l.c.K.$ manifolds.

Remark 5.2.2. *Warped products of types $M^n=N\times_f N_T$, $M^n=N_\perp\times_f N_T$ and $M^n=N_\theta\times_f N_T$ are $\mathcal{D}_2$-minimal warped product submanifolds in $l.c.K.$ manifolds, where $\mathcal{D}_2=\mathcal{D}_T$.*

Now, we turn our attention to almost contact manifolds, we are going to explain the natural existence of $\mathcal{D}_i$-minimal warped product submanifolds in almost contact manifolds, for both $i = 1$ and $i = 2$. Observe that all almost contact manifolds considered in this thesis satisfy $(\tilde{\nabla}_\xi \phi)\xi = 0$. Hence, it is convenient to state

Lemma 5.2.3. *Let M^n be a submanifold tangent to the characteristic vector field ξ in an almost contact manifold $\tilde{M}^{2l+1}$. If $(\tilde{\nabla}_\xi \phi)\xi = 0$ on $\tilde{M}^{2l+1}$, then $h(\xi, \xi) = 0$.*

In the third section of Chapter Three, contact CR-warped product submanifolds were introduced. Beginning with Sasakian manifolds, we consider a contact CR-warped product submanifold of type $M^n = N_T \times_f N_\perp$.

Corollary 5.2.2. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product submanifold in a Sasakian manifold $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. Then, the following hold*

(i) $h(X, \xi) = 0$*;*

(ii) $g(h(X, X), FZ) = 0$*;*

(iii) $g(h(X, X), \zeta) = -g(h(\phi X, \phi X), \zeta)$*,*

for every $X \in \Gamma(TN_T)$, $Z \in \Gamma(TN_\perp)$ and $\zeta \in \Gamma(\nu)$.

Proof. From (2.3.55) we obtain

$$X - \eta(X)\xi = -\phi\nabla_X \xi - \phi h(X, \xi).$$

Applying ϕ on the above equation, taking into consideration $\eta(\nabla_X \xi) = 0$, then it yields

$$\phi X = \nabla_X \xi + h(X, \xi).$$

By comparing the tangential and normal terms in the above equation we get (i). (ii) is well-known (see, for example (Mihai, 2004), (Munteanu, 2005) and (Mustafa et al., 2013)). For the last part, take an arbitrary $\zeta \in \Gamma(\nu)$, then by making use of (2.3.55) and (2.3.27), we obtain

$$\nabla_X \phi X + h(\phi X, X) - \phi\nabla_X X - \phi h(X, X) = -g(X, X)\xi + \eta(X)X,$$

taking the inner product with $\phi\zeta$ in the above equation, we deduce

$$g(h(\phi X, X), \phi\zeta) - g(h(X, X), \zeta) = 0, \tag{5.2.1}$$

interchanging X with ϕX in (5.2.1), gives

$$\begin{aligned} g(h(\phi X, \phi X), \zeta) &= g(h(\phi(\phi X), \phi X), \phi\zeta) = g(\tilde{\nabla}_{\phi X}\phi(\phi X), \phi\zeta) \\ &= -g(\tilde{\nabla}_{\phi X}X, \phi\zeta) + g(\tilde{\nabla}_{\phi X}(\eta(X)\xi), \phi\zeta) \\ &= -g(h(X, \phi X), \phi\zeta) + \eta(X)g(\tilde{\nabla}_{\phi X}\xi, \phi\zeta) \\ &= -g(h(X, \phi X), \phi\zeta) + \eta(X)g(h(\phi X, \xi), \phi\zeta). \end{aligned}$$

Making use of statement (i) in the above equation, we conclude that

$$g(h(\phi X, \phi X), \zeta) = -g(h(X, \phi X), \phi\zeta). \tag{5.2.2}$$

From $(5.2.1)$ and $(5.2.2)$, we obtain statement (iii). $\square$

In view of the above Lemma, and taking into account Lemma 5.2.3, it is straightforward to apply the same procedure as in the proof of Lemma 5.2.1 to verify the following

Lemma 5.2.4. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product submanifold in Sasakian manifolds $\tilde{M}^{2l+1}$ such that ξ is tangent to N_T. Then, M^n is $\mathcal{D}_1$-minimal warped product, where $\mathcal{D}_1 = \mathcal{D}_T \oplus \langle\xi\rangle$.*

For Kenmotsu manifolds, some parts of the next two results can be found in (Mustafa et al., 2015). By an analogous proof of Lemma 5.2.2, it is easy to show the following

Corollary 5.2.3. *For the warped products $M^n = N_T \times_f N$ and $M^n = N \times_f N_T$ in Kenmotsu manifolds, where N_T and N are respectively invariant and Reimannian submanifolds of $\tilde{M}^{2l+1}$ such that the characteristic vector field ξ is tangent to the first factor, the following hold*

(i) $g(h(X, Y), FZ) = 0$;

(ii) $g(h(X, X), \zeta) = -g(h(\phi X, \phi X), \zeta)$,

where X, Y are tangent to N_T, Z is tangent to N and $\zeta \in \Gamma(\nu)$.

Semi-slant warped product submanifolds were defined and extensively discussed in chapter four. Moreover, examples and a characterization theorem were given for warped product submanifolds of types $N_T \times_f N_\theta$ and $N_\theta \times_f N_T$ in Kenmotsu manifolds, such that ξ is tangent to the first factor. In view of the above corollary and Lemma 5.2.3, we have the next result.

Lemma 5.2.5. *Let $\mathcal{D}_1 = \mathcal{D}_T$. Then, the warped products $N_T \times_f N_\perp$, $N_T \times_f N_\theta$ and $N_T \times_f N$ are $\mathcal{D}_1$-minimal in Kenmotsu manifolds $\tilde{M}^{2l+1}$. While $N_\perp \times_f N_T$, $N_\theta \times_f N_T$ and $N \times_f N_T$ are $\mathcal{D}_2$-minimal in $\tilde{M}^{2l+1}$, where $\mathcal{D}_2 = \mathcal{D}_T$. In both categories, the characteristic vector field ξ is assumed to be tangential to the first factor, where N_T, $N_\perp$, N_θ and N are invariant, anti-invariant, slant and Riemannian submanifolds of $\tilde{M}^{2l+1}$.*

In general, we have the following result for nearly trans-Sasakian manifolds, where a special case of this result had been proven in (Mustafa et al.,2014).

Remark 5.2.3. *By similar discussion as above and putting $\mathcal{D}_1 = \mathcal{D}_T$, we can prove that any warped product of the type $N_T \times_f N$ is $\mathcal{D}_1$-minimal in nearly trans-Sasakian manifolds. By the contrary, reversing the two factors of such warped products gives $\mathcal{D}_2$-minimal warped products, where ξ is tangent to the first factor in both cases.*

Typically as warped product submanifolds of types $N_T \times_f N$ and $N \times_f N_T$, $\mathcal{D}_i$-minimality can be proved for generic warped product submanifolds. This is left to the reader.

Remark 5.2.4. *Generic warped product submanifolds are $\mathcal{D}_i$-minimal in almost Hermitian and almost contact manifolds considered in this thesis, where i refers to the holomorphic distribution.*

From all the above results, we notice that $\mathcal{D}_i$-minimality is possessed by CR, semi-slant and generic warped product submanifolds [*], in both almost Hermitian and almost contact manifolds of interest. In the following example, we construct a hemi-slant warped

[*]The notion of generic submanifolds were introduced in both almost Hermitian and almost contact manifolds (see, for example (Bejancu, 1986), (Khan & Khan, 2009) and references of (Chen, 2013)). In this thesis, we will not consider generic warped product submanifolds. This is to avoid confusion in terminology, and because of limited time and space also. Anyway, corresponding results to generic warped product submanifolds are identical to those of warped product submanifolds of types $N_T \times_f N$ and $N \times_f N_T$, where N is a Riemannian submanifold.

product submanifold which is $\mathcal{D}_1$-minimal warped product submanifold. This means that, the class of $\mathcal{D}_i$-minimal warped product submanifold is wide enough to be considered in further research.

Example 5.2.1. *Consider a submanifold M^3 in $\mathbb{R}^8$ given by the equations*

$$x_1 = u, \quad x_2 = v, \quad x_6 = u\sin\vartheta, \quad x_8 = u\cos\vartheta, \quad x_i = 0,\ i = 3, 4, 5, 7,$$

where $\vartheta \in (0, \frac{\pi}{2})$, $u \neq 0$ and $v \neq 0$. Then, the tangent bundle TM^3 is spanned by

$$Z_1 = \frac{\partial}{\partial x_1} + \sin\vartheta\frac{\partial}{\partial x_6} + \cos\vartheta\frac{\partial}{\partial x_8}, \quad Z_2 = \frac{\partial}{\partial x_2}, \quad Z_3 = u\cos\vartheta\frac{\partial}{\partial x_6} - u\sin\vartheta\frac{\partial}{\partial x_8}.$$

Then, it is easy to see that $\mathcal{D}_\theta = span\{Z_1, Z_2\}$ is a slant distribution with slant angle $\theta = \frac{\pi}{4}$. It is also easy to show that $\mathcal{D}_\perp = span\{Z_3\}$ is an anti-invariant distribution. Consequently, M^3 turns out to be hemi-slant submanifold. Moreover, one can directly check integrability of $\mathcal{D}_\theta$, so this permits us to denote the integral manifolds of $\mathcal{D}_\theta$ and $\mathcal{D}_\perp$ by N_θ and $N_\perp$, respectively. Therefor, the metric tensor of M^3 is computed by

$$g = 2du^2 + dv^2 + u^2 d\vartheta^2.$$

Equivalently,

$$g = g_{N_\theta} + u^2 g_{N_\perp}.$$

As a result, M^3 is a warped product hemi-slant submanifold of $\mathbb{R}^8$ with warping function $f = u$. Then using the Gauss formula, we have

$$h(Z_1, Z_1) = h(Z_2, Z_2) = 0.$$

hence, M^3 is a $\mathcal{D}_1$-minimal warped product submanifolds in $\mathbb{R}^8$.

Moreover, by easy computation we obtain $h(Z_1, Z_2) = 0$. Hence, N_θ is totally geodesic in $\mathbb{R}^8$. Also,

$$h(Z_3, Z_3) = -\frac{u}{2}\left(-\frac{\partial}{\partial x_1} + \sin\vartheta\frac{\partial}{\partial x_6} + \cos\vartheta\frac{\partial}{\partial x_8}\right).$$

Therefore, $N_\perp$ is totally umbilical in $\mathbb{R}^8$. However, M^3 is neither totally geodesic nor totally umbilical in $\mathbb{R}^8$.

Now, let us recall the following significant key result for warped product submanifolds $M^n = N_1 \times_f N_2$ from Chapter Two. It is clear that Proposition 2.3.2 (4) implies that the sectional curvature and the warping function are related by

$$\sum_{a=1}^{n_1} \sum_{A=n_1+1}^{n} K(e_a \wedge e_A) = \frac{n_2 \Delta f}{f}, \tag{5.2.3}$$

where $\{e_1, \cdots, e_{n_1}, e_{n_1+1}, \cdots, e_n\}$ are local fields of orthonormal frame of $\Gamma(TM^n)$ such that n_1, n_2 and n are the dimensions of N_1, N_2 and M^n, respectively. It is clear that $e_a \in \{e_1, \cdots, e_{n_1}\}$, and $e_A \in \{e_{n_1+1}, \cdots, e_n\}$. We point out that $(5.2.3)$ is a key ingredient of this work, which has also been used frequently in proving inequalities like that in (Chen, 2002).

Once and for all, we are going to prove two basic results which are frequently used in all coming sections, especially in dealing with the equality case of inequalities to come. The next theorem is a direct consequence of Theorem 6.3.1 in the next chapter. However, we prove it here in a different way.

Theorem 5.2.1. *Let φ be $\mathcal{D}_i$-minimal isometric immersion, for $i = 1$ or 2, from a warped product $M^n = N_1 \times_f N_2$ into any Riemannian manifold $\tilde{M}^m$. Then*

$$||h(\mathcal{D}_1, \mathcal{D}_2)||^2 = \tilde{\tau}(T_x M^n) - \tilde{\tau}(T_x N_1) - \tilde{\tau}(T_x N_2) - \frac{n_2\, \Delta(f)}{f},$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the distributions of the first and the second factors of $N_1 \times_f N_2$, respectively.

Proof. In virtue of the Gauss equation, we have

$$n^2 ||\vec{H}||^2 = ||h||^2 + 2\tau(T_x M^n) - 2\tilde{\tau}(T_x M^n). \tag{5.2.4}$$

Now, let $\{e_1, \cdots, e_{n_1}, e_{n_1+1}, \cdots, e_n = e_{n_1+n_2}\}$ and $\{e_{n+1}, \cdots, e_m\}$ be the local fields of orthonormal frames of $\Gamma(TM^n)$ and $\Gamma(T^\perp M^n)$, respectively, where $\{e_1, \cdots, e_{n_1}\}$ and $\{e_{n_1+1}, \cdots, e_n = e_{n_1+n_2}\}$ are the frames of $\Gamma(TN_1)$ and $\Gamma(TN_2)$, respectively. Then, and without loss of generality, choose e_{n+1} to be in the direction of the mean curvature vector $\vec{H}$.

Now, from $(2.3.45)$, we have

$$\tau(T_x M^n) = \sum_{1 \le i < j \le n} K_{ij} = \sum_{a=1}^{n_1} \sum_{A=n_1+1}^{n} K_{aA} + \sum_{1 \le a < b \le n_1} K_{ab} + \sum_{n_1+1 \le A < B \le n} K_{AB}. \tag{5.2.5}$$

Via (5.2.3) and (2.3.45), the above equation is congruent to

$$\tau\big(T_xM^n\big) = \frac{n_2\Delta f}{f} + \tau\big(T_xN_1\big) + \tau\big(T_xN_2\big). \tag{5.2.6}$$

In view of (2.3.43), it is common to have

$$\tau\big(T_xN_1\big) = \sum_{r=n+1}^{m}\sum_{1\le a<b\le n_1}\Big(h^r_{aa}h^r_{bb} - \big(h^r_{ab}\big)^2\Big) + \tilde{\tau}\big(T_xN_1\big), \tag{5.2.7}$$

and

$$\tau\big(T_xN_2\big) = \sum_{r=n+1}^{m}\sum_{n_1+1\le A<B\le n}\Big(h^r_{AA}h^r_{BB} - \big(h^r_{AB}\big)^2\Big) + \tilde{\tau}\big(T_xN_2\big). \tag{5.2.8}$$

By $(5.2.4)$-$(5.2.8)$, one directly obtains

$$\begin{aligned}\Big(\sum_{i=1}^{n}h^{n+1}_{ii}\Big)^2 &= \sum_{r=n+1}^{m}\sum_{i=1}^{n}(h^r_{ii})^2 + \sum_{r=n+1}^{m}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h^r_{ij})^2 + \frac{2n_2\Delta f}{f}\\
&+2\sum_{r=n+1}^{m}\sum_{1\le a<b\le n_1}\Big(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\Big) + 2\sum_{r=n+1}^{m}\sum_{n_1+1\le A<B\le n}\Big(h^r_{AA}h^r_{BB} - (h^r_{AB})^2\Big)\\
&+2\Big(\tilde{\tau}(T_xN_1) + \tilde{\tau}(T_xN_2) - \tilde{\tau}(T_xM^n)\Big).\end{aligned}$$

By rearranging the right hand side terms in an appropriate manner, we can obtain

$$\begin{aligned}\Big(\sum_{i=1}^{n}h^{n+1}_{ii}\Big)^2 &= \sum_{r=n+1}^{m}\sum_{a=1}^{n_1}(h^r_{aa})^2 + 2\sum_{r=n+1}^{m}\sum_{1\le a<b\le n_1}h^r_{aa}h^r_{bb}\\
&+ \sum_{r=n+1}^{m}\sum_{A=n_1+1}^{n}(h^r_{AA})^2 + 2\sum_{r=n+1}^{m}\sum_{n_1+1\le A<B\le n}h^r_{AA}h^r_{BB}\\
&+ \sum_{r=n+1}^{m}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h^r_{ij})^2 - 2\sum_{r=n+1}^{m}\sum_{1\le a<b\le n_1}(h^r_{ab})^2 - 2\sum_{r=n+1}^{m}\sum_{n_1+1\le A<B\le n}(h^r_{AB})^2\\
&+\frac{2n_2\Delta f}{f} + 2\Big(\tilde{\tau}(T_xN_1) + \tilde{\tau}(T_xN_2) - \tilde{\tau}(T_xM^n)\Big).\end{aligned} \tag{5.2.9}$$

If φ is $\mathcal{D}_1$-minimal, then it follows

$$\sum_{r=n+1}^{m}\sum_{a=1}^{n_1}(h^r_{aa})^2 + 2\sum_{r=n+1}^{m}\sum_{1\le a<b\le n_1}h^r_{aa}h^r_{bb} = 0,$$

$$\sum_{r=n+1}^{m}\sum_{A=n_1+1}^{n}(h^r_{AA})^2 + 2\sum_{r=n+1}^{m}\sum_{n_1+1\le A<B\le n}h^r_{AA}h^r_{BB} = \Big(\sum_{i=1}^{n}h^{n+1}_{ii}\Big)^2$$

and

$$\sum_{r=n+1}^{m}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{r})^{2}-2\sum_{r=n+1}^{m}\sum_{1\leq a<b\leq n_1}(h_{ab}^{r})^{2}-2\sum_{r=n+1}^{m}\sum_{n_1+1\leq A<B\leq n}(h_{AB}^{r})^{2}$$
$$=2\sum_{r=n+1}^{m}\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^{2}.$$

By substituting the above three equations in (5.2.9), we immediately reach

$$2\sum_{r=n+1}^{m}\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^{2}=-\frac{2n_2\Delta f}{f}-2\Big(\tilde{\tau}(T_xN_1)+\tilde{\tau}(T_xN_2)-\tilde{\tau}(T_xM^n)\Big).$$

Analogously, we get the same result when φ is $\mathcal{D}_2$-minimal. This gives the assertion. □

The second key result is

Lemma 5.2.6. *Let φ be a $\mathcal{D}_2$-minimal isometric immersion of a warped product submanifold $M^n = N_1 \times_f N_2$ into any Riemannian manifold $\tilde{M}^m$. If N_2 is totally umbilical in $\tilde{M}^m$, then φ is $\mathcal{D}_2$-totally geodesic.*

Proof. Let $\check{h}$ and $\hat{h}$ denote the second fundamental forms of N_2 in M^n and $\tilde{M}^m$, respectively. Then for every vector fields Z and W tangent to N_2 we have

$$h(Z,W)=\hat{h}(Z,W)-\check{h}(Z,W).$$

Notice that, Corollary 2.3.1 and the above hypothesis guarantee that N_2 is totally umbilical in both M^n and $\tilde{M}^m$. From this fact and part (iii) of Proposition 2.3.1, the above equation takes the form

$$h(Z,W)=g(Z,W)(\Psi+\nabla(\ln f)), \tag{5.2.10}$$

for some vector field $\Psi\in\Gamma(T\tilde{M}^m)$ such that Ψ is normal to $\Gamma(TN_2)$.

Considering the local field of orthonormal frames as in the above proof. Then, taking the summation over the orthonormal frame fields of $\Gamma(TN_2)$ in the above equation, we get

$$\sum_{A,B=n_1+1}^{n}h(e_A,e_B)=\sum_{A,B=n_1+1}^{n}g(e_A,e_B)(\Psi+\nabla(\ln f)).$$

Taking into account $\mathcal{D}_2$-minimality of φ, the left hand side of the above equation vanishes and we get

$$0=n_2\,(\Psi+\nabla(\ln f)).$$

Since N_2 is not empty, we obtain

$$\Psi = -\nabla(\ln f).$$

Making use of the above equation in (5.2.10), we obtain

$$h(Z, W) = 0,$$

for every vector fields $Z, W \in \Gamma(TN_2)$. Meaning that; φ is $\mathcal{D}_2$-totally geodesic. This completes the proof. $\square$

5.3 MODIFIED INEQUALITIES IN ALMOST HERMITIAN MANIFOLDS

In 2001, B.Y. Chen established an interesting basic inequality (Chen, 2001), for CR-warped product submanifolds in Kaehler manifolds; that is,

Theorem 5.3.1. *(Chen, 2001). Let $\varphi : M^n = N_T \times_f N_\perp \longrightarrow \tilde{M}^{2m}$ be an isometric immersion of an n-dimensional CR-warped product submanifold into a Kaehler manifold $\tilde{M}^{2m}$. Then, we have*

(i) $||h||^2 \geq 2n_2||\nabla \ln f||^2$.

(ii) If the equality in (i) holds, then N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2m}$, respectively.

In this chapter, a modification for the equality case of the above theorem will be proved. In fact, this modification can be carried out on all first inequalities of h in both almost Hermitian and almost contact manifolds. For this, we are going to prove a general result for warped product manifolds which also provides us with a strong link between all inequalities of the next chapter and Theorem 6.3.1. To this end, consider the warped product manifold $M^n = N_1 \times_f N_2$. Then, for any unit vectors X and Z tangent to N_1 and N_2 respectively, we use (2.3.4), (2.3.6) and (2.3.7) to write

$$\begin{aligned} K(X \wedge Z) = g(R(Z, X)X, Z) &= (\nabla_X X) \ln f g(Z, Z) - g(\nabla_X((X \ln f)Z), Z) \\ &= (\nabla_X X) \ln f g(Z, Z) - g(\nabla_X(X \ln f)Z + (X \ln f)\nabla_X Z, Z) \\ &= (\nabla_X X) \ln f - X(X \ln f) - (X \ln f)^2. \end{aligned}$$

Taking the summation over orthonormal frame fields yields

$$\sum_{A=n_1+1}^{n}\sum_{a=1}^{n_1} K(e_a\wedge e_A)=\sum_{A=n_1+1}^{n}\sum_{a=1}^{n_1}\Big((\nabla_{e_a}e_a)\ln f-e_a(e_a\ln f)-(e_a\ln f)^2\Big).$$

Applying (2.3.21) and (2.3.22) on the above equation results in

$$\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n} K(e_a\wedge e_A)=n_2\{\Delta\ln f-||\nabla\ln f||^2\}. \tag{5.3.1}$$

Consequently, joining (5.2.3) and (5.3.1), it turns out that

$$\frac{\Delta f}{f}=\Delta\ln f-||\nabla\ln f||^2. \tag{5.3.2}$$

Now, we present the following modification of Theorem 5.3.1, where we ascertain the necessary and sufficient conditions for the equality case. The proof of this theorem will be referred to several times in the rest of this work.

Theorem 5.3.2. *Let $\varphi : M^n = N_T \times_f N_\perp \longrightarrow \tilde{M}^{2m}$ be an isometric immersion of an n-dimensional CR-warped product submanifold into a Kaehler manifold $\tilde{M}^{2m}$. Then, we have*

(i) $||h||^2 \geq 2n_2||\nabla\ln f||^2$.

(ii) The equality in (i) holds if and only if N_T, $N_\perp$, M^n are respectively totally geodesic, totally umbilical, minimal submanifolds in $\tilde{M}^{2m}$, and

$$n_2\,\Delta(\ln f)=\tilde{\tau}(T_xM^n)-\tilde{\tau}(T_xN_T)-\tilde{\tau}(T_xN_\perp). \tag{5.3.3}$$

Proof. (i) was proved in (Chen, 2001). For (ii), by Lemma 5.2.1 we know that $N_T\times_f N_\perp$ is $\mathcal{D}_1$-minimal in Kaehler manifolds, then from Theorem 5.2.1 we have

$$||h(\mathcal{D}_T,\mathcal{D}_\perp)||^2=\tilde{\tau}(T_xM^n)-\tilde{\tau}(T_xN_T)-\tilde{\tau}(T_xN_\perp)-\frac{n_2\,\Delta(f)}{f},$$

which is equivalent to

$$||h(\mathcal{D}_T,\mathcal{D}_\perp)_{F\mathcal{D}_\perp}||^2+||h(\mathcal{D}_T,\mathcal{D}_\perp)_\nu||^2=\tilde{\tau}(T_xM^n)-\tilde{\tau}(T_xN_T)-\tilde{\tau}(T_xN_\perp)-\frac{n_2\,\Delta(f)}{f}.$$

In view of Proposition 3.2.1 (iv), notice that $PZ=0$ for CR-warped product submanifolds of the type $N_T\times_f N_\perp$, it is easy to show that (see, for example (Chen, 2003))

$$||h(\mathcal{D}_T,\mathcal{D}_\perp)_{F\mathcal{D}_\perp}||^2=n_2||\nabla\ln f||^2. \tag{5.3.4}$$

Combining the above two equations, and making use of $(5.3.2)$, we obtain

$$||h(\mathcal{D}_T, \mathcal{D}_\perp)_\nu||^2 = \tilde{\tau}(T_x M^n) - \tilde{\tau}(T_x N_T) - \tilde{\tau}(T_x N_\perp) - n_2\, \Delta(\ln f). \tag{5.3.5}$$

Now, for the sufficiency, assume that the equality holds in (i); i.e.,

$$||h||^2 = \ 2n_2\, ||\nabla \ln f||^2.$$

By the linearity of the Hermitian metric, the squared norm of the second fundamental form can be expanded as follows

$$||h||^2 =$$

$$||h(\mathcal{D}_T, \mathcal{D}_T)||^2 + 2\, ||h(\mathcal{D}_T, \mathcal{D}_\perp)||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)||^2 =$$

$$||h(\mathcal{D}_T, \mathcal{D}_T)||^2 + 2\, ||h(\mathcal{D}_T, \mathcal{D}_\perp)_{F\mathcal{D}_\perp}||^2 + 2\, ||h(\mathcal{D}_T, \mathcal{D}_\perp)_\nu||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)||^2.$$

$$= \ 2n_2\, ||\nabla \ln f||^2.$$

It follows from $(5.3.4)$ and the above equation that

$$||h(\mathcal{D}_T, \mathcal{D}_T)||^2 + 2\, ||h(\mathcal{D}_T, \mathcal{D}_\perp)_\nu||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)||^2 = 0.$$

Then, as in (Chen, 2001), we can show that N_T, $N_\perp$ and M^n are respectively totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2m}$. Moreover, the above equation also implies

$$||h(\mathcal{D}_T, \mathcal{D}_\perp)_\nu||^2 = 0.$$

The above equation and $(5.3.5)$ give

$$n_2 \Delta \ln f = \tilde{\tau}(T_x M^n) - \tilde{\tau}(T_x N_T) - \tilde{\tau}(T_x N_\perp). \tag{5.3.6}$$

Conversely, assume that $N_T \times_f N_\perp$ is a minimal warped product submanifold in Kaehler manifolds $\tilde{M}^{2m}$, where N_T and $N_\perp$ are respectively totally geodesic and totally umbilical submanifolds in $\tilde{M}^{2m}$, and

$$n_2\, \Delta(\ln f) = \tilde{\tau}(T_x M^n) - \tilde{\tau}(T_x N_T) - \tilde{\tau}(T_x N_\perp). \tag{5.3.7}$$

In view of (5.3.5) and the above equation, we deduce that

$$||h(\mathcal{D}_T, \mathcal{D}_\perp)_\nu||^2 = 0. \tag{5.3.8}$$

Since N_T is totally geodesic in both M^n and $\tilde{M}^{2m}$, it implies

$$h(\mathcal{D}_T, \mathcal{D}_T) = 0. \tag{5.3.9}$$

By Lemma 5.2.1, it is proved that M^n is always $\mathcal{D}_1$-minimal in $\tilde{M}^{2m}$. Since M^n is assumed to be a minimal submanifold in $\tilde{M}^{2m}$, M^n is also $\mathcal{D}_2$-minimal in $\tilde{M}^{2m}$. By the assumption, $N_\perp$ is totally umbilical in $\tilde{M}^{2m}$, then the hypothesis of Lemma 5.2.6 is satisfied. Hence, it implies that M^n is $\mathcal{D}_2$-totally geodesic in $\tilde{M}^{2m}$, which means

$$h(\mathcal{D}_\perp, \mathcal{D}_\perp) = 0. \tag{5.3.10}$$

Therefore, the equality in (ii) holds by (5.3.8)-(5.3.10). This completes the proof. $\square$

Now, let $\tilde{M}^{2m}$ be an almost Hermitian manifold and $M^n = N_\theta \times N_\perp$ be mixed totally geodesic hemi-slant submanifold of $\tilde{M}^{2m}$. If $dim N_\theta = 2s = n_1$ and $dim N_\perp = n_2$, then $n = 2s + n_2$. Suppose $\{e_1, \cdots, e_s, e_{s+1} = \sec\theta P e_1, \cdots, e_{2s} = \sec\theta P e_s = e_{n_1}\}$ is a local orthonormal frame for $\mathcal{D}_\theta$, and $\{e_{n_1+1}, \cdots, e_{n_1+n_2} = e_n\}$ is a local orthonormal frame of $\mathcal{D}_\perp$. Then, the local orthonormal frames of $F\mathcal{D}_\theta$ and $F\mathcal{D}_\perp$ subbundles are respectively $\{e_{n+1} = \bar{e}_1 = \csc\theta F e_1, \cdots, \bar{e}_s = \csc\theta F e_s, \bar{e}_{s+1} = \csc\theta\sec\theta F P e_1, \cdots, \bar{e}_{2s} = \csc\theta\sec\theta F P e_s = e_{4s+n_2}\}$ and $\{e_{4s+n_2+1} = \bar{e}_{2s+1}, \cdots, e_{4s+2n_2} = \bar{e}_{2s+n_2}\}$. We note here that, the normal subbundle ν is null because the submanifold M^n is mixed totally geodesic.

