CHAPTER 4

MALMQUIST PRODUCTIVITY INDEX AND DATA ENVELOPMENT ANALYSIS

This section clarifies the model adopted in measuring productivity growth for this study. The first part introduces the concept of the distance functions. Equipped with the distance functions, the subsequent part provides the theoretical foundation of Malmquist productivity index and its decompositions. Finally, this section presents the theoretical background of data envelopment analysis (DEA) that is used to calculate the Malmquist productivity index.

4.1 DISTANCE FUNCTIONS

The basic tools of Malmquist productivity indexes are the input and output distance functions, defined by radial scalings of inputs and outputs respectively. Malmquist (1953) defined his distance function as the radial contraction to an indifference curve, while Shephard (1953) defined it in terms of a production function. This paper defines the input distance function directly on the technology as in Sherphard (1970).

Let $x = (x_1,...,x_N) \in \Re_+^N$ denote a vector of inputs and $y = (y_1,...,y_M) \in \Re_+^M$ being an output vector. The production technology T is defined by

$$T = \{(x, y) : x \text{ can produce } y\}, \tag{4.1}$$

and it consists of all input-output vectors that are technically feasible. In the case of single output, the production function is often used to represent the technology. This function is defined by

$$F(x) = \max\{y : (x, y) \in T\}, \tag{4.2}$$

and T can be recovered from F as

$$T = \{(x, y) : F(x) \geq y, x \in \Re_{+}^{N}\} \tag{4.3}$$

The input distance function is defined on the technology T as

$$D_I(y, x) = \sup\left\{\lambda : \left(\frac{x}{y}, y\right) \in T\right\}, \tag{4.4}$$

i.e., as the "maximal" feasible contraction of x. From its definition, it follows that the input distance function is homogeneous of degree +1 in inputs, i.e.,

$$D_I(y, \lambda x) = \lambda D_I(y, x), \text{ for all } \lambda > 0. \tag{4.5}$$

Moreover, if inputs are weakly disposable[6], then it is a complete characterization of T, i.e.,

$$T = \{(x, y) : D_i(y, x) \geq 1\}. \tag{4.6}$$

The output distance function, due to Sherphard (1970) is defined by

$$D_o(x, y) = \inf\left\{\theta : \left(x, \frac{y}{\theta}\right) \in T\right\}, \tag{4.7}$$

$$= \{sup\ [\theta : (x, \theta y) \in T]\}^{-1} \tag{4.8}$$

This function is homogeneous of degree +1 in outputs, and it is a complete characterization of T provided outputs are weakly disposable, i.e,

$$T = \{(x, y) : D_o(x, y) \leq 1\}. \tag{4.9}$$

Thus if inputs and outputs are weakly disposable, then

$$D_o(x, y) \leq 1 \text{ if and only if } D_i(y, x) \geq 1 \tag{4.10}$$

[6] This term was introduced by Fare et al. (1994a), which mean a proportional decreased in outputs remains producible with no change in inputs.

If the technology T exhibits constant returns to scale, one can prove that the input and output distance functions are reciprocal. In technical terms, constant returns to scale (CRS) is defined as

$$\lambda T = T \text{ for all } \lambda > 0. \tag{4.11}$$

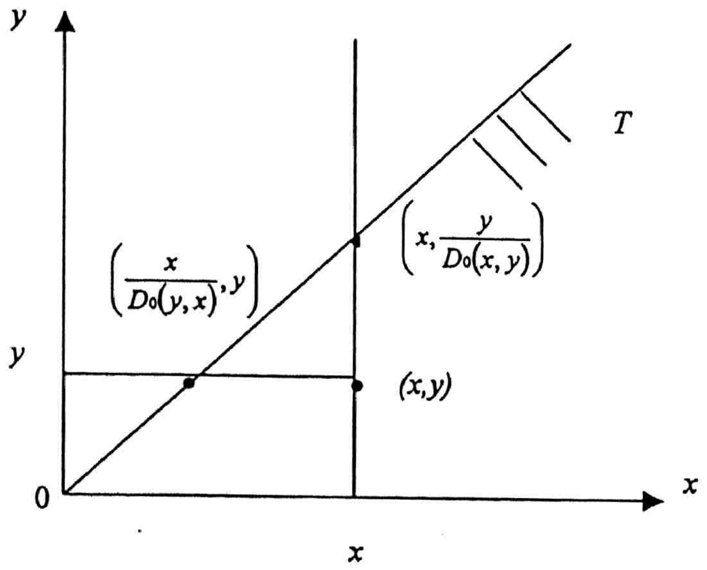
Under this condition, we have

$$D_0(x, y) = 1 / D_i(y, x), \tag{4.12}$$

i.e., the two distance functions are reciprocals. Moreover, the output distance function is homogeneous of degree -1 in inputs and the input distance function is homogeneous of degree -1 in outputs.