B. Sahin constructed a basic simple inequality for warped product hemi-slant submanifolds in Kaehler manifolds (Sahin, 2009), which contains the squared norm of the second fundamental form and the gradient of $\ln f$. Here, we extend this inequality to the setting of nearly Kaehler manifolds.

Theorem 5.3.3. *Let $M^n = N_\theta \times {}_f N_\perp$ be a mixed totally geodesic hemi-slant warped product submanifold into a nearly Kaehler manifold $\tilde{M}^{2m}$ such that N_θ and $N_\perp$ are proper slant and totally real submanifolds of dimensions $2s = n_1$ and n_2, respectively. Then*

(i) The second fundamental form of M^n satisfies the following inequality

$$||h||^2 \geq n_2 \cot^2\theta ||\nabla(\ln f)||^2.$$

(ii) If the equality in (i) holds, then N_θ is totally geodesic submanifold in $\tilde{M}^{2m}$.

Proof. Since M^n is mixed totally geodesic, the second fundamental form can be written as

$$||h||^2 = ||h(\mathcal{D}_\theta, \mathcal{D}_\theta)||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)||^2.$$

By the linearity of the Hermitian metric, the above formula is identical to

$$||h||^2 = ||h(\mathcal{D}_\theta, \mathcal{D}_\theta)_{F\mathcal{D}_\theta}||^2 + ||h(\mathcal{D}_\theta, \mathcal{D}_\theta)_{F\mathcal{D}_\perp}||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)_{F\mathcal{D}_\theta}||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)_{F\mathcal{D}_\perp}||^2.$$

The above equation implies

$$||h||^2 \geq ||h(\mathcal{D}_\theta, \mathcal{D}_\theta)_{F\mathcal{D}_\perp}||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)_{F\mathcal{D}_\theta}||^2.$$

In view of Lemma 4.3.1 (ii), taking in consideration the condition of mixed totally geodesy in our hypothesis, the first term on the right hand side of the above inequality vanishes identically. Hence, the above inequality becomes

$$||h||^2 \geq ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)_{F\mathcal{D}_\theta}||^2.$$

Thus, this can be written as

$$||h||^2 \geq \sum_{a=1}^{n_1} \sum_{A,B=n_1+1}^{n} g\big(h(e_A, e_B), Fe_a\big)^2 = \sum_{r=1}^{2s} \sum_{A,B=n_1+1}^{n} g\big(h(e_A, e_B), \bar{e}_r\big)^2.$$

In virtue of the adapted frame of $F\mathcal{D}_\theta$, the above inequality may be expressed as

$$||h||^2 \geq \csc^2\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} g\big(h(e_A, e_B), Fe_a\big)^2$$
$$+ \sec^2\theta \csc^2\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} g\big(h(e_A, e_B), FPe_a\big)^2.$$

Evaluating the right hand side from Lemma 4.3.1 (i), taking into consideration that our warped product submanifold is mixed totally geodesic, so we get

$$||h||^2 \geq \csc^2\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} \big(Pe_a \ln f\big)^2 g(e_A, e_B)^2$$
$$+ \sec^2\theta \csc^2\theta \cos^4\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} \big(e_a \ln f\big)^2 g(e_A, e_B)^2.$$

This directly gives

$$||h||^2 \geq \csc^2\theta \cos^2\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} \big(\sec\theta\, Pe_a \ln f\big)^2 g(e_A, e_B)^2$$

$$+\csc^2\theta\cos^2\theta\sum_{a=1}^{s}\sum_{A,B=n_1+1}^{n}\big(e_a\ln f\big)^2 g(e_A,e_B)^2.$$

Equivalently

$$||h||^2\geq\cot^2\theta\sum_{a=1}^{2s}\sum_{A,B=n_1+1}^{n}\big(e_a\ln f\big)^2 g(e_A,e_B)^2.$$

Making use of (2.3.21), the inequality of (i) follows immediately.

For statement (ii), notice that the inequality of statement (i) has been obtained by computing $||h(\mathcal{D}_\perp,\mathcal{D}_\perp)_{F\mathcal{D}_\theta}||^2$. Therefore, if the equality in (i) holds, then

$$||h(\mathcal{D}_\theta,\mathcal{D}_\theta)_{F\mathcal{D}_\theta}||^2=||h(\mathcal{D}_\theta,\mathcal{D}_\theta)_{F\mathcal{D}_\perp}||^2=||h(\mathcal{D}_\perp,\mathcal{D}_\perp)_{F\mathcal{D}_\perp}||^2=0.$$

Now, since the ν subbundle is null for mixed totally geodesic hemi-slant warped product submanifolds, the conditions

$$||h(\mathcal{D}_\theta,\mathcal{D}_\theta)_{F\mathcal{D}_\theta}||^2=||h(\mathcal{D}_\theta,\mathcal{D}_\theta)_{F\mathcal{D}_\perp}||^2=0$$

imply that

$$h(\mathcal{D}_\theta,\mathcal{D}_\theta)=0. \tag{5.3.11}$$

From Corollary 2.3.1, we know that N_θ is totally geodesic in M^n. Thus, Corollary 2.3.1 and (5.3.11) prove that N_θ is totally geodesic submanifold in $\tilde{M}^{2m}$. □

5.4 MODIFIED INEQUALITIES IN ALMOST CONTACT MANIFOLDS

Recently, we proved an inequality that generalizes all first inequalities of h in almost contact manifolds (Theorem 4.1, (Mustafa et al., 2013)).

Theorem 5.4.1. *(Mustafa et al., 2013). Let $M^n=N_T\times{}_fN_\perp$ be a contact CR-warped product submanifold of a nearly trans-Sasakian manifold such that ξ is tangent to the first factor, where N_T and $N_\perp$ are invariant and anti-invariant submanifolds, of dimensins n_1 and n_2, respectively. Then, we have*

(i) $||h||^2\geq 2n_2[||\nabla\ln f||^2+\alpha^2-\beta^2]$.

(ii) If the equality in (i) holds, then N_T, $N_\perp$ and M^n are respectively totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2l+1}$.

Putting $\beta = 0$ in the above theorem, a special case of the inequality in α-Sasakian and nearly α-Sasakian is obtained

$$||h||^2 \geq 2n_2[||\nabla \ln f||^2 + \alpha^2].$$

Taking $\alpha = 1$, the inequality of Sasakian manifolds in (Hesegawa & Mihai, 2003) and (Munteanu, 2005) follows immediately with equality sign typically as in the above theorem.

Similarly, for β-Kenmotsu and nearly β-Kenmotsu manifolds, we derive the following inequality from the above theorem

$$||h||^2 \geq 2n_2[||\nabla \ln f||^2 - \beta^2], \tag{5.4.1}$$

which gives the first inequality of Kenmotsu in (Arsalan et al., 2005).

Analogously, if we take both functions α and β to be zeros, we can successfully use Theorem 5.4.1 to obtain similar inequalities for cosymplectic and nearly cosymplectic manifolds; that is

$$||h||^2 \geq 2n_2||\nabla \ln f||^2.$$

It is straightforward to follow analogous scheme as that in the proof of Theorem 5.3.2 to get the following modification for the Theorem 5.4.1.

Theorem 5.4.2. *Let $M^n = N_T \times {}_f N_\perp$ be a contact CR-warped product submanifold of a nearly trans-Sasakian manifold such that ξ is tangent to the first factor, where N_T and $N_\perp$ are invariant and anti-invariant submanifolds, of dimensions $n_1 = 2s + 1$ and n_2, respectively. Then, we have*

(i) $||h||^2 \geq 2n_2[||\nabla \ln f||^2 + \alpha^2 - \beta^2]$.

(ii) The equality sign in (i) holds identically if and only if N_T, $N_\perp$ and M^n are respectively totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2l+1}$, and

$$n_2[\Delta(\ln f) + \alpha^2 - \beta^2] = \tilde{\tau}(T_x M^n) - \tilde{\tau}(T_x N_T) - \tilde{\tau}(T_x N_\perp). \tag{5.4.2}$$

It is worth pointing out that, particular case inequalities for Sasakian, Kenmotsu and cosymplectics can easily be derived from the above inequality. This can be achieved typically as we did for Theorem 5.4.1. These inequalities are valid for the nearly structures of Sasakian, Kenmotsu and cosymplectic also.

As an answer to Problem 1.4.8, we established a geometric inequality for semi-slant warped product submanifolds in nearly trans-Sasakian manifolds (Mustafa et al.,2014). In fact the next two theorems are extensions of Theorems 5.4.1 and 5.4.2, respectively. For this, the local orthonormal frame of the contact semi-slant submanifolds is constructed in the following way: Let $\tilde{M}^{2l+1}$ be an almost contact manifold and $M^n = N_T \times N_\theta$ be a semi-slant submanifold of $\tilde{M}^{2l+1}$ such that the characteristic vector field ξ is tangent to N_T. If $dimN_T = 2s+1 = n_1$ and $dimN_\theta = 2q = n_2$, then $n = 2s+1+2q = n_1+n_2$. Suppose $\{e_o = \xi, e_1, \cdots, e_s, e_{s+1} = \phi e_1, \cdots, e_{2s} = \phi e_s\}$ is a local orthonormal frame of $\mathcal{D}_T$ and $\{e_{n_1+1} = e_1^\star, \cdots, e_q^\star, e_{q+1}^\star = \sec\theta P e_1^\star, \cdots, e_{2q}^\star = \sec\theta P e_q^\star = e_n\}$ is a local orthonormal frame for $\mathcal{D}_\theta$. Then the local orthonormal frames in the normal bundle $T^\perp M^n$ of $F\mathcal{D}_\theta$ and the invariant normal subbundle ν are respectively $\{e_{n+1} = \bar{e}_1 = \csc\theta F e_1^\star, \cdots, \bar{e}_q = \csc\theta F e_q^\star, \bar{e}_{q+1} = \csc\theta\sec\theta F P e_1^\star, \cdots, \bar{e}_{2q} = \csc\theta\sec\theta F P e_q^\star = e_{n+2q}\}$ and $\{e_{n+2q+1} = \bar{e}_{2q+1}, \cdots, e_{2l+1} = \bar{e}_{2(q+\gamma)}\}$. It is obvious that the dimensions of $F\mathcal{D}_\theta$ and ν are $2q$ and 2γ, respectively. This comes from the fact that the dimension of slant and invariant submanifolds is always even.

If ν is the maximal invariant subbundle of the normal bundle $T^\perp M^n$, then in the case of semi-slant submanifold, the normal bundle $T^\perp M^n$ has the following decomposition

$$T^\perp M^n = F\mathcal{D}_\theta \oplus \nu. \tag{5.4.3}$$

Now, we are going to state and prove the following inequality.

Theorem 5.4.3. *Let $M^n = N_T \times {}_f N_\theta$ be a semi-slant warped product submanifold into a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ such that N_T and N_θ are invariant and proper slant submanifolds of dimensions $2s+1$ and $2q$ respectively, where ξ is tangent to the first factor. Then,*

(i) The second fundamental form of M^n satisfies the following inequality

$$||h||^2 \geq 2n_2\{(\frac{1}{9}\cot^2\theta + \csc^2\theta)[||\nabla \ln f||^2 - \beta^2] + \alpha^2\}.$$

(ii) If the equality in (i) holds, then N_T, N_θ and M^n are totally geodesic, totally umbilical and minimal in $\tilde{M}^{2l+1}$, respectively.

Proof. In view of the above adapted frame, and the definition of the second fundamental form, it is straightforward to get the following expansion

$$\begin{aligned}
||h||^2 &= \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n} g(h(e_i,e_j),e_r)^2 \\
&= \sum_{r=1}^{2q}\sum_{i,j=1}^{n} g(h(e_i,e_j),\bar{e}_r)^2 + \sum_{r=2q+1}^{2(q+\gamma)}\sum_{i,j=1}^{n} g(h(e_i,e_j),\bar{e}_r)^2 \\
&\geq \sum_{r=1}^{2q}\sum_{i,j=1}^{n} g(h(e_i,e_j),\bar{e}_r)^2 \\
&= \sum_{r=1}^{2q}\sum_{a,b=1}^{2s+1} g(h(e_a,e_b),\bar{e}_r)^2 + 2\sum_{r=1}^{2q}\sum_{a=1}^{2s+1}\sum_{A=1}^{2q} g(h(e_a,e_A^\star),\bar{e}_r)^2 \\
&\quad + \sum_{r=1}^{2q}\sum_{D,B=1}^{2q} g(h(e_D^\star,e_B^\star),\bar{e}_r)^2.
\end{aligned}$$

In view of Lemma 4.2.3 (ii), the first term in the right hand side of the last equality is identically zero, so let us compute the next term

$$\begin{aligned}
2\sum_{r=1}^{2q}\sum_{a=1}^{2s+1}\sum_{A=1}^{2q} g(h(e_a,e_A^\star),\bar{e}_r)^2 &= 2\sum_{r=1}^{2q}\sum_{a=1}^{2s}\sum_{A=1}^{2q} g(h(e_a,e_A^\star),\bar{e}_r)^2 \\
&\quad + \ 2\sum_{r=1}^{2q}\sum_{A=1}^{2q} g(h(\xi,e_A^\star),\bar{e}_r)^2.
\end{aligned}$$

By Lemma 4.2.3 (iii), we can evaluate the second term in the right hand side of the above relation, while applying the local orthonormal frame to the first term, the above expression takes the following form

$$\begin{aligned}
2\sum_{r=1}^{2q}\sum_{a=1}^{2s}\sum_{A=1}^{2q} g(h(e_a,e_A^\star),\bar{e}_r)^2 &= 2\csc^2\theta \sum_{A,B=1}^{q}\sum_{a=1}^{2s} g(h(e_a,e_A^\star),Fe_B^\star)^2 \\
&\quad + \ 2\csc^2\theta\sec^2\theta \sum_{A,B=1}^{q}\sum_{a=1}^{2s} g(h(e_a,Pe_A^\star),Fe_B^\star)^2 \\
&\quad + \ 2\csc^2\theta\sec^2\theta \sum_{A,B=1}^{q}\sum_{a=1}^{2s} g(h(e_a,e_A^\star),FPe_B^\star)^2 \\
&\quad + \ 2\csc^2\theta\sec^4\theta \sum_{A,B=1}^{q}\sum_{a=1}^{2s} g(h(e_a,Pe_A^\star),FPe_B^\star)^2 \\
&\quad + 2\sum_{A,B=2s+1}^{n} (-\alpha g(e_A^\star,e_B^\star))^2.
\end{aligned}$$

Now, the first four inner products on the right hand side can be evaluated by means of the last four parts of Lemma 4.2.3. Hence, we get

$$2\sum_{r=1}^{2q}\sum_{a=1}^{2s}\sum_{A=1}^{2q} g(h(e_a, e_A^\star), \bar{e}_r)^2$$

$$= 2\csc^2\theta \sum_{A,B=1}^{q}\sum_{a=1}^{2s}\Big(\frac{1}{3}\{(e_a \ln f) - \beta\eta(e_a)\}g(Pe_A^\star, e_B^\star)$$

$$-\{(\phi e_a \ln f) + \alpha\eta(e_a)\}g(e_A^\star, e_B^\star)\Big)^2$$

$$+ 2\csc^2\theta\sec^2\theta \sum_{A,B=1}^{q}\sum_{a=1}^{2s}\Big(\frac{1}{3}\cos^2\theta\{(e_a \ln f) - \beta\eta(e_a)\}g(e_A^\star, e_B^\star)$$

$$-\{(\phi e_a \ln f) + \alpha\eta(e_a)\}g(Pe_A^\star, e_B^\star)\Big)^2$$

$$+ 2\csc^2\theta\sec^2\theta \sum_{A,B=1}^{q}\sum_{a=1}^{2s}\Big(\frac{1}{3}\cos^2\theta\{(e_a \ln f) - \beta\eta(e_a)\}g(e_A^\star, e_B^\star)$$

$$-\{(\phi e_a \ln f) + \alpha\eta(e_a)\}g(e_A^\star, Pe_B^\star)\Big)^2$$

$$+2\csc^2\theta\sec^4\theta \sum_{A,B=1}^{q}\sum_{a=1}^{2s}\Big(-\frac{1}{3}\cos^2\theta\{(e_a \ln f) - \beta\eta(e_a)\}g(e_A^\star, Pe_B^\star)$$

$$-\cos^2\theta\{(\phi e_a \ln f) + \alpha\eta(e_a)\}g(e_A^\star, e_B^\star)\Big)^2$$

$$+ 4q\alpha^2.$$

On one hand, all terms which have $\eta(e_a)$ cancel. This is because all orthonormal vector fields from the set $\{e_1, \cdots, e_{2s}\}$ are orthogonal to ξ. On the other hand, the terms which contain $g(Pe_A^\star, e_B^\star)$ also vanish, since in view of the local fields of orthonormal frame of $\Gamma(TN_\theta)$, we know that every $Pe_A^\star$ and $e_B^\star$ are orthogonal, where A, B run over $\{e_1^\star, \cdots, e_q^\star\}$. Consequently, the above equation descends to

$$2\sum_{r=1}^{2q}\sum_{a=1}^{2s}\sum_{A=1}^{2q} g(h(e_a, e_A^\star), \bar{e}_r)^2 = 2\csc^2\theta \sum_{A=1}^{q}\sum_{a=1}^{2s}(-(\phi e_a \ln f)||e_A^\star||^2)^2$$

$$+\ 2\csc^2\theta\sec^2\theta \sum_{A=1}^{q}\sum_{a=1}^{2s}(\frac{1}{3}\cos^2\theta(e_a \ln f)||e_A^\star||^2)^2$$

$$+\ 2\csc^2\theta\sec^2\theta \sum_{A=1}^{q}\sum_{a=1}^{2s}(\frac{1}{3}\cos^2\theta(e_a \ln f)||e_A^\star||^2)^2$$

$$+ \ 2\csc^2\theta\sec^4\theta\sum_{A=1}^{q}\sum_{a=1}^{2s}(-\cos^2\theta(\phi e_a\ln f)||e_A^\star||^2)^2$$
$$+4q\alpha^2.$$

Observe that a runs from 1 to $2s$, thus in order to fulfill the formula of $||\nabla\ln f||^2$ obtained in (2.3.23), we should add the term $(\xi\ln f)^2$ with appropriate coefficients. Hence, adding and subtracting such terms, taking into account Lemma 4.2.3 (i), by straightforward computations, we can reach

$$||h||^2 \geq 2n_2\{(\frac{1}{9}\cot^2\theta+\csc^2\theta)[||\nabla\ln f||^2-\beta^2]+\alpha^2\}.$$

For part (ii), one can refer to the proof of the inequality in (Mustafa et al., 2013), or similar inequalities in this field. □

It is trivial to get the inequality in (Uddin et al., 2014) from the above one. Moreover, many other inequalities of the same type can be derived from this inequality, specially in Kenmotsu manifolds.

Now, we will modify the above theorem by figuring out the necessary and sufficient conditions of the equality case. Thus, we have

Theorem 5.4.4. *Let $M^n = N_T \times {}_f N_\theta$ be a semi-slant warped product submanifold into a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ such that N_T and N_θ are invariant and proper slant submanifolds of dimensions n_1 and n_2 respectively, where ξ is tangent to the first factor. Then,*

(i) The second fundamental form of M^n satisfies the following inequality

$$||h||^2 \geq 2n_2\{(\frac{1}{9}\cot^2\theta+\csc^2\theta)[||\nabla\ln f||^2-\beta^2]+\alpha^2\}.$$

(ii) The equality in (i) holds if and only if N_T, N_θ and M^n are totally geodesic, totally umbilical and minimal in $\tilde{M}^{2l+1}$, respectively, and

$$n_2[\Delta(\ln f)+\alpha^2-\beta^2] = \tilde{\tau}(T_xM^n)-\tilde{\tau}(T_xN_T)-\tilde{\tau}(T_xN_\perp). \tag{5.4.4}$$

Proof. Statement (i) was proved in Theorem 5.4.3. For (ii), the proof is as similar as Theorem 5.3.2. □

We notice that, the above theorem is a natural generalization of Theorem 5.4.2, since it is valid for any proper semi-slant warped product submanifold. Hence if we let θ to be $\frac{\pi}{2}$, then Theorem 5.4.2 follows directly.

At the end of this section, we present a contact extension of Theorem 5.3.3.

Theorem 5.4.5. *Let $M^n = N_\theta \times {}_f N_\perp$ be a mixed totally geodesic hemi-slant warped product submanifold into a nearly trans-Sasakian manifold $\tilde{M}^{2l+1}$ such that N_θ and $N_\perp$ are proper slant and totally real submanifolds of dimensions n_1 and n_2, respectively. Then,*

(i) The second fundamental form of M^n satisfies the following inequality

$$||h||^2 \geq n_2 \cot^2\theta\{||\nabla(\ln f)||^2 - \beta^2\}.$$

(ii) If the equality in (i) holds, then N_θ is totally geodesic submanifold in $\tilde{M}^{2l+1}$.

Proof. Let $\tilde{M}^{2l+1}$ be an almost contact manifold and $M^n = N_\theta \times N_\perp$ be mixed totally geodesic hemi-slant submanifold of $\tilde{M}^{2l+1}$ such that ξ is tangent to the first factor. If $dimN_\theta = 2s+1 = n_1$ and $dimN_\perp = n_2$, then $n = 2s+1+n_2$. Suppose $\{e_0 = \xi, e_1, \cdots, e_s, e_{s+1} = \sec\theta Pe_1, \cdots, e_{2s} = \sec\theta Pe_s = e_{n_1}\}$ is a local orthonormal frame for $\mathcal{D}_\theta$, and $\{e_{n_1+1}, \cdots$
$, e_{n_1+n_2} = e_n\}$ is a local orthonormal frame of $\mathcal{D}_\perp$. Then, the local orthonormal frames of $F\mathcal{D}_\theta$ and $F\mathcal{D}_\perp$ are respectively $\{e_{n+1} = \bar{e}_1 = \csc\theta Fe_1, \cdots, \bar{e}_s = \csc\theta Fe_s,$
$\bar{e}_{s+1} = \csc\theta\sec\theta FPe_1, \cdots, \bar{e}_{2s} = \csc\theta\sec\theta FPe_s = e_{4s+1+n_2}\}$ and $\{e_{4s+n_2+2} = \bar{e}_{2s+1}, \cdots, e_{4s+2n_2+1} = \bar{e}_{2s+n_2}\}$. It is not difficult to show that, the normal subbundle ν is empty because the submanifold M^n is mixed totally geodesic.