To illustrate the two distance functions, assume that one input is used to produce one output. Given observed input-output combination *(x,y)*, the two distance functions are illustrated as can be seen from Figure 4.1, which is reproduced from Fare et al. (1998). The output distance function scales the observed point due north (in the output direction) reaching the boundary of technology. The input distance function scales the observed point due west (in the input direction) until the boundary is attained.

Figure 4.1: Input and Output Distance Functions



Source: Fare et al. (1998)

4.2 MALMQUIST PRODUCTIVITY INDEXES: DEFINITION

Caves, Christensen and Diewert (1982) introduced two theoretical indexes which they named Malmquist input and output productivity indexes. These indexes follow the spirit of Sten Malmquist's (1953) quantity index. Malmquist constructed his quantity index by comparing two quantity vectors to an arbitrary indifference curve using radial scaling. Caves, Christensen and Diewert compare two input-output vectors to a reference technology using radial input and output scaling, for the input and output productivity indexes respectively.

Since this study measures productivity using the output-based method, only the corresponding theoretical are introduced in this chapter. To define the output-based Malmquist index of productivity change, let t and $t+1$ denote two time periods and let $D_o^t\left(x^t, y^t\right)$ be the value of the output distance function for the technology from period t and the input-output vector from the same period. Let $D_o^t\left(x^{t+1}, y^{t+1}\right)$ be the value of the distance function for the input-output vector of period $t+1$ and the technology at t. The t-period output-oriented Malmquist productivity index due to Caves, Christensen and Diewert is defined as[7]

$$M_o^t = \frac{D_o^t\left(x^{t+1}, y^{t+1}\right)}{D_o^t\left(x^t, y^t\right)} \tag{4.13}$$

This index is illustrated in Figure 4.2.

The t-period technology is represented by T^t and the two input-output vectors by $\left(x^t, y^t\right)$ and $\left(x^{t+1}, y^{t+1}\right)$. $\left(x^t, y^t\right)$ is feasible, but $\left(x^{t+1}, y^{t+1}\right)$ is outside T^t; thus $D_o^t\left(x^t, y^t\right) \leq 1$ and $D_o^t\left(x^{t+1}, y^{t+1}\right) > 1$. In terms of the distances on the y-axis, the productivity index is

$$M_o^t = \frac{0d}{0e} / \frac{0a}{0b}, \tag{4.14}$$

and a productivity improvement would be signaled since this value is greater than one.

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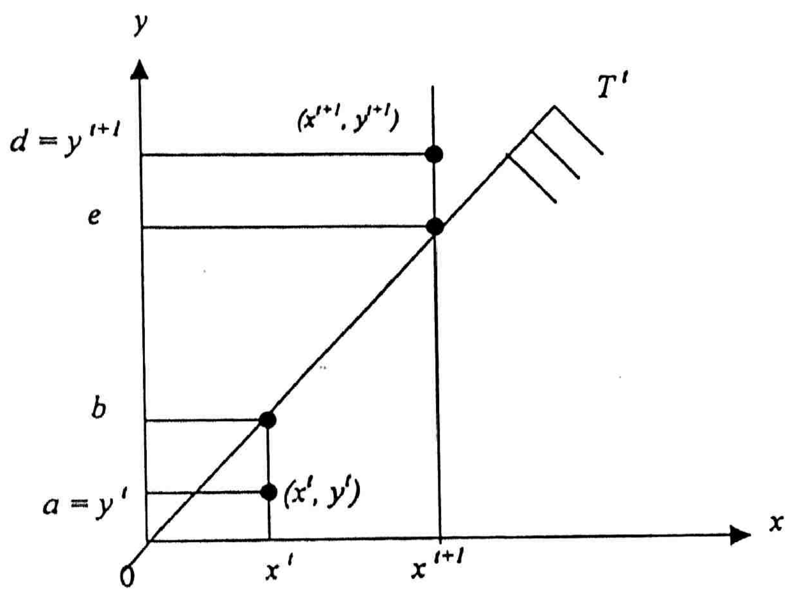
[7] Subscript o refers to output-oriented measures.