Since M^n is mixed totally geodesic, the second fundamental form can be written as

$$||h||^2 = ||h(\mathcal{D}_\theta, \mathcal{D}_\theta)||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)||^2.$$

The above formula can be extended to the following form

$$||h||^2 = ||h(\mathcal{D}_\theta, \mathcal{D}_\theta)_{F\mathcal{D}_\theta}||^2 + ||h(\mathcal{D}_\theta, \mathcal{D}_\theta)_{F\mathcal{D}_\perp}||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)_{F\mathcal{D}_\theta}||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)_{F\mathcal{D}_\perp}||^2.$$

The above equation implies

$$||h||^2 \geq ||h(\mathcal{D}_\theta, \mathcal{D}_\theta)_{F\mathcal{D}_\perp}||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)_{F\mathcal{D}_\theta}||^2.$$

Making use of Lemma 4.3.2 (ii), and notice that our warped product submanifold is mixed totally geodesic, then the first term on the right hand side of the above inequality vanishes identically. Hence, the above inequality descends to

$$||h||^2 \geq ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)_{F\mathcal{D}_\theta}||^2.$$

Thus, this can be written as

$$||h||^2 \geq \sum_{a=1}^{n_1} \sum_{A,B=n_1+1}^{n} g\big(h(e_A, e_B), Fe_a\big)^2 = \sum_{r=1}^{2s} \sum_{A,B=n_1+1}^{n} g\big(h(e_A, e_B), \bar{e}_r\big)^2.$$

In virtue of the adapted frame of $F\mathcal{D}_\theta$, the above inequality may be expressed as

$$||h||^2 \geq \csc^2\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} g\big(h(e_A, e_B), Fe_a\big)^2$$

$$+ \sec^2\theta \csc^2\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} g\big(h(e_A, e_B), FPe_a\big)^2.$$

Evaluating the right hand side from Lemma 4.3.2 (i), taking into consideration that our warped product submanifold is mixed totally geodesic, so we get

$$||h||^2 \geq \csc^2\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} \big((Pe_a \ln f) + \alpha\eta(e_a)\big)^2 g(e_A, e_B)^2$$

$$+ \sec^2\theta \csc^2\theta \cos^4\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} \big(e_a \ln f\big)^2 g(e_A, e_B)^2.$$

Since all vector fields e_a, where $a \in \{1, \cdots, 2s\}$, are orthogonal to ξ, the above inequality takes the form

$$||h||^2 \geq \csc^2\theta \cos^2\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} \big(\sec\theta\; Pe_a \ln f\big)^2 g(e_A, e_B)^2$$

$$+ \csc^2\theta \cos^2\theta \sum_{a=1}^{s} \sum_{A,B=n_1+1}^{n} \big(e_a \ln f\big)^2 g(e_A, e_B)^2.$$

Equivalently

$$||h||^2 \geq \cot^2\theta \sum_{a=1}^{2s} \sum_{A,B=n_1+1}^{n} \big(e_a \ln f\big)^2 g(e_A, e_B)^2.$$

From Theorem 3.3.5 (i), we know that $\xi \ln f = \beta$ for any warped product submanifold in nearly trans-Sasakian manifold. Thus, adding and subtracting $(\xi \ln f)^2$ with appropriate coefficients from the above inequality, we obtain (i) immediately, while (ii) follows exactly as same as that of Theorem 5.3.3. $\square$

Observe that the function α does not appear in the above inequality. This is coherent with Proposition 3.3.1 which says that: mixed totally geodesic warped product submanifolds do not exist in α-Sasakian manifolds. Therefore, the above inequality is not valid for Sasakian manifolds also.

5.4.1 CONCLUSION

In what follows we present three tables of inequalities. The first two summarize all first inequalities of h for CR-warped product and semi-slant warped product submanifolds, respectively, whereas the third table presents another type of inequalities for hemi-slant submanifolds in both almost Hermitian and almost contact manifolds.

Firstly, the first inequalities of h for CR-warped product submanifolds are summarized in the following table:

Manifold	Inequality
Kaehler	$\|h\|\|^2 \geq \left(2n_2\|\|\nabla(\ln f)\|\|^2\right)$
Nearly Kaehler	$\|h\|\|^2 \geq \left(2n_2\|\|\nabla(\ln f)\|\|^2\right)$
Nearly trans-Sasakian	$\|\|h\|\|^2 \geq 2n_2\left(\|\|\nabla \ln f\|\|^2 + \alpha^2 - \beta^2\right)$
Nearly α-Sasakian	$\|\|h\|\|^2 \geq 2n_2\left(\|\|\nabla(\ln f)\|\|^2 + \alpha^2\right)$
Sasakian	$\|\|h\|\|^2 \geq 2n_2\left(\|\|\nabla(\ln f)\|\|^2 + 1\right)$
Nearly β-Kenmotsu	$\|\|h\|\|^2 \geq 2n_2\left(\|\|\nabla \ln f\|\|^2 - \beta^2\right)$
Kenmotsu	$\|\|h\|\|^2 \geq 2n_2\left(\|\|\nabla \ln f\|\|^2 - 1\right)$
Nearly cosymplectic	$\|\|h\|\|^2 \geq \left(2n_2\|\|\nabla(\ln f)\|\|^2\right)$
Cosymplectic	$\|\|h\|\|^2 \geq \left(2n_2\|\|\nabla(\ln f)\|\|^2\right)$

Table 5.1: First inequality of h for CR-warped product submanifolds of type $N_T \times_f N_\perp$.

Next, the first inequalities of h for semi-slant warped product submanifolds of type $N_T \times_f N_\theta$ and the inequality of mixed totally geodesic hemi-slant warped product submanifolds of the type $N_\theta \times_f N_\perp$ are, respectively, summarized in the following tables:

Manifold	Inequality
Nearly Kaehler	$|h||^2 \geq 2n_2\left(\frac{1}{9}\cot^2\theta + \csc^2\theta\right)||\nabla \ln f||^2$
Nearly trans-Sasakian	$||h||^2 \geq 2n_2\left\{\left(\frac{1}{9}\cot^2\theta + \csc^2\theta\right)\left[||\nabla \ln f||^2 - \beta^2\right] + \alpha^2\right\}$
Nearly α-Sasakian	$||h||^2 \geq 2n_2\left\{\left(\frac{1}{9}\cot^2\theta + \csc^2\theta\right)||\nabla \ln f||^2 + \alpha^2\right\}$
Nearly β-Kenmotsu	$||h||^2 \geq 2n_2\left(\frac{1}{9}\cot^2\theta + \csc^2\theta\right)\left[||\nabla \ln f||^2 - \beta^2\right]$
Kenmotsu	$||h||^2 \geq 2n_2\left(\frac{1}{9}\cot^2\theta + \csc^2\theta\right)\left[||\nabla \ln f||^2 - 1\right]$
Nearly cosymplectic	$||h||^2 \geq 2n_2\left(\frac{1}{9}\cot^2\theta + \csc^2\theta\right)||\nabla \ln f||^2$

Table 5.2: First inequality of h for semi-slant warped product submanifolds of type $N_T \times_f N_\theta$.

Manifold	Inequality
Kaehler	$|h||^2 \geq \left(n_2\cot^2\theta||\nabla(\ln f)||^2\right)$
Nearly Kaehler	$|h||^2 \geq \left(n_2\cot^2\theta||\nabla(\ln f)||^2\right)$
Nearly trans-Sasakian	$||h||^2 \geq n_2\cot^2\theta\left\{||\nabla(\ln f)||^2 - \beta^2\right\}$
Nearly β-Kenmotsu	$||h||^2 \geq n_2\cot^2\theta\left\{||\nabla(\ln f)||^2 - \beta^2\right\}$
Kenmotsu	$||h||^2 \geq n_2\cot^2\theta\left\{||\nabla(\ln f)||^2 - 1\right\}$
Nearly cosymplectic	$||h||^2 \geq \left(n_2\cot^2\theta||\nabla(\ln f)||^2\right)$
Cosymplectic	$||h||^2 \geq \left(n_2\cot^2\theta||\nabla(\ln f)||^2\right)$

Table 5.3: An inequality of h for mixed totally geodesic hemi-slant warped product submanifolds of type $N_\theta \times_f N_\perp$.

CHAPTER 6: $\mathcal{D}_i$-MINIMALITY FOR GENERAL SECOND INEQUALITY OF h

6.1 INTRODUCTION

In the previous chapter we proved several kinds of inequalities in terms of h. Since those inequalities come from computing the $F\mathcal{D}_\perp$-component of $h(\mathcal{D}_T, \mathcal{D}_\perp)$, they were classified under the first inequality of h category. In the current chapter, we intend to construct another two inequalities of h by computing both components of $h(\mathcal{D}_T, \mathcal{D}_\perp)$; $F\mathcal{D}_\perp$-component and ν-component. To distinguish them from the first category, we call the inequalities of this chapter the second inequalities of h.

It is worth pointing out that, the second inequality of h was first proved for CR-warped product submanifolds in complex space forms (Chen, 2003). After that, it was extended to contact CR-warped product submanifolds in Sasakian space forms (Munteanu, 2005).

In spite of proving that these two inequalities are optimal inequalities, which is done by concrete examples satisfying the equality cases and characterization theorems, the equality case was, surprisingly, not discussed in both papers. Even though the direction of sufficiency was clear, the direction of necessity was not. This is because of $\mathcal{D}_i$-minimality which was not proved at that time. Since we showed that warped product submanifolds of these two papers are $\mathcal{D}_1$-minimal, it becomes possible to discuss the equality case in both directions.

Inspired by (Chen, 2003) and (Munteanu, 2005), we first extend the second inequality of h for Kenmotsu space forms by means of Codazzi equation in the second section. Another more general proof of this inequality is presented in the next section using the Gauss equation, where the inequality of Kenmotsu manifolds becomes a special case of this general inequality. This makes the two methods more precise and coherent.

In the third section, a general inequality of h is established for any $\mathcal{D}_i$-minimal warped product submanifold $M^n = N_1 \times_f N_2$ isometrically immersed in an arbitrary Riemannian manifold $\tilde{M}^m$. This inequality generalizes all inequalities in the second section of this chapter. Moreover, this inequality provides us with enough new inequalities which we, and other geometers, couldn't prove by Chen's method in (Chen, 2003). In Table 6.2, we list some of these inequalities, and we show how to derive them there. By the same way, this inequality can be derived for any $\mathcal{D}_i$-minimal warped product submanifold. For example, similar inequalities may be derived for any nearly structure in both almost Hermitian and almost contact ambient manifolds, and for CR, semi-slant and generic warped product submanifolds.

6.2 THE SECOND INEQUALITY OF h

By means of Codazzi equation, Chen proved the following inequality of CR-warped product submanifolds in a complex space form (Chen, 2003). As mentioned before, Chen provided an example showing that this inequality is optimal (Chen, 2008). More over, warped product submanifolds satisfying the equality case were completely characterized in (Chen, 2003).

On contrast, the equality case was not discussed. This is because the direction of necessity had been not clear yet. This because of $\mathcal{D}_i$-minimality which was not proved at that time. Since we showed that CR-warped product submanifolds of the type $N_T \times_f N_\perp$ are $\mathcal{D}_1$-minimal, it becomes possible to discuss the equality case in both directions. Here, we figure out necessary and sufficient conditions for the equality case.

Theorem 6.2.1. *Let $M^n = N_T \times_f N_\perp$ be a CR-warped product submanifold in a complex space form $\tilde{M}^{2m}(c_{Ka})$. Then, the following hold*

(i) $\frac{1}{2}||h||^2 \geq n_1 n_2 \frac{c_{Ka}}{4} + n_2||\nabla \ln f||^2 - n_2\, \Delta(\ln f)$.

(ii) The equality sign in (i) holds if and only if N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2m}(c_{Ka})$, respectively.

Proof. (i) was proved in (Chen, 2003). For (ii), if we first put $PZ = 0$ in Proposition 3.2.1 (iv), then it can be proved, as in (Chen, 2003), that

$$||h(\mathcal{D}_T, \mathcal{D}_\perp)_{F\mathcal{D}_\perp}||^2 = n_2||\nabla \ln f||^2. \tag{6.2.1}$$

Moreover, it is explicitly shown in (Chen, 2003), by a method derived via Codazzi equation, that

$$||h(\mathcal{D}_T, \mathcal{D}_\perp)_\nu||^2 = n_1 n_2 \frac{c_{Ka}}{4} - n_2\, \Delta(\ln f). \tag{6.2.2}$$

Combining $(6.2.1)$ and $(6.2.2)$ together, it obviously yields

$$||h(\mathcal{D}_T, \mathcal{D}_\perp)||^2 = n_1 n_2 \frac{c_{Ka}}{4} - n_2\, \Delta(\ln f) + n_2 ||\nabla \ln f||^2. \tag{6.2.3}$$

Now, it is straightforward to expand $||h||^2$ as

$$||h||^2 = ||h(\mathcal{D}_T, \mathcal{D}_T)||^2 + 2\, ||h(\mathcal{D}_T, \mathcal{D}_\perp)||^2 + ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)||^2. \tag{6.2.4}$$

Now, for the sufficiency, assume that the equality holds in (i); i.e.,

$$\frac{1}{2}||h||^2 = n_1 n_2 \frac{c_{Ka}}{4} + n_2 ||\nabla \ln f||^2 - n_2\, \Delta(\ln f). \tag{6.2.5}$$

From $(6.2.3)$-$(6.2.5)$, it automatically gives

$$||h(\mathcal{D}_T, \mathcal{D}_T)||^2 = ||h(\mathcal{D}_\perp, \mathcal{D}_\perp)||^2 = 0.$$

Therefore, as in (Chen, 2001), we can show that N_T, $N_\perp$ and M^n are respectively totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2m}(c_{Ka})$.

Conversely, assume that $N_T \times_f N_\perp$ is a minimal warped product submanifold in Kaehler manifolds $\tilde{M}^{2m}(c_{Ka})$, where N_T and $N_\perp$ are respectively totally geodesic and totally umbilical submanifolds in $\tilde{M}^{2m}(c_{Ka})$.

From the above assumption and Corollary 2.3.1, we conclude that N_T is totally geodesic in both M^n and $\tilde{M}^{2m}(c_{Ka})$. This implies

$$h(\mathcal{D}_T, \mathcal{D}_T) = 0. \tag{6.2.6}$$

By Lemma 5.2.1, it is proved that M^n is always $\mathcal{D}_1$-minimal in $\tilde{M}^{2m}(c_{Ka})$. Since M^n is assumed to be a minimal submanifold in $\tilde{M}^{2m}(c_{Ka})$, M^n is also $\mathcal{D}_2$-minimal in $\tilde{M}^{2m}(c_{Ka})$. By the assumption, $N_\perp$ is totally umbilical in $\tilde{M}^{2m}(c_{Ka})$, then the hypothesis of Lemma 5.2.6 is satisfied. Hence, it implies that M^n is $\mathcal{D}_2$-totally geodesic in $\tilde{M}^{2m}(c_{Ka})$, which means

$$h(\mathcal{D}_\perp, \mathcal{D}_\perp) = 0. \tag{6.2.7}$$

Therefore, the equality in (ii) holds by $(6.2.6)$ and $(6.2.7)$. This completes the proof.

□

In the sequel, M.I. Munteanu proved a similar inequality for contact CR-warped products in Sasakian space forms (see Theorem 3.3 in (Munteanu, 2005)). Moreover, a solid example constructed there shows that the next inequality is sharp inequality. However, he did not discuss necessary conditions for the equality case of this inequality. Fortunately, necessary and sufficient conditions for the equality case can be easily demonstrated by using Lemma 5.2.6. This is because contact CR-warped product submanifold of type $N_T \times_f N_\perp$ is $\mathcal{D}_1$-minimal. Hence, statement (ii) of the next theorem is proved in the same way as (ii) of the previous theorem.

Theorem 6.2.2. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product of a Sasakian space form $\tilde{M}^{2l+1}(c_S)$ and let $h = 2s + 1 = dim N_T$ and $n_2 = dim N_\perp$. Then, the following hold*

(i) $||h||^2 \geq 2n_2\bigg(||\nabla \ln f||^2 - \Delta \ln f + \frac{c_S+3}{2}s + 1\bigg).$

(ii) The equality sign in (i) holds if and only if N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2l+1}(c_S)$, respectively.

Now, we are going to extend the above inequality for contact CR-warped products in Kenmotsu space forms. It is worth pointing out that, our next theorem corrects the statement and the proof of the second inequality of (Arsalan et al., 2005). Also, another proof of this inequality is provided in the next section where the following inequality becomes a special case of Theorem 6.3.1.

For this end, we present some preparatory lemmas which were not proved and considered in (Arsalan et al., 2005). Errors of their statement and proof are due to neglecting these lemmas.

We point out that, the missing terms in (Arsalan et al., 2005) come from not considering Lemmas 6.2.1 and 6.2.4, while Lemmas 6.2.2 and 6.2.3 show where the ν-component of h comes from.

Lemma 6.2.1. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a Kenmotsu manifold $\tilde{M}^{2l+1}$. Then, we have*

(i) $\nabla_X \phi X = \phi \nabla_X X;$

(ii) $g(\nabla_X X, \xi) = -||X||^2,$

where $X \in \Gamma(TN_T)$ and orthogonal to ξ.

Proof. Given any $X \in \Gamma(TN_T)$ orthogonal to ξ. One may use (2.3.27) and (2.3.57) to derive

$$\nabla_X \phi X + h(\phi X, X) = \tilde{\nabla}_X \phi X = \phi \tilde{\nabla}_X X + (\tilde{\nabla}_X \phi) X = \phi \nabla_X X + \phi h(X, X).$$

Hence, we have

$$\nabla_X \phi X + h(\phi X, X) = \phi \nabla_X X + \phi h(X, X).$$

Taking the inner product with ϕZ in the above equation, and taking in consideration Corollary 5.2.3 (i), one can deduce that $\phi \nabla_X X \in \Gamma(TN_T)$. Again, from Corollary 5.2.3 (i), we know that $\phi h(X, X) \in \Gamma(\nu)$. Hence, comparing tangential and normal components of the above equation, it yields $\nabla_X \phi X = \phi \nabla_X X$, which is statement (i).

On the other hand, with straightforward computations we have

$$-\phi X = (\tilde{\nabla}_X \phi)\xi = -\phi \tilde{\nabla}_X \xi = -\phi \nabla_X \xi - \phi h(X, \xi).$$

Taking the inner product with ϕX in the above equation directly gives $g(\nabla_X X, \xi) = -||X||^2$. This is statement (ii) which completes the proof. □

The next couple of lemmas show that $||h_\nu(X, Z)||^2 = g(\phi h(X, Z), h(\phi X, Z))$.

Lemma 6.2.2. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a Kenmotsu manifold $\tilde{M}^{2l+1}$. Then, we have*

$$A_{\phi\zeta} \phi X = A_\zeta X,$$

where $X \in \Gamma(TN_T)$ and orthogonal to ξ. The normal vector field ζ belongs to the normal subbundle ν.

Proof. Taking X as in the above proof. Then, for any $U \in \Gamma(TM^n)$, we can use (2.3.57) and (2.3.27) to show that

$$g(\phi U, X)\xi = (\tilde{\nabla}_U \phi)X = \tilde{\nabla}_U \phi X - \phi \tilde{\nabla}_U X = \nabla_U \phi X + h(\phi X, U) - \phi \nabla_U X - \phi h(X, U).$$

By taking the inner product with $\zeta \in \Gamma(\nu)$ in the above equation, and taking into account (2.3.29), we deduce that

$$A_{\phi\zeta} \phi X = A_\zeta X.$$

This gives the assertion. □

Hence, in virtue of the preceding lemma, we get the following one

Lemma 6.2.3. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a Kenmotsu manifold $\tilde{M}^{2l+1}$. Then, we have*

$$||h_\nu(X,Z)||^2 = g(\phi h(X,Z), h(\phi X, Z)),$$

where X and Z are tangent to the first and the second factors, respectively. Here, X is orthogonal to ξ and h_ν denotes the ν-component of h.

Proof. In view of Lemma 6.2.2, it is possible to write

$$A_{\phi h_\nu(X,Z)}\phi X = A_{h_\nu(X,Z)}X. \tag{6.2.8}$$

Also, the norm of the ν-component of h can be written as

$$||h_\nu(X,Z)||^2 = g(h_\nu(X,Z), h(X,Z)) = g(A_{h_\nu(X,Z)}X, Z).$$

Making use of $(6.2.8)$ in the above equation, it follows

$$||h_\nu(X,Z)||^2 = g(\phi h(X,Z) - \phi h_{F\mathcal{D}_\perp}(X,Z), h(\phi X, Z)). \tag{6.2.9}$$

We know that $\phi h_{F\mathcal{D}_\perp}(X,Z) \in \mathcal{D}_\perp$. Consequently, the above equation takes the form

$$||h_\nu(X,Z)||^2 = g(\phi h(X,Z), h(\phi X, Z)).$$

This is the desired result. $\square$

Finally, we have

Lemma 6.2.4. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a Kenmotsu manifold $\tilde{M}^{2l+1}$. Then, we have*

$$\phi \nabla_{\phi X} X = \nabla_{\phi X}\phi X + ||X||^2 \xi,$$

where X is tangent to the first factor and orthogonal to ξ.

Proof. We use $(2.3.27)$, $(2.3.34)$ and $(2.3.57)$ to show that

$$\nabla_{\phi X}\phi X + h(\phi X, \phi X) = \tilde{\nabla}_{\phi X}\phi X = \phi \tilde{\nabla}_{\phi X} X + (\tilde{\nabla}_{\phi X}\phi)X$$

$$= \phi\nabla_{\phi X}X + \phi h(X, \phi X) - ||X||^2\xi.$$

Comparing tangential and normal components in the above equation, as $\phi h(X, \phi X)$ is normal in view of Corollary 5.2.3 (i), we deduce that

$$\phi\nabla_{\phi X}X = \nabla_{\phi X}\phi X + ||X||^2\xi.$$

□

Based on the above lemmas, and inspired by proofs in (Chen, 2003) and (Munteanu, 2005), we present the main theorem of this section:

Theorem 6.2.3. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of Kenmotsu space forms $\tilde{M}^{2l+1}(c_{Ke})$. Then, we have*

(i) $||h||^2 \geq 2n_2\Big\{||\nabla \ln f||^2 - \Delta(\ln f) + \frac{c_{Ke}-3}{2}s - 1\Big\}.$

(ii) The equality sign in (i) holds if and only if N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2l+1}(c_{Ke})$, respectively.

Proof. Given any $X \in \Gamma(TN_T)$ orthogonal to ξ, and $Z \in \Gamma(TN_\perp)$. By using (2.3.27), (2.3.57) and Lemma 6.2.1 (i) we obtain

$$T_1 = g(h(\nabla_X\phi X, Z), \phi Z) = g(\tilde{\nabla}_Z\nabla_X\phi X, \phi Z) = g(\tilde{\nabla}_Z\phi\nabla_X X, \phi Z)$$

$$= g(\phi\tilde{\nabla}_Z\nabla_X X, \phi Z) + g((\tilde{\nabla}_Z\phi)\nabla_X X, \phi Z) = (\nabla_X X(\ln f))||Z||^2 - \eta(\nabla_X X)||Z||^2.$$

Via Lemma 6.2.1 (ii) the above equation takes the form

$$T_1 = \Big((\nabla_X X(\ln f)) + ||X||^2\Big)||Z||^2.$$

First, notice that X is orthogonal with ξ, if we set $\alpha = 0$ in Proposition (3.3.2) (3), then by (2.3.2) and (2.3.27) one can derive the following

$$T_3 = -g(\nabla^\perp_X h(\phi X, Z), FZ) = -X\Big((X\ln f)g(Z, Z)\Big) + g(h(\phi X, Z), \tilde{\nabla}_X\phi Z)$$

$$= -(X^2\ln f)g(Z, Z) - 2(X\ln f)^2 g(Z, Z) + g(h(\phi X, Z), \phi\nabla_X Z)$$

$$+ g(h(\phi X, Z), \phi h(X, Z))$$

$$= -(X^2\ln f)g(Z, Z) - 2(X\ln f)^2 g(Z, Z) + (X\ln f)\, g(h(\phi X, Z), \phi Z)$$

$$+g(h(\phi X, Z), \phi h(X, Z))$$

$$= -(X^2 \ln f) g(Z, Z) - (X \ln f)^2 g(Z, Z) + g(h(\phi X, Z), \phi h(X, Z)). \tag{6.2.10}$$

Applying Lemma 6.2.3 on the last term in the right hand side of (6.2.10), we then reach

$$T_3 = -g(\nabla^{\perp}_X h(\phi X, Z), FZ) = -(X^2 \ln f) g(Z, Z) - (X \ln f)^2 g(Z, Z) + ||h_v(X, Z)||^2.$$

Similarly,

$$T_4 = g(\nabla^{\perp}_{\phi X} h(X, Z), FZ) = -((\phi X)^2 \ln f) g(Z, Z) - (\phi X \ln f)^2 g(Z, Z) + ||h_v(X, Z)||^2.$$

Putting $\alpha = 0$ in statement (3) of Proposition 3.3.2, then it is not difficult to prove

$$T_5 = g(h(\phi X, \nabla_X Z), FZ) = (X \ln f)^2 g(Z, Z) \tag{6.2.11}$$

and

$$T_6 = -g(h(X, \nabla_{\phi X} Z), FZ) = (\phi X \ln f)^2 g(Z, Z). \tag{6.2.12}$$

Finally, we can use (2.3.27) and (2.3.57) to compute

$$T_2 = -g(h(\nabla_{\phi X} X, Z), \phi Z) = -g(\tilde{\nabla}_Z \nabla_{\phi X} X, \phi Z) = g(\phi \tilde{\nabla}_Z \nabla_{\phi X} X, Z)$$

$$= g(\tilde{\nabla}_Z \phi \nabla_{\phi X} X, Z) - g((\tilde{\nabla}_Z \phi) \nabla_{\phi X} X, Z) = (\phi \nabla_{\phi X} X(\ln f)) ||Z||^2. \tag{6.2.13}$$

By combining Lemma 6.2.4 with (6.2.13), it gives

$$T_2 = ((\phi \nabla_{\phi X} X(\ln f)) ||Z||^2 = (\nabla_{\phi X} \phi X(\ln f)) ||Z||^2 + ||X||^2 ||Z||^2 (\xi \ln f)$$

$$= \Big((\nabla_{\phi X} \phi X(\ln f)) + ||X||^2 \Big) ||Z||^2.$$

Now, from (2.3.58) we have

$$\tilde{R}(X, \phi X, Z, \phi Z) = \frac{c_{Ke} + 1}{2} g(X, X) g(Z, Z). \tag{6.2.14}$$

Also, in virtue of (2.3.30) and (2.3.40), we obtain

$$\tilde{R}(X, \phi X, Z, \phi Z) = -g(\nabla^{\perp}_X h(\phi X, Z) - h(\nabla_X \phi X, Z) - h(\phi X, \nabla_X Z), FZ)$$

$$+ g(\nabla^{\perp}_{\phi X} h(X, Z) - h(\nabla_{\phi X} X, Z) - h(X, \nabla_{\phi X} Z), FZ). \tag{6.2.15}$$

Thus, substituting T_i for $i = 1, \cdots, 6$ in the above equation gives

$$\tilde{R}(X, \phi X, Z, \phi Z) = 2||h_v(X, Z)||^2 - (X^2 \ln f)||Z||^2 + ((\nabla_X X) \ln f)||Z||^2$$

$$-((\phi X)^2 \ln f)||Z||^2 + ((\nabla_{\phi X} \phi X) \ln f)||Z||^2 + 2||X||^2||Z||^2.$$

Equivalently,

$$2||h_v(X, Z)||^2 = \Bigg(\frac{c_{Ke} + 1}{2}||X||^2 + (X^2 \ln f) - ((\nabla_X X) \ln f)$$

$$+ ((\phi X)^2 \ln f) - ((\nabla_{\phi X} \phi X) \ln f) - 2||X||^2\Bigg)||Z||^2.$$

Now, let $\{e_0 = \xi, e_1, \cdots, e_s, e_{s+1} = \phi e_1, \cdots, e_{2s} = \phi e_s, e_{n_1+1} = e_1^\star, \cdots, e_n = e_q^\star\}$ be local fields of orthonormal frame of $\Gamma(TM^n)$. Putting $X = e_a$ and $Z = e_A$ in the above equation, and take the summation over this adapted frame. Then, we get

$$\sum_{a=1}^{n_1} \sum_{A=n_1+1}^{n} 2||h_v(e_a, e_A)||^2 = \sum_{a=1}^{n_1} \sum_{A=n_1+1}^{n} 2\Bigg(\frac{c_{Ke} + 1}{2}||e_a||^2 + (e_a^2 \ln f) - ((\nabla_{e_a} e_a) \ln f)$$

$$+ ((\phi e_a)^2 \ln f) - ((\nabla_{\phi e_a} \phi e_a) \ln f) - 2||e_a||^2\Bigg)||e_A||^2.$$

As $h(\xi, Z) = 0$ and $\xi \ln f = 1$, we apply (2.3.22) to get

$$||h_\nu(\mathcal{D}_T, \mathcal{D}_\perp)||^2 = 2n_2\Bigg\{\frac{c_{Ke} - 3}{2}s - \Delta(\ln f)\Bigg\}. \tag{6.2.16}$$

From (5.4.1), the first inequality of h for nearly β-Kenmotsu is derived from a more general setting. This inequality is valid also for β-Kenmotsu manifolds. Thus, putting $\beta = 1$ in (5.4.1), the first inequality of h for Kenmotsu manifolds follows; namely,

$$||h||^2 \geq 2n_2\Bigg\{||\nabla \ln f||^2 - 1\Bigg\}. \tag{6.2.17}$$

From the proof of Theorem 4.1 in (Mustafa et al., 2013), we know that the above inequality comes from calculating the $F\mathcal{D}_\perp$-component of h and putting $\alpha = 0$, $\beta = 1$. More precisely,

$$||h_{F\mathcal{D}_\perp}(\mathcal{D}_T, \mathcal{D}_\perp)||^2 = 2n_2\Bigg\{||\nabla \ln f||^2 - 1\Bigg\}. \tag{6.2.18}$$

Consequently, combining (6.2.16) and (6.2.18) together gives statement (i) immediately. Statement (ii) can be proved in the same way as that of Theorem 6.2.1. The proof is complete. □

Following the same procedure as above, but in a simpler way, the second inequality of h can be derived for contact CR-warped product submanifolds of type $N_T \times_f N_\perp$ of cosymplectic space forms $\tilde{M}^{2l+1}(c_c)$. The verification is left to the reader.