Two time periods are involved in definition 4.13 suggesting a *t+1* productivity index namely

$$M_o^{t+1} = \frac{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}{D_o^{t+1}\left(x^t, y^t\right)} \tag{4.15}$$

Figure 4.2: The *t* period Malmquist Output Oriented Productivity Index



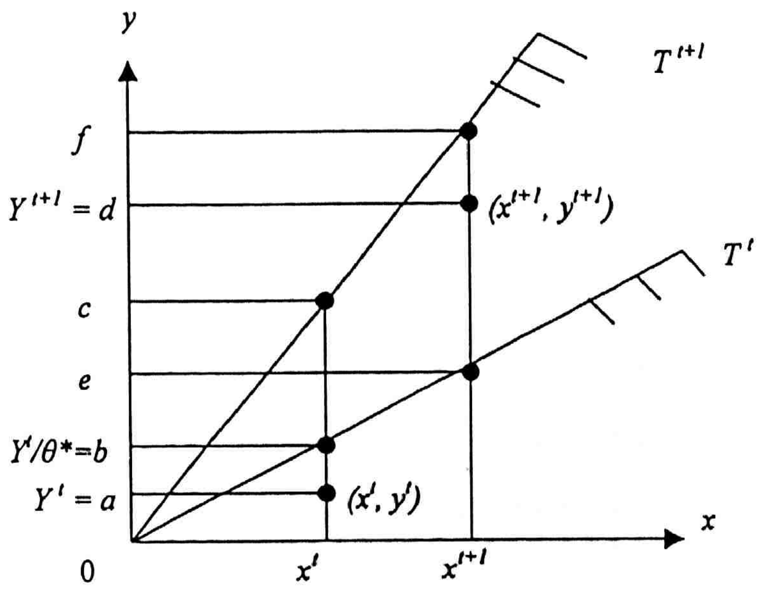
Source: Fare et al. (1998)

Inspired by Caves, Christensen and Diewert (1982), Fare, Grosskopf, Lindgren and Roos (1989), defined their output-oriented Malmquist index as the geometric mean of (4.13) and (4.15);i.e.,

$$M_o\left(x^t, y^t, x^{t+1}, y^{t+1}\right) = \left(\frac{D_o^t\left(x^{t+1}, y^{t+1}\right)}{D_o^t\left(x^t, y^t\right)} \frac{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}{D_o^{t+1}\left(x^t, y^t\right)}\right)^{\frac{1}{2}} \tag{4.16}$$

The productivity index (4.16) may be illustrated in Figure 4.3.

Figure 4.3: The Malmquist Output-Based Index of TFP and Output Distance Function



Source: Fare et al. (1994a)

In this figure, two technologies are involved, one for period t and one for period $t+1$. The latter contains the former, i.e., technical progress has occurred. The productivity changes for the two input-output vectors (x^t, y^t) and (x^{t+1}, y^{t+1}), based on y-distances in Figure 4.3 is thus,

$$M_o(x^t, y^t, x^{t+1}, y^{t+1}) = \left(\frac{0d}{0e} \frac{0b}{0a} \frac{0c}{0a} \frac{0d}{0f} \right)^{\frac{1}{2}} \quad (4.17)$$

4.3 DECOMPOSITIONS OF THE MALMQUIST PRODUCTIVITY INDEX

Following Fare et al. (1989) an equivalent way of writing the Malmquist productivity index (4.16) is

$$M_o\left(x^t, y^t, x^{t+1}, y^{t+1}\right) = \frac{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}{D_o^t\left(x^t, y^t\right)} \times \left[\left(\frac{D_o^t\left(x^{t+1}, y^{t+1}\right)}{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}\right)\left(\frac{D_o^t\left(x^t, y^t\right)}{D_o^{t+1}\left(x^t, y^t\right)}\right)\right]^{\frac{1}{2}} \tag{4.18}$$

where the ratio outside the brackets measures the change in relative efficiency (i.e., the change in how far observed production is from maximum potential production) between years t and $t+1$. The geometric mean of the two ratios inside the brackets captures the shift in technology between the two periods evaluated at x^t and x^{t+1}, that is

$$\text{Efficiency change} = \frac{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}{D_o^t\left(x^t, y^t\right)} \tag{4.19}$$

$$\text{Technical change} = \left[\left(\frac{D_o^t\left(x^{t+1}, y^{t+1}\right)}{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}\right)\left(\frac{D_o^t\left(x^t, y^t\right)}{D_o^{t+1}\left(x^t, y^t\right)}\right)\right]^{\frac{1}{2}} \tag{4.20}$$

Note that if $x^t = x^{t+1}$ and $y^t = y^{t+1}$ (i.e., there has been no change in inputs and outputs between the periods), the productivity index (4.18) signals no change: $M(\bullet) = 1$. In this case, the component measures of efficiency change and technical change are reciprocals, but not necessarily equal to one.

The decomposition is illustrated in Figure 4.3 for constant returns to scale technology. Technical advance has occurred in the sense that $T^t \subset T^{t+1}$. Note that $(x^t, y^t) \in T^t$ and $(x^{t+1}, y^{t+1}) \in T^{t+1}$; however $(x^{t+1}, y^{t+1}) \notin T^t$ (i.e., technical progress has occurred). In terms of the distances along the y-axis, the index (4.18) has becomes

$$M_o(x^t, y^t, x^{t+1}, y^{t+1}) = \left(\frac{0d}{0f}\right)\left(\frac{0b}{0a}\right)\left[\left(\frac{0d/0e}{0d/0f}\right)\left(\frac{0a/0b}{0a/0c}\right)\right]^{\frac{1}{2}} \quad (4.21)$$

$$= \left(\frac{0d}{0f}\right)\left(\frac{0b}{0a}\right)\left[\left(\frac{0f}{0e}\right)\left(\frac{0c}{0b}\right)\right]^{\frac{1}{2}} \quad (4.22)$$

The last expression shows that the ratios inside the brackets measure shifts in technology at input level x^t and x^{t+1}, respectively; thus technical change is measured as the geometric mean of those two shifts. The terms outside the brackets measure relative technical at t and $t+1$, capturing changes in relative efficiency over time, that is, whether production is getting closer (catching up) or farther from the frontier.

According to Fare et al. (1994a), improvements in productivity yield Malmquist index values greater than unity. Deterioration in performance over time is associated with a Malmquist index less than unity. The same interpretation applies to the values taken by the components of the overall TFP index. Improvement in the efficiency component yielded index values greater than one and is considered to be evidence of catching up (to the frontier). Values of the technical change component greater than one are considered to be evidence of technological progress.

4.4 DATA ENVELOPMENT ANALYSIS (DEA)

The Malmquist index can be calculated in several ways[8]. This study used the DEA approach outlined by Fare et al. (1994a) to construct the best-practice frontier at each time period for the five mobile service providers. Comparing each operator to the best-practice frontier gives a measure of its catching up in efficiency to that frontier and a measure of shift in the frontier (or innovation in technology). Then, the Malmquist indexes, which measure the changes in TFP, are calculated as a product of these two components.

Assume that there are $k = 1,\dots, K$ firms that produce $m = 1,\dots, M$ outputs $y^t_{k,m}$ using $n = 1,\dots, N$ inputs $x^t_{k,n}$ at each time period $t = 1, \dots, T$. Under DEA, the reference technology with constant returns to scale (CRS) at each time period t from the data can be defined as

$$G^t = \Big[(x^t, y^t) : y^t_m \le \sum_{k=1}^{K} z^t_k y^t_{k,m} \qquad m = 1,\dots, M,$$

$$\sum_{k=1}^{K} z^t_k x^t_{k,n} \le x^t_n \qquad n = 1,\dots, N, \qquad (4.23)$$

$$z^t_k \ge 0 \qquad k = 1,\dots, K \Big],$$

where z^t_k refers to weight on each specific cross-sectional observation. Following Afriat (1972), the assumption of constant returns to scale may be relaxed to allow variable returns to scales by adding the following restriction:

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[8] There are four different approaches: 1. Data Envelopment Analysis 2. Stochastic Activity Analysis 3. Aigner-Chu and 4. Stochastic Production Frontiers. Refer Fare et al. (1998) for more details.

$$\sum_{k=1}^{K} z_k^t = 1 \qquad (VRS) \tag{4.24}$$

Following Fare et al. (1994a), this study used an enhanced decomposition of the Malmquist index by decomposing the efficiency change component calculated relative to the constant returns to scale technology into a pure efficiency change component (calculated relative to the VRS technology) and a scale change component which captures changes in the deviation between the VRS and CRS technology.