Theorem 6.2.4. *Let $M^n = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of cosymplectic space forms $\tilde{M}^{2l+1}(c_c)$. Then, we have*

(i) $||h||^2 \geq 2n_2\left(||\nabla(\ln f)||^2 - \Delta(\ln f) + (n_1 - 1)\frac{c_c}{4}\right).$

(ii) The equality sign in (i) holds if and only if N_T, $N_\perp$ and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^{2l+1}(c_c)$, respectively.

We point out that, all inequalities of this section are obtained as particular cases from a more general inequality in the next section. For this, it will be enough to calculate the scalar curvature from the corresponding curvature tensor formula.

The following table summarizes the second inequality of the CR-warped products in some space forms.

Manifold	Inequality
complex space form	$\|\|h\|\|^2 \geq 2n_2\left(\|\|\nabla(\ln f)\|\|^2 - \Delta(\ln f) + n_1\frac{c_{Ka}}{4}\right)$
Sasakian space form	$\|\|h\|\|^2 \geq 2n_2\left(\|\|\nabla(\ln f)\|\|^2 - \Delta(\ln f) + \frac{c_S+3}{2}s + 1\right)$
Kenmotsu space form	$\|\|h\|\|^2 \geq 2n_2\left(\|\|\nabla \ln f\|\|^2 - \Delta(\ln f) + \frac{c_{Ke}-3}{2}s - 1\right)$
cosymplectic space form	$\|\|h\|\|^2 \geq 2n_2\left(\|\|\nabla(\ln f)\|\|^2 - \Delta(\ln f) + (n_1 - 1)\frac{c_c}{4}\right)$

Table 6.1: The second inequality of h for CR-warped product submanifolds of type $N_T \times_f N_\perp$ in some space forms.

6.3 A NEW METHOD FOR A GENERAL INEQUALITY OF h

By making use of the Gauss equation, we construct a new general inequality for $\mathcal{D}_i$-minimal warped product submanifolds in arbitrary Riemannian manifolds. This inequality generalizes all inequalities of the previous section.

The following direct, but significant, result is another key lemma for this section which will also be frequently used in subsequent chapters.

Lemma 6.3.1. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m$ be an isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian manifold $\tilde{M}^m$. Then, we have*

$$\tau\big(T_xM^n\big) = \frac{n_2\Delta f}{f} + \sum_{r=n+1}^{m}\Bigg\{\sum_{1\leq a<b\leq n_1}\Big(h^r_{aa}h^r_{bb} - \big(h^r_{ab}\big)^2\Big) + \sum_{n_1+1\leq A<B\leq n}\Big(h^r_{AA}h^r_{BB} - \big(h^r_{AB}\big)^2\Big)\Bigg\} + \tilde{\tau}\big(T_xN_1\big) + \tilde{\tau}\big(T_xN_2\big), \tag{6.3.1}$$

where $n_1,\ n_2,\ n$ and m are the dimensions of $N_1,\ N_2,\ M^n$ and $\tilde{M}^m$, respectively.

Proof. From (2.3.45), we have

$$\tau\big(T_xM^n\big) = \sum_{1\leq i<j\leq n} K_{ij} = \sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n} K_{aA} + \sum_{1\leq a<b\leq n_1} K_{ab} + \sum_{n_1+1\leq A<B\leq n} K_{AB}. \tag{6.3.2}$$

Via (5.2.3) and (2.3.45), the above equation is congruent to

$$\tau\big(T_xM^n\big) = \frac{n_2\Delta f}{f} + \tau\big(T_xN_1\big) + \tau\big(T_xN_2\big). \tag{6.3.3}$$

In view of (2.3.43), it is common to have

$$\tau\big(T_xN_1\big) = \sum_{r=n+1}^{m}\sum_{1\leq a<b\leq n_1}\Big(h^r_{aa}h^r_{bb} - \big(h^r_{ab}\big)^2\Big) + \tilde{\tau}\big(T_xN_1\big), \tag{6.3.4}$$

and

$$\tau\big(T_xN_2\big) = \sum_{r=n+1}^{m}\sum_{n_1+1\leq A<B\leq n}\Big(h^r_{AA}h^r_{BB} - \big(h^r_{AB}\big)^2\Big) + \tilde{\tau}\big(T_xN_2\big). \tag{6.3.5}$$

By joining (6.3.3), (6.3.4) and (6.3.5) together, we get the result. □

From now on, we should be familiar with the following two formulas of the mean curvature vector, according to $\mathcal{D}_i$-minimality property. From Definition 2.3.3 we can distinguish two cases, if φ is $\mathcal{D}_1$-minimal isometric immersion of $M^n = N_1 \times_f N_2$ into any Riemannian manifold $\tilde{M}^m$, then following our adapted orthonormal frame, we derive the following formula of the squared norm of the mean curvature vector of M^n

$$||\vec{H}||^2 = \frac{1}{n^2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + ... + h^r_{nn})^2. \tag{6.3.6}$$

Similarly, for a $\mathcal{D}_2$-minimal isometric immersion φ, we have

$$||\vec{H}||^2 = \frac{1}{n^2}\sum_{r=n+1}^{m}(h^r_{11} + ... + h^r_{n_1n_1})^2. \tag{6.3.7}$$

Now, we present the main theorem of this chapter.

Theorem 6.3.1. *For $i = 1$ or 2, let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m$ be a $\mathcal{D}_i$-minimal isometric immersion of a warped product submanifold M^n into a Riemannian manifold $\tilde{M}^m$. Then, we have*

(i) $\frac{1}{2}||h||^2 \geq \tilde{\tau}(T_xM^n) - \tilde{\tau}(T_xN_1) - \tilde{\tau}(T_xN_2) - \frac{n_2\Delta f}{f}$.

(ii) The equality in (i) holds identically if and only if N_1, N_2 and M^n are totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^m$, respectively.

Proof. Via (2.3.44), we first have

$$||h||^2 = -2\tau(T_xM^n) + 2\tilde{\tau}(T_xM^n) + n^2||\vec{H}||^2.$$

In view of Lemma 6.3.1, the above equation takes the following form

$$\begin{aligned} ||h||^2 = {} & 2\tilde{\tau}(T_xM^n) - 2\tilde{\tau}(T_xN_1) - 2\tilde{\tau}(T_xN_2) - 2\frac{n_2\Delta f}{f} + n^2||\vec{H}||^2 \\ & -2\left(\sum_{r=n+1}^{m}\sum_{1\leq a<b\leq n_1}\left(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\right)\right) \\ & -2\left(\sum_{r=n+1}^{m}\sum_{n_1+1\leq A<B\leq n}\left(h^r_{AA}h^r_{BB} - (h^r_{AB})^2\right)\right). \end{aligned}$$

This is equivalent to

$$\begin{aligned} ||h||^2 = {} & 2\tilde{\tau}(T_xM^n) - 2\tilde{\tau}(T_xN_1) - 2\tilde{\tau}(T_xN_2) - 2\frac{n_2\Delta f}{f} + n^2||\vec{H}||^2 \\ & -\left(\sum_{r=n+1}^{m}\sum_{1\leq a\neq b\leq n_1}\left(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\right)\right) \\ & -\left(\sum_{r=n+1}^{m}\sum_{n_1+1\leq A\neq B\leq n}\left(h^r_{AA}h^r_{BB} - (h^r_{AB})^2\right)\right). \end{aligned} \tag{6.3.8}$$

There are two natural ways to proceed our proof, corresponding to whether one considers the immersed warped product as $\mathcal{D}_1$-minimal or instead as $\mathcal{D}_2$-minimal. Accordingly, we distinguish two cases:

Case (1): If $i = 1$; that is, φ is $\mathcal{D}_1$-minimal immersion, then

$$-\left(\sum_{r=n+1}^{m}\sum_{1\leq a\neq b\leq n_1}\left(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\right)\right) =$$

$$\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}(h^r_{ab})^2-\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}h^r_{aa}h^r_{bb}=$$

$$\overbrace{\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}(h^r_{ab})^2+\left(\sum_{r=n+1}^{m}\left((h^r_{11})^2+\cdots+(h^r_{n_1n_1})^2\right)\right)}$$

$$\overbrace{-\left(\sum_{r=n+1}^{m}\left((h^r_{11})^2+\cdots+(h^r_{n_1n_1})^2\right)\right)-\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}h^r_{aa}h^r_{bb}}\,.$$

By means of the binomial theorem, we deduce that

$$\overbrace{\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}(h^r_{ab})^2+\left(\sum_{r=n+1}^{m}\left((h^r_{11})^2+\cdots+(h^r_{n_1n_1})^2\right)\right)}=\sum_{r=n+1}^{m}\sum_{a,b=1}^{n_1}(h^r_{ab})^2,$$

and

$$\overbrace{-\left(\sum_{r=n+1}^{m}\left((h^r_{11})^2+\cdots+(h^r_{n_1n_1})^2\right)\right)-\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}h^r_{aa}h^r_{bb}}=$$

$$-\sum_{r=n+1}^{m}(h^r_{11}+\cdots+h^r_{n_1n_1})^2.$$

Next, by combining the last three equations together we obtain

$$-\left(\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}\left(h^r_{aa}h^r_{bb}-(h^r_{ab})^2\right)\right)=\sum_{r=n+1}^{m}\sum_{a,b=1}^{n_1}(h^r_{ab})^2-\sum_{r=n+1}^{m}(h^r_{11}+\cdots+h^r_{n_1n_1})^2. \tag{6.3.9}$$

By Definition 2.3.3, the second term in the right hand side vanishes whenever φ is $\mathcal{D}_1$-minimal, consequently (6.3.9) reduces to

$$-\left(\sum_{r=n+1}^{m}\sum_{1\le a\ne b\le n_1}\left(h^r_{aa}h^r_{bb}-(h^r_{ab})^2\right)\right)=\sum_{r=n+1}^{m}\sum_{a,b=1}^{n_1}(h^r_{ab})^2. \tag{6.3.10}$$

Combining (6.3.10) and (6.3.8), it yields to

$$||h||^2=2\tilde{\tau}(T_xM^n)-2\tilde{\tau}(T_xN_1)-2\tilde{\tau}(T_xN_2)-2\frac{n_2\Delta f}{f}+n^2||\vec{H}||^2$$

$$+\sum_{r=n+1}^{m}\sum_{a,b=1}^{n_1}(h^r_{ab})^2$$

$$-\left(\sum_{r=n+1}^{m}\sum_{n_1+1\le A\ne B\le n}\left(h^r_{AA}h^r_{BB}-(h^r_{AB})^2\right)\right).$$

Equivalently,

$$||h||^2\ge 2\tilde{\tau}(T_xM^n)-2\tilde{\tau}(T_xN_1)-2\tilde{\tau}(T_xN_2)-2\frac{n_2\Delta f}{f}+n^2||\vec{H}||^2$$

$$-\left(\sum_{r=n+1}^{m}\sum_{n_1+1\leq A\neq B\leq n}\left(h^r_{AA}h^r_{BB}-(h^r_{AB})^2\right)\right).$$

Again, by adding and subtracting similar term technique, the above inequality becomes

$$||h||^2 \geq 2\tilde{\tau}(T_xM^n) - 2\tilde{\tau}(T_xN_1) - 2\tilde{\tau}(T_xN_2) - 2\frac{n_2\Delta f}{f} + n^2||\vec{H}||^2$$

$$-\sum_{r=n+1}^{m}\left((h^r_{n_1+1n_1+1})^2+\cdots+(h^r_{nn})^2+\sum_{n_1+1\leq A\neq B\leq n}h^r_{AA}h^r_{BB}\right)$$

$$+\sum_{r=n+1}^{m}\left((h^r_{n_1+1n_1+1})^2+\cdots+(h^r_{nn})^2+\sum_{n_1+1\leq A\neq B\leq n}(h^r_{AB})^2\right).$$

Applying the binomial theorem on the last two terms of the above equation, we derive that

$$||h||^2 \geq 2\tilde{\tau}(T_xM^n) - 2\tilde{\tau}(T_xN_1) - 2\tilde{\tau}(T_xN_2) - 2\frac{n_2\Delta f}{f} + n^2||\vec{H}||^2$$

$$-\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn})^2$$

$$+\sum_{r=n+1}^{m}\sum_{A,B=n_1+1}^{n}(h^r_{AB})^2.$$

Hence, we reach

$$||h||^2 \geq 2\tilde{\tau}(T_xM^n) - 2\tilde{\tau}(T_xN_1) - 2\tilde{\tau}(T_xN_2) - 2\frac{n_2\Delta f}{f} + n^2||\vec{H}||^2$$

$$-\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn})^2.$$

In view of (6.3.6), we know that the last term in the right hand side of the above inequality is equal to $-n^2||\vec{H}||^2$ for $\mathcal{D}_1$-minimal warped product submanifolds. By this fact, the inequality of statement (i) follows immediately from the above inequality. Analogously, but using (6.3.7) instead of (6.3.6), we get same inequality for $\mathcal{D}_2$-minimal warped product submanifolds.

Now, the equality sign of the inequality in (i) holds if and only if

$$(a)\ h(\mathcal{D}_1,\mathcal{D}_1)=0, \qquad (b)\ h(\mathcal{D}_2,\mathcal{D}_2)=0.$$

Hence, we need to show that (a) and (b) hold if and only if N_1, N_2 and M^n are respectively totally geodesic, totally umbilical and minimal submanifolds in $\tilde{M}^m$.

First, assume that (a) and (b) are satisfied. Since $M^n = N_1 \times_f N_2$ is a warped product, Corollary 2.3.1 asserts that N_1 and N_2 are totally geodesic and totally umbilical in M^n, respectively. Therefore, part (a) above implies that the first factor is a totally geodesic submanifold in $\tilde{M}^m$. The second factor is totally umbilical in $\tilde{M}^m$ because of part (b). Moreover, (b) and (a) together imply that M^n is minimal in $\tilde{M}^m$.

For the converse, (a) is clear. To obtain (b), we first notice that minimality and $\mathcal{D}_1$-minimality of M^n in $\tilde{M}^m$ yield to $\mathcal{D}_2$-minimality of M^n in $\tilde{M}^m$. Hence, Lemma 5.2.6 proves (b) for either $i = 1$, or $i = 2$. $\square$

The first direct and important consequence of Theorem 6.3.1 is the following:

As we promised in Chapter Five, Theorem 5.2.1 has two proofs, one was given in Chapter Five, while the other proof is obtained via the next remark.

Remark 6.3.1. *An important consequence of Theorem 6.3.1 is Theorem 5.2.1. Because statement (ii) of Theorem 6.3.1 informs us that, the equality sign of statement (i) holds identically if and only if $||h||^2$ reduces to the form $||h(\mathcal{D}_1, \mathcal{D}_2)||^2$, which gives Theorem 5.2.1 immediately.*

6.4 SPECIAL INEQUALITIES AND APPLICATIONS

The purpose of this section is to derive some particular case inequalities from Theorem 6.3.1. A lot of second inequalities of h can be simply obtained form this theorem. For this, all what we need is to compute the following expression via the corresponding curvature tensor formula

$$2\bigg(\tilde{\tau}(T_xM^n) - \tilde{\tau}(T_xN_1) - \tilde{\tau}(T_xN_2)\bigg).$$

As the first example, we embark on this process by considering CR-warped product submanifolds of type $N_T \times_f N_\perp$ in complex space forms. Since the ambient manifold $\tilde{M}^m$ of Theorem 6.3.1 is an arbitrary Riemannian manifold, we can consider $\tilde{M}^m$ to be a Kaehler manifold. Hence, for every CR-warped product $M^n = N_T \times_f N_\perp$ in a complex space form $\tilde{M}^{2m}(c_{Ka})$, we just use the curvature tensor of complex space forms (2.3.49) to compute the following

$$2\bigg(\tilde{\tau}(T_xM^n) - \tilde{\tau}(T_xN_1) - \tilde{\tau}(T_xN_2)\bigg) = \frac{c_{Ka}}{4}\bigg(n(n-1) + 3n_1 - n_1(n_1-1) - 3n_1 - n_2(n_2-1)\bigg)$$

$$= \frac{c_{Ka} n_1 n_2}{2}.$$

Substituting the above expression in Theorem 6.3.1, taking into account Lemma 5.2.1 and (5.3.2). Theorem 6.2.1 can be obtained as a special case.

Remark 6.4.1. *Inequalities of Theorems 4.1, 5.1 and 6.1 in (Chen, 2003) are special cases of Theorem 6.3.1, where the ambient manifold is a complex Euclidean, a complex projective and a complex hyperbolic space, respectively.*

In general, we have now a clear procedure of deriving any space form second inequality of h which is applicable for any $\mathcal{D}_i$-minimal warped product submanifold in arbitrary Riemannian manifolds. For instance, the following shows the direction of deriving a special case from a more general case in Kaehler manifolds:

Theorem 6.3.1 $\Longrightarrow$ Theorem 6.2.1$\Longrightarrow$ Theorems in the above remark.

In the sequel, Theorem 6.3.1 can be applied to get an important consequence in generalized complex space forms. More precisely, for both CR and semi-slant warped product submanifolds in generalized complex space forms Theorem 6.3.1 implies

$$\frac{1}{2}||h||^2 \geq 2n_1 n_2 \frac{c_{RK} + 3\gamma}{4} - \frac{n_2 \Delta f}{f}.$$

We notice that, it was difficult to prove the above inequality by old methods. Moreover, this inequality generalizes all inequalities in the previous remark. However, this is just to show how Theorem 6.3.1 works for deriving any second inequality of h for any $\mathcal{D}_i$-minimal warped product submanifold in almost Hermitian manifolds. This will be clear from the next table which provides the second inequalities of h for any $\mathcal{D}_i$-minimal warped product submanifold in complex and generalized complex space forms.

In almost contact counter part, Theorems 6.2.2, 6.2.3 and 6.2.4 in the previous section are direct consequences of Theorem 6.3.1. Thus, if the ambient manifold is a Sasakian space form and the immersion is a contact CR-warped product, then we may use (2.3.56) to evaluate $\tilde{\tau}(T_x M^n)$, $\tilde{\tau}(T_x N_T)$ and $\tilde{\tau}(T_x N_\perp)$ in Theorem 6.3.1 (i). Further using (5.3.2), we obtain

$$||h||^2 \geq \Bigg\{ 2n_2 \Big(||\nabla(\ln f)||^2 - \Delta(\ln f) \Big) + \frac{c_S + 3}{4} n(n-1) - \frac{c_S - 1}{4} \Big(2(n-1) - 3(n_1 - 1) \Big)$$
$$- n_1(n_1 - 1)\frac{c_S + 3}{4} - (n_1 - 1)\frac{c_S - 1}{4} - n_2(n_2 - 1)\frac{c_S + 3}{4} \Bigg\}.$$

Simple calculations on the integer coefficient of c_S gives

$$||h||^2 \geq 2n_2\Big\{||\nabla(\ln f)||^2 - \Delta(\ln f) + \frac{c_S+3}{4}n_1 - \frac{c_S-1}{4}\Big\}.$$

Straightforward calculations on the above inequality shows that it is identical to that in Theorem 6.2.2.

Similarly, it is direct to follow the above procedure to derive the inequality of Theorem 6.2.3 for contact CR-warped product of type $N_T \times_f N_\perp$ in Kenmotsu space forms. Using $(2.3.58)$, we directly obtain

$$||h||^2 \geq 2n_2\Big\{||\nabla(\ln f)||^2 - \Delta(\ln f) + \frac{c_{Ke}-3}{4}n_1 - \frac{c_{Ke}+1}{4}\Big\}.$$

Simple calculations show that the above inequality is congruent to

$$||h||^2 \geq 2n_2\Big\{\frac{c_{Ke}-3}{2}s - \Delta \ln f + ||\nabla \ln f||^2 - 1\Big\}.$$

It is clear that the above inequality is identical to that in Theorem 6.2.3. This makes the two methods of proving this inequality coherent and precise. On the contrary, comparing the above inequality with the second one in (Arsalan et al., 2005) shows the differences. Although it looks as a small difference, but, as we saw in section two of the current chapter, it needs deep results to fix the errors. For this, one can compare the proofs of Theorem 6.2.3 and that of the second inequality in (Arsalan et al., 2005).

By following similar arguments as above, we list more particular case of inequalities for different kinds of ambient manifolds in the final section of this chapter by

As another application of Theorem 6.3.1, we have

Corollary 6.4.1. *Let $M^n = N_1 \times_f N_2$ be a $\mathcal{D}_i$-minimal warped product in a Riemannian manifold $\tilde{M}^m$ and suppose N_1 is compact. Denote by dv_1 and $vol(N_1)$ the volume element and the volume on N_1. Let λ_1 be the first non zero eigenvalue of the Laplacian on N_1. Then*

$$\frac{1}{2}\int_{N_1} ||h||^2 dv_1 \geq n_1\Big(\tilde{\tau}(T_xM) - \tilde{\tau}(T_xN_1) - \tilde{\tau}(T_xN_2)\Big)vol(N_1) + n_1\lambda_1\int_{N_1}(\ln f)^2 dv_1.$$

Proof. From the minimum principle we have

$$\int_{N_1} ||\nabla \ln f||^2 dv_1 \geq \lambda_1 \int_{N_1} (\ln f)^2 dv_1.$$

Now we have to integrate on N_1 the inequality of Theorem 6.3.1 which is satisfied by the norm of h, and then we obtain immediately the result. □

6.4.1 CONCLUSION

Now, we give some extensions of our main inequality of this chapter for different kinds of space forms of interest. Using the abbreviation, g. c. s. f. $\equiv$ generalized complex space form, then we have

Manifold	Inequality
Real Space Form	$\|\|h\|\|^2 \geq 2n_2\Big(\|\|\nabla(\ln f)\|\|^2 - \Delta(\ln f) + n_1 c\Big)$
Complex Space Form	$\|\|h\|\|^2 \geq 2n_2\Big(\|\|\nabla(\ln f)\|\|^2 - \Delta(\ln f) + n_1 \frac{c_{Ka}}{4}\Big)$
g. c. s. f.	$\|\|h\|\|^2 \geq 2n_2\Big(\|\|\nabla(\ln f)\|\|^2 - \Delta(\ln f) + n_1 \frac{c_{RK}+3\gamma}{4}\Big)$
Sasakian Space Form	$\|\|h\|\|^2 \geq 2n_2\Big(\|\|\nabla(\ln f)\|\|^2 - \Delta(\ln f) + \frac{c_S+3}{2}s + 1\Big)$
Kenmotsu Space Form	$\|\|h\|\|^2 \geq 2n_2\Big(\|\|\nabla(\ln f)\|\|^2 - \Delta(\ln f) + \frac{c_{Ke}-3}{2}s - 1\Big)$
Cosymplectic Space Form	$\|\|h\|\|^2 \geq 2n_2\Big(\|\|\nabla(\ln f)\|\|^2 - \Delta(\ln f) + (n_1 - 1)\frac{c_c}{4}\Big)$

Table 6.2: General second inequality of h for $\mathcal{D}_i$-minimal warped product submanifolds satisfying $g(PX, Z) = 0 \;\forall\; X \in TN_1,\; Z \in TN_2$.

This table is of particular value because of the wide class it contains. It provides a sample of inequalities for $\mathcal{D}_i$-minimal warped product submanifolds $N_1 \times_f N_2$ in different kinds of space forms, satisfying $g(PX, Z) = 0 \;\forall\; X \in TN_1,\; Z \in TN_2$. Notice that the condition $g(PX, Z) = 0$ holds for CR, semi-slant, hemi-slant warped product submanifolds and for any warped product submanifold with a holomorphic factor in both almost Hermitian and almost contact manifolds.