To construct the Malmquist productivity index of firm k' between t and $t+1$, the following four distance functions are calculated using DEA approach: $D_o^t(x^t, y^t)$, $D_o^{t+1}(x^t, y^t)$, $D_o^t(x^{t+1}, y^{t+1}), D_o^{t+1}(x^{t+1}, y^{t+1})$. These distance functions are the reciprocals of the output-based Farrell's measure of technical efficiency. The non-parametric programming models used to calculate the output-based Farrell measure of technical efficiency for each firm $k' = 1, \ldots, K$, is expressed as

$$\left[D_o^t(x_{k'}^t, y_{k'}^t)\right]^{-1} = \max \lambda^{k'} \tag{4.25}$$

subject to

$$\lambda^{k'} y_{k,m}^t \leq \sum_{k=1}^{K} z_k^t y_{k,m}^t \qquad m = 1, \ldots, M,$$

$$\sum_{k=1}^{K} z_k^t x_{k,n}^t \leq x_{k',n}^t \qquad n = 1,\dots, N, \qquad (4.26)$$

$$\sum_{k=1}^{K} z_k^t = 1 \qquad (VRS)$$

$$z_k^t \geq 0 \qquad k = 1,\dots, K].$$

The computation of $D_o^{t+1}(x^{t+1}, y^{t+1})$ is similar to (4.26), where $t+1$ is substituted for t.

Construction of the Malmquist index also requires calculation of two mixed-distance functions, which is computed by comparing observations in one time period with the best practice frontier of another time period. The inverse of the mixed-distance function for observation k' can be obtained from

$$\left[D_o^t\left(x_{k'}^{t+1}, y_{k'}^{t+1}\right)\right]^{-1} = \max \lambda^{k'} \qquad (4.27)$$

subject to

$$\lambda^{k'} y_{k',m}^{t+1} \leq \sum_{k=1}^{K} z_k^t y_{k,m}^t \qquad m = 1,\dots M,$$

$$\sum_{k=1}^{K} z_k^t x_{k,n}^t \leq x_{k',n}^{t+1} \qquad n = 1,\dots, N, \qquad (4.28)$$

$$\sum_{k=1}^{K} z_k^t = 1 \qquad (VRS)$$

$$z_k^t \geq 0 \qquad k = 1,\dots, K].$$

To measure changes in scale efficiency, the inverse output distance functions under the VRS technology are also calculated by adding (4.24) into the constraints in (4.26) and (4.28). Technical change is calculated relative to the CRS technology. Scale efficiency change in each time period is constructed as the ratio of the distance function satisfying CRS to the distance function under VRS, while the pure efficiency change is defined as the ratio of the own-period distance functions in each period under VRS. With these two distance functions with respect to the VRS technology, the decomposition of (4.18) becomes[9]

$$M_o\left(x^t, y^t, x^{t+1}, y^{t+1}\right) = \left(\frac{D_o^{t+1}\left(x^t, y^t\right)}{D_o^t\left(x^t, y^t\right)}\right)\left(\frac{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}{D_o^t\left(x^{t+1}, y^{t+1}\right)}\right)^{\frac{1}{2}} \times \left(\frac{D_o^t\left(x^t, y^t\right)}{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}\right) \times \left(\frac{D_{oc}^{t+1}\left(x^t, y^t\right)}{D_o^{t+1}\left(x^t, y^t\right)} \frac{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}{D_{oc}^{t+1}\left(x^{t+1}, y^{t+1}\right)} \frac{D_{oc}^t\left(x^t, y^t\right)}{D_o^t\left(x^t, y^t\right)} \frac{D_o^t\left(x^{t+1}, y^{t+1}\right)}{D_{oc}^t\left(x^{t+1}, y^{t+1}\right)}\right)^{\frac{1}{2}} \tag{4.29}$$

where

$$\left(\frac{D_o^{t+1}\left(x^t, y^t\right)}{D_o^t\left(x^t, y^t\right)}\right)\left(\frac{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}{D_o^t\left(x^{t+1}, y^{t+1}\right)}\right)^{\frac{1}{2}} = \text{Technical Change}$$

$$\left(\frac{D_o^t\left(x^t, y^t\right)}{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}\right) = \text{Pure Efficiency Change}$$

$$\left(\frac{D_{oc}^{t+1}\left(x^t, y^t\right)}{D_o^{t+1}\left(x^t, y^t\right)} \frac{D_o^{t+1}\left(x^{t+1}, y^{t+1}\right)}{D_{oc}^{t+1}\left(x^{t+1}, y^{t+1}\right)} \frac{D_{oc}^t\left(x^t, y^t\right)}{D_o^t\left(x^t, y^t\right)} \frac{D_o^t\left(x^{t+1}, y^{t+1}\right)}{D_{oc}^t\left(x^{t+1}, y^{t+1}\right)}\right)^{\frac{1}{2}} = \text{Scale Efficiency Change}$$

Note that when the technology in fact exhibits CRS, the scale change factor equals to one and it is the same decomposition as (4.18).

[9] Subscript c under the distance function denotes that it is measured with reference to CRS technology while unsubscripted distance functions are with reference to VRS technology.