CHAPTER 7: A GENERAL GEOMETRIC INEQUALITY OF RICCI CURVATURE AND THE MEAN CURVATURE VECTOR FOR $\mathcal{D}_i$-MINIMAL WARPED PRODUCT SUBMANIFOLDS

7.1 INTRODUCTION

In this chapter, we first establish a basic general inequality for $\mathcal{D}_i$- minimal warped product submanifolds in Riemannian space forms. This inequality involves the Ricci curvature, the mean curvature vector and the warping function. Afterwards, the equality cases are discussed.

In the other section, some extensions are discussed, by deriving many special case inequalities for different kinds of space forms. This was achieved by taking a middle step from the proof of Theorem 7.3.1 which is valid for an arbitrary Riemannian ambient manifold. Then following almost similar argument like that in the preceding chapter, we calculate some terms by tensor curvature formulas of different space forms.

Many applications are derived from the main theorem of this chapter, the most important one was separately stated in a short section, since it is one of our goals of this thesis. In general, this chapter provides a lot of special answers for several problems discussed in chapter one.

7.2 SOME TECHNICAL LEMMAS

This section presents some results which are useful for the next section. As we will see, these results are simplifications for some terms which appear in the proof of the next section. The first two lemmas are for $\mathcal{D}_1$-minimal warped product submanifolds, whereas the last lemma is for an arbitrary warped product submanifold.

Firstly, we have

Lemma 7.2.1. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m$ be a $\mathcal{D}_1$-minimal isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian manifold $\tilde{M}^m$. Then,*

$$\sum_{r=n+1}^{m} \sum_{b=2}^{n_1} h_{11}^r h_{bb}^r = - \sum_{r=n+1}^{m} \left(h_{11}^r\right)^2, \tag{7.2.1}$$

and
$$\sum_{r=n+1}^{m}\sum_{a=2}^{n_1}\sum_{A=n_1+1}^{n} h^r_{aa}h^r_{AA} = -\sum_{r=n+1}^{m}\sum_{A=n_1+1}^{n} h^r_{11}h^r_{AA}, \tag{7.2.2}$$
where $n_1,\ n_2,\ n$ and m are the dimensions of $N_1,\ N_2,\ M^n$ and $\tilde{M}^m$, respectively.

Proof. For a $\mathcal{D}_1$-minimal warped product submanifold M^n of $\tilde{M}^m$, we can carry out the following simplifications
$$\sum_{r=n+1}^{m}\sum_{b=2}^{n_1} h^r_{11}h^r_{bb} = \sum_{r=n+1}^{m} h^r_{11}\Big(g\big(h(e_2,e_2),e_r\big)+\cdots+g\big(h(e_{n_1},e_{n_1}),e_r\big)\Big)$$
$$= \sum_{r=n+1}^{m} h^r_{11}\Big(g\big(h(e_1,e_1),e_r\big)+\cdots+g\big(h(e_{n_1},e_{n_1}),e_r\big)-g\big(h(e_1,e_1),e_r\big)\Big)$$
$$= -\sum_{r=n+1}^{m}\big(h^r_{11}\big)^2. \tag{7.2.3}$$

This gives $(7.2.1)$. By the same procedure, $(7.2.2)$ follows directly, whenever M^n is a $\mathcal{D}_1$-minimal, which completes the proof □

Secondly, we state the following

Lemma 7.2.2. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m$ be a $\mathcal{D}_1$-minimal isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian manifold $\tilde{M}^m$. Then, we have*
$$\frac{1}{2}\sum_{r=n+1}^{m}\Big(2h^r_{11}-\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)\Big)^2 =$$
$$2\sum_{r=n+1}^{m}(h^r_{11})^2+\frac{1}{2}n^2||\vec{H}||^2-2\sum_{r=n+1}^{m}\sum_{A=n_1+1}^{n}h^r_{11}h^r_{AA}, \tag{7.2.4}$$
where $n_1,\ n_2,\ n$ and m are the dimensions of $N_1,\ N_2,\ M^n$ and $\tilde{M}^m$, respectively.

Proof. Since M^n is $\mathcal{D}_1$-minimal, it is obvious that
$$\frac{1}{2}\sum_{r=n+1}^{m}\Big(2h^r_{11}-\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)\Big)^2$$
$$= 2\sum_{r=n+1}^{m}(h^r_{11})^2+\frac{1}{2}\sum_{r=n+1}^{m}\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)^2$$
$$-2\sum_{r=n+1}^{m}h^r_{11}\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big),$$
$$= 2\sum_{r=n+1}^{m}(h^r_{11})^2+\frac{1}{2}n^2||\vec{H}||^2-2\sum_{r=n+1}^{m}\sum_{A=n_1+1}^{n}h^r_{11}h^r_{AA}. \tag{7.2.5}$$
□

Finally, we prove the following general lemma.

Lemma 7.2.3. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m$ be an isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian manifold $\tilde{M}^m$. Then, we have*

$$\sum_{r=n+1}^{m}\left\{\frac{1}{2}\Big(\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)-2h^r_{nn}\Big)^2+\sum_{B=n+1}^{n-1}h^r_{nn}h^r_{BB}\right\}$$
$$=\sum_{r=n+1}^{m}\left\{\frac{1}{2}\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)^2+\big(h^r_{nn}\big)^2-\sum_{A=n_1+1}^{n}h^r_{nn}h^r_{AA}\right\}, \qquad (7.2.6)$$

where $n_1,\ n_2,\ n$ and m are the dimensions of $N_1,\ N_2,\ M^n$ and $\tilde{M}^m$, respectively.

Proof. By simple calculation, we deduce that

$$\frac{1}{2}\sum_{r=n+1}^{m}\Big(\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)-2h^r_{nn}\Big)^2+\sum_{r=n+1}^{m}\sum_{B=n+1}^{n-1}h^r_{nn}h^r_{BB}$$
$$=\sum_{r=n+1}^{m}\left\{\frac{1}{2}\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)^2+2\big(h^r_{nn}\big)^2-\sum_{A=n_1+1}^{n}h^r_{nn}h^r_{AA}\right.$$
$$\left.-\sum_{A=n_1+1}^{n}h^r_{nn}h^r_{AA}+\sum_{B=n_1+1}^{n-1}h^r_{nn}h^r_{BB}\right\}, \qquad (7.2.7)$$

and

$$\sum_{r=n+1}^{m}\left\{\sum_{A=n_1+1}^{n}-h^r_{nn}h^r_{AA}+\sum_{B=n_1+1}^{n-1}h^r_{nn}h^r_{BB}\right\}$$
$$=\sum_{r=n+1}^{m}\left\{-h^r_{nn}\Big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\Big)+h^r_{nn}\Big(h^r_{n_1+1n_1+1}+\cdots+h^r_{n-1n-1}\Big)\right\}$$
$$=\sum_{r=n+1}^{m}\Big(h^r_{nn}\big(-h^r_{n_1+1n_1+1}-\cdots-h^r_{n-1n-1}-h^r_{nn}+h^r_{n_1+1n_1+1}+\cdots+h^r_{n-1n-1}\big)\Big)$$
$$=-\sum_{r=n+1}^{m}\big(h^r_{nn}\big)^2. \qquad (7.2.8)$$

From (7.2.8) and (7.2.7), we get the assertion. $\square$

7.3 A GENERAL GEOMETRIC INEQUALITY OF $\mathcal{D}_i$-MINIMAL WARPED PRODUCT SUBMANIFOLDS IN A RIEMANNIAN SPACE FORM

The current section is devoted to provide one of the most important theorems in this work. It is remarkable to say that, most of the Riemannian invariants are used throughout the next proof. As a result, we gain a very rich geometry in equality cases. Therefore, geometric concepts like totally geodesic, mixed totally geodesic, $\mathcal{D}_i$-totally geodesic, minimal, $\mathcal{D}_i$-minimal, totally umbilical and $\mathcal{D}_i$-totally umbilical submanifolds are included in the equality discussion. Hence, the inequality is a geometric one and so special for warped product submanifolds.

The natural existence of $\mathcal{D}_i$-minimal warped products, for $i = 1, 2$, was shown in both almost Hermitian and almost contact manifolds in chapter five. For this large class of warped product submanifolds we provide the main theorem in this work.

Theorem 7.3.1. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m(c)$, for $i = 1$ or $i = 2$, be a $\mathcal{D}_i$-minimal isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian space form $\tilde{M}^m(c)$. Then the following hold:*

(1) For each unit vector $e_o \in T_xM^n$, we have the following inequality

$$\frac{1}{4}\big(n_1 + n_2\big)^2 ||\vec{H}||^2 \geq Ric(e_o) + \frac{n_2\,\Delta f}{f} - c\big(n_1 + n_2 + n_1n_2 - 1\big) \tag{7.3.1}$$

where n_1, n_2 are the dimensions of N_1 and N_2, respectively.

(2) If $\vec{H}(x) = 0$, then at each $x \in M^n$ there is a unit tangent vector e_o satisfies the equality case of (7.3.1) if and only if M^n is mixed totally geodesic and e_o lies in the relative null space $\mathcal{N}_x$ at x.

(3) If M^n is $\mathcal{D}_1$-minimal, then

(a) the equality case of (7.3.1) holds identically for all unit tangent vectors to N_1 at each $x \in M^n$ if and only if M^n is mixed totally geodesic and $\mathcal{D}_1$-totally geodesic warped product submanifold in $\tilde{M}^m(c)$,

(b) the equality case of (7.3.1) holds identically for all unit tangent vectors to N_2 at each $x \in M^n$ if and only if M^n is mixed totally geodesic and either a $\mathcal{D}_2$-

totally geodesic warped product submanifold, or M^n is a $\mathcal{D}_2$-totally umbilical warped product submanifold in $\tilde{M}^m(c)$ with $\dim N_2 = 2$,

(c) *the equality case of (7.3.1) holds identically for all unit tangent vectors to M^n at each $x \in M^n$ if and only if either M^n is a totally geodesic submanifold, or M^n is a mixed totally geodesic, totally umbilical and $\mathcal{D}_1$-totally geodesic warped product submanifold with $\dim N_2 = 2$.*

For the $\mathcal{D}_2$-minimal case, (3) holds analogously.

Proof. First of all, let us assume M^n to be a $\mathcal{D}_1$-minimal warped product submanifold, for the $\mathcal{D}_2$-minimal case, we can apply similar scheme to the following proof. If we start from (2.3.44), then we have

$$n^2||\vec{H}||^2 = 2\tau(T_xM^n) + ||h||^2 - 2\tilde{\tau}(T_xM^n). \tag{7.3.2}$$

Let $\{e_1, \cdots, e_{n_1}, e_{n_1+1}, \cdots, e_n, e_{n+1}, \cdots, e_m\}$ to be a local orthonormal frame fields of $\Gamma(T\tilde{M}^m(c))$ such that $\{e_1, \cdots, e_{n_1}\}$ are tangent to N_1, and $\{e_{n_1+1}, \cdots, e_n\}$ are tangent to N_2. Hence, $\{e_{n+1}, \cdots, e_m\}$ are normal to M^n. By similar technique as in (Chen, 1999), but for arbitrary unit tangent vector $e_o \in \{e_1, \cdots, e_n\}$, and for arbitrary Riemannian manifold [*], we can expand (7.3.2) as the following

$$\begin{aligned} n^2\|\vec{H}\|^2 = 2\tau(T_xM^n) + \sum_{r=n+1}^{m}\Big(\big(h^r_{oo}\big)^2 + \big(h^r_{11}+\cdots+h^r_{nn}-h^r_{oo}\big)^2 + 2\sum_{1\le i<j\le n}\big(h^r_{ij}\big)^2\Big) \\ - 2\sum_{r=n+1}^{m}\sum_{\substack{1\le i<j\le n\\ i,j\neq o}} h^r_{ii}h^r_{jj} - 2\tilde{\tau}\big(T_xM^n\big) \\ = 2\tau(T_xM^n) + \frac{1}{2}\sum_{r=n+1}^{m}\Big(\big(h^r_{11}+\cdots+h^r_{nn}\big)^2 + \big(2h^r_{oo} - (h^r_{11}+\cdots+h^r_{nn})\big)^2\Big) \\ + 2\sum_{r=n+1}^{m}\sum_{1\le i<j\le n}(h^r_{ij})^2 - 2\sum_{r=n+1}^{m}\sum_{\substack{1\le i<j\le n\\ i,j\neq o}} h^r_{ii}h^r_{jj} - 2\tilde{\tau}\big(T_xM^n\big). \end{aligned} \tag{7.3.3}$$

First, because of $\mathcal{D}_1$-minimality, the above equation becomes

$$n^2||\vec{H}||^2 = 2\tau(T_xM^n) + \frac{1}{2}\sum_{r=n+1}^{m}\Big(\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)^2$$

[*] At this stage of proof, we use arbitrary tangent vector e_o in order to distinguish the factor that e_o is tangent to. For the sake of generalization, our proof is valid also for an arbitrary Riemannian manifold. This is useful to derive all inequalities of the same kind from equation (7.3.3), as we will see sooner.

$$+\Big(2h^r_{oo} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})\Big)^2\Bigg)$$

$$+ \sum_{r=n+1}^{m} \sum_{1\le i<j\le n} (h^r_{ij})^2 - \sum_{r=n+1}^{m} \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} h^r_{ii}h^r_{jj} - 2\tilde{\tau}\big(T_xM^n\big)$$

$$+ \sum_{r=n+1}^{m} \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} (h^r_{ij})^2 - \sum_{r=n+1}^{m} \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} h^r_{ii}h^r_{jj} + \sum_{r=n+1}^{m} \sum_{\substack{j=1\\ j\neq o}}^{n} (h^r_{oj})^2. \tag{7.3.4}$$

By using (2.3.43), one may write

$$\sum_{r=n+1}^{m} \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} (h^r_{ij})^2 - \sum_{r=n+1}^{m} \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} h^r_{ii}h^r_{jj} = \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} \tilde{K}_{ij} - \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} K_{ij}. \tag{7.3.5}$$

By Definition 2.3.3, and due to $\mathcal{D}_1$-minimality, we also have

$$\sum_{r=n+1}^{m} \big(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\big)^2 = n^2||\vec{H}||^2. \tag{7.3.6}$$

From (7.3.4)-(7.3.6), one will get the following

$$\frac{1}{2}n^2||\vec{H}||^2 = 2\tau(T_xM^n) + \frac{1}{2}\sum_{r=n+1}^{m}\bigg(2h^r_{oo} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})\bigg)^2$$

$$+ \sum_{r=n+1}^{m} \sum_{1\le i<j\le n} (h^r_{ij})^2 - \sum_{r=n+1}^{m} \sum_{\substack{1\le i<j\le n\\ i,\, j\neq o}} h^r_{ii}h^r_{jj} - \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} K_{ij} + \sum_{r=n+1}^{m} \sum_{\substack{j=1\\ j\neq o}}^{n} (h^r_{oj})^2$$

$$-2\tilde{\tau}\big(T_xM^n\big) + \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} \tilde{K}_{ij}. \tag{7.3.7}$$

Now, by applying Lemma 6.3.1 in (7.3.7), we directly obtain

$$\frac{1}{2}n^2||\vec{H}||^2 = \tau(T_xM^n) + \frac{1}{2}\sum_{r=n+1}^{m}\bigg(2h^r_{oo} - \Big(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\Big)\bigg)^2$$

$$+ \sum_{r=n+1}^{m}\bigg\{\sum_{1\le i<j\le n} \big(h^r_{ij}\big)^2 - \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} h^r_{ii}h^r_{jj}\bigg\} - \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} K_{ij} + \sum_{r=n+1}^{m} \sum_{\substack{j=1\\ j\neq o}}^{n} \big(h^r_{oj}\big)^2$$

$$-2\tilde{\tau}\big(T_xM^n\big) + \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} \tilde{K}_{ij} + \frac{n_2\Delta f}{f} + \sum_{r=n+1}^{m}\bigg\{\sum_{1\le a<b\le n_1}\Big(h^r_{aa}h^r_{bb} - \big(h^r_{ab}\big)^2\Big)$$

$$+ \sum_{n_1+1\le A<B\le n}\Big(h^r_{AA}h^r_{BB} - \big(h^r_{AB}\big)^2\Big)\bigg\} + \tilde{\tau}\big(T_xN_1\big) + \tilde{\tau}\big(T_xN_2\big). \tag{7.3.8}$$

For the sake of simplicity, we put

$$\varpi = \frac{n_2\Delta f}{f} - 2\tilde{\tau}\big(T_xM^n\big) + \sum_{\substack{1\le i<j\le n\\ i,j\neq o}} \tilde{K}_{ij} + \tilde{\tau}\big(T_xN_1\big) + \tilde{\tau}\big(T_xN_2\big). \tag{7.3.9}$$

So that, equation (7.3.8) takes the form

$$\frac{1}{2}n^2||\vec{H}||^2 = \varpi+\tau(T_xM^n)-\sum_{\substack{1\le i<j\le n\\ i,j\neq o}} K_{ij}+\frac{1}{2}\sum_{r=n+1}^{m}\Big(2h^r_{oo}-\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)\Big)^2$$

$$+\sum_{r=n+1}^{m}\Big\{\sum_{1\le i<j\le n}\big(h^r_{ij}\big)^2-\sum_{\substack{1\le i<j\le n\\ i,j\neq o}} h^r_{ii}h^r_{jj}\Big\}+\sum_{r=n+1}^{m}\sum_{\substack{j=1\\ j\neq o}}^{n}\big(h^r_{oj}\big)^2$$

$$+\sum_{r=n+1}^{m}\Big\{\sum_{1\le a<b\le n_1}\Big(h^r_{aa}h^r_{bb}-\big(h^r_{ab}\big)^2\Big)+\sum_{n_1+1\le A<B\le n}\Big(h^r_{AA}h^r_{BB}-\big(h^r_{AB}\big)^2\Big)\Big\}. \quad (7.3.10)$$

According to the choice of the unit tangent vector e_o, we know that it is either tangent to the first factor, or to the second, hence we distinguish the two cases:

Case (i): If e_o is tangent to N_1, then we fix a unit tangent vector from $e_1, \cdots, e_{n_1}$ to be e_o, without loss of generality let $e_o = e_1$. Hence, from (2.3.11) and (7.3.10) one can deduce that

$$\frac{1}{2}n^2\|\vec{H}\|^2 \ge Ric(e_1)+\varpi_1+\frac{1}{2}\sum_{r=n+1}^{m}\Big(2h^r_{11}-\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)\Big)^2$$

$$+\sum_{r=n+1}^{m}\Big\{\sum_{1\le i<j\le n}(h^r_{ij})^2-\Big(\sum_{1\le a<b\le n_1}(h^r_{ab})^2+\sum_{n_1+1\le A<B\le n}\big(h^r_{AB}\big)^2\Big)\Big\}$$

$$+\sum_{r=n+1}^{m}\Big\{\sum_{1\le a<b\le n_1}h^r_{aa}h^r_{bb}+\sum_{r=n+1}^{m}\sum_{n_1+1\le A<B\le n}h^r_{AA}h^r_{BB}-\sum_{2\le i<j\le n}h^r_{ii}h^r_{jj}\Big\}, \quad (7.3.11)$$

where $\varpi_1 = \varpi$ for $o = 1$.

Straightforward computations lead to

$$\sum_{r=n+1}^{m}\Big\{\sum_{1\le i<j\le n}(h^r_{ij})^2-\Big(\sum_{1\le a<b\le n_1}(h^r_{ab})^2+\sum_{n_1+1\le A<B\le n}\big(h^r_{AB}\big)^2\Big)\Big\}$$

$$=\sum_{r=n+1}^{m}\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}\big(h^r_{aA}\big)^2, \quad (7.3.12)$$

and

$$\sum_{r=n+1}^{m}\Big\{\sum_{1\le a<b\le n_1}h^r_{aa}h^r_{bb}+\sum_{r=n+1}^{m}\sum_{n_1+1\le A<B\le n}h^r_{AA}h^r_{BB}-\sum_{2\le i<j\le n}h^r_{ii}h^r_{jj}\Big\}$$

$$=\sum_{r=n+1}^{m}\Big\{\sum_{b=2}^{n_1}h^r_{11}h^r_{bb}-\sum_{a=2}^{n_1}\sum_{A=n_1+1}^{n}h^r_{aa}h^r_{AA}\Big\}. \quad (7.3.13)$$

From (7.3.11)-(7.3.13), it implies

$$\frac{1}{2}n^2\|\vec{H}\|^2 \ge Ric(e_1)+\varpi_1+\frac{1}{2}\sum_{r=n+1}^{m}\Big(2h^r_{11}-\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)\Big)^2$$

$$+\sum_{r=n+1}^{m}\Big\{\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}\big(h^r_{aA}\big)^2+\sum_{b=2}^{n_1}h^r_{11}h^r_{bb}-\sum_{a=2}^{n_1}\sum_{A=n_1+1}^{n}h^r_{aa}h^r_{AA}\Big\}. \tag{7.3.14}$$

If we use equations (7.2.1) and (7.2.2) of Lemma (7.2.1) to evaluate the last two terms of the right hand side in (7.3.14), then the above inequality descends to the following form

$$\frac{1}{2}n^2||\vec{H}||^2\geq Ric(e_1)+\varpi_1+\frac{1}{2}\sum_{r=n+1}^{m}\Big(2h^r_{11}-\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)\Big)^2$$

$$-\sum_{r=n+1}^{m}\big(h^r_{11}\big)^2+\sum_{r=n+1}^{m}\sum_{A=n_1+1}^{n}h^r_{11}h^r_{AA}. \tag{7.3.15}$$

In virtue of Lemma 7.2.2, the inequality in (7.3.15) yields

$$\frac{1}{2}n^2||\vec{H}||^2\geq Ric(e_1)+\varpi_1+\sum_{r=n+1}^{m}\Big\{\big(h^r_{11}\big)^2-\sum_{A=n_1+1}^{n}h^r_{11}h^r_{AA}+\frac{1}{4}\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)^2\Big\}$$

$$+\frac{1}{4}\sum_{r=n+1}^{m}\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)^2. \tag{7.3.16}$$

As discussed above, and in view of our hypothesis, the last term of the right hand side in the above inequality equals $\frac{1}{4}n^2||\vec{H}||^2$. Therefor, by straightforward computations, taking into account $(7.3.9)$ and $(2.3.9)$, we finally reach

$$\frac{1}{4}n^2||\vec{H}||^2\geq Ric(e_1)+\sum_{r=n+1}^{m}\Big(h^r_{11}-\frac{1}{2}\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)\Big)^2+\frac{n_2\,\Delta f}{f}-c\big(n_1n_2+n-1\big),$$

which gives (7.3.1) immediately.

Case (ii): If e_o is tangent to N_2, then we fix a unit tangent vector from $e_{n_1+1},\cdots,e_n$ to be e_o, say $e_o=e_n$. Then from (2.3.11) and (7.3.10), we derive

$$\frac{1}{2}n^2\|\vec{H}\|^2\geq Ric(e_n)+\varpi_n+\frac{1}{2}\sum_{r=n+1}^{m}\Big(2h^r_{nn}-\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)\Big)^2$$

$$+\sum_{r=n+1}^{m}\Big\{\sum_{1\leq i<j\leq n}(h^r_{ij})^2-\Big(\sum_{1\leq a<b\leq n_1}(h^r_{ab})^2+\sum_{n_1+1\leq A<B\leq n}\big(h^r_{AB}\big)^2\Big)\Big\}$$

$$+\sum_{r=n+1}^{m}\Big\{\sum_{1\leq a<b\leq n_1}h^r_{aa}h^r_{bb}+\sum_{r=n+1}^{m}\sum_{n_1+1\leq A<B\leq n}h^r_{AA}h^r_{BB}-\sum_{1\leq i<j\leq n-1}h^r_{ii}h^r_{jj}\Big\}, \tag{7.3.17}$$

where $\varpi_n=\varpi$ for $o=n$.

By similar analogue to *case (i)*, we obtain

$$\frac{1}{2}n^2\|\vec{H}\|^2\geq Ric(e_n)+\varpi_n+\frac{1}{2}\sum_{r=n+1}^{m}\Big(2h^r_{nn}-\big(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}\big)\Big)^2$$

$$+\sum_{r=n+1}^{m}\Big\{\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}\big(h^r_{aA}\big)^2+\sum_{B=n+1}^{n-1}h^r_{nn}h^r_{BB}-\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n-1}h^r_{aa}h^r_{AA}\Big\}. \tag{7.3.18}$$

Since M^n is a $\mathcal{D}_1$-minimal warped product submanifold, we have

$$\sum_{r=n+1}^{m}\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n-1} h^r_{aa}h^r_{AA} = 0. \tag{7.3.19}$$

Hence, in view of (7.3.19), the inequality in (7.3.18) reduces to this form

$$\frac{1}{2}n^2\|\vec{H}\|^2 \geq Ric(e_n) + \varpi_n + \frac{1}{2}\sum_{r=n+1}^{m}\Big(2h^r_{nn} - \big(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\big)\Big)^2$$

$$+ \sum_{r=n+1}^{m}\Big\{\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}\big(h^r_{aA}\big)^2 + \sum_{B=n+1}^{n-1} h^r_{nn}h^r_{BB}\Big\}. \tag{7.3.20}$$

In view of Lemma 7.2.3, the inequality of (7.3.20) takes the following form

$$\frac{1}{2}n^2||\vec{H}||^2 \geq Ric(e_n) + \varpi_n + \sum_{r=n+1}^{m}\Big\{\big(h^r_{nn}\big)^2 - \sum_{A=n_1+1}^{n} h^r_{nn}h^r_{AA} + \frac{1}{4}\big(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\big)^2\Big\}$$

$$+\frac{1}{4}\sum_{r=n+1}^{m}\big(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\big)^2. \tag{7.3.21}$$

Similar strategy as in the proof of case (i) leads to

$$\frac{1}{4}n^2||\vec{H}||^2 \geq Ric(e_n) + \sum_{r=n+1}^{m}\Big(h^r_{nn} - \frac{1}{2}\big(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}\big)\Big)^2 + \frac{n_2\Delta f}{f} - c\big(n_1n_2 + n - 1\big),$$

which directly gives (7.3.1).

Analogously, it is straightforward to derive a typical inequality as in (7.3.1) by following similar procedure as in case (i), when M^n is a $\mathcal{D}_2$-minimal warped product submanifold. Therefore, for $i = 1$ and 2 the inequality in (7.3.1) holds for a $\mathcal{D}_i$-minimal isometric immersion. Since e_o is an arbitrary unit tangent vector to M^n at x, (7.3.1) follows directly.

Now, we are going to discuss the equality cases of this inequality. Firstly, let us recall the notion of the relative null space, $\mathcal{N}_x$, of the submanifold M^n in the Riemannian manifold $\tilde{M}^m$ at a point $x \in M^n$, which was defined in (Chen, 1999). That is,

$$\mathcal{N}_x = \{X \in T_xM^n : h(X, Y) = 0 \quad \forall\, Y \in T_xM^n\}. \tag{7.3.22}$$

For $o \in \{1, \cdots, n\}$, a unit tangent vector e_o to M^n at x satisfies the equality sign of (7.3.1) identically if and only if the following three conditions hold

$$\left.\begin{aligned} &\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}(h^r_{aA})^2 = 0;\\ &\sum_{\substack{j=1\\ j\neq o}}^{n}(h^r_{oj})^2 = 0;\\ &2h^r_{oo} = h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}, \end{aligned}\right\} \tag{7.3.23}$$

where $r \in \{n+1, \cdots, m\}$. The mixed totally geodesy follows from the first condition in (7.3.23), whereas minimality and the other two conditions imply that e_o lies in the relative null space $\mathcal{N}_x$. Since the converse is trivial, this proves statement (2).

For a $\mathcal{D}_1$-minimal warped product submanifold, the equality sign of (7.3.1) holds identically for all unit tangent vectors to N_1 at x if and only if

$$\left.\begin{aligned} \sum_{a=1}^{n_1} \sum_{A=n_1+1}^{n} (h^r_{aA})^2 = 0; \\ \sum_{j=1}^{n} \sum_{\substack{a=1 \\ (j \neq a)}}^{n_1} (h^r_{aj})^2 = 0; \\ 2h^r_{aa} = h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}, \end{aligned}\right\} \tag{7.3.24}$$

where $a \in \{1, \cdots, n_1\}$ and $r \in \{n+1, \cdots, m\}$. Since M is $\mathcal{D}_1$-minimal, the third condition above implies

$$h^r_{aa} = 0, \quad \forall a \in \{1, \cdots, n_1\}.$$

Joining the above equation with the second condition in (7.3.24), we can show that M^n is $\mathcal{D}_1$-totally geodesic warped product in $\tilde{M}^m(c)$, while mixed totally geodesy follows from the first condition of (7.3.24), which proves (a) in statement (3).

For a $\mathcal{D}_1$-minimal warped product submanifold, the equality sign of (7.3.1) holds identically for all unit tangent vectors to N_2 at x if and only if the following are satisfied

$$\left.\begin{aligned} \sum_{a=1}^{n_1} \sum_{A=n_1+1}^{n} (h^r_{aA})^2 = 0; \\ \sum_{j=1}^{n} \sum_{\substack{A=n_1+1 \\ (j \neq A)}}^{n} (h^r_{Aj})^2 = 0; \\ 2h^r_{AA} = h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}, \end{aligned}\right\} \tag{7.3.25}$$

where $A \in \{n_1+1, \cdots, n\}$ and $r \in \{n+1, \cdots, m\}$.

The first condition above always means that M^n is mixed totally geodesic submanifold of $\tilde{M}^m(c)$.

From the third condition above two possibilities arise, either

$$h^r_{AA} = 0, \quad \forall A \in \{n_1+1, \cdots, n\}, \quad r \in \{n+1, \cdots, m\}, \tag{7.3.26}$$

or, $\dim N_2 = 2$.

If (7.3.26) holds, then in view of the second condition in (7.3.25), we conclude that M^n is a $\mathcal{D}_2$-totally geodesic warped product submanifold in $\tilde{M}^m(c)$. This is the first situation of part (b) of statement (3) of the theorem.

For the other situation, assume that M^n is not $\mathcal{D}_2$-totally geodesic warped product submanifold in $\tilde{M}^m(c)$ and $\dim N_2 = 2$. Therefore, from the second condition of (7.3.25) again, we conclude that M^n is a $\mathcal{D}_2$-totally umbilical warped product submanifold in $\tilde{M}^m(c)$, which is the second situation of this part. Thus, this proves this part completely.

To show (c), we first combine (7.3.24) and (7.3.25) together. Thus, we can use parts (a) and (b) of (3).

For the first situation of this part, assume firstly that $\dim N_2 \neq 2$. Because parts (a) and (b) of statement (3) respectively imply that M^n is $\mathcal{D}_1$-totally geodesic and $\mathcal{D}_2$-totally geodesic submanifold in $\tilde{M}^m(c)$. Hence, M^n is a totally geodesic submanifold in $\tilde{M}^m(c)$.

For the other situation, assume that the first situation does not hold. As a result, parts (a) and (b) directly give that M^n is mixed totally geodesic and $\mathcal{D}_1$-totally geodesic submanifold in $\tilde{M}^m(c)$ with $\dim N_2 = 2$.

To show that M^n is a totally umbilical submanifold in $\tilde{M}^m(c)$, it is enough to know that M^n is $\mathcal{D}_2$-totally umbilical warped product submanifold in $\tilde{M}^m(c)$ from (b), and it is $\mathcal{D}_1$-totally geodesic from (a). Hence, this gives the assertion of part (c).

Following similar procedure as above, the equality cases can be proved whenever M^n is a $\mathcal{D}_2$minimal warped product submanifold in $\tilde{M}^m(c)$. Hence, the proof is completed. $\square$

Now, we are going to discuss some applications and generalizations that the above theorem and proof give. We point out that, this kind of inequalities was proved for Riemannian space forms (Chen, 1999). On one hand, neither Chen nor others proved such inequality for warped product submanifolds. This was enough to motivate us for carrying out the preceding long proof. On the other hand, so many ”extensions” of the original inequality that Chen proved in (Chen, 1999) were published, (see, for example references of (Chen, 2008)). In fact, all these inequalities, including the original one, are direct consequences of the next inequality

$$\frac{n^2}{4}\|\vec{H}\|^2 \geq Ric(e_o) + \sum_{\substack{1\leq i<j\leq n\\ i,j\neq o}} \tilde{K}_{ij} - \tilde{\tau}\big(T_xM^n\big). \tag{7.3.27}$$

One can easily figure out that the above inequality comes from $(7.3.3)$. To obtain all inequalities of the same kind, it is enough to use the corresponding curvature tensor formula to calculate

$$\sum_{\substack{1\leq i<j\leq n\\ i,j\neq o}} \tilde{K}_{ij} - \tilde{\tau}\big(T_xM^n\big).$$

The above technique enables us to derive any inequality of this kind, hence the proof above is a multi-task proof. Another significant application of this proof comes in the next few lines.

In the second section of (Yoon, 2006), this kind of inequalities were proved for different classes of submanifolds; invariant, anti-invariant, slant, bi-slant and semi-slant submanifolds of cosymplectic manifolds, but not for warped products also. Some of these results can be obtained from $(7.3.27)$. For semi-slant and bi-slant submanifolds, take f to be constant, then one should follow similar procedure as we will explain in section four.

7.3.1 A NECESSARY CONDITION FOR THE MINIMALITY OF WARPED PRODUCT SUBMANIFOLDS

As a first answer of Problems 1.4.11 and 1.4.12, we apply the inequality of Theorem 7.3.1 to give a necessary condition for a warped product to be $\mathcal{D}_i$-minimal (for both $i = 1, 2$) in a Euclidean m-space $\mathbb{E}^m$. It is obvious that assuming $\mathcal{D}_i$-minimality for both $i = 1, 2$ implies the minimality of M^n in $\mathbb{E}^m$.

Corollary 7.3.1. *If $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \mathbb{E}^m$ is a $\mathcal{D}_i$-minimal isometric immersion (for both $i = 1, 2$), from a warped product submanifold M^n into a Euclidean m-space, then*

$$Ric(e_o) \leq -\frac{n_2\,\Delta f}{f}, \tag{7.3.28}$$

for each unit vector $e_o \in T_xM^n$, where n_2 is the dimension of N_2.

In view of $(2.3.10)$ and $(2.3.11)$, one can conclude that the Ricci curvatures determine the Ricci tensor completely. Taking the warping function, f, to be constant. Then, it will be clear that this corollary is coherent with the second condition stated in chapter one; namely,

Condition 2: If $\varphi : M^n \to \mathbb{E}^m$ is a minimal immersion from a manifold of positive dimension into a Euclidean m-space, then the Ricci tensor of M^n is negative semi-definite.

7.4 SOME EXTENSIONS OF THEOREM 7.3.1

In this section, we discuss some extensions of Theorem 7.3.1 to ambient spaces with other geometric structures.

For any $\mathcal{D}_i$-minimal warped product submanifold we use (7.3.16) to derive the inequality when the unit vector e_o is tangent to the first factor, and (7.3.21) when it is tangent to the second. Thus, all what we need is to use the specific form of the curvature tensor to evaluate the expressions

$$\varpi_1 - \frac{n_2 \Delta f}{f} = -2\tilde{\tau}(T_x M^n) + \sum_{2 \leq i < j \leq n} \tilde{K}_{ij} + \tilde{\tau}(T_x N_1) + \tilde{\tau}(T_x N_2) \tag{7.4.1}$$

for the first case, and

$$\varpi_n - \frac{n_2 \Delta f}{f} = -2\tilde{\tau}(T_x M^n) + \sum_{1 \leq i < j \leq n-1} \tilde{K}_{ij} + \tilde{\tau}(T_x N_1) + \tilde{\tau}(T_x N_2) \tag{7.4.2}$$

for the second.

Firstly, let $M^n = N_\theta \times_f N_\perp$ be a $\mathcal{D}_\theta$-minimal warped product submanifold in a complex space form $\tilde{M}^{2m}(c_{Ka})$. Then, we have

$$\frac{1}{4} n^2 ||\vec{H}||^2 \geq Ric(e_o) + \frac{n_2 \, \Delta f}{f} - \frac{c_{Ka}}{4}[n_1 n_2 + n - 1 + \frac{3}{2}\cos^2\theta], \tag{7.4.3}$$

when the unit vector $e_o \in T_x N_\theta$, and

$$\frac{1}{4} n^2 ||\vec{H}||^2 \geq Ric(e_o) + \frac{n_2 \, \Delta f}{f} - \frac{c_{Ka}}{4}[n_1 n_2 + n - 1], \tag{7.4.4}$$

when it is tangent to $N_\perp$, whereas the equality case is the same as that in the Theorem 7.3.1.

To see how we can get the above two inequalities, one just needs to use the tensor formula of complex space form (2.3.49). Therefore, using (2.3.49) we compute (7.4.1). As a result, the inequality in (7.4.3) follows directly. That is;

$$-\frac{c_{Ka}}{4}[n(n-1) + 3\cos^2\theta n_1] + \frac{c_{Ka}}{8}[(n-1)(n-2) + 3\cos^2\theta(n_1 - 1)]$$

$$+\frac{c_{Ka}}{8}[n_1(n_1-1)+3\cos^2\theta n_1]+\frac{c_{Ka}}{8}[n_2(n_2-1)]=-\frac{c_{Ka}}{4}[n_1n_2+n-1+\frac{3}{2}\cos^2\theta].$$

Similarly, in the second case, we obtain

$$-\frac{c_{Ka}}{4}[n(n-1)+3\cos^2\theta n_1]+\frac{c_{Ka}}{8}[(n-1)(n-2)+3\cos^2\theta n_1]$$
$$+\frac{c_{Ka}}{8}[n_1(n_1-1)+3\cos^2\theta n_1]+\frac{c_{Ka}}{8}[n_2(n_2-1)]=-\frac{c_{Ka}}{4}[n_1n_2+n-1].$$

For Kenmotsu ambient space, it is easy to derive this inequalities for the warped products $N_T\times_f N_\perp$, $N_T\times_f N_T$ and $N_T\times_f N_\theta$ in a Kenmotsu space form where ξ is tangent to the first factor. For example, we will compute them for the last warped product, then the others are similar. For each unit vector $e_o\in T_xM^n$ orthogonal to ξ, we have the following

$$\frac{1}{4}n^2H^2\geq Ric(e_o)+\frac{n_2\Delta f}{f}-\frac{(c_{Ke}-3)}{4}[n+n_1n_2-1]+\frac{(c_{Ke}+1)}{4}[n_2-\frac{1}{2}],$$

in case e_o is tangent to N_T, and

$$\frac{1}{4}n^2H^2\geq Ric(e_o)+\frac{n_2\Delta f}{f}-\frac{(c_{Ke}-3)}{4}[n+n_1n_2-1]$$
$$+\frac{(c_{Ke}+1)}{4}[n_2+1-\frac{3}{2}\cos^2\theta],$$

for the other case.

Analogously, for other particular cases, (7.3.16) and (7.3.21) are useful to obtain this inequalities for any $\mathcal{D}_i$-minimal warped product submanifold. For example, in generalized complex space forms, we have

$$\frac{1}{4}n^2||\vec{H}||^2\geq Ric(e_o)+\frac{n_2\Delta f}{f}-\frac{c_{RK}}{4}[n_1n_2+n+\frac{1}{2}]$$

for the first case, and

$$\frac{1}{4}n^2||\vec{H}||^2\geq Ric(e_o)+\frac{n_2\Delta f}{f}-\frac{c_{RK}}{4}[n_1n_2+n-1]$$

for the second.

A lot of other important inequalities can be obtained from Theorem 7.3.1, for instance $N_T\times_f N_\perp$ as a $\mathcal{D}_1$-minimal and $N_\perp\times_f N_T$ as a $\mathcal{D}_2$-minimal warped product submanifolds in $l.c.K$ manifolds, semi-slant and generic warped products in $l.c.K$ also. We leave these examples and others in cosymplectic manifolds for the reader.

Chen obtained some estimates of the squared mean curvature for isometrically immersed submanifolds in some spaces (Chen, 1999). Here, we apply Theorem 7.3.1 to generalize those estimates for warped product submanifolds.

Corollary 7.4.1. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m(c)$ be a $\mathcal{D}_i$-minimal isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian space form $\tilde{M}^m(c)$. Then*

$$||\vec{H}(x)||^2 \geq (\frac{4}{(n_1+n_2)^2})\{\max_{e_o} Ric(e_o) + \frac{n_2\,\Delta f}{f} - c[n_1+n_2+n_1n_2-1]\},$$

for any unit tangent vectors e_o at x.

Corollary 7.4.2. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m(c)$ be a $\mathcal{D}_i$-minimal isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian space form $\tilde{M}^m(c)$. Then, for any integer k, $2 \leq k \leq n$, we have*

$$||\vec{H}(x)||^2 \geq (\frac{4}{(n_1+n_2)^2})\{(\frac{(n-1)\Theta_k(x)}{k-1}) + \frac{n_2\,\Delta f}{f} - c[n_1+n_2+n_1n_2-1]\},$$

where Θ_k is the Riemannian invariant on M^n introduced in (2.3.19).

7.5 CONCLUSION

In the next two tables, we list most of cases under consideration that can be proved by applying similar procedure as above. In the second column, two inequalities are given for every ambient manifold, the upper is for case (i), and the lower one is for case (ii); i. e., $e_o \in T_xN_1$ and $e_o \in T_xN_2$, respectively. It is significant to say that, the inequality of case (i) coincides with that of case (ii) in real space forms.

For simplicity's sake, put $\mathcal{I} = \frac{n^2}{4}||\vec{H}||^2 \geq Ric(e_o) + \frac{n_2\Delta f}{f}$. Here, R. S. F $\equiv$ real space form, C. S. F. $\equiv$ complex space form, G. C. S. F. $\equiv$ generalized complex space form, Sas. S. F. $\equiv$ Sasakian space form, Co. S. F. $\equiv$ cosymplectic space form and Ke. S. F. $\equiv$ Kenmotsu space form.

Manifold	Inequality
$\mathcal{D}_i$-minimal R. S. F.	$\frac{n^2}{4}\|\|\vec{H}\|\|^2 \geq Ric(e_o) + \frac{n_2\Delta f}{f} - c[n_1n_2 + n - 1]$
$N_T \times_f N_\perp$ C. S. F.	$\mathcal{I} - \frac{c_{Ka}}{4}\left(n + n_1n_2 + \frac{1}{2}\right)$ $\mathcal{I} - \frac{c_{Ka}}{4}\left(n + n_1n_2 - 1\right)$
$N_T \times_f N_\perp$ G. C. S. F.	$\mathcal{I} - \frac{c_{RK}+3\gamma}{4}\left(n + n_1n_2 - 1\right) - \frac{c_{RK}-\gamma}{4}\left(\frac{3}{2}\right)$ $\mathcal{I} - \frac{c_{RK}+3\gamma}{4}\left(n + n_1n_2 - 1\right)$
$N_T \times_f N_\perp$ Sas. S. F.	$\mathcal{I} - \frac{c_S+3}{4}\left(n + n_1n_2 - 1\right) + \frac{c_S-1}{4}\left(n_2 - \frac{1}{2}\right)$ $\mathcal{I} - \frac{c_S+3}{4}\left(n + n_1n_2 - 1\right) + \frac{c_S-1}{4}\left(n_2\right)$
$N_T \times_f N_\perp$ Co. S. F.	$\mathcal{I} - \frac{c_c}{4}\left(n_1 + n_1n_2 - \frac{1}{2}\right)$ $\mathcal{I} - \frac{c_c}{4}\left(n_1 + n_1n_2 - 1\right)$
$N_T \times_f N_\theta$ G. C. S. F.	$\mathcal{I} - \frac{c_{RK}+3\gamma}{4}\left(n + n_1n_2 - 1\right) - \frac{c_{RK}-\gamma}{4}\left(\frac{3}{2}\right)$ $\mathcal{I} - \frac{c_{RK}+3\gamma}{4}\left(n + n_1n_2 - 1\right) - \frac{c_{RK}-\gamma}{4}\left(\frac{3}{2}\cos^2\theta\right)$
$N_T \times_f N_\perp$ Ke. S. F.	$\mathcal{I} - \frac{c_{Ke}-3}{4}\left(n + n_1n_2 - 1\right) + \frac{c_{Ke}+1}{4}\left(n_2 - \frac{1}{2}\right)$ $\mathcal{I} - \frac{c_{Ke}-3}{4}\left(n + n_1n_2 - 1\right) + \frac{c_{Ke}+1}{4}\left(n_2\right)$
$N_\perp \times_f N_T$ Ke. S. F.	$\mathcal{I} - \frac{c_{Ke}-3}{4}\left(n + n_1n_2 - 1\right) + \frac{c_{Ke}+1}{4}\left(n_2 + 1\right)$ $\mathcal{I} - \frac{c_{Ke}-3}{4}\left(n + n_1n_2 - 1\right) + \frac{c_{Ke}+1}{4}\left(n_2 - \frac{3}{2}\right)$

Table 7.1: A general inequality for $\mathcal{D}_i$-minimal warped product submanifolds in terms of Ricci curvature and the mean curvature vector.

Manifold	Inequality
$N_T \times_f N_\theta$ Ke. S. F.	$\mathcal{I} - \frac{c_{Ke}-3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_{Ke}+1}{4}\Big(n_2 - \frac{1}{2}\Big)$ $\mathcal{I} - \frac{c_{Ke}-3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_{Ke}+1}{4}\Big(n_2 - \frac{3}{2}\cos^2\theta\Big)$
$N_\theta \times_f N_T$ Ke. S. F.	$\mathcal{I} - \frac{c_{Ke}-3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_{Ke}+1}{4}\Big(n_2 - \frac{3}{2}\cos^2\theta\Big)$ $\mathcal{I} - \frac{c_{Ke}-3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_{Ke}+1}{4}\Big(n_2 - \frac{3}{2}\Big)$
$N_\perp \times_f N_\theta$ Ke. S. F.	$\mathcal{I} - \frac{c_{Ke}-3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_{Ke}+1}{4}\Big(n_2 + 1\Big)$ $\mathcal{I} - \frac{c_{Ke}-3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_{Ke}+1}{4}\Big(n_2 - \frac{3}{2}\cos^2\theta\Big)$
$N_\theta \times_f N_\perp$ Ke. S. F.	$\mathcal{I} - \frac{c_{Ke}-3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_{Ke}+1}{4}\Big(n_2 + (1 - \frac{3}{2}\cos^2\theta)\Big)$ $\mathcal{I} - \frac{c_{Ke}-3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_{Ke}+1}{4}\Big(n_2\Big)$
$N_\theta \times_f N_\perp$ Sas. S. F.	$\mathcal{I} - \frac{c_S+3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_S-1}{4}\Big(n_2 + (1 - \frac{3}{2}\cos^2\theta)\Big)$ $\mathcal{I} - \frac{c_S+3}{4}\Big(n + n_1 n_2 - 1\Big) + \frac{c_S-1}{4}\Big(n_2\Big)$
$N_\theta \times_f N_\perp$ Co. S. F.	$\mathcal{I} - \frac{c_c}{4}\Big(n_1 + n_1 n_2 - 2 + \frac{3}{2}\cos^2\theta\Big)$ $\mathcal{I} - \frac{c_c}{4}\Big(n_1 + n_1 n_2 - 1\Big)$
$N_\theta \times_f N_\perp$ C. S. F.	$\mathcal{I} - \frac{c_{Ka}}{4}\Big(n + n_1 n_2 - 1 + \frac{3}{2}\cos^2\theta\Big)$ $\mathcal{I} - \frac{c_{Ka}}{4}\Big(n + n_1 n_2 - 1\Big)$
$N_\theta \times_f N_\perp$ G. C. S. F.	$\mathcal{I} - \frac{c_{RK}+3\gamma}{4}\Big(n + n_1 n_2 - 1\Big) - \frac{c_{RK}-\gamma}{4}\Big(\frac{3}{2}\cos^2\theta)\Big)$ $\mathcal{I} - \frac{c_{RK}+3\gamma}{4}\Big(n + n_1 n_2 - 1\Big)$
$N_\perp \times_f N_\theta$ G. C. S. F.	$\mathcal{I} - \frac{c_{RK}+3\gamma}{4}\Big(n + n_1 n_2 - 1\Big)$ $\mathcal{I} - \frac{c_{RK}+3\gamma}{4}\Big(n + n_1 n_2 - 1\Big) - \frac{c_{RK}-\gamma}{4}\Big(\frac{3}{2}\cos^2\theta\Big)$

Table 7.2: A general inequality for $\mathcal{D}_i$-minimal warped product submanifolds in terms of Ricci curvature and the mean curvature vector.

CHAPTER 8: THE δ-INVARIANT INEQUALITIES OF WARPED PRODUCT SUBMANIFOLDS

8.1 INTRODUCTION

In (Chen, 1993), Chen initiated a significant inequality in terms of the intrinsic invariant δ-invariant. This celebrated inequality drew attention of several authors (Chen, 2008). Motivated by the result of Chen, we construct two new general inequalities in terms of δ-invariants, but this time for warped product submanifolds of Riemannian manifolds.

We recall the following algebraic lemma:

Lemma 8.1.1. *Let $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta$ be $(n+1)$ $(n \geq 2)$ real numbers such that*

$$(\sum_{i=1}^{n} \alpha_i)^2 = (n-1)(\sum_{i=1}^{n} \alpha_i^2 + \beta).$$

Then $2\alpha_1\alpha_2 \geq \beta$, with equality holds if and only if $\alpha_1 + \alpha_2 = \alpha_3 = \cdots = \alpha_n$.

Also, we recall the following definition of the *Chen first invariant*, which was defined in Chapter Two as

$$\delta_{\tilde{M}^m}(x) = \tilde{\tau}(T_x\tilde{M}^m) - \inf\{\tilde{K}(\pi) : \pi \subset T_x\tilde{M}^m, x \in \tilde{M}^m, \dim \pi = 2\}. \tag{8.1.1}$$

8.2 SOME TECHNICAL LEMMAS

For simplicity's sake, we present three computational lemmas in this section. Once they are verified, the proofs of inequalities in the next two sections become straightforward.

The following result is useful in proofs of the next two inequalities.

Lemma 8.2.1. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m$ be an isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian manifold $\tilde{M}^m$. Then, we have*

$$\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \sum_{r=n+2}^{m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{m}(h_{12}^r)^2 =$$

$$\frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{m}(h_{11}^r + h_{22}^r)^2 + \sum_{r=n+1}^{m}\sum_{j=3}^{n}\Big((h_{1j}^r)^2 + (h_{2j}^r)^2\Big), \tag{8.2.1}$$

where n and m are the dimensions of M^n and $\tilde{M}^m$, respectively.

Proof. If we start from the left hand side of $(8.2.1)$, then the first two terms can be respectively expanded in this way

$$\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2=\frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2+\sum_{j=3}^{n}(h_{1j}^{n+1})^2+(h_{12}^{n+1})^2+\sum_{j=3}^{n}(h_{2j}^{n+1})^2, \tag{8.2.2}$$

and

$$\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2=\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2+\sum_{r=n+2}^{m}\sum_{j=3}^{n}(h_{1j}^{r})^2+\sum_{r=n+2}^{m}\sum_{j=3}^{n}(h_{2j}^{r})^2$$
$$+\sum_{r=n+2}^{m}(h_{12}^{r})^2+\frac{1}{2}\sum_{r=n+2}^{m}\Big((h_{11}^{r})^2+(h_{22}^{r})^2\Big). \tag{8.2.3}$$

Using equations $(8.2.2)$ and $(8.2.3)$ to substitute the first two terms on the left hand side of $(8.2.1)$, taking into consideration the following relations

$$\sum_{r=n+2}^{m}h_{11}^{r}h_{22}^{r}+\frac{1}{2}\sum_{r=n+2}^{m}\Big((h_{11}^{r})^2+(h_{22}^{r})^2\Big)=\frac{1}{2}\sum_{r=n+2}^{m}(h_{11}^{r}+h_{22}^{r})^2,$$

$$\sum_{j=3}^{n}(h_{1j}^{n+1})^2+\sum_{r=n+2}^{m}\sum_{j=3}^{n}(h_{1j}^{r})^2+\sum_{j=3}^{n}(h_{2j}^{n+1})^2+\sum_{r=n+2}^{m}\sum_{j=3}^{n}(h_{2j}^{r})^2$$
$$=\sum_{r=n+1}^{m}\sum_{j=3}^{n}\Big((h_{1j}^{r})^2+(h_{2j}^{r})^2\Big)$$

and

$$(h_{12}^{n+1})^2+\sum_{r=n+2}^{m}(h_{12}^{r})^2=\sum_{r=n+1}^{m}(h_{12}^{r})^2,$$

the right hand side of $(8.2.1)$ follows immediately, and completes the proof. $\square$

For the first inequality of this chapter, we prove the following computational lemma.

Lemma 8.2.2. *Let $\varphi : M^n = N_1\times_f N_2 \longrightarrow \tilde{M}^m$ be an isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian manifold $\tilde{M}^m$. Then, the following holds*

$$\frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2+\sum_{r=n+1}^{m}\sum_{j=3}^{n}\Big((h_{1j}^{r})^2+(h_{2j}^{r})^2\Big)=$$
$$\frac{1}{2}\sum_{\substack{a,b=3\\ a\neq b}}^{n_1}(h_{ab}^{n+1})^2+\frac{1}{2}\sum_{\substack{A,B=n_1+1\\ A\neq B}}^{n}(h_{AB}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{a,b=3}^{n_1}(h_{ab}^{r})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{A,B=n_1+1}^{n}(h_{AB}^{r})^2$$
$$+\sum_{r=n+1}^{m}\sum_{a=3}^{n_1}\Big((h_{1a}^{r})^2+(h_{2a}^{r})^2\Big)+\sum_{r=n+1}^{m}\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^2, \tag{8.2.4}$$

where $n_1,\ n$ and m are the dimensions of $N_1,\ M^n$ and $\tilde{M}^m$, respectively.

Proof. It is not difficult to expand the following expressions as

$$\frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2=\frac{1}{2}\sum_{\substack{a,b=3\\ a\neq b}}^{n_1}(h_{ab}^{n+1})^2+\frac{1}{2}\sum_{\substack{A,B=n_1+1\\ A\neq B}}^{n}(h_{AB}^{n+1})^2+\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{n+1})^2, \quad (8.2.5)$$

and

$$\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2=\frac{1}{2}\sum_{r=n+2}^{m}\sum_{a,b=3}^{n_1}(h_{ab}^{r})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{A,B=n_1+1}^{n}(h_{AB}^{r})^2$$
$$+\sum_{r=n+2}^{m}\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^2. \quad (8.2.6)$$

If we use the above two equations to substitute the first two terms in the left hand side of $(8.2.4)$, taking into account the following equation

$$\sum_{r=n+1}^{m}\sum_{j=3}^{n}\Big((h_{1j}^{r})^2+(h_{2j}^{r})^2\Big)+\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{n+1})^2+\sum_{r=n+2}^{m}\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^2=$$
$$\sum_{r=n+1}^{m}\sum_{a=3}^{n_1}\Big((h_{1a}^{r})^2+(h_{2a}^{r})^2\Big)+\sum_{r=n+1}^{m}\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^2,$$

then the right hand side of $(8.2.4)$ automatically follows. □

Finally, this lemma is necessary for the final inequality of this work.

Lemma 8.2.3. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m$ be an isometric immersion of an n-dimensional warped product submanifold M^n into a Riemannian manifold $\tilde{M}^m$. Then, the following equation holds*

$$\frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2+\frac{1}{2}\sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2=$$
$$\frac{1}{2}\sum_{r=n+2}^{m}\sum_{a,b=3}^{n_1}(h_{ab}^{r})^2+\sum_{r=n+2}^{m}\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^2+\frac{1}{2}\sum_{\substack{a,b=3\\ a\neq b}}^{n_1}(h_{ab}^{n+1})^2$$
$$+\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{n+1})^2+\frac{1}{2}\sum_{r=n+1}^{m}\sum_{A,B=n_1+1}^{n}(h_{AB}^{r})^2,$$

where $n_1,\ n$ and m are the dimensions of $N_1,\ M^n$ and $\tilde{M}^m$, respectively.

Proof. First, observe that

$$\frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2=\frac{1}{2}\sum_{\substack{a,b=3\\ a\neq b}}^{n_1}(h_{ab}^{n+1})^2+\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{n+1})^2+\frac{1}{2}\sum_{\substack{A,B=n_1+1\\ A\neq B}}^{n}(h_{AB}^{n+1})^2,$$

and

$$\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2=\frac{1}{2}\sum_{r=n+2}^{m}\sum_{a,b=3}^{n_1}(h_{ab}^{r})^2$$

$$+\sum_{r=n+2}^{m}\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{A,B=n_1+1}^{n}(h_{AB}^{r})^2.$$

By the preceding couple of equations, with help of the following two

$$\frac{1}{2}\sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2+\frac{1}{2}\sum_{\substack{A,B=n_1+1\\ A\neq B}}^{n}(h_{AB}^{n+1})^2=\frac{1}{2}\sum_{A,B=n_1+1}^{n}(h_{AB}^{n+1})^2$$

and

$$\frac{1}{2}\sum_{A,B=n_1+1}^{n}(h_{AB}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{A,B=n_1+1}^{n}(h_{AB}^{r})^2=\frac{1}{2}\sum_{r=n+1}^{m}\sum_{A,B=n_1+1}^{n}(h_{AB}^{r})^2,$$

the assertion follows directly. □

8.3 THE δ-INVARIANT INEQUALITY FOR $\mathcal{D}_i$-MINIMAL WARPED PRODUCT SUBMANIFOLDS

For the class of $\mathcal{D}_i$-minimal warped product submanifolds, we establish the following relationship

Theorem 8.3.1. *For $i=1$ or 2, let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m(c)$ be a $\mathcal{D}_i$-minimal isometric immersion of a warped product submanifold M^n, for $n \geq 2$, into a Riemannian space form $\tilde{M}^m(c)$. Then, for each point $x \in M^n$ and each plane $\pi \subset T_xM^n$, we have*

$$\delta_{M^n}(x)-\frac{n_2\,\Delta f}{f}\leq\frac{n^2(n-2)}{2(n-1)}||\vec{H}||^2-(n_1n_2+1-\frac{1}{2}n^2+\frac{1}{2}n)c. \qquad (8.3.1)$$

Equality in (8.3.1) holds at $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, \cdots, e_m\}$ of $T_x^{\perp}M^n$ such that (a) $\pi = Span\{e_1, e_2\}$ and (b) the shape operators take the following forms:

If $r = n+1$, *then*

$$A_{e_{n+1}} = \begin{pmatrix} \mu_1 & h_{12}^{n+1} & 0_{13} & 0_{1n_1} & h_{1n_1+1}^{n+1} & \cdots & \cdots & \cdots & h_{1n}^{n+1} \\ h_{21}^{n+1} & \mu_2 & 0_{23} & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{31} & 0_{32} & \mu & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{n_11} & \cdots & 0 & \mu & h_{n_1n_1+1}^{n+1} & \cdots & \cdots & \cdots & h_{n_1n}^{n+1} \\ h_{n_1+11}^{n+1} & \cdots & \cdots & h_{n_1+1n_1}^{n+1} & \ddots & 0 & \cdots & \cdots & 0_{n_1+1n} \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ h_{n1}^{n+1} & \cdots & \cdots & h_{nn_1}^{n+1} & 0_{nn_1+1} & 0 & \cdots & 0 & \mu \end{pmatrix},$$

$\mu = \mu_1 + \mu_2$.

If $r \in \{n+2, \cdots, m\}$, *then*

$$A_{e_r} = \left(\begin{array}{cccc|ccccc} h_{11}^r & h_{12}^r & 0_{13} & 0_{1n_1} & h_{1n_1+1}^r & \cdots & \cdots & \cdots & h_{1n}^r \\ h_{12}^r & -h_{11}^r & 0_{23} & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{31} & 0_{32} & 0_{33} & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{n_11} & \cdots & \cdots & 0_{n_1n_1} & h_{n_1n_1+1}^r & \cdots & \cdots & \cdots & h_{n_1n}^r \\ \hline h_{n_1+11}^r & \cdots & \cdots & h_{n_1+1n_1}^r & \ddots & 0 & 0 & \cdots & 0_{n_1+1n} \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ h_{n1}^r & \cdots & \cdots & h_{nn_1}^r & 0_{nn_1+1} & 0 & \cdots & 0 & 0_{nn} \end{array}\right).$$

Proof. Let $x \in M^n$ and $\pi \subset T_xM^n$ a 2-plane. We choose an orthonormal basis $\{e_1, \cdots, e_{n_1}, e_{n_1+1}, \cdots, e_n\}$ of T_xM^n, where $n_1 + n_2 = n$, and $\{e_{n+1}, \cdots, e_m\}$ is an orthonormal basis of $T_x^{\perp}M^n$, such that $\pi = Span\{e_1, e_2\}$ and the normal vector e_{n+1} is in the direction of the mean curvature vector $\vec{H}$, hence $e_{n+1} = \frac{\vec{H}}{||\vec{H}||}$.

In virtue of (2.3.56) and (2.3.44), we obtain

$$n^2||\vec{H}||^2 = 2\tau(T_xM^n) + ||h||^2 - n(n-1)c. \tag{8.3.2}$$

The above equation may be written as

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\tau(T_xM^n) + ||h||^2 - n(n-1)c. \tag{8.3.3}$$

Now, if we put

$$\Xi = 2\tau(T_xM^n) - \frac{n^2(n-2)}{n-1}||\vec{H}||^2 - n(n-1)c, \tag{8.3.4}$$

then from (8.3.3) and (8.3.4), we have

$$n^2||\vec{H}||^2 = (n-1)\big(\Xi + ||h||^2\big). \tag{8.3.5}$$

Moreover, it is obvious that

$$||h||^2 = \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2. \tag{8.3.6}$$

Thus, combining $(8.3.5)$ and $(8.3.6)$ together, we get

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left(\Xi + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right). \tag{8.3.7}$$

Applying Lemma 8.1.1 on the above relation for

$$\alpha_i = h_{ii}^{n+1}, \quad \forall\, i \in \{1, \cdots, n\}$$

and

$$\beta = \Xi + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2,$$

we derive

$$h_{11}^{n+1}h_{22}^{n+1} \geq \frac{1}{2}\left(\Xi + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right). \tag{8.3.8}$$

In virtue of (2.3.9) and (2.3.44), it is possible to write

$$K(\pi) = c + \sum_{r=n+1}^{m}\left(h_{11}^r h_{22}^r - (h_{12}^r)^2\right). \tag{8.3.9}$$

Equivalently,

$$K(\pi) = c + \sum_{r=n+2}^{m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{m}(h_{12}^r)^2 + h_{11}^{n+1}h_{22}^{n+1}. \tag{8.3.10}$$

From (8.3.8) and (8.3.10), we obtain

$$\begin{aligned} K(\pi) \geq{}& c + \frac{1}{2}\,\Xi + \sum_{r=n+2}^{m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{m}(h_{12}^r)^2 \\ &+ \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2. \end{aligned} \tag{8.3.11}$$

By means of Lemma 8.2.1, it is clear that the above inequality is congruent to

$$K(\pi) \geq c + \frac{1}{2}\Xi + \frac{1}{2}\sum_{\substack{i,j=3\\i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2$$

$$+ \frac{1}{2}\sum_{r=n+2}^{m}(h_{11}^{r} + h_{22}^{r})^2 + \sum_{r=n+1}^{m}\sum_{j=3}^{n}\Big((h_{1j}^{r})^2 + (h_{2j}^{r})^2\Big). \qquad (8.3.12)$$

A help may be taken from Lemma 8.2.2, to show that the above inequality is identical to

$$K(\pi) \geq c + \frac{1}{2}\Xi + \frac{1}{2}\sum_{\substack{a,b=3\\a\neq b}}^{n_1}(h_{ab}^{n+1})^2 + \frac{1}{2}\sum_{\substack{A,B=n_1+1\\A\neq B}}^{n}(h_{AB}^{n+1})^2$$

$$+ \frac{1}{2}\sum_{r=n+2}^{m}\sum_{a,b=3}^{n_1}(h_{ab}^{r})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{A,B=n_1+1}^{n}(h_{AB}^{r})^2 + \frac{1}{2}\sum_{r=n+2}^{m}(h_{11}^{r} + h_{22}^{r})^2$$

$$+ \sum_{r=n+1}^{m}\sum_{a=3}^{n_1}\Big((h_{1a}^{r})^2 + (h_{2a}^{r})^2\Big) + \sum_{r=n+1}^{m}\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^2.$$

In virtue of Theorem 5.2.1, the last term of the right hand side in the above equation can be evaluated to obtain

$$K(\pi) \geq c + \frac{1}{2}\Xi + n_1 n_2\, c - \frac{n_2\,\Delta(f)}{f}.$$

Hence, the above inequality takes the following form via (8.3.4)

$$K(\pi) \geq \tau(T_x M^n) - \frac{n^2(n-2)}{2(n-1)}||\vec{H}||^2 - \frac{n_2\,\Delta f}{f} + (n_1 n_2 + 1 - \frac{1}{2}n^2 + \frac{1}{2}n)c,$$

which gives the inequality directly.

The equality sign of (8.3.1) holds at a point $x \in M^n$ if and only if all equalities of the inequalities of the above proof hold. Form the above proof, we see that this occurs if and only if some conditions are satisfied. These conditions can be classified into two categories.

Firstly, if $r = n + 1$, then we have

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1},$$

and

$$\sum_{\substack{a,b=3\\a\neq b}}^{n_1}\left(h_{ab}^{n+1}\right)^2 = \sum_{\substack{A,B=n_1+1\\A\neq B}}^{n}\left(h_{AB}^{n+1}\right)^2 = h_{1a}^{n+1} = h_{2a}^{n+1} = 0,$$

where $a \in \{3, \cdots, n_1\}$. This implies

$$A_{e_{n+1}} = \begin{pmatrix} \mu_1 & h_{12}^{n+1} & 0_{13} & 0_{1n_1} & h_{1n_1+1}^{n+1} & \cdots & \cdots & \cdots & h_{1n}^{n+1} \\ h_{21}^{n+1} & \mu_2 & 0_{23} & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{31} & 0_{32} & \mu & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{n_11} & \cdots & 0 & \mu & h_{n_1n_1+1}^{n+1} & \cdots & \cdots & \cdots & h_{n_1n}^{n+1} \\ h_{n_1+11}^{n+1} & \cdots & \cdots & h_{n_1+1n_1}^{n+1} & \ddots & 0 & \cdots & \cdots & 0_{n_1+1n} \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & 0 & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ h_{n1}^{n+1} & \cdots & \cdots & h_{nn_1}^{n+1} & 0_{nn_1+1} & 0 & \cdots & 0 & \mu \end{pmatrix},$$

$\mu = \mu_1 + \mu_2$.

Secondly, if $r \in \{n+2, \cdots, m\}$, then we have the following

$$\sum_{a,b=3}^{n_1} \left(h_{ab}^r\right)^2 = \sum_{A,B=n_1+1}^{n} \left(h_{AB}^r\right)^2 = h_{1a}^r = h_{2a}^r = h_{11}^r + h_{22}^r = 0, \forall\, a \in \{3, \cdots, n_1\}.$$

Equivalently,

$$A_{e_r} = \left(\begin{array}{cccc|ccccc} h_{11}^r & h_{12}^r & 0_{13} & 0_{1n_1} & h_{1n_1+1}^r & \cdots & \cdots & \cdots & h_{1n}^r \\ h_{12}^r & -h_{11}^r & 0_{23} & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{31} & 0_{32} & 0_{33} & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{n_11} & \cdots & \cdots & 0_{n_1n_1} & h_{n_1n_1+1}^r & \cdots & \cdots & \cdots & h_{n_1n}^r \\ \hline h_{n_1+11}^r & \cdots & \cdots & h_{n_1+1n_1}^r & \ddots & 0 & 0 & \cdots & 0_{n_1+1n} \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ h_{n1}^r & \cdots & \cdots & h_{nn_1}^r & 0_{nn_1+1} & 0 & \cdots & 0 & 0_{nn} \end{array}\right).$$

□

At the end of this section, we provide a table which includes some of almost Hermitian and almost contact manifolds of interest. Following almost similar techniques of previous chapters, the next table can be verified. For simplicity's sakes, we set

$$\mathcal{I} = \delta_{M^n}(x) - \frac{n_2\,\Delta f}{f} \leq \frac{n^2(n-2)}{2(n-1)}||\vec{H}||^2, \tag{8.3.13}$$

then we have

Manifold	Inequality
R. S. F.	$\mathcal{I}+\frac{1}{2}c\Big(n_1^2+n_2^2-n-2\Big)$
C. S. F.	$\mathcal{I}+\frac{1}{2}\frac{c_{Ka}}{4}\Big(n_1^2+n_2^2-n-2\Big)$
G. C. S. F.	$\mathcal{I}+\frac{1}{2}\frac{c_{RK}+3\gamma}{4}\Big(n_1^2+n_2^2-n-2\Big)$
Sas. S. F.	$\mathcal{I}+\frac{1}{2}\frac{c_S+3}{4}\Big(n_1^2+n_2^2-n-2\Big)+\frac{3}{2}\frac{c_S-1}{4}\Big(-2n_1+2\Big)$
Ken. S. F.	$\mathcal{I}+\frac{1}{2}\frac{c_{Ke}-3}{4}\Big(n_1^2+n_2^2-n-2\Big)+\frac{3}{2}\frac{c_{Ke}+1}{4}\Big(-2n_1+2\Big)$
Co. S. F.	$\mathcal{I}+\frac{1}{2}\frac{c_c}{4}\Big(n_1^2+n_2^2-n-2\Big)+\frac{3}{2}\frac{c_c}{4}\Big(-2n_1+2\Big)$

Table 8.1: General inequalities involving the δ-invariant and the mean curvature vector for $\mathcal{D}_i$-minimal warped product submanifolds satisfying $g(PX, Z) = 0\, \forall\, X \in TN_1,\ Z \in TN_2$.

8.4 THE δ-INVARIANT INEQUALITY FOR GENERAL WARPED PRODUCT SUBMANIFOLDS

In the previous section, we proved an inequality in terms of δ-invariant for $\mathcal{D}_i$-minimal warped product submanifolds. Here, we present another inequality in a more general setting, it is for general warped product submanifolds.

Theorem 8.4.1. *Let $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \tilde{M}^m(c)$ be an isometric immersion of a warped product submanifold M^n into a Riemannian space form $\tilde{M}^m(c)$. Then, for each point $x \in M^n$ and each plane section $\pi_i \subset T_x N_i^{n_i}$, $n_i \geq 2$, for $i = 1,\ 2$, we have:*

(i) if $\pi_1 \subset T_x N_1$, then

$$\delta_{N_1^{n_1}}(x) \leq \frac{n^2}{2}||\vec{H}||^2 - \frac{n_2 \Delta f}{f} + \frac{1}{2}n_1(n_1 + 2n_2 - 1)c - c; \tag{8.4.1}$$

(ii) if $\pi_2 \subset T_x N_2$, then

$$\delta_{N_2^{n_2}}(x) \leq \frac{n^2}{2}||\vec{H}||^2 - \frac{n_2 \Delta f}{f} + \frac{1}{2}n_2(n_2 + 2n_1 - 1)c - c. \tag{8.4.2}$$

Equalities of the above two inequalities hold at $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_x M^n$ and an orthonormal basis $\{e_{n+1}, \cdots, e_m\}$ of $T_x^{\perp} M^n$ such that (a) $\pi = Span\{e_1, e_2\}$ and (b) the shape operators take the following forms:

(ì) *If $\pi_1 \subset T_xN_1$, then for $r = n+1$, we have*

$$A_{e_{n+1}} = \left(\begin{array}{ccccc|ccc}
\mu_1 & h^{n+1}_{12} & 0 & \cdots & 0_{1n_1} & 0_{1n_1+1} & \cdots & 0_{1n} \\
h^{n+1}_{21} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0_{n_11} & 0 & 0 & \cdots & \mu & 0_{n_1n_1+1} & \cdots & 0_{n_1n} \\
\hline
0_{n_1+11} & \cdots & \cdots & \cdots & 0_{n_1+1n_1} & h^{n+1}_{n_1+1n_1+1} & \cdots & h^{n+1}_{n_1+1n} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0_{n1} & \cdots & \cdots & \cdots & 0_{nn_1} & h^{n+1}_{nn_1+1} & \cdots & h^{n+1}_{nn}
\end{array}\right),$$

$\mu = \mu_1 + \mu_2$.

If $r \in \{n+2, \cdots, m\}$, then we have

$$A_{e_r} = \left(\begin{array}{ccccc|ccc}
h^r_{11} & h^r_{12} & 0 & \cdots & 0_{1n_1} & 0_{1n_1+1} & \cdots & 0_{1n} \\
h^r_{21} & -h^r_{11} & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0_{n_11} & 0 & 0 & \cdots & 0_{n_1n_1} & 0_{n_1n_1+1} & \cdots & 0_{n_1n} \\
\hline
0_{n_1+11} & \cdots & \cdots & \cdots & 0_{n_1+1n_1} & h^r_{n_1+1n_1+1} & \cdots & h^r_{n_1+1n} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0_{n1} & \cdots & \cdots & \cdots & 0_{nn_1} & h^r_{nn_1+1} & \cdots & h^r_{nn}
\end{array}\right).$$

(ìi) *If $\pi_2 \subset T_xN_2$, then for $r = n+1$, we have*

$$A_{e_{n+1}} =$$

$$\left(\begin{array}{cccc|ccccc}
h^{n+1}_{11} & \cdots & \cdots & h^{n+1}_{1n_1} & 0_{1n_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
h^{n+1}_{n_11} & \cdots & \cdots & h^{n+1}_{n_1n_1} & 0_{n_1n_1+1} & \cdots & \cdots & \cdots & 0_{n_1n} \\
\hline
0_{n_1+11} & \cdots & \cdots & 0_{n_1+1n_1} & \mu_1 & h^{n+1}_{n_1+1n_1+2} & 0 & \cdots & 0_{n_1+1n} \\
\vdots & \ddots & \ddots & \vdots & h^{n+1}_{n_1+2n_1+1} & \mu_2 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
0_{n1} & \cdots & \cdots & 0_{nn_1} & 0_{nn_1+1} & 0 & \cdots & 0 & \mu
\end{array}\right),$$

where $\mu = \mu_1 + \mu_2$.

If $r \in \{n+2, \cdots, m\}$, then we have

$A_{e_r} =$

$$\left(\begin{array}{cccc|ccccc}
h^r_{11} & \cdots & \cdots & h^r_{1n_1} & 0_{1n_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\
\vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
h^r_{n_11} & \cdots & \cdots & h^r_{n_1n_1} & 0_{n_1n_1+1} & \cdots & \cdots & \cdots & 0_{n_1n} \\
\hline
0_{n_1+11} & \cdots & \cdots & 0_{n_1+1n_1} & h^r_{n_1+1n_1+1} & h^r_{n_1+1n_1+2} & 0 & \cdots & 0_{n_1+1n} \\
\vdots & \ddots & \ddots & \vdots & h^r_{n_1+2n_1+1} & -h^r_{n_1+1n_1+1} & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\
0_{n1} & \cdots & \cdots & 0_{nn_1} & 0_{nn_1+1} & 0 & \cdots & 0 & 0
\end{array}\right).$$

(iii) *If the equality of (i) or (ii) holds, then $N_1 \times_f N_2$ is mixed totally geodesic in $\tilde{M}^m(c)$. Moreover, $N_1 \times_f N_2$ is both $\mathcal{D}_1$-minimal and $\mathcal{D}_2$-minimal. Thus, $N_1 \times_f N_2$ is a minimal warped product submanifold in $\tilde{M}^m(c)$.*

Proof. For $x \in M^n$, let $\pi_1 \subset T_xN_1$ be a 2-plane. We choose an orthonormal basis $\{e_1, \cdots, e_{n_1}, e_{n_1+1}, \cdots, e_n\}$ of T_xM^n, where $\{e_1, \cdots, e_{n_1}\}$ is an orthonormal basis for T_xN_1 and $\{e_{n_1}, e_{n_1+1}, \cdots, e_n\}$ is for T_xN_2. Hence, $\{e_{n+1}, \cdots, e_m\}$ is an orthonormal basis of $T_x^\perp M^n$. First, put $\pi_1 = Span\{e_1, e_2\}$ such that the normal vector e_{n+1} is in the direction of the mean curvature vector $\vec{H}$. By $(2.3.39)$ and $(2.3.9)$ we have

$$n^2||\vec{H}||^2 = 2\tau(T_xM^n) + ||h||^2 - n(n-1)c. \tag{8.4.3}$$

Equivalently,

$$\left(\sum_{a=1}^{n_1} h^{n+1}_{aa}\right)^2 = 2\tau(T_xM^n)+||h||^2-n(n-1)c-\left(\sum_{A=n_1+1}^{n} h^{n+1}_{AA}\right)^2-2\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n} h^{n+1}_{aa}h^{n+1}_{AA}.$$

Putting

$$\Upsilon_1 = 2\tau(T_xM^n) - \frac{n_1-2}{n_1-1}\left(\sum_{a=1}^{n_1} h^{n+1}_{aa}\right)^2$$
$$-\left(\sum_{A=n_1+1}^{n} h^{n+1}_{AA}\right)^2 - 2\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n} h^{n+1}_{aa}h^{n+1}_{AA} - n(n-1)c. \tag{8.4.4}$$

Thus, from the above two equations we may write

$$\Big(\sum_{a=1}^{n_1} h_{aa}^{n+1}\Big)^2 = (n_1-1)\Big(\Upsilon_1 + ||h||^2\Big), \tag{8.4.5}$$

i.e.,

$$\begin{aligned}\Big(\sum_{a=1}^{n_1} h_{aa}^{n+1}\Big)^2 = (n_1-1)\Big(\Upsilon_1 &+ \sum_{a=1}^{n_1}(h_{aa}^{n+1})^2 + \sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2 \\ &+ \sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\Big).\end{aligned} \tag{8.4.6}$$

Applying Lemma 8.1.1 on the above equation for

$$\alpha_a = h_{aa}^{n+1}, \quad \forall\, a \in \{1, \cdots, n_1\}$$

and

$$\beta = \Upsilon_1 + \sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2,$$

then we derive

$$h_{11}^{n+1}h_{22}^{n+1} \geq \frac{1}{2}\Big(\Upsilon_1 + \sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\Big). \tag{8.4.7}$$

From (2.3.9) and (2.3.39) we also have

$$K(\pi_1) = c + \sum_{r=n+1}^{m}\Big(h_{11}^r h_{22}^r - (h_{12}^r)^2\Big).$$

Therefore, by combining the above two relations together, we get

$$\begin{aligned}K(\pi_1) \geq c + \frac{1}{2}\Upsilon_1 + \frac{1}{2}\sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2 + \sum_{r=n+2}^{m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{m}(h_{12}^r)^2 \\ +\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2.\end{aligned}$$

From Lemma 8.2.1, it is obvious that the above inequality is identical to

$$K(\pi_1) \geq c + \frac{1}{2}\Upsilon_1 + \frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2$$

$$+\frac{1}{2}\sum_{r=n+2}^{m}(h_{11}^r + h_{22}^r)^2 + \sum_{r=n+1}^{m}\sum_{j=3}^{n}\Big((h_{1j}^r)^2 + (h_{2j}^r)^2\Big) + \frac{1}{2}\sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2.$$

Hence, the above inequality yields to

$$K(\pi_1) \geq c + \frac{1}{2}\Upsilon_1 + \frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2 + \frac{1}{2}\sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2. \quad (8.4.8)$$

By $(8.4.4)$ and the above equation, we obtain

$$K(\pi_1) \geq c + \tau(T_xM^n) + \frac{1}{2(n_1-1)}\left(\sum_{a=1}^{n_1} h_{aa}^{n+1}\right)^2 - \frac{n^2}{2}||\vec{H}||^2 - \frac{1}{2}n(n-1)c$$

$$+\frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2 + \frac{1}{2}\sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2.$$

Making use of $(6.3.3)$, we get

$$\tau_1(T_xN_1) - K(\pi_1) \leq \frac{n^2}{2}||\vec{H}||^2 - \frac{n_2\Delta f}{f} + (\frac{n^2}{2} - \frac{n}{2} - 1)c - \tau_2(T_xN_2)$$

$$-\frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 - \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2 - \frac{1}{2}\sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2. \quad (8.4.9)$$

Applying the Gauss equation on $\tau_2(T_xN_2)$, gives

$$-\tau_2(T_xN_2) = -\tilde{\tau}_2(T_xN_2) + \frac{1}{2}\sum_{r=n+1}^{m}\sum_{A,B=n_1+1}^{n}(h_{AB}^{r})^2 - \frac{1}{2}\sum_{r=n+1}^{m}(h_{n_1+1n_1+1}^{r} + \cdots + h_{nn}^{r})^2. \quad (8.4.10)$$

In view of the above two relations, we can write

$$\tau_1(T_xN_1) - K(\pi_1) \leq \frac{n^2}{2}||\vec{H}||^2 - \frac{n_2\Delta f}{f} + (\frac{n^2}{2} - \frac{n}{2} - 1)c - \tilde{\tau}_2(T_xN_2)$$

$$-\frac{1}{2}\sum_{\substack{i,j=3\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 - \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=3}^{n}(h_{ij}^{r})^2 - \frac{1}{2}\sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2 + \frac{1}{2}\sum_{r=n+1}^{m}\sum_{A,B=n_1+1}^{n}(h_{AB}^{r})^2. \quad (8.4.11)$$

Lemma 8.2.3 is useful to show that $(8.4.11)$ is equivalent to the following

$$\tau_1(T_xN_1) - K(\pi_1) \leq \frac{n^2}{2}||\vec{H}||^2 - \frac{n_2\Delta f}{f} + (\frac{n^2}{2} - \frac{n}{2} - 1)c - \tilde{\tau}_2(T_xN_2)$$

$$-\frac{1}{2}\sum_{r=n+2}^{m}\sum_{a,b=3}^{n_1}(h_{ab}^{r})^2 - \sum_{r=n+2}^{m}\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{r})^2$$

$$-\frac{1}{2}\sum_{\substack{a,b=3\\ a\neq b}}^{n_1}(h_{ab}^{n+1})^2 - \sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{n+1})^2. \quad (8.4.12)$$

Hence, the inequality in (i) follows directly from the above one.

If $\pi_2 \subset T_xN_2$, then put $\pi_2 = Span\{e_{n_1+1}, e_{n_1+2}\}$. Now, following similar analogy like the first case, we can write

$$\left(\sum_{A=n_1+1}^{n} h_{AA}^{n+1}\right)^2 = 2\tau(T_xM^n) + ||h||^2 - n(n-1)c - \left(\sum_{a=1}^{n_1} h_{aa}^{n+1}\right)^2 - 2\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n} h_{aa}^{n+1}h_{AA}^{n+1}.$$

Putting

$$\Upsilon_2 = 2\tau(T_xM^n) - \frac{n_2-2}{n_2-1}\left(\sum_{A=n_1+1}^{n} h_{AA}^{n+1}\right)^2 - \left(\sum_{a=1}^{n_1} h_{aa}^{n+1}\right)^2 - 2\sum_{a=1}^{n_1}\sum_{A=n_1+1}^{n} h_{aa}^{n+1}h_{AA}^{n+1} - n(n-1)c. \tag{8.4.13}$$

Thus, from the above two equations we may write

$$\left(\sum_{A=n_1+1}^{n} h_{AA}^{n+1}\right)^2 = (n_2-1)\left(\Upsilon_2 + ||h||^2\right), \tag{8.4.14}$$

i.e.,

$$\left(\sum_{A=n_1+1}^{n} h_{AA}^{n+1}\right)^2 = (n_2-1)\Bigg(\Upsilon_2 + \sum_{a=1}^{n_1}(h_{aa}^{n+1})^2 + \sum_{A=n_1+1}^{n}(h_{AA}^{n+1})^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\Bigg). \tag{8.4.15}$$

Applying Lemma 8.1.1 on the above equation for

$$\alpha_a = h_{AA}^{n+1}, \quad \forall\, a \in \{n_1+1, \cdots, n\}$$

and

$$\beta = \Upsilon_2 + \sum_{a=1}^{n_1}(h_{aa}^{n+1})^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2,$$

then we derive

$$h_{n_1+1n_1+1}^{n+1}h_{n_1+2n_1+2}^{n+1} \geq \frac{1}{2}\Bigg(\Upsilon_2 + \sum_{a=1}^{n_1}(h_{aa}^{n+1})^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\Bigg). \tag{8.4.16}$$

From (2.3.9) and (2.3.39) we also have

$$K(\pi_2) = c + \sum_{r=n+1}^{m}\left(h_{n_1+1n_1+1}^{r}h_{n_1+2n_1+2}^{r} - (h_{n_1+1n_1+2}^{r})^2\right).$$

Therefore, by combining the above two relations we reach

$$K(\pi_2) \geq c + \sum_{r=n+2}^{m} h_{n_1+1n_1+1}^{r}h_{n_1+2n_1+2}^{r} - \sum_{r=n+1}^{m}(h_{n_1+1n_1+2}^{r})^2 + \frac{1}{2}\Upsilon_2$$

$$+\frac{1}{2}\sum_{a=1}^{n_1}(h_{aa}^{n+1})^2+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2.$$

Now, following similar procedure as the first case, the inequality of statement (ii) follows immediately.

For the equality case, we also distinguish two cases based on whether the 2-plane π_i is tangent to the first factor or to the second. In statement $(\grave{i})$, we consider $\pi_1 \subset T_xN_1$, then the equality holds if and only if all equalities of (8.4.7), (8.4.8), (8.4.9), (8.4.10) and (8.4.12) hold. One can see that these equalities hold if and only if the following conditions are satisfied, respectively.

(i) $h_{11}^{n+1}+h_{22}^{n+1}=h_{33}^{n+1}=\cdots=h_{n_1n_1}^{n+1}$,

(ii) $\displaystyle\sum_{r=n+2}^{m}(h_{11}^r+h_{22}^r)^2+\sum_{r=n+1}^{m}\sum_{j=3}^{n}\Big((h_{1j}^r)^2+(h_{2j}^r)^2\Big)=0,$

(iii) $\displaystyle\left(\sum_{a=1}^{n_1}h_{aa}^{n+1}\right)^2=\sum_{r=n+1}^{m}(h_{n_1+1n_1+1}^r+\cdots+h_{nn}^r)^2=0,$

(iv) $$\sum_{r=n+2}^{m}\sum_{a,b=3}^{n_1}(h_{ab}^r)^2+\sum_{r=n+2}^{m}\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^r)^2$$
$$+\sum_{\substack{a,b=3\\ a\neq b}}^{n_1}(h_{ab}^{n+1})^2+\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}(h_{aA}^{n+1})^2=0.$$

From condition (iii), it is clear that $N_1\times_f N_2$ is both $\mathcal{D}_1$-minimal and $\mathcal{D}_2$-minimal warped product submanifold in $\tilde{M}^m(c)$. This implies that $N_1\times_f N_2$ is minimal in $\tilde{M}^m(c)$.

Now, we are going to classify the other conditions in two categories, according to the normal vector field r. Firstly, if $r=n+1$, then we have

$$h_{11}^{n+1}+h_{22}^{n+1}=h_{33}^{n+1}=\cdots=h_{n_1n_1}^{n+1},$$

and

$$\sum_{j=3}^{n}h_{1j}^{n+1}=\sum_{j=3}^{n}h_{2j}^{n+1}=\sum_{\substack{a,b=3\\ a\neq b}}^{n_1}h_{ab}^{n+1}=\sum_{a=3}^{n_1}\sum_{A=n_1+1}^{n}h_{aA}^{n+1}=0.$$

Equivalently,

$$A_{e_{n+1}} = \left(\begin{array}{ccccc|ccc} \mu_1 & h_{12}^{n+1} & 0 & \cdots & 0_{1n_1} & 0_{1n_1+1} & \cdots & 0_{1n} \\ h_{21}^{n+1} & \mu_2 & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \mu & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_{n_1 1} & 0 & 0 & \cdots & \mu & 0_{n_1 n_1+1} & \cdots & 0_{n_1 n} \\ \hline 0_{n_1+1 1} & \cdots & \cdots & \cdots & 0_{n_1+1 n_1} & h_{n_1+1 n_1+1}^{n+1} & \cdots & h_{n_1+1 n}^{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{nn_1} & h_{nn_1+1}^{n+1} & \cdots & h_{nn}^{n+1} \end{array}\right),$$

$\mu = \mu_1 + \mu_2$.

Secondly, if $r \in \{n+2, \cdots, m\}$, then the conditions above imply

$$h_{11}^r + h_{22}^r = \sum_{j=3}^{n} h_{1j}^r = \sum_{j=3}^{n} h_{2j}^r = \sum_{a,b=3}^{n_1} h_{ab}^r = \sum_{a=3}^{n_1} \sum_{A=n_1+1}^{n} h_{aA}^r = 0.$$

Equivalently,

$$A_{e_r} = \left(\begin{array}{ccccc|ccc} h_{11}^r & h_{12}^r & 0 & \cdots & 0_{1n_1} & 0_{1n_1+1} & \cdots & 0_{1n} \\ h_{21}^r & -h_{11}^r & 0 & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0_{33} & \cdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0_{n_1 1} & 0 & 0 & \cdots & 0_{n_1 n_1} & 0_{n_1 n_1+1} & \cdots & 0_{n_1 n} \\ \hline 0_{n_1+1 1} & \cdots & \cdots & \cdots & 0_{n_1+1 n_1} & h_{n_1+1 n_1+1}^r & \cdots & h_{n_1+1 n}^r \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n1} & \cdots & \cdots & \cdots & 0_{nn_1} & h_{nn_1+1}^r & \cdots & h_{nn}^r \end{array}\right).$$

Obviously, the above two matrices show that $N_1 \times_f N_2$ is mixed totally geodesic submanifold in $\tilde{M}^m(c)$.

Analogously, the equality sign in $(\grave{ii})$ holds if and only if the following are satisfied

$$A_{e_{n+1}} = \left(\begin{array}{cccc|ccccc} h^{n+1}_{11} & \cdots & \cdots & h^{n+1}_{1n_1} & 0_{1n_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ h^{n+1}_{n_11} & \cdots & \cdots & h^{n+1}_{n_1n_1} & 0_{n_1n_1+1} & \cdots & \cdots & \cdots & 0_{n_1n} \\ \hline 0_{n_1+11} & \cdots & \cdots & 0_{n_1+1n_1} & \mu_1 & h^{n+1}_{n_1+1n_1+2} & 0 & \cdots & 0_{n_1+1n} \\ \vdots & \ddots & \ddots & \vdots & h^{n+1}_{n_1+2n_1+1} & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & \mu & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{nn_1} & 0_{nn_1+1} & 0 & \cdots & 0 & \mu \end{array}\right),$$

where $\mu = \mu_1 + \mu_2$.

Also,

$$A_{e_r} = \left(\begin{array}{cccc|ccccc} h^{r}_{11} & \cdots & \cdots & h^{r}_{1n_1} & 0_{1n_1+1} & \cdots & \cdots & \cdots & 0_{1n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ h^{r}_{n_11} & \cdots & \cdots & h^{r}_{n_1n_1} & 0_{n_1n_1+1} & \cdots & \cdots & \cdots & 0_{n_1n} \\ \hline 0_{n_1+11} & \cdots & \cdots & 0_{n_1+1n_1} & h^{r}_{n_1+1n_1+1} & h^{r}_{n_1+1n_1+2} & 0 & \cdots & 0_{n_1+1n} \\ \vdots & \ddots & \ddots & \vdots & h^{r}_{n_1+2n_1+1} & -h^{r}_{n_1+1n_1+1} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0_{n1} & \cdots & \cdots & 0_{nn_1} & 0_{nn_1+1} & 0 & \cdots & 0 & 0 \end{array}\right).$$

Clearly, $N_1 \times_f N_2$ is mixed totally geodesic in $\tilde{M}^m(c)$. Also, it is not difficult to show that $N_1 \times_f N_2$ is both $\mathcal{D}_1$-minimal and $\mathcal{D}_2$-minimal, which implies the minimality of $N_1 \times_f N_2$ in $\tilde{M}^m(c)$. $\square$

At the end of this section, we leave the following remark to the reader, since it can be verified by the same techniques as in previous tables.

Remark 8.4.1. *Extensions of the preceding inequality for any $\mathcal{D}_i$-minimal warped product submanifold in space forms can be obtained by following similar techniques as in the previous four tables.*

8.5 ANOTHER TWO NECESSARY CONDITIONS FOR THE MINIMALITY OF WARPED PRODUCT SUBMANIFOLDS

In the previous chapter, we have found a necessary condition for a warped product submanifold to be $\mathcal{D}_i$-minimal (for both $i = 1,\ 2$) in a Euclidean m-space $\mathbb{E}^m$ (see Corollary 7.3.1). As a second answer of Problems 1.4.11 and 1.4.12, we apply Theorem 8.3.1 to give a necessary condition for a warped product to be $\mathcal{D}_i$-minimal (for both $i = 1,\ 2$) in a Euclidean m-space $\mathbb{E}^m$.

Corollary 8.5.1. *If $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \mathbb{E}^m$ is a $\mathcal{D}_i$-minimal isometric immersion for both $i = 1, 2$, from a warped product submanifold M^n into a Euclidean m-space, then*

$$\delta_{M^n}(x) \leq \frac{n_2\, \Delta f}{f}, \tag{8.5.1}$$

where $\delta_{M^n}(x)$ is the Chen first invariant and n_2 is the dimension of N_2.

Finally, and more generally, Theorem 8.4.1 guarantees the following necessary condition for any warped product submanifold to be minimal in Euclidean spaces.

Corollary 8.5.2. *If $\varphi : M^n = N_1 \times_f N_2 \longrightarrow \mathbb{E}^m$ is a minimal isometric immersion from a warped product submanifold M^n into a Euclidean m-space, then for each point $x \in M^n$ and each plane section $\pi \subset T_x M^n$, we have:*

(i) if $\pi_1 \subset T_x N_1$, then

$$\delta_{N_1^{n_1}}(x) \leq -\frac{n_2 \Delta f}{f}; \tag{8.5.2}$$

(ii) if $\pi_2 \subset T_x N_2$, then

$$\delta_{N_2^{n_2}}(x) \leq -\frac{n_2 \Delta f}{f}. \tag{8.5.3}$$

CHAPTER 9: SOME CONCLUSIONS AND FURTHER RESEARCH PROBLEMS

9.1 INTRODUCTION

This chapter is mainly devoted to present some open problems related to this thesis. We organize this chapter to have three main sections. The first section offers those problems which could not be solved by this work. We should notice that most of these problems are special cases of those stated in Chapter One of this thesis.

The other two sections provide new open problems in this field. It is fair to say, that all problems of these two sections are due to the current thesis. However, all problems of this chapter can also be, in some sense, considered as conclusions as we will soon see. At the very least, they could guide our programs of research for many years.

9.2 SOME PROBLEMS WHICH ARE NOT SOLVED IN THIS THESIS

Taking a quick look at the existence and nonexistence tables of Chapter Three, one can conclude that many of warped product submanifolds are not proved whether they exist or not. We refer to this situation by ?. Since neither a positive nor a negative answer is obtained, it is reasonable to ask the following questions:

Problem 9.2.1. *Do proper warped product submanifolds of types* $N_T \times_f N_\theta$*,* $N_T \times_f N$ *and* $N_\perp \times_f N_\theta$ *exist in nearly Kaehler manifolds?*

Analogously,

Problem 9.2.2. *Do proper warped product submanifolds of types* $N_T \times_f N_\theta$*,* $N_T \times_f N$ *and* $N_\perp \times_f N_\theta$ *exist in nearly Sasakian and nearly cosymplectic manifolds?*

For the next problem, we believe that warped product hemi-slant submanifolds of the type $N_\theta \times_f N_\perp$ exists in Sasakian and cosymplectic manifolds. However, it is not proved yet. Thus, it is interesting to state

Problem 9.2.3. *Does a proper warped product submanifold of the type* $N_\theta \times_f N_\perp$ *exist in Sasakian, cosymplectic, nearly Sasakian and nearly cosymplectic manifolds?*

For non-Kaehler nearly Kaehler manifolds, many well-known examples were given by famous geometers for CR-submanifolds (see, for example (Bejancu, 1986)). On the contrary, solid examples can not be found in the literature of warped product submanifolds in nearly Kaehler manifolds. Thus, we have the following problem

Problem 9.2.4. *Construct concrete examples for CR-warped product submanifolds of non-Kaehler nearly Kaehler manifolds?*

Likewise

Remark 9.2.1. *The last problem can be analogously stated for all warped product submanifolds of the first three problems of this section.*

As we mentioned ealier in thesis, the hope of extending the second inequality of h for warped product submanifolds, other than CR-warped product submanifolds, had completely failed when we used a method based on Codazzi equation (Chen, 2003). That is, the second inequality of h was not proved for semi-slant, generic *, $\mathcal{D}_i$-minimal hemi-slant or any $\mathcal{D}_i$-minimal warped product submanifold other than CR-warped product submanifolds. However, and via the Gauss equation, this hope has been completely achieved in Chapter Six of the current work (see Theorem 6.3.1).

Analogously, the first inequality have been proved firstly for CR-warped product submanifolds in Kaehler manifolds by Chen (Chen, 2001). Soon it was extended to most structures of interest (see references in (Chen, 2013)). Recently, the most general version of this inequality is proved in almost contact manifolds (Mustafa et al., 2013). After that, the first inequality of h has been proved for a semi-slant setting which can be considered a new impulse given for warped product semi-slant submanifolds (Uddin et al., 2014). Moreover, the most general version of such inequalities is given in (Mustafa et al.,2014).

Now, based on our results of Chapter Five, CR, semi-slant and generic warped product submanifolds are $\mathcal{D}_i$-minimal warped product submanifolds in all almost contact and almost Hermitian manifolds of interest of this thesis. In this work, we show that CR and

* As mentioned in the margins of chapter five, generic submanifolds were defined for both almost Hermitian and almost contact manifolds. To avoid confusion, we did not give the definition of generic submanifolds. For simplicity's sake, we consider warped product of types $N_T \times_f N$ and $N \times_f N_T$, where N is a Riemannian submanifold.

semi-slant do possess the first inequality of h in most structures of interest, while we are not sure about generic warped product submanifolds structure. Hence we ask

Problem 9.2.5. *Does a generic warped product submanifold of nearly Kaehler and nearly trans-Sasakian manifolds have the first inequality of h?*

Hint: preparatory lemmas are proven in Chapter Three for a general case, i. e., $N \times_f N_T$, the reader can easily prove that these lemmas are true for the generic case also. Some problems related to the local field of orthonormal frame may be encountered. So, all what you need is to consider such problems in the adapted frame.

One of our future goals is to prove or to extend the inequality constructed by Chen in (Chen, 1999), from Riemannain submanifolds to warped product submanifolds. Hence we ask:

Problem 9.2.6. *For warped product submanifolds, can we prove similar inequalities like Theorem 1 of section 3 in (Chen, 1999)?*

It should be known that, the term "similar inequality" means simple inequality involving similar extrinsic and intrinsic invariants, with equality case discussed completely. Otherwise, it will not be similar inequality.

9.3 MORE NEW PROBLEMS THAT AROSE FROM RESULTS AND PROOFS OF THIS THESIS

As mentioned in Chapter Six, Chen and Munteanu have constructed many examples to show that the first and the second inequalities of h are optimal and sharp inequalities in Kaehler and Sasakian manifolds, (see (Chen, 2001), (Munteanu, 2005), (Chen, 2003) and (Chen, 2008)). Since it is a considerable geometrical contribution, we are now working in this direction. Of course it is not easy to construct examples satisfying equalities of some inequalities of this work, such as inequalities of chapter seven and eight, but it may be easier to achieve it for the general second inequality of h in Chapter Six, Theorem 6.3.1. Therefore, it is remarkable to ask the following

Problem 9.3.1. *Are the inequalities of Chapters Six, Seven and Eight optimal inequalities?*

In another line of thought, few, but significant, applications are derived in this thesis by applying the constructed inequalities, especially in Chapters Six, Seven and Eight. Those applications are special case solutions for Problems 1.4.11 and 1.4.12. However, it is still possible to derive more applications from these inequalities. This direction is suitable for postgraduate students in this field.

Problem 9.3.2. *Can we give other new special case answers for Problems 1.4.11 and 1.4.12?*

Next, we offer a third interesting direction of this field; that is, discussing space form cases in more details. It is clear that we provide some new special case solutions for Problems 2.3.1, 2.3.29 and 2.3.32. Anyway, more space form solutions should be derived. Moreover, most of our results are for general cases, so they can be useful if discussed for space form cases. For this, we take this direction into account in our potential research.

Problem 9.3.3. *Carry out comparisons between different space form cases based on general theorems of this thesis.*

For the above three open directions of research, we recommend researches, specially postgraduate students to include these problems into their research programs, because they are easy, important and straightforward.

On the contrary, other directions seem to be more complicated. For example, it is not an easy task to classify warped products that satisfy equality cases of our inequalities. For those who are interested and qualified to prove such classifications, plenty of inequalities are constructed in this thesis. Hence, it is a challenging and fruitful topic for research.

Problem 9.3.4. *Classify warped product submanifolds that satisfy the equality cases of inequalities of this thesis.*

Hint: Following B. Y. Chen in his classifications may be helpful and interesting (see (Chen, 2001), (Chen, 2003), (Chen, 2008) and (Chen, 2013)).

The above four directions of research can be thought of to be a continuation of the work started by this thesis.

9.4 OBSERVATIONS, PROBLEMS AND SUGGESTIONS FOR SOLVING THESE PROBLEMS WHICH NATURALLY AROSE FROM THIS THESIS

Finally, we consider two important issues of this field. The first one relates to hemi-slant warped product submanifolds, while the other concerns of some generalizations from $\mathcal{D}_i$-minimal to general warped product submanifolds.

9.4.1 HEMI-SLANT WARPED PRODUCT SUBMANIFOLDS OF THE TYPE $N_\theta \times_f N_\perp$

It is well-known that CR-structures are very important from a mathematical point of view and also in physics, specially in general relativity, (Bejancu, 1986). In warped product submanifolds, semi-slant warped product submanifolds failed to generalize CR-warped product submanifolds in Kaehlerian, Sasakian and cosymplectic manifolds (see chapter three of this thesis). By contrast, we have succeeded to generalize both types of CR-warped product submanifolds of Kenmotsu manifolds and for both types of semi-slant warped product submanifolds also, this was achieved by giving examples for these four types (see chapters three and four).

However, warped product semi-slant submanifolds failed to generalize CR-warped product submanifolds in the most important manifolds of the current thesis *, the Kaehlerian and the Sasakian cases. Fortunately, hemi-slant warped product submanifolds successfully generalizes CR-warped product sbmanifolds in Kaehler, we expect that to be true in Sasakian also. Based on our results and proofs, we believe that hemi-slant warped product submanifolds are the "strongest" warped product submanifolds other than the general case, taking into consideration that bi-slant warped products are not defined until the moment, this is because of the fact that $g(PX, Z) \neq 0$, where X and Z are tangent to the first and the second factors respectively.

In Chapter Five, two significant observations related to hemi-slant warped product submanifolds should be mentioned. These observations relate to the first and the second inequalities of h.

Observation 1: We notice that the second inequality of h holds for $\mathcal{D}_i$-minimal hemi-slant warped product submanifolds, but not for the general case.

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*We note that these two facts were recently proved in Kaehler and Sasakian manifolds. In the current thesis we give another proof of such facts. Moreover, we totally solved the confusion caused by Kenmotsu manifolds by asserting the existence of both types of CR-warped product submanifolds and semi-slant also.

Observation 2: This observation comes form comparisons between different kinds of warped product submanifolds. The first inequality of h for hemi-slant warped product submanifolds could not been proved in both almost Hermitian and almost contact manifolds in a natural way. Thus, we extended another inequality of h, but for mixed totally geodesic hemi-slant warped product submanifolds. Therefore, this inequality which was first proved by Sahin does not generalize the first inequality of h for CR-warped product sbmanifolds, in Kaehler manifolds for example.

Thus, we conclude that

Conclusion: From the above two observations, we conclude that the known procedures that used by all geometers in this field are not suitable for hemi-slant warped product submanifolds, because it is "stronger" than other kinds of warped products.

Hence, we address the following problem

Problem 9.4.1. *Find new methods to prove the first and the second inequalities of h for general hemi-slant warped product submanifolds.*

9.4.2 GENERALIZING RESULTS FROM $\mathcal{D}_i$-MINIMAL TO GENERAL WARPED PRODUCT SUBMANIFOLDS

In this work, new special case solutions were found for all problems addressed in chapter one. In particular, Theorems 6.3.1, 7.3.1 and 8.3.1 were proved for $\mathcal{D}_i$-minimal warped product submanifolds. So, we have

Problem 9.4.2. *Can "similar inequalities" be proved for arbitrary warped product submanifolds which are not $\mathcal{D}_i$-minimal submanifolds?*

We explained the meaning of "similar inequalities" above. Moreover, by intuition we guess the answer of the above problem is yes for Theorem 7.3.1 and no for Theorem 6.3.1, while it is still not obvious for Theorem 8.3.1.

In fact, the above negative answer for Theorem 6.3.1 can be easily proved, that was given by my supervisor Dr. Loo Tee How. That is, by considering the the partial mean curvatures of N_1 and N_2, we see that some inner products of these curvatures do not cancel. Therefore, either an assumption is imposed or a a little bit different inequality may appear.

Nevertheless, one can clearly discover that Theorem 6.3.1, and thus second inequality of h, is valid only for $\mathcal{D}_i$-minimal warped product submanifolds and it is the final version of such inequalities. Moreover, one can see that slant angles does not appear in the final form of this inequality. For this, one can refer to the hemi-slant case in the second table of chapter five. For further research, one can define the notion of bi-slant warped product submanifolds, to see whether this inequality remains the same or gets a new shape.

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